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ORIGINAL PAPER

Improvements on the infinity norm bound for the inverse of Nekrasov matrices

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Abstract New bounds for the infinity norm of the inverse of Nekrasov matrices, which involve a parameter, are given. And then we determine the optimal value of the parameter such that the new bounds are better than those in Cvetković et al. (Appl. Math. Comput. **219**, 5020–5024, 2013). Numerical examples are given to illustrate the corresponding results.

Keywords Infinity norm · Nekrasov matrices · H-matrices

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1 Introduction

A matrix $A = [a_{ij}] \in C^{n,n}$ is called an H-matrix if its comparison matrix $< A >= [m_{ij}]$ defined by

$$< A >= [m_{ij}] \in C^{n,n}, m_{ij} = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \neq j, \end{cases}$$

is an M-matrix, i.e., $< A >^{-1} \geq 0$, [1, 3, 4]. H-matrices are widely used in many subjects such as computational mathematics, mathematical physics, economics and dynamical system theory [10]. A special interesting problem among them is to find upper bounds of the infinity norm of H-matrices, since it can be used to prove the convergence of matrix splitting and matrix multi-splitting iterative methods for solving large sparse systems of linear equations, see [1, 5–7]. Many researchers have obtained some bounds. In 1975, J.M. Varah [11] provided the following upper bound for strictly diagonally dominant (SDD) matrices as one of the most important subclass of H-matrices. Here a matrix $A = [a_{ij}] \in C^{n,n}$ is called SDD if for each $i \in N = \{1, 2, \ldots, n\}$,

$$|a_{ii}| > r_i(A),$$

where $r_i(A) = \sum\limits_{j \neq i} |a_{ij}|$.

Theorem 1 [11] *Let $A = [a_{ij}] \in C^{n,n}$ be SDD. Then*

$$||A^{-1}||_\infty \leq \frac{1}{\min\limits_{i \in N}(|a_{ii}| - r_i(A))}.$$

We call the bound in Theorem 1 the Varah's bound. As Cvetković et al. [5] pointed out, the Varah's bound works only for SDD matrices, and even then it is not always good enough. Hence, it can be useful to obtain new upper bounds for a wider class of matrices which sometimes are tighter in the SDD case. In 2013, Cvetković et al. [5] study the class of Nekrasov matrices which contains SDD matrices and is a subclass of H-matrices, and give the following bounds by applying the Varah's bound to the matrix $C = I - (|D| - |L|)^{-1}|U|$, where D is the diagonal part, $-L$ is the strict lower triangular part, and $-U$ is the strict upper triangular part of a Nekrasov matrix.

Definition 1 [4, 5] A matrix $A = [a_{ij}] \in C^{n,n}$ is called a Nekrasov matrix if for each $i \in N$,

$$|a_{ii}| > h_i(A),$$

where $h_1(A) = r_1(A) = \sum\limits_{j \neq 1} |a_{1j}|$ and $h_i(A) = \sum\limits_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j(A) + \sum\limits_{j=i+1}^{n} |a_{ij}|$, $i = 2, 3, \ldots, n$.

Theorem 2 [5, Theorem 2] *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix. Then*

$$||A^{-1}||_\infty \leq \frac{\max\limits_{i\in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max\limits_{i\in N} \frac{h_i(A)}{|a_{ii}|}}, \tag{1}$$

and

$$||A^{-1}||_\infty \leq \frac{\max\limits_{i\in N} z_i(A)}{\min\limits_{i\in N}(|a_{ii}| - h_i(A))}, \tag{2}$$

where $z_1(A) = 1$ and $z_i(A) = \sum\limits_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A) + 1$, $i = 2, 3, \ldots, n$.

Since a SDD matrix is a Nekrasov matrix [4, 8], the bounds (1) and (2) can be also applied to SDD matrices. However, the Varah's bound cannot be used to estimate the infinity norm of the inverse of Nekrasov matrices. Furthermore bounds (1) or (2) are tighter than Varah's bound when $\min\limits_{i\in N}(|a_{ii}| - r_i(A))$ is very small for the SDD matrix A (for details, see [5]).

As shown in [5], each bound (1) or (2) can work better than the other one. So, in general case, for Nekrasov matrices, one can take the smallest estimation of these two.

To estimate the infinity norm of the inverse of Nekrasov matrices more precisely, we in this paper give new bounds which involve a parameter μ based on the bounds in Theorem 2, and then determine the optimal value of μ such that the new bounds are better than bounds (1) or (2) in Theorem 2 (Theorem 2 in [5]). Numerical examples are given to illustrate the corresponding results.

2 New bounds for the infinity norm of the inverse of Nekrasov matrices

First, some lemmas and notation are listed. Given a matrix $A = [a_{ij}]$, by $A = D - L - U$ we denote the standard splitting of A into its diagonal (D), strictly lower ($-L$) and strictly upper ($-U$) triangular parts. And by $[A]_{ij}$ we denote the (i, j)-entry of A, that is, $[A]_{ij} = a_{ij}$.

Lemma 1 [2] *Let $A = [a_{ij}] \in C^{n,n}$ be a nonsingular H-matrix. Then*

$$|A^{-1}| \leq \ < A >^{-1} .$$

Lemma 2 [9] *Given any matrix $A = [a_{ij}] \in C^{n,n}$, $n \geq 2$, with $a_{ii} \neq 0$ for all $i \in N$, then*

$$h_i(A) = |a_{ii}| \left[(|D| - |L|)^{-1} |U| e \right]_i ,$$

where $e \in C^{n,n}$ is the vector with all components equal to 1.

