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Semilocal convergence of multi-point improved super-Halley-type methods without the second derivative under generalized weak condition

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Abstract In this paper, we consider the semilocal convergence of multi-point improved super-Halley-type methods in Banach space. Different from the results of super-Halley method studied in reference Gutiérrez, J.M. and Hernández, M.A. (Comput. Math. Appl. **36**,1–8, 1998) these methods do not require second derivative of an operator, the *R*-order is improved and the convergence condition is also relaxed. We prove a convergence theorem to show existence and uniqueness of the solution.

Keywords Semilocal convergence · Super-Halley-type method *R*-order of convergence · Nonlinear equations in Banach spaces · Generalized weak condition

1 Introduction

Many problems arisen from scientific and engineering computing areas need to solve the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where $F : \Omega \subseteq X \rightarrow Y$ is a nonlinear operator, X and Y are Banach spaces, Ω is a non-empty open convex subset of X. The second-order Newton's method [1] is widely used to solve (1.1). In recent years, some third-order

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methods have been developed since their rapid convergence speed, for example, the Chebyshev-Halley methods [2], which are given by

$$x_{n+1} = x_n - \left(I + \frac{1}{2} L_F(x_n)[I - \alpha L_F(x_n)]^{-1} \right) \Gamma_n F(x_n), \tag{1.2}$$

where I is the identity operator in a Banach space X, $\Gamma_n = F'(x_n)^{-1}$ and $L_F(x_n) = \Gamma_n F''(x_n)\Gamma_n F(x_n)$. This family of methods contains some classical third-order methods, such as Chebyshev method ($\alpha = 0$), Halley method ($\alpha = 1/2$) and super-Halley method ($\alpha = 1$). Some results on Chebyshev-Halley methods and their variants can be found in references[2–8, 12, 13] where in the reference [6], J.M. Gutiérrez and M.A. Hernández have studied the convergence of super-Halley method. By assuming that

(A1) $\|\Gamma_0\| \leqslant B$,
(A2) $\|\Gamma_0 F(x_0)\| \leqslant \eta$,
(A3) $\|F''(x)\| \leqslant k_1, \quad x \in \Omega_0$,
(A4) $\|F''(x) - F''(y)\| \leqslant k_2\|x - y\|, \quad x, y \in \Omega_0$,

where $\Gamma_0 = [F'(x_0)]^{-1}$, Ω_0 is an open convex subset of Ω. J.M. Gutiérrez and M.A. Hernández have proven that the R-order of super-Halley method is at least three.

Notice that under the conditions (A1)-(A4), the solution of some equations can not be studied. Such as the nonlinear integral equation of mixed Hammerstein type [9], which is given by

$$x(s) + \sum_{i=1}^{m} \int_a^b G_i(s,t) H_i(x(t))dt = u(s), s \in [a,b], \tag{1.3}$$

where x is a solution to be found, u, G_i and H_i are known functions ($i = 1, 2, \ldots, m$), $-\infty < a < b < +\infty$. To find the solution of (1.3), it needs to solve the following equation

$$[F(x)](s) = x(s) + \sum_{i=1}^{m} \int_a^b G_i(s,t) H_i(x(t))dt - u(s), s \in [a,b]. \tag{1.4}$$

On the condition that $H_i''(x(t))$ is (L_i, q_i)-Hölder continuous in Ω_0, $i = 1, 2, \ldots, m$, considering the max-norm, then

$$\|F''(x) - F''(y)\| \le \sum_{i=1}^{m} L_i \|x - y\|^{q_i}, L_i \ge 0, q_i \in [0,1], \quad \forall x, y \in \Omega_0. \tag{1.5}$$

this shows that for $q_i \in (0, 1)$, F'' is neither Lipschitz continuous nor Hölder continuous in Ω_0. So the operator given by (1.4) does not satisfy the Lipschitz condition (A4), the solution of this equation can not be studied by super-Halley method under the conditions (A1)-(A4). Since the importance for nonlinear integral equation of mixed Hammerstein type, it has been studied in many papers, such as the references [8–11]. In reference [8], J.A. Ezquerro and M.A. Hernández have replaced the assumption (A4) by the following conditions

(B4)$\|F''(x) - F''(y)\| \leqslant \omega(\|x - y\|), \quad x, y \in \Omega$, where $\omega(z)$ is a non-decreasing continuous real function for $z > 0$ and $\omega(0) \geqslant 0$.

(B5) there exists a positive real function $\nu \in C[0,1]$, with $\nu(t) \leqslant 1$, such that $\omega(tz) \leqslant \nu(t)\omega(z)$, for $t \in [0,1]$, $z \in (0,+\infty)$.

Under the conditions (A1)-(A3), (B4)-(B5), J.A. Ezquerro and M.A. Hernández have proven that Halley method converges with R-order at least two. Choosing $\omega(z) = \sum_{i=1}^{m} L_i z^{q_i}$, they also proved that the R-order of Halley method is at least $2+q$, where $q = \min\{q_1, q_2..., q_m\}$, $q_i \in [0,1]$, $i = 1, 2...m$.

In reference [12], a family of modified super-Halley methods with fourth-order convergence is studied. This family of methods focus on finding a simple root of nonlinear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function, D is an open interval. Extending one family of the methods to Banach space, then the corresponding formula can be written as

$$x_{n+1} = x_n - \left(I + \frac{1}{2} K_F(x_n) + \frac{1}{2} K_F(x_n)^2 (I - \delta_1 K_F(x_n))^{-1} \right) \Gamma_n F(x_n), \tag{1.6}$$

where I is the identity operator in a Banach space X, $\delta_1 \in [0,1]$, $\Gamma_n = F'(x_n)^{-1}$ and $K_F(x_n)$is an operator defined by

$$K_F(x_n) = \Gamma_n F'' \left(x_n - \frac{1}{3}\Gamma_n F(x_n) \right) \Gamma_n F(x_n).$$

Notice that the Chebyshev-Halley methods and the methods (1.6) require second Fréchet derivative of operator F, when F'' is hard to compute or the computational cost is large, then the Chebyshev-Halley methods and the methods (1.6) become less useful. In reference [13], M.A. Hernández has introduced a modified Chebyshev method free from second Fréchet derivative given by

$$\begin{cases} y_n = x_n - \Gamma_n F(x_n), \\ z_n = x_n + (1/2)(y_n - x_n), \\ x_{n+1} = y_n - \Gamma_n [F'(z_n) - F'(x_n)](y_n - x_n), \quad n \geqslant 0, \end{cases} \tag{1.7}$$

where $\Gamma_n = F'(x_n)^{-1}$. Under the conditions (A1)-(A4), M.A. Hernández has proven that the R-order of method (1.7) is three.

