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An improved full-Newton step $O(n)$ infeasible interior-point method for horizontal linear complementarity problem

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Abstract We present an improved version of an infeasible interior-point method for horizontal linear complementarity problem (J. Optim. Theory Appl.,**161**(3),853–869, 2014). In the earlier version, each iteration consisted of one so-called feasibility and a few centering steps. Here, each iteration consists of only a feasibility step, whereas the iteration bound improves the earlier bound.

Keywords Horizontal linear complementarity problem · Infeasible interior-point method · Polynomial complexity

Mathematics Subject Classification (2010) MSC 90C33 · MSC 90C51

1 Introduction

Various extensions and generalizations of solution approaches for horizontal linear complementarity problem (HLCP) have been investigated. Among them, after the popular paper of Karmarkar [5], the interior-point methods (IPMs) earned more attention than other methods. IPMs are divided into two categories: feasible IPMs and infeasible IPMs (IIPMs). Feasible IPMs start with a strictly feasible interior-point and maintain feasibility during the solution process. IIPMs begin from a positive point and feasibility is reached as the optimality is approached. In [1, 3, 4, 7] the authors

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proposed some feasible IPMs for solving HLCP. Zhang [14] presented a class of IIPMs for HLCP and showed that the algorithm has $O(n^2 \log \frac{1}{\epsilon})$ iteration complexity. Stoer et al. [13] described a short-step IIPM of predictor-corrector type for solving HLCP.

The primal-dual full-Newton step feasible IPM for linear optimization (LO) was first analyzed by Roos et al. [11] and was later extended to infeasible version by Roos [10]. For a comprehensive study of IIPM and a motivation for using full-Newton steps methods we refer to [10, 11]. Kheirfam [6] extended both versions of the method to HLCP based on a new proximity measure, and developed a different analysis from the mentioned works for full-Newton step feasible IPMs and IIPMs. The obtained iteration bounds coincide with the currently best known iteration bounds for HLCP.

Motivated by recent developments on IIPMs, we present an improved and simplified version of an IIPM with full-Newton step for solving HLCP introduced by Kheirfam [6]. The new algorithm starts from an infeasible point, located in a small neighborhood of the central path of a perturbed HLCP. Then, after a full-Newton step the new iterate is well-centered for the new perturbed HLCP. This kind of strategy reduces the number of iterations and the resulting complexity coincides with the best known bound.

The following lemma is fundamental for the analysis of the IIPM proposed in this paper, its proof is exactly similar to the proof of the lemma A.1 in [12].

Lemma 1 *Let $a, b \in R^n$ and*

$$f(a,b) = \sum_{i=1}^{n} u(a_i b_i),$$

where $u(x) = 1 + x + \frac{1}{1+x} - 2,\ x > -1$. If $\|a\|^2 + \|b\|^2 = 2r^2$, with $r \in [0, 1)$, then

$$f(a,b) \leq (n-1)u(0) + \max\{u(r^2), u(-r^2)\}.$$

Above Lemma enables us to obtain an upper bound for the proximity measure after a full-Newton step. In the terminologies of [6], it means that after a feasibility step the new iterates are well-centered. The main advantage of the new algorithm is that each iteration requires only one feasibility step, whereas the previous algorithm needed three additional centering steps in each (main) iteration.

Given two matrices $Q, R \in R^{n\times n}$, and a vector $q \in R^n$, the HLCP consists in finding a pair of vectors $(x, s) \in R^{2n}$ such that

$$(P) \qquad Qx + Rs = q,\ \ x, s \geq 0,\ \ xs = 0.$$

Let $\kappa \geq 0$ be a given constant. We assume that the HLCP is $P_*(\kappa)$ in the sense that

$$Qx + Rs = 0 \text{ implies } x^T s \geq -4\kappa \sum_{i \in I_+} x_i s_i,$$

where $I_+ = \{i : x_i s_i \geq 0\}$. If the above condition is satisfied, then we say that (Q, R) is a $P_*(\kappa)$-pair. In accordance with the available results on IIPMs, e.g., see [6], it is assumed that there exists a solution (x^*, s^*) such that $\|x^*\|_\infty \leq \rho_p$ and $\|s^*\|_\infty \leq \rho_d$, where ρ_p and ρ_d are sufficiently large positive constants. In our algorithm, the initial iterates will be $(x^0, s^0) = (\rho_p e, \rho_d e)$. In this case, we have

$$0 \leq x^0 - x^* \leq \rho_p e, \quad 0 \leq s^0 - s^* \leq \rho_d e.$$

The remainder of our work is organized as follows: After a brief introduction to the perturbed problem and its central path, the algorithm is presented in Section 2. Section 3 contains the analysis of the new algorithm. In Section 3.1 we derive an upper bound for the proximity measure after a step. This result depends on Lemma 1 and expresses this bound in terms of a quantity $\omega(v)$. One should note that the definition of $\omega(v)$ slightly differs from $\omega(v)$ in [6]. The Sections 3.2, 3.3 and 3.4 serve to derive an upper bound for $\omega(v)$ and complexity analysis. Finally, we reported some numerical results in Section 4.