Lemma 3 [9] *A matrix $A = [a_{ij}] \in C^{n,n}$, $n \geq 2$ is a Nekrasov matrix if and only if*

$$(|D| - |L|)^{-1}|U|e < e,$$

i.e., then $I - (|D| - |L|)^{-1}|U|$ is a SDD matrix, where I is the identity matrix.

Let

$$C = I - (|D| - |L|)^{-1}|U| = [c_{ij}]$$

and

$$B = |D|C = |D| - |D|(|D| - |L|)^{-1}|U| = [b_{ij}],$$

and then from Lemma 3, B and C are SDD when A is a Nekrasov matrix. Note that $c_{11} = 1$, $c_{k1} = 0$, $k = 2, 3, \ldots, n$, and $c_{1k} = -\frac{|a_{1k}|}{|a_{11}|}$, $k = 2, 3, \ldots, n$, and that $b_{11} = |a_{11}|$, $b_{k1} = 0$, $k = 2, 3, \ldots, n$, and $b_{1k} = -|a_{1k}|$, $k = 2, 3, \ldots, n$, which lead to the following lemma.

Lemma 4 *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix, and let $\mu > \frac{r_1(A)}{|a_{11}|}$. Then the matrices*

$$C(\mu) = CD(\mu) = \left(I - (|D| - |L|)^{-1}|U|\right) D(\mu), \tag{3}$$

and

$$B(\mu) = BD(\mu) = \left(|D| - |D|(|D| - |L|)^{-1}|U|\right) D(\mu) \tag{4}$$

are SDD, where $D(\mu) = diag(\mu, 1, \cdots, 1)$. In addition,

$$||C(\mu)^{-1}||_\infty \leq \max\left\{\frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max\limits_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}\right\}, \tag{5}$$

and

$$||B(\mu)^{-1}||_\infty \leq \frac{1}{\min\left\{\mu|a_{11}| - h_1(A), \min\limits_{i \neq 1}(|a_{ii}| - h_i(A))\right\}}. \tag{6}$$

Proof It is not difficult from (3) to see that $[C(\mu)]_{k1} = \mu c_{k1}$ for all $k \in N$ and $[C(\mu)]_{kj} = c_{kj}$ for all $k \in N$ and $j \neq 1$. Hence

$$[C(\mu)]_{11} = \mu, \ r_1(C(\mu)) = r_1(C) = \frac{r_1(A)}{|a_{11}|}$$

and for $i = 2, \ldots, n$,

$$[C(\mu)]_{ii} = c_{ii}, \ r_i(C(\mu)) = r_i(C).$$

Since C is SDD and $\mu > \frac{r_1(A)}{|a_{11}|}$, we have that $C(\mu)$ is SDD.

Moreover, by applying the Varah's bound to estimate the infinity norm of $C(\mu)^{-1}$, we obtain

$$||C(\mu)^{-1}||_\infty \leq \max_{i \in N} \frac{1}{|[C(\mu)]_{ii}| - r_i(C(\mu))} = \max\left\{\frac{1}{\mu - r_1(C)}, \max_{i \neq 1} \frac{1}{|c_{ii}| - r_i(C)}\right\}.$$

Note that $C = I - (|D| - |L|)^{-1}|U| = [c_{ij}]$ and all diagonal entries of matrix $(|D| - |L|)^{-1}|U|$ are less than 1. Then we have that for $i \in N,\ i \neq 1$,

$$|c_{ii}| = 1 - \left[(|D| - |L|)^{-1}|U|\right]_{ii}$$

and that for each $i \in N$,

$$r_i(C) = \sum_{k \neq i} \left[(|D| - |L|)^{-1}|U|\right]_{ik}.$$

Then (also see the proof of Theorem 2 in [5]) for $i \in N,\ i \neq 1$,

$$|c_{ii}| - r_i(C) = 1 - \sum_{k \in N} \left[(|D| - |L|)^{-1}|U|\right]_{ik} = 1 - \left[(|D| - |L|)^{-1}|U|e\right]_i = 1 - \frac{h_i(A)}{|a_{ii}|}.$$

Since $r_1(C) = \frac{r_1(A)}{|a_{11}|} = \frac{h_1(A)}{|a_{11}|}$, we have

$$\begin{aligned} ||C(\mu)^{-1}||_\infty &\leq \max\left\{\frac{1}{\mu - r_1(C)}, \max_{i \neq 1} \frac{1}{|c_{ii}| - r_i(C)}\right\} \\ &= \max\left\{\frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \max_{i \neq 1} \frac{1}{1 - \frac{h_i(A)}{|a_{ii}|}}\right\} \\ &= \max\left\{\frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max\limits_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}\right\}. \end{aligned}$$

Inequality (6) can be proved analogously. □

Now, we give the main result of this paper.

Theorem 3 *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix. Then for $\mu > \frac{r_1(A)}{|a_{11}|}$,*

$$||A^{-1}||_\infty \leq \max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max\left\{\frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max\limits_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}}\right\}, \tag{7}$$

and

$$||A^{-1}||_\infty \leq \frac{\max\{\mu, 1\} \max\limits_{i \in n} z_i(A)}{\min\left\{\mu|a_{11}| - h_1(A), \min\limits_{i \neq 1}(|a_{ii}| - h_i(A))\right\}}, \tag{8}$$

where $z_i(A)$ is defined in Theorem 2.