In order to reduce the computational cost of second Fréchet derivative , to improve the R-order of convergence, also in order to relax the convergence condition used in reference [6], in this paper, we consider the semilocal convergence for multi-point improved super-Halley-type methods in Banach space given by

$$\begin{cases} z_n = x_n - \left[I + \frac{1}{2} Q(x_n) + \frac{1}{2} Q(x_n)^2 (I - \delta_1 Q(x_n))^{-1} \right] \Gamma_n F(x_n), \\ x_{n+1} = z_n - \left[I + Q(x_n) + \delta_2 Q(x_n)^2 \right] \Gamma_n F(z_n), \quad n \geqslant 0, \end{cases} \tag{1.8}$$

where $\delta_1 \in [0,1]$, $\delta_2 \in [-1,1]$, $\Gamma_n = F'(x_n)^{-1}$, I is the identity operator in Banach space X, $Q(x_n) = 3\Gamma_n[F'(x_n) - F'(u_n)]$ and $u_n = x_n - \frac{1}{3}\Gamma_n F(x_n)$. The first step of these methods can be viewed as variants of the methods (1.6), where $K_F(x_n)$ is approximated by $Q(x_n)$.

The derivation of this approximation can be stated as:
Since $u_n = x_n - \frac{1}{3}\Gamma_n F(x_n)$, $F(x_n) \to F(x^*) = 0$ as $n \to \infty$, where x^* is a solution of $F(x) = 0$, then $x_n - u_n \to 0$ as $n \to \infty$.

By Taylor expansion, it follows that

$$F'(x_n) = F'(u_n) + F''(u_n)(x_n - u_n) + O((x_n - u_n)^2),$$

Omitting $O((x_n - u_n)^2)$, it holds that

$$F'(x_n) \approx F'(u_n) + F''(u_n)(x_n - u_n).$$

So

$$F'(x_n) - F'(u_n) \approx F''(u_n)(x_n - u_n).$$

Notice that $u_n = x_n - \frac{1}{3}\Gamma_n F(x_n)$, then

$$F''(u_n)\Gamma_n F(x_n) \approx 3[F'(x_n) - F'(u_n)].$$

Moreover

$$\Gamma_n F''(u_n)\Gamma_n F(x_n) \approx 3\Gamma_n[F'(x_n) - F'(u_n)].$$

That is $K_F(x_n) \approx Q(x_n)$. This approximation is a small modification of the one used in reference [13], where in the reference [13], a third-order variant of Chebyshev method free from second Fréchet derivative given by (1.7) is introduced. Applying $Q(x_n)$ to replace $K_F(x_n)$ in the methods (1.6), the first step of methods (1.8) can be obtained. The second step adds the function evaluation at another point. Consequently, under the conditions (A1)-(A4), the R-order of methods (1.8) is increased to five, which is higher than the ones of super-Halley method, the methods (1.6) and method (1.7).

Applying a condition similar to the one used in reference [8], suppose that:
(C4) $\|F''(x) - F''(y)\| \leqslant \omega(\|x - y\|), \quad x, y \in \Omega_0,$
where $\omega(\mu)$ is a continuous and non-decreasing real function for $\mu > 0$ and satisfies $\omega(0) \geqslant 0, \omega(t\mu) \leqslant t^q\omega(\mu)$, for $\mu \in (0, +\infty), t \in [0, 1], q \in [0, 1]$.
Notice that the condition (C4) is weaker than the assumption (A4), since it contains the Lipschitz continuity (assumption (A4)) and Hölder continuity as its special cases, and it is effective for many problems where neither Lipschitz nor Hölder continuity is effective, such as the nonlinear integral equation of mixed Hammerstein type given by (1.3).

Under the conditions (A1)-(A3) and (C4), the semilocal convergence of methods (1.8) is analyzed and a existence-uniqueness theorem is proved to show the R-order of these methods. Finally, the efficiency index analysis and numerical results are also carried out.

2 Some preliminary results for convergence analysis

Define $B(x, r) = \{y \in X : \|y-x\| < r\}$ and $\overline{B(x, r)} = \{y \in X : \|y-x\| \leq r\}$ in this paper. Let the nonlinear operator $F : \Omega \subseteq X \to Y$ be twice Fréchet differentiable in a non-empty open convex subset $\Omega_0 \subseteq \Omega$, where X and Y are Banach spaces. Moreover, let $x_0 \in \Omega_0$ and assume that
(C1) Γ_0 exists and $\|\Gamma_0\| \leqslant \beta$,
(C2) $\|\Gamma_0 F(x_0)\| \leqslant \eta$,
(C3) $\|F''(x)\| \leqslant M, \quad x \in \Omega_0, \quad M \geqslant 0$,
(C4) $\|F''(x) - F''(y)\| \leq \omega(\|x - y\|), \quad x, y \in \Omega_0$, where $\omega(\mu)$ is a continuous

non-decreasing real function for $\mu > 0$ and satisfies $\omega(0) \geq 0$, $\omega(t\mu) \leq t^q \omega(\mu)$, for $\mu \in (0, +\infty)$, $t \in [0, 1]$, $q \in [0, 1]$.

Now we define the following functions as:

$$g(t) = p(t) + \frac{t}{2}\left(1 + t + |\delta_2| t^2\right)\left[1 + \frac{t}{1 - \delta_1 t} + p(t)^2\right], \tag{2.1}$$

$$h(t) = \frac{1}{1 - g(t)t}, \tag{2.2}$$

$$\begin{aligned}\varphi(t, u) = &\left[\frac{u}{(q+1)3^q} + t^2(1 + |\delta_2| + |\delta_2| t) + \frac{1}{q+1}(1 + t + |\delta_2| t^2)u\right]\phi(t, u)\\ &+\frac{t^2}{2}\left(1 + \frac{t}{1 - \delta_1 t}\right)\left(1 + t + |\delta_2| t^2\right)\phi(t, u)\\ &+\frac{t}{2}\left(1 + t + |\delta_2| t^2\right)^2 \phi(t, u)^2,\end{aligned} \tag{2.3}$$

where

$$p(t) = 1 + \frac{1}{2}t + \frac{t^2}{2(1 - \delta_1 t)}, \tag{2.4}$$

$$\begin{aligned}\phi(t, u) = &\frac{t^2}{2}\left(1 + \frac{1}{1 - \delta_1 t} + \frac{t}{1 - \delta_1 t}\right) + \frac{t^3}{8}\left(1 + \frac{t}{1 - \delta_1 t}\right)^2\\ &+\frac{u}{2(q+1)3^q} + \frac{u}{(q+1)(q+2)}.\end{aligned} \tag{2.5}$$

Let $f(t) = g(t)t - 1$, since $f(0) = -1 < 0$, $f\left(\frac{1}{2}\right) \geqslant \frac{27}{256} > 0$, we can know that $f(t) = 0$ has at least a root in $\left(0, \frac{1}{2}\right)$. Let s^* be the smallest positive root of equation $g(t)t - 1 = 0$, then $s^* < \frac{1}{2}$.

Lemma 1 *Let the functions g, h and φ be defined by (2.1)-(2.3) and s^* be the smallest positive root of equation $g(t)t - 1 = 0$. Then we have (a) $g(t)$ and $h(t)$ are increasing and $g(t) > 1$, $h(t) > 1$ for $t \in (0, s^*)$; (b) for $t \in (0, s^*)$ and a fixed $u > 0$, $\varphi(t, u)$ is increasing as the function of t; for $u > 0$ and a fixed $t \in (0, s^*)$, $\varphi(t, u)$ is increasing as the function of u.*

Lemma 2 *Let $\theta \in (0, 1)$, the definitions of functions g, h and φ be given by (2.1)-(2.3). Then $g(\theta t) < g(t)$, $h(\theta t) < h(t)$, $\varphi(\theta t, \theta^{(1+q)}u) < \theta^{(2+2q)}\varphi(t, u)$ for $t \in (0, s^*)$, $u > 0$, where s^* is the smallest positive root of equation $g(t)t - 1 = 0$.*

Define the following sequences as:

$$\eta_{n+1} = d_n \eta_n, \tag{2.6}$$

$$\beta_{n+1} = h(a_n)\beta_n, \tag{2.7}$$

$$a_{n+1} = M\beta_{n+1}\eta_{n+1}, \tag{2.8}$$

$$b_{n+1} = \beta_{n+1}\eta_{n+1}\omega(\eta_{n+1}), \tag{2.9}$$

$$d_{n+1} = h(a_{n+1})\varphi(a_{n+1}, b_{n+1}), \tag{2.10}$$

where $n \geqslant 0$. Here, we choose $\eta_0 = \eta$, $\beta_0 = \beta$, $a_0 = M\beta\eta$, $b_0 = \beta\eta\omega(\eta)$ and $d_0 = h(a_0)\varphi(a_0, b_0)$.

Lemma 3 *Let s^* be the smallest positive root of equation $g(t)t - 1 = 0$. If*

$$a_0 < s^* \quad and \quad h(a_0)d_0 < 1, \tag{2.11}$$

then
(a) $h(a_n) > 1$ and $d_n < 1$ for $n \geqslant 0$,
(b) the sequences $\{\eta_n\}$, $\{a_n\}$, $\{b_n\}$ and $\{d_n\}$ are decreasing,
(c) $g(a_n)a_n < 1$ and $h(a_n)d_n < 1$ for $n \geqslant 0$.

The following lemma will be used in latter developments.

Lemma 4 *Assume that the nonlinear operator $F : \Omega \subseteq X \to Y$ is twice Fréchet differentiable in a non-empty open convex subset $\Omega_0 \subseteq \Omega$, where X and Y are Banach spaces. Then*

$$\begin{aligned} F(x_{n+1}) = & \; 3\int_0^1 \left[F''\left(x_n + t(u_n - x_n)\right) - F''(x_n)\right](u_n - x_n)dt\Gamma_n F(z_n) \\ & + 3\delta_2\left[F'(u_n) - F'(x_n)\right]Q(x_n)\Gamma_n F(z_n) \\ & - F''(x_n)(y_n - x_n)\left[Q(x_n) + \delta_2 Q(x_n)^2\right]\Gamma_n F(z_n) \\ & + \int_0^1 \left[F''\left(x_n + t(y_n - x_n)\right) - F''(x_n)\right](y_n - x_n)dt(x_{n+1} - z_n) \\ & + \int_0^1 F''\left(y_n + t(z_n - y_n)\right)(z_n - y_n)dt(x_{n+1} - z_n) \\ & + \int_0^1 \left[F'\left(z_n + t(x_{n+1} - z_n)\right) - F'(z_n)\right](x_{n+1} - z_n)dt, \end{aligned} \tag{2.12}$$

where $y_n = x_n - \Gamma_n F(x_n)$, z_n and x_{n+1} are given in (1.8), $\delta_2 \in [-1, 1]$, the definitions of $\Gamma_n, u_n, Q(x_n)$ are same to the ones in (1.8) .

Proof By Taylor expansion, it holds that

$$\begin{aligned} F(x_{n+1}) = & \; F(z_n) + F'(z_n)(x_{n+1} - z_n) \\ & + \int_0^1 \left[F'\left(z_n + t(x_{n+1} - z_n)\right) - F'(z_n)\right](x_{n+1} - z_n)dt, \end{aligned} \tag{2.13}$$

$$F'(z_n) = F'(y_n) + \int_0^1 F''(y_n + t(z_n - y_n))(z_n - y_n)dt, \tag{2.14}$$

$$\begin{aligned} F'(y_n) = {} & F'(x_n) + F''(x_n)(y_n - x_n) \\ & + \int_0^1 \left[F''(x_n + t(y_n - x_n)) - F''(x_n)\right](y_n - x_n)dt. \end{aligned} \tag{2.15}$$

Then we have

$$\begin{aligned} F(x_{n+1}) = {} & F(z_n) - F'(x_n)\left[I + Q(x_n) + \delta_2 Q(x_n)^2\right]\Gamma_n F(z_n) \\ & - F''(x_n)(y_n - x_n)\left[I + Q(x_n) + \delta_2 Q(x_n)^2\right]\Gamma_n F(z_n) \\ & + \int_0^1 \left[F''(x_n + t(y_n - x_n)) - F''(x_n)\right](y_n - x_n)dt(x_{n+1} - z_n) \\ & + \int_0^1 F''(y_n + t(z_n - y_n))(z_n - y_n)dt(x_{n+1} - z_n) \\ & + \int_0^1 \left[F'(z_n + t(x_{n+1} - z_n)) - F'(z_n)\right](x_{n+1} - z_n)dt \\ = {} & 3\left[F'(u_n) - F'(x_n)\right]\Gamma_n F(z_n) + F''(x_n)\Gamma_n F(x_n)\Gamma_n F(z_n) \\ & + 3\delta_2\left[F'(u_n) - F'(x_n)\right]Q(x_n)\Gamma_n F(z_n) \\ & - F''(x_n)(y_n - x_n)\left[Q(x_n) + \delta_2 Q(x_n)^2\right]\Gamma_n F(z_n) \\ & + \int_0^1 \left[F''(x_n + t(y_n - x_n)) - F''(x_n)\right](y_n - x_n)dt(x_{n+1} - z_n) \\ & + \int_0^1 F''(y_n + t(z_n - y_n))(z_n - y_n)dt(x_{n+1} - z_n) \\ & + \int_0^1 \left[F'(z_n + t(x_{n+1} - z_n)) - F'(z_n)\right](x_{n+1} - z_n)dt. \end{aligned} \tag{2.16}$$