Proof We only prove that (7) holds, and in an analogous way, (8) is proved easily. Let $C(\mu) = CD(\mu) = \left(I - (|D| - |L|)^{-1}|U|\right) D(\mu)$, where $D(\mu) = diag(\mu, 1, \cdots, 1)$. From (3), we have

$$C(\mu) = \left(I - (|D| - |L|)^{-1}|U|\right) D(\mu) = (|D| - |L|)^{-1} < A > D(\mu),$$

which implies that

$$< A >= (|D| - |L|)C(\mu)D(\mu)^{-1}.$$

Furthermore, since a Nekrasov matrix is an H-matrix, we have from Lemma 1,

$$||A^{-1}||_\infty \leq || < A >^{-1} ||_\infty \leq ||D(\mu)||_\infty ||C(\mu)^{-1}||_\infty ||(|D| - |L|)^{-1}||_\infty. \quad (9)$$

Note that $|D| - |L|$ is an M-matrix, and then similar to the proof of Theorem 2 in [5], we easily obtain

$$||(|D| - |L|)^{-1}||_\infty = ||y||_\infty = \max_{i \in n} \frac{z_i(A)}{|a_{ii}|}, \quad (10)$$

where $y = (|D| - |L|)^{-1}e = [y_1, y_2, \ldots, y_n]^T$ and $z_i(A) = |a_{ii}|y_i$, i.e.,

$$z_1(A) = 1, \ and \ z_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A) + 1, i = 2, \ldots, n.$$

From (5), (9), (10) and the fact that $||D(\mu)||_\infty = \max\{\mu, 1\}$, we obtain

$$||A^{-1}||_\infty \leq \max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\}.$$

The conclusions follows. □

Example 1 Consider the Nekrasov matrix A_1 in [5], where

$$A_1 = \begin{bmatrix} -7 & 1 & -0.2 & 2 \\ 7 & 88 & 2 & -3 \\ 2 & 0.5 & 13 & -2 \\ 0.5 & 3.0 & 1 & 6 \end{bmatrix}.$$

By computation, $h_1(A) = 3.2000$, $h_2(A) = 8.2000$, $h_3(A) = 2.9609$, $h_4(A) = 0.7359$, $z_1(A) = 1$, $z_2(A) = 2$, $z_3(A) = 1.2971$ and $z_4(A) = 1.2394$, and $||A_1^{-1}||_\infty = 0.1921$. By Theorem 2 (Theorem 2 in [5]), we have

$$||A_1^{-1}||_\infty \leq 0.3805, \ (The\ bound\ (1)\ of\ Theorem\ 2)$$

and

$$||A_1^{-1}||_\infty \leq 0.5263. \ (The\ bound\ (2)\ of\ Theorem\ 2)$$

By Theorem 3, we have

	μ=0.5	μ=0.8	μ=1.1	μ=1.4	μ=1.7
bound(7)	4.8198	0.6025	0.3535	0.3745	0.4547
	μ=0.6	μ=0.9	μ=1.2	μ=1.5	μ=1.8
bound(8)	2.0000	0.6452	0.4615	0.5699	0.6839

Remark 1 Example 1 shows that for some values of μ, bounds (7) and (8) of Theorem 3 are better than bounds (1) and (2) of Theorem 2 respectively. Figures 1 and 2 show that there is an interval such that for any μ in this interval, the bound (7) ((8), resp.) of Theorem 3 for the matrix A_1 is always smaller than the bound (1) ((2), resp.) of Theorem 2.

An interesting problem arises: whether there is an interval of μ such that the bound (7) ((8), resp.) of Theorem 3 for any Nekrasov matrix is smaller than the bound (1) ((2), resp.) of Theorem 2? In the following section, we will study this problem.

3 Optimal values of the parameter μ

In this section, we determine the values of μ in the bounds (7) and (8) such that the bounds (7) and (8) for $||A^{-1}||_\infty$ are less or equal to the bounds (1) and (2), respectively.

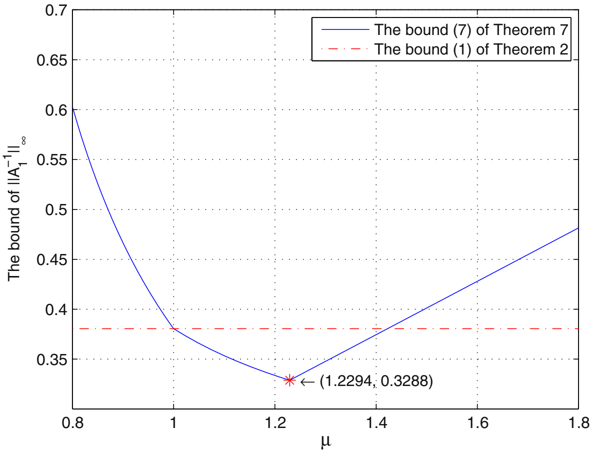

Fig. 1 The bounds (1) and (7)

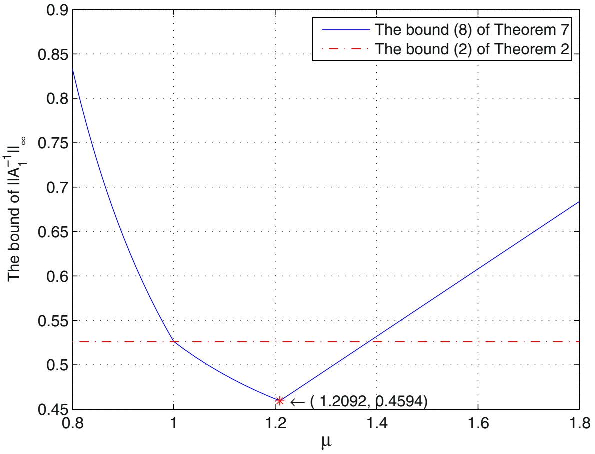

Fig. 2 The bounds (2) and (8)

3.1 Optimal value of μ for the bound (7)

We distinguish two cases:

$$\frac{h_1(A)}{|a_{11}|} > \max_{i\neq 1} \frac{h_i(A)}{|a_{ii}|}, \ \textit{and}\ \frac{h_1(A)}{|a_{11}|} \leq \max_{i\neq 1} \frac{h_i(A)}{|a_{ii}|}$$

to determine the optimal value for the bound (7).

Lemma 5 *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix with*

$$\frac{h_1(A)}{|a_{11}|} > \max_{i\neq 1} \frac{h_i(A)}{|a_{ii}|}. \tag{11}$$

Then

$$1 < \eta_1 < \eta_2, \tag{12}$$

where $\eta_1 = 1 + \frac{h_1(A)}{|a_{11}|} - \max\limits_{i\neq 1} \frac{h_i(A)}{|a_{ii}|}$, *and* $\eta_2 = \dfrac{1-\max\limits_{i\neq 1} \frac{h_i(A)}{|a_{ii}|}}{1-\frac{h_1(A)}{|a_{11}|}}$.

Proof Obviously, the first inequality in (12) holds. We only prove that the second holds. From Inequality (11), we have that

$$\frac{h_1(A)}{|a_{11}|} \max_{i\neq 1} \frac{h_i(A)}{|a_{ii}|} - \left(\frac{h_1(A)}{|a_{11}|}\right)^2 < 0.$$

Equivalently,

$$1-\max_{i\neq 1} \frac{h_i(A)}{|a_{ii}|} + \frac{h_1(A)}{|a_{11}|} - \frac{h_1(A)}{|a_{11}|} + \frac{h_1(A)}{|a_{11}|} \max_{i\neq 1} \frac{h_i(A)}{|a_{ii}|} - \left(\frac{h_1(A)}{|a_{11}|}\right)^2 < 1-\max_{i\neq 1} \frac{h_i(A)}{|a_{ii}|},$$

i.e.,

$$\left(1-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}+\frac{h_1(A)}{|a_{11}|}\right)\left(1-\frac{h_1(A)}{|a_{11}|}\right)<1-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}.$$

Note that $1-\frac{h_1(A)}{|a_{11}|}>0$, then

$$1-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}+\frac{h_1(A)}{|a_{11}|}<\frac{1-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}{1-\frac{h_1(A)}{|a_{11}|}},$$

that is, $\eta_1<\eta_2$. The conclusion follows. □

We now give an interval of μ such that the bound (7) of Theorem 3 is less than the bound (1) of Theorem 2.

Lemma 6 *Let $A=[a_{ij}]\in C^{n,n}$ be a Nekrasov matrix with*

$$\frac{h_1(A)}{|a_{11}|}>\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}.$$

Then for each $\mu\in(1,\eta_2)$,

$$\begin{aligned}||A^{-1}||_\infty &\le \max\{\mu,1\}\max_{i\in N}\frac{z_i(A)}{|a_{ii}|}\max\left\{\frac{1}{\mu-\frac{h_1(A)}{|a_{11}|}},\frac{1}{1-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}\right\}\\ &<\frac{\max_{i\in N}\frac{z_i(A)}{|a_{ii}|}}{1-\max_{i\in N}\frac{h_i(A)}{|a_{ii}|}}.\end{aligned}$$

Proof From Lemma 5, we have $\mu\in(1,\eta_1]\bigcup[\eta_1,\eta_2)$, and $\max\{\mu,1\}=\mu$.

(I) For $\mu\in(1,\eta_1]$, then

$$\mu-\frac{h_1(A)}{|a_{11}|}\le 1-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|},$$

that is,

$$\frac{1}{\mu-\frac{h_1(A)}{|a_{11}|}}\ge\frac{1}{1-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}.$$

Therefore,

$$\max\{\mu,1\}\max\left\{\frac{1}{\mu-\frac{h_1(A)}{|a_{11}|}},\frac{1}{1-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}\right\}=\frac{\mu}{\mu-\frac{h_1(A)}{|a_{11}|}}.$$