Notice that

$$\begin{aligned} F'(u_n) = {} & F'(x_n) + F''(x_n)(u_n - x_n) \\ & + \int_0^1 \left[F''(x_n + t(u_n - x_n)) - F''(x_n)\right](u_n - x_n)dt. \end{aligned} \tag{2.17}$$

Substituting (2.17) into (2.16), then (2.12) can be obtained. □

3 Semilocal convergence for the method

For $n = 0$, the existence of Γ_0 shows that y_0, u_0 exist. Furthermore,

$$\|y_0 - x_0\| = \| - \Gamma_0 F(x_0)\| \leqslant \eta_0. \tag{3.1}$$

$$\|u_0 - x_0\| = \left\| -\frac{1}{3}\Gamma_0 F(x_0) \right\| \leqslant \frac{1}{3}\eta_0. \quad (3.2)$$

This means that $y_0 \in B(x_0, R\eta)$ and $u_0 \in B(x_0, R\eta)$, where $R = \frac{g(a_0)}{1-d_0}$. Moreover,

$$\|Q(x_0)\| \leqslant 3M\beta\|x_0 - u_0\| \leqslant M\beta\eta = a_0. \quad (3.3)$$

Since $\delta_1 \in [0, 1]$ and $a_0 < s^* < 1/2$, we have $\delta_1 a_0 < 1$. By the Banach lemma, it follows that $[I - \delta_1 Q(x_0)]^{-1}$ exists and

$$\|(I - \delta_1 Q(x_0))^{-1}\| \leqslant \frac{1}{1 - \delta_1 a_0}. \quad (3.4)$$

Then

$$\begin{aligned} \|z_0 - x_0\| &\leqslant \left[1 + \frac{1}{2}a_0 + \frac{a_0^2}{2(1 - \delta_1 a_0)}\right] \|\Gamma_0 F(x_0)\| \\ &= p(a_0)\|\Gamma_0 F(x_0)\| \leqslant p(a_0)\eta_0 \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} \|z_0 - y_0\| &\leqslant \left[\frac{1}{2}a_0 + \frac{a_0^2}{2(1 - \delta_1 a_0)}\right] \|\Gamma_0 F(x_0)\| \\ &\leqslant \left[\frac{1}{2}a_0 + \frac{a_0^2}{2(1 - \delta_1 a_0)}\right] \eta_0. \end{aligned} \quad (3.6)$$

From Taylor expression, it holds that

$$\begin{aligned} F(z_n) &= F(x_n) + F'(x_n)(z_n - x_n) + \int_0^1 \left[F'\left(x_n + t(z_n - x_n)\right) - F'(x_n)\right](z_n - x_n)dt \\ &= -\frac{3}{2}\left[F'(x_n) - F'(u_n)\right]\Gamma_n F(x_n) \\ &\quad -\frac{3}{2}\left[F'(x_n) - F'(u_n)\right]Q(x_n)(I - \delta_1 Q(x_n))^{-1}\Gamma_n F(x_n) \\ &\quad +\int_0^1 \left[F'\left(x_n + t(z_n - x_n)\right) - F'(x_n)\right](z_n - x_n)dt. \end{aligned} \quad (3.7)$$

Then we have

$$\begin{aligned} \|F(z_0)\| &\leqslant \frac{1}{2}\left[1 + \frac{a_0}{1 - \delta_1 a_0} + p(a_0)^2\right] M\|\Gamma_0 F(x_0)\|\eta_0 \\ &\leqslant \frac{1}{2}\left[1 + \frac{a_0}{1 - \delta_1 a_0} + p(a_0)^2\right] M\eta_0^2 \end{aligned} \quad (3.8)$$

and

$$\begin{aligned} \beta_0\|F(z_0)\| &\leqslant \frac{a_0}{2}\left[1 + \frac{a_0}{1 - \delta_1 a_0} + p(a_0)^2\right] \|\Gamma_0 F(x_0)\| \\ &\leqslant \frac{a_0}{2}\left[1 + \frac{a_0}{1 - \delta_1 a_0} + p(a_0)^2\right] \eta_0. \end{aligned} \quad (3.9)$$

Notice that

$$\|x_1 - z_0\| \leqslant \left(1 + a_0 + |\delta_2|a_0^2\right)\beta_0\|F(z_0)\|$$

$$\leqslant \frac{a_0}{2}\left(1 + a_0 + |\delta_2|a_0^2\right)\left[1 + \frac{a_0}{1-\delta_1 a_0} + p(a_0)^2\right]\|\Gamma_0 F(x_0)\|$$

$$\leqslant \frac{a_0}{2}\left(1 + a_0 + |\delta_2|a_0^2\right)\left[1 + \frac{a_0}{1-\delta_1 a_0} + p(a_0)^2\right]\eta_0, \qquad (3.10)$$

so

$$\|x_1 - x_0\| \leqslant \|x_1 - z_0\| + \|z_0 - x_0\| \leqslant g(a_0)\|\Gamma_0 F(x_0)\| \leqslant g(a_0)\eta_0. \quad (3.11)$$

From the assumption $d_0 < 1/h(a_0) < 1$, it follows that $x_1 \in B(x_0, R\eta)$.
Since $a_0 < s^*$ and $g(a_0) < g(s^*)$, we have

$$\|I - \Gamma_0 F'(x_1)\| \leqslant \|\Gamma_0\|\|F'(x_0) - F'(x_1)\|$$
$$\leqslant M\beta_0\|x_1 - x_0\|$$
$$\leqslant g(a_0)a_0 < 1.$$

It follows by the Banach lemma that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\|\Gamma_1\| \leqslant \frac{\|\Gamma_0\|}{1 - \|\Gamma_0\|\|F'(x_0) - F'(x_1)\|}$$

$$\leqslant \frac{\|\Gamma_0\|}{1 - g(a_0)a_0} = h(a_0)\|\Gamma_0\| \leqslant h(a_0)\beta_0 = \beta_1. \qquad (3.12)$$

From Lemma 4, we have

$$\|F(x_1)\| \leqslant \left[\frac{1}{(q+1)3^q}\eta_0\omega(\eta_0) + a_0(1 + |\delta_2| + |\delta_2|a_0)M\eta_0\right]\beta_0\|F(z_0)\|$$

$$+\frac{1}{q+1}(1 + a_0 + |\delta_2|a_0^2)\eta_0\omega(\eta_0)\beta_0\|F(z_0)\|$$

$$+\frac{a_0}{2}\left(1 + \frac{a_0}{1-\delta_1 a_0}\right)\left(1 + a_0 + |\delta_2|a_0^2\right)M\eta_0\beta_0\|F(z_0)\|$$

$$+\frac{1}{2}\left(1 + a_0 + |\delta_2|a_0^2\right)^2 M[\beta_0\|F(z_0)\|]^2. \qquad (3.13)$$

Since

$$F(z_n) = \frac{3}{2}\int_0^1 \left[F''(x_n + t(u_n - x_n)) - F''(x_n)\right](u_n - x_n)dt\,\Gamma_n F(x_n)$$

$$-\frac{3}{2}\left[F'(x_n) - F'(u_n)\right]Q(x_n)\left(I - \delta_1 Q(x_n)\right)^{-1}\Gamma_n F(x_n)$$

$$+\int_0^1 \left[F''(x_n + t(y_n - x_n)) - F''(x_n)\right](y_n - x_n)^2(1-t)dt$$

$$+\int_0^1 F''(x_n + t(y_n - x_n))(y_n - x_n)dt(z_n - y_n)$$

$$+\int_0^1 \left[F'(y_n + t(z_n - y_n)) - F'(y_n)\right](z_n - y_n)dt,$$

then

$$\begin{aligned}\|F(z_0)\| \leqslant\ & \frac{a_0}{2}\left(1+\frac{1}{1-\delta_1 a_0}+\frac{a_0}{1-\delta_1 a_0}\right)M\|\Gamma_0 F(x_0)\|\eta_0\\ &+\frac{a_0^2}{8}\left(1+\frac{a_0}{1-\delta_1 a_0}\right)^2 M\|\Gamma_0 F(x_0)\|\eta_0\\ &+\frac{1}{2(q+1)3^q}\|\Gamma_0 F(x_0)\|\eta_0\omega(\eta_0)\\ &+\frac{1}{(q+1)(q+2)}\|\Gamma_0 F(x_0)\|\eta_0\omega(\eta_0)\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\beta_0\|F(z_0)\| \leqslant\ & \frac{a_0^2}{2}\left(1+\frac{1}{1-\delta_1 a_0}+\frac{a_0}{1-\delta_1 a_0}\right)\|\Gamma_0 F(x_0)\|\\ &+\frac{a_0^3}{8}\left(1+\frac{a_0}{1-\delta_1 a_0}\right)^2 \|\Gamma_0 F(x_0)\|\\ &+\frac{b_0}{2(q+1)3^q}\|\Gamma_0 F(x_0)\|+\frac{b_0}{(q+1)(q+2)}\|\Gamma_0 F(x_0)\|\\ =\ & \phi(a_0,b_0)\|\Gamma_0 F(x_0)\| \leqslant \phi(a_0,b_0)\eta_0.\end{aligned} \tag{3.15}$$

Substituting (3.15) into (3.13), it holds that

$$\begin{aligned}\|F(x_1)\| \leqslant\ & \left[\frac{1}{(q+1)3^q}\eta_0\omega(\eta_0)+a_0(1+|\delta_2|+|\delta_2|a_0)M\eta_0\right]\phi(a_0,b_0)\|\Gamma_0 F(x_0)\|\\ &+\frac{1}{q+1}(1+a_0+|\delta_2|a_0^2)\eta_0\omega(\eta_0)\phi(a_0,b_0)\|\Gamma_0 F(x_0)\|\\ &+\frac{a_0}{2}\left(1+\frac{a_0}{1-\delta_1 a_0}\right)\left(1+a_0+|\delta_2|a_0^2\right)M\eta_0\phi(a_0,b_0)\|\Gamma_0 F(x_0)\|\\ &+\frac{1}{2}\left(1+a_0+|\delta_2|a_0^2\right)^2 M\eta_0\phi(a_0,b_0)^2\|\Gamma_0 F(x_0)\|\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\beta_0\|F(x_1)\| \leqslant\ & \left[\frac{b_0}{(q+1)3^q}+a_0^2(1+|\delta_2|+|\delta_2|a_0)\right]\phi(a_0,b_0)\|\Gamma_0 F(x_0)\|\\ &+\frac{1}{q+1}(1+a_0+|\delta_2|a_0^2)b_0\phi(a_0,b_0)\|\Gamma_0 F(x_0)\|\\ &+\frac{a_0^2}{2}\left(1+\frac{a_0}{1-\delta_1 a_0}\right)\left(1+a_0+|\delta_2|a_0^2\right)\phi(a_0,b_0)\|\Gamma_0 F(x_0)\|\\ &+\frac{a_0}{2}\left(1+a_0+|\delta_2|a_0^2\right)^2 \phi(a_0,b_0)^2\|\Gamma_0 F(x_0)\|\\ =\ & \varphi(a_0,b_0)\|\Gamma_0 F(x_0)\| \leqslant \varphi(a_0,b_0)\eta_0.\end{aligned} \tag{3.17}$$

From (3.12) and (3.17), it follows that

$$\begin{aligned}\|y_1 - x_1\| &= \| - \Gamma_1 F(x_1)\| \leqslant \|\Gamma_1\|\|F(x_1)\| \\ &\leqslant h(a_0)\beta_0\|F(x_1)\| \leqslant h(a_0)\varphi(a_0, b_0)\|\Gamma_0 F(x_0)\| \\ &= d_0\|\Gamma_0 F(x_0)\| \leqslant d_0\eta_0 = \eta_1. \end{aligned} \tag{3.18}$$

Since $g(a_0) > 1$, we have

$$\begin{aligned}\|y_1 - x_0\| &\leqslant \|y_1 - x_1\| + \|x_1 - x_0\| \\ &\leqslant (g(a_0) + d_0)\eta_0 \\ &< g(a_0)(1 + d_0)\eta < R\eta, \end{aligned} \tag{3.19}$$

which shows $y_1 \in B(x_0, R\eta)$. Similarly, it can obtain that $u_1 \in B(x_0, R\eta)$. In addition, we have

$$M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \leqslant h(a_0)d_0a_0 = a_1, \tag{3.20}$$

$$\|\Gamma_1\|\|\Gamma_1 F(x_1)\|\omega(\|\Gamma_1 F(x_1)\|) \leqslant \beta_1\eta_1\omega(\eta_1) = b_1. \tag{3.21}$$

Applying induction, the existence of $\Gamma_n = [F'(x_n)]^{-1}$ and the following items can be obtained.