Consider the function $f(x)=\frac{x}{x-\frac{h_1(A)}{|a_{11}|}}$, $x\in[1,\eta_1]$. It is easy from $\frac{h_1(A)}{|a_{11}|}<1$ to prove that $f(x)$ is a monotonically decreasing function of x. Hence, for any $\mu\in(1,\eta_1]$,

$$f(\mu)<f(1),$$

i.e.,

$$\frac{\mu}{\mu-\frac{h_1(A)}{|a_{11}|}} < \frac{1}{1-\frac{h_1(A)}{|a_{11}|}} = \frac{1}{1-\max\limits_{i\in N}\frac{h_i(A)}{|a_{ii}|}},$$

which implies that

$$\frac{\mu\max\limits_{i\in N}\frac{z_i(A)}{|a_{ii}|}}{\mu-\frac{h_1(A)}{|a_{11}|}} < \frac{\max\limits_{i\in N}\frac{z_i(A)}{|a_{ii}|}}{1-\max\limits_{i\in N}\frac{h_i(A)}{|a_{ii}|}}.$$

Hence,

$$\max\{\mu,1\}\max_{i\in N}\frac{z_i(A)}{|a_{ii}|}\max\left\{\frac{1}{\mu-\frac{h_1(A)}{|a_{11}|}},\frac{1}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}\right\} < \frac{\max\limits_{i\in N}\frac{z_i(A)}{|a_{ii}|}}{1-\max\limits_{i\in N}\frac{h_i(A)}{|a_{ii}|}}.$$

(II) For $\mu\in[\eta_1,\eta_2)$, then

$$\mu-\frac{h_1(A)}{|a_{11}|}\geq 1-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|},$$

that is,

$$\frac{1}{\mu-\frac{h_1(A)}{|a_{11}|}} \leq \frac{1}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}.$$

Therefore,

$$\max\{\mu,1\}\max\left\{\frac{1}{\mu-\frac{h_1(A)}{|a_{11}|}},\frac{1}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}\right\} = \frac{\mu}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}.$$

Consider the function $g(x)=\frac{x}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}},\ x\in[\eta_1,\eta_2]$. Obviously, $g(x)$ is a monotonically increasing function of x. Hence, for any $\mu\in[\eta_1,\eta_2)$,

$$g(\mu) < g\left(\frac{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}{1-\frac{h_1(A)}{|a_{11}|}}\right),$$

that is,

$$\frac{\mu}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}} < \frac{1}{1-\frac{h_1(A)}{|a_{11}|}} = \frac{1}{1-\max\limits_{i\in N}\frac{h_i(A)}{|a_{ii}|}},$$

which implies that

$$\frac{\mu\max\limits_{i\in N}\frac{z_i(A)}{|a_{ii}|}}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}} < \frac{\max\limits_{i\in N}\frac{z_i(A)}{|a_{ii}|}}{1-\max\limits_{i\in N}\frac{h_i(A)}{|a_{ii}|}}.$$

Hence,

$$\max\{\mu, 1\} \max_{i\in N} \frac{z_i(A)}{|a_{ii}|} \max\left\{\frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}\right\} < \frac{\max\limits_{i\in N}\frac{z_i(A)}{|a_{ii}|}}{1-\max\limits_{i\in N}\frac{h_i(A)}{|a_{ii}|}}.$$

The conclusion follows from (I) and (II).

□

Lemma 6 provides an interval of μ such that the bound (7) in Theorem 3 is better than the bound (1) in Theorem 2. Moreover, we can determine the optimal value of μ by the following theorem.

Theorem 4 *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix with*

$$\frac{h_1(A)}{|a_{11}|} > \max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}.$$

Then

$$\min_{\mu\in(1,\eta_2)}\left\{\max\{\mu, 1\}\max\left\{\frac{1}{\mu-\frac{h_1(A)}{|a_{11}|}}, \frac{1}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}\right\}\right\} = \frac{1+\frac{h_1(A)}{|a_{11}|}-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}. \quad (13)$$

Furthermore,

$$||A^{-1}||_\infty \le \frac{\max\limits_{i\in N}\frac{z_i(A)}{|a_{ii}|}\left(1+\frac{h_1(A)}{|a_{11}|}-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}\right)}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}} < \frac{\max\limits_{i\in N}\frac{z_i(A)}{|a_{ii}|}}{1-\max\limits_{i\in N}\frac{h_i(A)}{|a_{ii}|}}. \quad (14)$$

Proof From the proof of Lemma 6, we have that

$$f(x) = \frac{x}{x-\frac{h_1(A)}{|a_{11}|}},\ x\in[1,\eta_1]$$

is decreasing, and that

$$g(x) = \frac{x}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}},\ x\in[\eta_1,\eta_2]$$

is increasing. Therefore, the minimum of $f(x)$, which is equal to that of $g(x)$, is

$$\begin{aligned} f\left(1+\frac{h_1(A)}{|a_{11}|}-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}\right) &= g\left(1+\frac{h_1(A)}{|a_{11}|}-\max_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}\right) \\ &= \frac{1+\frac{h_1(A)}{|a_{11}|}-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}{1-\max\limits_{i\neq 1}\frac{h_i(A)}{|a_{ii}|}}, \end{aligned}$$

which implies that (13) holds. Again by Lemma 6, (14) follows easily. □

Remark 2 Theorem 4 provides a method to determine the optimal value of μ for a Nekrasov matrix $A = [a_{ij}] \in C^{n,n}$ with

$$\frac{h_1(A)}{|a_{11}|} > \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}.$$

Also consider the matrix A_1. By computation, we get

$$\frac{h_1(A_1)}{|a_{11}|} = 0.4571 > 0.2278 = \max_{i \neq 1} \frac{h_i(A_1)}{|a_{ii}|}.$$

Hence, by Theorem 4, we can obtain that the bound (7) in Theorem 3 reaches its minimum

$$\frac{\max\limits_{i \in N} \frac{z_i(A_1)}{|a_{ii}|} \left(1 + \frac{h_1(A_1)}{|a_{11}|} - \max\limits_{i \neq 1} \frac{h_i(A_1)}{|a_{ii}|}\right)}{1 - \max\limits_{i \neq 1} \frac{h_i(A_1)}{|a_{ii}|}} = 0.3288$$

at $\mu = \eta_1 = 1.2294$ (also see Fig. 1).