(I) $\|\Gamma_n\| \leqslant h(a_{n-1})\|\Gamma_{n-1}\| \leqslant h(a_{n-1})\beta_{n-1} = \beta_n$,

(II) $\|\Gamma_n F(x_n)\| \leqslant h(a_{n-1})\varphi(a_{n-1}, b_{n-1})\|\Gamma_{n-1}F(x_{n-1})\| \leqslant d_{n-1}\eta_{n-1} = \eta_n$,

(III) $M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \leqslant a_n$,

(IV) $\|\Gamma_n\|\|\Gamma_n F(x_n)\|\omega(\|\Gamma_n F(x_n)\|) \leqslant b_n$,

(V) $\|z_n - x_n\| \leqslant p(a_n)\|\Gamma_n F(x_n)\| \leqslant p(a_n)\eta_n$,

(VI) $\|x_{n+1} - x_n\| \leqslant g(a_n)\|\Gamma_n F(x_n)\| \leqslant g(a_n)\eta_n$.

Moreover, we have the following lemma.

Lemma 5 *Let the assumptions of Lemma 3 and the conditions (C1)-(C4) hold. Then* $\|u_n - x_0\| \leqslant R\eta$, $\|z_n - x_0\| \leqslant R\eta$, $\|x_{n+1} - x_0\| \leqslant R\eta$, *where* $R = \frac{g(a_0)}{1-d_0}$.

Proof By (II), (V) and (VI), it follows that

$$\begin{aligned}\|u_n - x_0\| &\leqslant \|u_n - x_n\| + \|x_n - x_0\| \leqslant \frac{1}{3}\eta_n + \sum_{i=0}^{n-1}\|x_{i+1} - x_i\| \\ &\leqslant \frac{1}{3}\eta_n + \sum_{i=0}^{n-1} g(a_i)\eta_i \leqslant g(a_0)\sum_{i=0}^{n}\eta_i,\end{aligned}$$

$$\begin{aligned}\|z_n - x_0\| &\leqslant \|z_n - x_n\| + \|x_n - x_0\| \leqslant p(a_n)\eta_n + \sum_{i=0}^{n-1}\|x_{i+1} - x_i\| \\ &\leqslant p(a_n)\eta_n + g(a_0)\sum_{i=0}^{n-1}\eta_i \leqslant g(a_0)\sum_{i=0}^{n}\eta_i\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1}-x_0\| &\leqslant \sum_{i=0}^{n}\|x_{i+1}-x_i\| \leqslant g(a_0)\sum_{i=0}^{n}\eta_i\\ &= g(a_0)\eta_0 + g(a_0)\sum_{i=1}^{n}\eta_0\left(\prod_{j=0}^{i-1}d_j\right). \end{aligned}\tag{3.22}$$

Let $\gamma = h(a_0)d_0, \lambda = 1/h(a_0)$. Since $a_1 = \gamma a_0$, $b_1 = \beta_1\eta_1\omega(\eta_1) = h(a_0)\beta_0 d_0\eta_0\omega(d_0\eta_0) \leqslant h(a_0)d_0^{(1+q)}b_0 < \gamma^{(1+q)}b_0$, by Lemma 2, it holds that

$$d_1 < h(\gamma a_0)\varphi(\gamma a_0, \gamma^{(1+q)}b_0) < \gamma^{(2+2q)}d_0 = \gamma^{(3+2q)^1-1}d_0 = \lambda\gamma^{(3+2q)^1}.$$

Suppose that $d_k \leqslant \lambda\gamma^{(3+2q)^k}, \quad k \geqslant 1$. From Lemma 3, we have $h(a_k) > 1$, $a_{k+1} < a_k$ and $h(a_k)d_k < 1$. Then

$$\begin{aligned}d_{k+1} &< h(a_k)\varphi\left(h(a_k)d_k a_k, h(a_k)d_k^{(1+q)}b_k\right) < h(a_k)\varphi\left(h(a_k)d_k a_k, h(a_k)^{(1+q)}d_k^{(1+q)}b_k\right)\\ &< h(a_k)^{(2+2q)}d_k^{(3+2q)} < \lambda\gamma^{(3+2q)^{k+1}}.\end{aligned}$$

Therefore $d_j \leqslant \lambda\gamma^{(3+2q)^j}, \quad j \geqslant 0$.
Furthermore,

$$\prod_{j=0}^{i-1}d_j \leqslant \prod_{j=0}^{i-1}\lambda\gamma^{(3+2q)^j} = \lambda^i\gamma^{\sum_{j=0}^{(i-1)}(3+2q)^j} = \lambda^i\gamma^{\frac{(3+2q)^i-1}{2+2q}}, \quad i \geq 1. \tag{3.23}$$

Substituting (3.23) into (3.22), it follows that

$$\begin{aligned}\|x_{n+1}-x_0\| &\leqslant g(a_0)\sum_{i=0}^{n}\eta_0\lambda^i\gamma^{\frac{(3+2q)^i-1}{2+2q}}\\ &\leqslant g(a_0)\eta\frac{1-\lambda^{n+1}\gamma^{\frac{(3+2q)^n+1+2q}{2+2q}}}{1-d_0} < R\eta.\end{aligned}$$

Similarly, $\|u_n - x_0\| \leqslant R\eta$, $\|z_n - x_0\| \leqslant R\eta$. The proof is completed. □

Lemma 6 *Let* $R = \frac{g(a_0)}{1-d_0}$. *If* $a_0 < s^*$ *and* $h(a_0)d_0 < 1$, *then* $R < 1/a_0$.

The following theorem shows the existence-uniqueness of a solution.

Theorem 1 *Let* $F : \Omega \subseteq X \to Y$ *be twice Fréchet differentiable in a non-empty open convex subset* Ω_0, *where* X *and* Y *are two Banach spaces. Assume that* $x_0 \in \Omega_0$ *and all conditions (C1)-(C4) hold. Let* $a_0 = M\beta\eta$, $b_0 = \beta\eta\omega(\eta)$, $d_0 = h(a_0)\varphi(a_0, b_0)$ *satisfy* $a_0 < s^*$ *and* $h(a_0)d_0 < 1$, *where* s^* *is the smallest positive root of the equation* $g(t)t - 1 = 0$ *and* g, h, φ *are defined by* (2.1)-(2.3). *Let*

$\overline{B(x_0, R\eta)} \subseteq \Omega_0$, *where* $R = \frac{g(a_0)}{1-d_0}$, *then starting from* x_0, *the sequence* $\{x_n\}$ *generated by methods* (1.8) *converges to a solution* x^* *of* $F(x) = 0$ *with* x_n, x^* *belong to* $\overline{B(x_0, R\eta)}$ *and* x^* *is the unique solution of* $F(x) = 0$ *in* $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega_0$.

Furthermore, a priori error estimate is given by

$$\|x_n - x^*\| \leqslant g(a_0)\eta\lambda^n \gamma^{\frac{(3+2q)^n-1}{2+2q}} \frac{1}{1-\lambda\gamma^{(3+2q)^n}}, \tag{3.24}$$

where $\gamma = h(a_0)d_0$ *and* $\lambda = 1/h(a_0)$.