Next, we study the bound in Theorem 3 for a Nekrasov matrix $A = [a_{ij}] \in C^{n,n}$ with

$$\frac{h_1(A)}{|a_{11}|} \leq \max_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}. \tag{15}$$

Theorem 5 *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix with (15) holds. Then we can take $\mu = \eta_1$ such that*

$$\begin{aligned} ||A^{-1}||_\infty &\leq \max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max\limits_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} \\ &= \frac{\max\limits_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max\limits_{i \in N} \frac{h_i(A)}{|a_{ii}|}}. \end{aligned}$$

Proof Since $\frac{h_1(A)}{|a_{11}|} \leq \max\limits_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}$, we have $\mu = \eta_1 \leq 1$, $\max\{\mu, 1\} = 1$ and

$$\max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max\limits_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} = \frac{1}{1 - \max\limits_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} = \frac{1}{1 - \max\limits_{i \in N} \frac{h_i(A)}{|a_{ii}|}}.$$

Hence,

$$\max\{\mu, 1\} \max_{i \in N} \frac{z_i(A)}{|a_{ii}|} \max \left\{ \frac{1}{\mu - \frac{h_1(A)}{|a_{11}|}}, \frac{1}{1 - \max\limits_{i \neq 1} \frac{h_i(A)}{|a_{ii}|}} \right\} = \frac{\max\limits_{i \in N} \frac{z_i(A)}{|a_{ii}|}}{1 - \max\limits_{i \in N} \frac{h_i(A)}{|a_{ii}|}}.$$

The proof is completed. □

3.2 Optimal value of μ for the bound (8)

We also distinguish two cases:

$$|a_{11}| - h_1(A) < \min_{i\neq 1}(|a_{ii}| - h_i(A)),\ and\ |a_{11}| - h_1(A) \geq \min_{i\neq 1}(|a_{ii}| - h_i(A))$$

to determine the optimal value for the bound (7). Before that we give a lemma which is proved easily.

Lemma 7 *Let a, b and c be positive real numbers, and $0 < a - b < c$. Then*

$$\frac{b+c}{a} < \frac{c}{a-b}.$$

Lemma 8 *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix with*

$$|a_{11}| - h_1(A) < \min_{i\neq 1}(|a_{ii}| - h_i(A)).$$

Then

$$1 < \eta_3 < \eta_4, \tag{16}$$

where $\eta_3 = \frac{\min\limits_{i\neq 1}(|a_{ii}|-h_i(A))+h_1(A)}{|a_{11}|}$, *and* $\eta_4 = \frac{\min\limits_{i\neq 1}(|a_{ii}|-h_i(A))}{|a_{11}|-h_1(A)}$.

Proof Since A is a Nekrasov matrix, we have $|a_{11}| - h_1(A) > 0$, consequently, the first inequality in (16) holds. Moreover, Let $a = |a_{11}|$, $b = h_1(A)$ and $c = \min\limits_{i\neq 1}(|a_{ii}| - h_i(A))$. Then from Lemma 7, the second holds. □

We now give an interval of μ such that the bound (8) of Theorem 3 is less than the bound (2) of Theorem 2.

Lemma 9 *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix with*

$$|a_{11}| - h_1(A) < \min_{i\neq 1}(|a_{ii}| - h_i(A)).$$

Then for each $\mu \in (1, \eta_4)$,

$$\begin{aligned} ||A^{-1}||_\infty &\leq \frac{\max\{\mu, 1\}\max\limits_{i\in n} z_i(A)}{\min\left\{\mu|a_{11}| - h_1(A), \min\limits_{i\neq 1}(|a_{ii}| - h_i(A))\right\}} \\ &< \frac{\max\limits_{i\in N} z_i(A)}{\min\limits_{i\in N}(|a_{ii}| - h_i(A))}. \end{aligned}$$

Proof From Lemma 8, we have $\mu \in (1, \eta_3] \bigcup [\eta_3, \eta_4)$, and $\max\{\mu, 1\} = \mu$.