Proof By Lemma 5, it follows that the sequence $\{x_n\}$ is well-defined in $\overline{B(x_0, R\eta)}$. Now we prove that $\{x_n\}$ is a Cauchy sequence. Since

$$\begin{aligned}\|x_{n+m} - x_n\| &\leqslant \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \leqslant g(a_0) \sum_{i=n}^{n+m-1} \eta_i \\ &\leqslant g(a_0)\eta\lambda^n \gamma^{\frac{(3+2q)^n-1}{2+2q}} \frac{1-\lambda^m \gamma^{\frac{(3+2q)^n((3+2q)^{m-1}+(1+2q))}{2+2q}}}{1-\lambda\gamma^{(3+2q)^n}},\end{aligned} \tag{3.25}$$

we have that $\{x_n\}$ is a Cauchy sequence. So there exists a x^* such that $\lim_{n\to\infty} x_n = x^*$.

Let $n = 0,\ \ m \to \infty$ in (3.25), then

$$\| x^* - x_0 \| \leqslant R\eta. \tag{3.26}$$

It shows that $x^* \in \overline{B(x_0, R\eta)}$.

Next we prove that x^* is a solution of $F(x) = 0$. From Lemma 4, we have

$$\begin{aligned}\|F(x_{n+1})\| \leqslant &\left[\frac{1}{(q+1)3^q}\omega(\eta_0) + a_0(1 + |\delta_2| + |\delta_2|a_0)M\right]\phi(a_0, b_0)\eta_n^2 \\ &+\frac{1}{q+1}(1 + a_0 + |\delta_2|a_0^2)\omega(\eta_0)\phi(a_0, b_0)\eta_n^2 \\ &+\frac{a_0}{2}\left(1 + \frac{a_0}{1-\delta_1 a_0}\right)\left(1 + a_0 + |\delta_2|a_0^2\right) M\phi(a_0, b_0)\eta_n^2 \\ &+\frac{1}{2}\left(1 + a_0 + |\delta_2|a_0^2\right)^2 M\phi(a_0, b_0)^2\eta_n^2.\end{aligned} \tag{3.27}$$

Letting $n \to \infty$ in (3.27), then it follows that $\|F(x_n)\| \to 0$ since $\eta_n \to 0$. By the continuity of F in Ω_0, then $F(x^*) = 0$. Now we prove the uniqueness of x^* in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega_0$. From Lemma 6, we have that

$$\frac{2}{M\beta} - R\eta = \left(\frac{2}{a_0} - R\right)\eta > \frac{1}{a_0}\eta > R\eta$$

and $\overline{B(x_0, R\eta)} \subseteq B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega_0$. Then $x^* \in B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega_0$. Let x^{**} be another root of $F(x) = 0$ in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega_0$. Define $H = \int_0^1 F'((1-t)x^* + tx^{**})dt$. Notice that

$$0 = F(x^{**}) - F(x^*) = H(x^{**} - x^*). \tag{3.28}$$

Since

$$\begin{aligned} &\|\Gamma_0\| \left\| H - F'(x_0) \right\| \\ &\leqslant M\beta \int_0^1 [(1-t)\|x^* - x_0\| + t\|x^{**} - x_0\|]dt \\ &< \frac{M\beta}{2}\left[R\eta + \frac{2}{M\beta} - R\eta\right] = 1, \end{aligned} \tag{3.29}$$

it follows by the Banach lemma that H is invertible, hence $x^{**} = x^*$.

Finally, by letting $m \to \infty$ in (3.25), then (3.24) can be obtained. Moreover, it holds that

$$\|x_n - x^*\| \leqslant \frac{g(a_0)\eta}{\gamma^{1/(2+2q)}(1-d_0)}(\gamma^{1/(2+2q)})^{(3+2q)^n}, \tag{3.30}$$

where $\gamma = h(a_0)d_0$ and $\lambda = 1/h(a_0)$. □

From (3.30), we know that the R-order of methods (1.8) is at least $3 + 2q$ under the conditions (C1)-(C4). Especially, when $q = 1$, The R-order becomes five, which is higher than the one of super-Halley method.

4 Numerical results and efficiency analysis

Example 1 We consider the solution for a integral equation of mixed Hammerstein type which is given by

$$x(s) = 1 + \int_0^1 G(s,t)\left(\frac{3}{5}x(t)^{7/3} + \frac{6}{15}x(t)^3\right)dt, s \in [0,1], \tag{4.1}$$

where $x \in C[0,1]$, $t \in [0,1]$ and G is the Green function

$$G(s,t) = \begin{cases} (1-s)t, & t \leqslant s, \\ s(1-t), & s \leqslant t. \end{cases}$$

Solving the equation (4.1) is equivalent to find the solution for $F(x) = 0$, where $F : \Omega \subseteq C[0,1] \to C[0,1]$,

$$[F(x)](s) = x(s) - 1 - \int_0^1 G(s,t)\left(\frac{3}{5}x(t)^{7/3} + \frac{6}{15}x(t)^3\right)dt, s \in [0,1]. \tag{4.2}$$

Taking $\Omega_0 = B(0,2)$. Obviously, $F(x)$ is twice Fréchet differentiable in $\Omega_0 \subseteq \Omega$. The Fréchet derivatives of F are given by

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)\left(\frac{7}{5}x(t)^{4/3} + \frac{6}{5}x(t)^2\right)y(t)dt, \quad y \in \Omega_0,$$

$$F''(x)yz(s) = -\int_0^1 G(s,t)\left(\frac{28}{15}x(t)^{1/3} + \frac{12}{5}x(t)\right)y(t)z(t)dt, \quad y, z \in \Omega_0.$$

Notice that F'' can not satisfy the assumption (A4), but it can satisfy the condition (C4). Here, we define

$$\omega(\mu) = \frac{7}{30}\mu^{\frac{1}{3}} + \frac{3}{10}\mu,$$

and

$$q = \frac{1}{3}.$$

Then F'' can satisfy the condition (C4). Taking the constant function $x_0(t) = 1$ as the initial approximate solution, then we have that

$$\|F(x_0)\| = \frac{1}{8},$$

$$\|\Gamma_0\| \leqslant \frac{40}{27} \equiv \beta,$$

$$\|\Gamma_0 F(x_0)\| \leqslant \frac{5}{27} \equiv \eta,$$

$$\|F''(x)\| \leqslant 0.894\cdots \equiv M.$$

Here, the max norm is used. Moreover, we compute

$$a_0 \simeq 0.2453.$$

Since

$$g(a_0) = p(a_0) + \frac{a_0}{2}\left(1 + a_0 + |\delta_2|a_0^2\right)\left[1 + \frac{a_0}{1-\delta_1 a_0} + p(a_0)^2\right],$$

where

$$p(a_0) = 1 + \frac{1}{2}a_0 + \frac{a_0^2}{2(1-\delta_1 a_0)},$$

we write

$$\widetilde{g}(\delta_1, \delta_2) = g(a_0),$$

then

$$g(a_0)a_0 \le \widetilde{g}(1,1)a_0 \simeq 0.3902 < 1,$$

this shows that $a_0 < s^*$. Since

$$\begin{aligned}\varphi(a_0, b_0) = & \left[\frac{b_0}{(q+1)3^q} + a_0^2(1 + |\delta_2| + |\delta_2|a_0) + \frac{1}{q+1}(1 + a_0 + |\delta_2|a_0^2)b_0\right]\phi(a_0, b_0)\\ & + \frac{a_0^2}{2}\left(1 + \frac{a_0}{1-\delta_1 a_0}\right)\left(1 + a_0 + |\delta_2|a_0^2\right)\phi(a_0, b_0)\\ & + \frac{a_0}{2}\left(1 + a_0 + |\delta_2|a_0^2\right)^2\phi(a_0, b_0)^2, \qquad (4.3)\end{aligned}$$

Table 1 Existence ball and uniqueness ball of the solution

δ_1	δ_2	existence ball	uniqueness ball
0	0	$\overline{B(1, 0.295\cdots)}$	$B(1, 1.214\cdots) \cap \Omega_0$
0 .5	1	$\overline{B(1, 0.306\cdots)}$	$B(1, 1.203\cdots) \cap \Omega_0$
1	1	$\overline{B(1, 0.311\cdots)}$	$B(1, 1.198\cdots) \cap \Omega_0$

where

$$\phi(a_0, b_0) = \frac{a_0^2}{2}\left(1 + \frac{1}{1-\delta_1 a_0} + \frac{a_0}{1-\delta_1 a_0}\right) + \frac{a_0^3}{8}\left(1 + \frac{a_0}{1-\delta_1 a_0}\right)^2 + \frac{b_0}{2(q+1)3^q} + \frac{b_0}{(q+1)(q+2)}, \tag{4.4}$$

define

$$\widetilde{\varphi}(\delta_1, \delta_2) = \varphi(a_0, b_0), \tag{4.5}$$

then

$$h(a_0)d_0 \leqslant \left[\frac{1}{1-\widetilde{g}(1,1)a_0}\right]^2 \widetilde{\varphi}(1,1) \simeq 0.0876 < 1.$$

As a result, the conditions of Theorem 1 are satisfied. Taking different parameters δ_1 and δ_2, the existence ball and uniqueness ball of the solution are listed in Table 1

From Table 1, one can know that as tested here, choosing $\delta_1 = \delta_2 = 1$, the radius of existence ball is larger than the others.

Example 2 Consider the following problem that find the minimizer of the chained Rosenbrock function [14]

Table 2 The iterative errors ($\|\mathbf{x}_k - \mathbf{x}^*\|_2$) of various methods

k	NM	CM	HM	SHM	KM	VCM	OPM
1	5.62e-1	3.26e-1	2.27e-1	6.01e-2	6.59e-2	3.36e-1	4.54e-2
2	4.81e-2	1.31e-3	4.51e-3	1.91e-5	9.97e-6	1.72e-2	5.70e-7
3	4.67e-3	1.10e-7	2.05e-8	1.46e-14	4.71e-16	4.70e-6	0
4	2.52e-6	0	0	0	0	0	0
5	1.08e-10	0	0	0	0	0	0

Table 3 The comparison of various methods

	order	F	F'	F''	Efficiency index
Newton method	2	1	1	0	$2^{1/(m+m^2)}$
Chebyshev-Halley methods	3	1	1	1	$3^{1/(m+m^2+m^2(m+1)/2)}$
Methods (1.6)	4	1	1	1	$4^{1/(m+m^2+m^2(m+1)/2)}$
Method (1.7)	3	1	2	0	$3^{1/(m+2m^2)}$
Methods (1.8)	5	2	2	0	$5^{1/(2m+2m^2)}$

$$g(\mathbf{x}) = \sum_{i=1}^{m}[4(x_i - x_{i+1}^2)^2 + (1 - x_{i+1})^2], \quad \mathbf{x} \in \mathbb{R}^m.$$

Finding the minimum of g needs to solve the nonlinear system $F(\mathbf{x}) = 0$, where $F(\mathbf{x}) = \nabla g(\mathbf{x})$. Here, apply methods (1.8) with $\delta_1 = 1, \delta_2 = -1$ (OPM) , and compare it with Newton method (NM), Chebyshev method (CM),

Halley method (HM), super-Halley method (SHM), the methods given by (1.6) with $\delta_1 = 0.9$ (KM) and method given by (1.7) (VCM). In numerical tests, the stopping criterion of each method is $\|\mathbf{x}_k - \mathbf{x}^*\|_2 \leqslant 1e - 15$, where $\mathbf{x}^* = (1, 1, \cdots, 1)^T$ is the exact solution. We choose $m = 50$ and $\mathbf{x}_0 = 1.2\mathbf{x}^*$. Listed in Table 2 are the iterative errors ($\|\mathbf{x}_k - \mathbf{x}^*\|_2$) of various methods. From Table 2 , we know that as tested here, OPM converges more rapidly than the others.

For the nonlinear systems of m equations in m variables, considering the efficiency index $\sigma^{1/\tau}$ [1], where σ is the order of method and τ represents the number of evaluations for the required scalar component functions per iteration, we list the efficiency index for various methods in Table 3. Moreover, we also compare the order of various methods, the numbers for the required function, first derivative and second derivative evaluations per iteration.

From Table 3, we can know that the order of methods (1.8) is higher than the others. The efficiency index of methods (1.8) is always higher than the ones of Newton method, method (1.7) and Chebyshev-Halley methods (including Chebyshev method, Halley method, super-Halley method). When $m \geq 2$, the efficiency index of methods (1.8) is higher than the one of methods (1.6). Since methods (1.8)do not need to compute the second derivative, when the computational cost of F'' is large or F'' is hard to compute, methods (1.8) are more efficient than methods (1.6).

5 Conclusions

This paper considers the semilocal convergence for multi-point improved super-Halley-type methods in Banach space. To solve the problem that F'' is neither Lipschitz nor Hölder continuous, the Lipschitz continuity of F'' used in reference [6] is replaced by its generalized condition, and the latter is weaker than the former. The

R-order of these methods is proved to be at least $3+2q$ with the generalized continuity condition of F'', where $q \in [0, 1]$. Especially, when F'' is Lipschitz continuous, The R-order becomes five, which is higher than the one of super-Halley method.

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