(I) For $\mu \in (1, \eta_3]$, then

$$\mu|a_{11}| \leq \min_{i\neq 1}(|a_{ii}| - h_i(A)) + h_1(A),$$

that is,

$$\mu|a_{11}| - h_1(A) \leq \min_{i \neq 1}(|a_{ii}| - h_i(A)).$$

Therefore,

$$\frac{\max\{\mu, 1\}}{\min\left\{\mu|a_{11}| - h_1(A), \min\limits_{i \neq 1}(|a_{ii}| - h_i(A))\right\}} = \frac{\mu}{\mu|a_{11}| - h_1(A)}.$$

Consider the function $f(x) = \frac{x}{|a_{11}|x - h_1(A)}$, $x \in [1, \eta_3]$. It is easy to prove that $f(x)$ is a monotonically decreasing function of x. Hence, for any $\mu \in (1, \eta_3]$,

$$f(\mu) < f(1),$$

i.e.,

$$\frac{\mu}{\mu|a_{11}| - h_1(A)} < \frac{1}{|a_{11}| - h_1(A)} = \frac{1}{\min\limits_{i \in N}(|a_{ii}| - h_i(A))},$$

which implies that

$$\frac{\mu \max\limits_{i \in N} z_i(A)}{\mu|a_{11}| - h_1(A)} < \frac{\max\limits_{i \in N} z_i(A)}{\min\limits_{i \in N}(|a_{ii}| - h_i(A))}.$$

Hence,

$$\frac{\max\{\mu, 1\} \max\limits_{i \in n} z_i(A)}{\min\left\{\mu|a_{11}| - h_1(A), \min\limits_{i \neq 1}(|a_{ii}| - h_i(A))\right\}} < \frac{\max\limits_{i \in N} z_i(A)}{\min\limits_{i \in N}(|a_{ii}| - h_i(A))}.$$

(II) For $\mu \in [\eta_3, \eta_4)$, then

$$\mu|a_{11}| \geq \min_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A),$$

that is,

$$\mu|a_{11}| - h_1(A) \geq \min_{i \neq 1}(|a_{ii}| - h_i(A)).$$

Therefore,

$$\frac{\max\{\mu, 1\}}{\min\left\{\mu|a_{11}| - h_1(A), \min\limits_{i \neq 1}(|a_{ii}| - h_i(A))\right\}} = \frac{\mu}{\min\limits_{i \neq 1}(|a_{ii}| - h_i(A))}.$$

Consider the function $g(x) = \frac{x}{\min\limits_{i \neq 1}(|a_{ii}| - h_i(A))}$, $x \in [\eta_3, \eta_4]$. Obviously, $g(x)$ is a monotonically increasing function of x. Hence, for any $\mu \in [\eta_3, \eta_4)$,

$$g(\mu) < g\left(\frac{\min\limits_{i \neq 1}(|a_{ii}| - h_i(A))}{|a_{11}| - h_1(A)}\right),$$

that is,

$$\frac{\mu}{\min\limits_{i \neq 1}(|a_{ii}| - h_i(A))} < \frac{1}{|a_{11}| - h_1(A)} = \frac{1}{\min\limits_{i \in N}(|a_{ii}| - h_i(A))},$$

which implies that

$$\frac{\mu \max\limits_{i\in N} z_i(A)}{\min\limits_{i\neq 1}(|a_{ii}| - h_i(A))} < \frac{\max\limits_{i\in N} z_i(A)}{\min\limits_{i\in N}(|a_{ii}| - h_i(A))}.$$

Hence,

$$\frac{\max\{\mu, 1\} \max\limits_{i\in n} z_i(A)}{\min\left\{\mu|a_{11}| - h_1(A), \min\limits_{i\neq 1}(|a_{ii}| - h_i(A))\right\}} < \frac{\max\limits_{i\in N} z_i(A)}{\min\limits_{i\in N}(|a_{ii}| - h_i(A))}.$$

The conclusion follows from (I) and (II).

□

Similar to the proof of Theorem 4, we can easily obtain the following theorem by Lemma 9.

Theorem 6 *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix with*

$$|a_{11}| - h_1(A) < \min_{i\neq 1}(|a_{ii}| - h_i(A)).$$

Then

$$\min_{\mu\in(1,\eta_4)} \left\{ \frac{\max\{\mu, 1\}}{\min\left\{\mu|a_{11}| - h_1(A), \min\limits_{i\neq 1}(|a_{ii}| - h_i(A))\right\}} \right\} = \frac{\min\limits_{i\neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}| \min\limits_{i\neq 1}(|a_{ii}| - h_i(A))}. \tag{17}$$

Furthermore,

$$||A^{-1}||_\infty \le \frac{\max\limits_{i\in N} z_i(A)\left(\min\limits_{i\neq 1}(|a_{ii}| - h_i(A)) + h_1(A)\right)}{|a_{11}| \min\limits_{i\neq 1}(|a_{ii}| - h_i(A))} < \frac{\max\limits_{i\in N} z_i(A)}{\min\limits_{i\in N}(|a_{ii}| - h_i(A))}. \tag{18}$$

Remark 3 Theorem 6 provides a method to determine the optimal value of μ for a Nekrasov matrix $A = [a_{ij}] \in C^{n,n}$ with

$$|a_{11}| - h_1(A) < \min_{i\neq 1}(|a_{ii}| - h_i(A)).$$

Also consider the matrix A_1. By computation, we get

$$|a_{11}| - h_1(A) = 3.8000 < 5.2641 = \min_{i\neq 1}(|a_{ii}| - h_i(A)).$$

Hence, by Theorem 6, we can obtain that the bound (8) in Theorem 3 reaches its minimum

$$\frac{\max\limits_{i\in N} z_i(A) \min\limits_{i\neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}| \min\limits_{i\neq 1}(|a_{ii}| - h_i(A))} = 0.4594$$

at $\mu = \eta_3 = 1.2092$ (also see Fig. 2).

Next, we study the bound (8) in Theorem 3 for a Nekrasov matrix $A = [a_{ij}] \in C^{n,n}$ with

$$|a_{11}| - h_1(A) \geq \min_{i\neq 1}(|a_{ii}| - h_i(A)).$$

Theorem 7 *Let $A = [a_{ij}] \in C^{n,n}$ be a Nekrasov matrix with*

$$|a_{11}| - h_1(A) \geq \min_{i\neq 1}(|a_{ii}| - h_i(A)).$$

Then we can take $\mu = \eta_3$ such that

$$\begin{aligned} ||A^{-1}||_\infty &\leq \frac{\max\{\mu, 1\} \max_{i\in n} z_i(A)}{\min\left\{\mu|a_{11}| - h_1(A), \min_{i\neq 1}(|a_{ii}| - h_i(A))\right\}} \\ &= \frac{\max_{i\in N} z_i(A)}{\min_{i\in N}(|a_{ii}| - h_i(A))}. \end{aligned}$$

Proof since $|a_{11}| - h_1(A) \geq \min_{i\neq 1}(|a_{ii}| - h_i(A))$, we have

$$\mu = \eta_3 = \frac{\min_{i\neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|} \leq 1,$$

$\max\{\mu,\ 1\} = 1$, and

$$\frac{\max\{\mu, 1\}}{\min\left\{\mu|a_{11}| - h_1(A), \min_{i\neq 1}(|a_{ii}| - h_i(A))\right\}} = \frac{1}{\min_{i\in N}(|a_{ii}| - h_i(A))}.$$

Hence,

$$\frac{\max\{\mu, 1\} \max_{i\in n} z_i(A)}{\min\left\{\mu|a_{11}| - h_1(A), \min_{i\neq 1}(|a_{ii}| - h_i(A))\right\}} = \frac{\max_{i\in N} z_i(A)}{\min_{i\in N}(|a_{ii}| - h_i(A))}.$$

The proof is completed. □

Remark 4

(I) Theorems 4 and 5 provide the value of μ, i.e.,

$$\mu = \eta_1 = 1 + \frac{h_1(A)}{|a_{11}|} - \max_{i\neq 1} \frac{h_i(A)}{|a_{ii}|}$$

such that the bound (7) in Theorem 3 is not worse than the bound (1) in Theorem 2 for a Nekrasov matrix $A = [a_{ij}] \in C^{n,n}$. In particular, for a Nekrasov matrix A with $\frac{h_1(A)}{|a_{11}|} > \max_{i\neq 1} \frac{h_i(A)}{|a_{ii}|}$, the bound (7) is better than the bound (1).

(II) Theorems 6 and 7 provide the value of μ, i.e.,

$$\mu = \eta_3 = \frac{\min\limits_{i \neq 1}(|a_{ii}| - h_i(A)) + h_1(A)}{|a_{11}|}$$

such that the bound (8) in Theorem 3 is not worse than the bound (2) in Theorem 2 for a Nekrasov matrix A. In particular, for a Nekrasov matrix A with $|a_{11}| - h_1(A) < \min\limits_{i \neq 1}(|a_{ii}| - h_i(A))$, the bound (8) is better than the bound (2).

4 Numerical examples

Example 2 Consider the following five Nekrasov matrices in [5]:

$$A_2 = \begin{bmatrix} 8 & 1 & -0.2 & 3.3 \\ 7 & 13 & 2 & -3 \\ -1.3 & 6.7 & 13 & -2 \\ 0.5 & 3 & 1 & 6 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 21 & -9.1 & -4.2 & -2.1 \\ -0.7 & 9.1 & -4.2 & -2.1 \\ -0.7 & -0.7 & 4.9 & -2.1 \\ -0.7 & -0.7 & -0.7 & 2.8 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 5 & 1 & 0.2 & 2 \\ 1 & 21 & 1 & -3 \\ 2 & 0.5 & 6.4 & -2 \\ 0.5 & -1 & 1 & 9 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 6 & -3 & -2 \\ -1 & 11 & -8 \\ -7 & -3 & 10 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 8 & -0.5 & -0.5 & -0.5 \\ -9 & 16 & -5 & -5 \\ -6 & -4 & 15 & -3 \\ -4.9 & -0.9 & -0.9 & 6 \end{bmatrix}.$$

Obviously, A_2, A_3 and A_4 are SDD. And it is not difficult to verify that A_4, A_5 satisfy the conditions in Theorems 4 and 6 and A_2, A_3, A_6 satisfy the conditions in Theorems 5 and 7. We compute by Matlab 7.0 the upper bounds for the infinity norm of the inverse of A_i, $i = 2, \ldots, 6$, which are showed in Table 1. It is easy to see from Table 1 that this example illustrates Theorems 4, 5, 6 and 7.

Table 1 The upper bounds for $||A_i^{-1}||_\infty$, $i = 2, \ldots, 6$

Matrix	A_2	A_3	A_4	A_5	A_6
Exact $\|\|A^{-1}\|\|_\infty$	0.2390	0.8759	0.2707	1.1519	0.4474
Varah	1	1.4286	0.5556	–	–
The bound (1)	0.8848	1.8076	0.6200	1.4909	1.1557
Theorems 4 or 5	0.8848	1.8076	0.5270	1.4266	1.1557
The bound (2)	0.6885	0.9676	0.7937	2.4848	0.5702
Theorems 6 or 7	0.6885	0.9676	0.5895	1.5923	0.5702

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