2 Infeasible full-Newton step IPM

In the case of an IIPM, we call the pair (x, s) an ϵ-solution of $P_*(\kappa)$-HLCP if the 2-norm of the residual vector $q - Qx - Rs$ does not exceed ϵ, and also $x^T s \leq \epsilon$. In this section, we present an infeasible-start algorithm that guarantees an ϵ-solution of $P_*(\kappa)$-HLCP, if it exists, or establishes that no such solution exists.

2.1 The perturbed problem

Denote the initial residual vector r_q^0, as $r_q^0 := q - Qx^0 - Rs^0$. In general, $r_q^0 \neq 0$. For any ν, with $0 < \nu \leq 1$, we consider the perturbed problem (P_ν), defined by

$$(P_\nu) \qquad q - Qx - Rs = \nu r_q^0, \quad (x, s) \geq 0.$$

Note that if $\nu = 1$, then $(x, s) = (x^0, s^0)$ yields a strictly feasible solution of (P_ν), which implies that if $\nu = 1$, then (P_ν) satisfies the interior point condition (IPC). Without proof we cite the following result ([6], Lemma 3.1).

Lemma 2 *Let the original problem* (P) *be feasible and* $0 < \nu \leq 1$. *Then, the perturbed problem* (P_ν) *satisfies the IPC.*

2.2 The central path of the perturbed problem

Let (P) be feasible and $0 < \nu \leq 1$. Then, Lemma 2 implies that the problem (P_ν) satisfies the IPC, for $0 < \nu \leq 1$, and hence its central path exists. This means that the system

$$q - Qx - Rs = \nu r_q^0, \quad x, s \geq 0, \tag{1}$$

$$xs = \mu e, \tag{2}$$

has a unique solution for any $\mu > 0$, as the μ-center of the perturbed problem (P_ν). In what follows, the parameters μ and ν always satisfy the relation $\mu = \nu\mu^0$.

2.3 An iteration of our algorithm

We just established that if $\nu = 1$ and $\mu = \mu^0$, then $(x, s) = (x^0, s^0)$ is the μ-center of the perturbed problem (P_ν). As stated before the initial iterates are given by

$$x^0 = \rho_p e,\ \ s^0 = \rho_d e,\ \ \mu^0 = \rho_p \rho_d.$$

Let (x, s) be a feasible solution of (P_ν), and $\mu = \nu\mu^0$. Then, we measure proximity to the μ-center of the perturbed problem (P_ν) by the quantity

$$\delta(x, s; \mu) := \delta(v) := \frac{1}{2}\|v^{-1} - v\|, \quad \text{where} \quad v := \sqrt{\frac{xs}{\mu}}. \tag{3}$$

As an immediate consequence, we have the following lemma.

Lemma 3 (Lemma II.62 in [11]) *Let $\delta := \delta(v)$ be given by (3). Then, for each i,*

$$-\delta + \sqrt{1+\delta^2} \le v_i \le \delta + \sqrt{1+\delta^2}.$$

Initially, we have $\delta(x, s; \mu) = \delta(x^0, s^0; \mu^0) = 0$. In what follows, we assume that at the start of each iteration, just before the μ-update, $\delta(x, s; \mu) \le \tau$ with $\tau > 0$. This certainly holds at the start of the first iteration.

Now we briefly describe one (main) iteration of our algorithm. Suppose that for some $\mu \in (0, \mu^0]$ we have x and s satisfying the feasibility condition (1) for $\nu = \frac{\mu}{\mu^0}$ and such that $\delta(x, s; \mu) \le \tau$. We reduce μ to $\mu^+ = (1-\theta)\mu$ with $\theta \in (0, 1)$, and find new iterates x^+ and s^+ satisfying (1), with μ replaced by μ^+ and ν by $\nu^+ = \frac{\mu^+}{\mu^0}$, such that $\delta(x^+, s^+; \mu^+) \le \tau$. Note that $\nu^+ = (1-\theta)\nu$. So, the relation $\mu = \nu\mu^0$ is maintained in every iteration.

We proceed by describing the search direction in the algorithm, which is the same as in [6], namely the unique solution of the following system

$$\begin{aligned} Q\Delta x + R\Delta s &= \theta\nu r_q^0, \\ s\Delta x + x\Delta s &= (1-\theta)\mu e - xs, \end{aligned} \tag{4}$$

where r_q^0 denotes the initial residual vector. After a full Newton step the iterates are given by

$$x^+ := x + \Delta x, \quad s^+ := s + \Delta s. \tag{5}$$

We conclude that iterates (x^+, s^+) satisfy the affine equation in (1), with ν replaced by ν^+. In the analysis, we should also guarantee that x^+ and s^+ are positive and satisfy $\delta(x^+, s^+; \mu^+) \le \tau$. Proving this is the crucial part in the analysis of the

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Primal – Dual Infeasible IPM

Input :
  Accuracy parameter $\epsilon > 0$;
  barrier update parameter $\theta,\ 0 < \theta < 1$;
  threshold parameter $\tau > 0$.
begin
  $x := \rho_p e;\ s := \rho_d e;\ \mu := \nu \rho_p \rho_d;\ \nu = 1;$
  while $\max(x^T s, \|r_q\|) > \epsilon$ **do**
  begin
    Newton step :
      $(x, s) := (x, s) + (\Delta x, \Delta s);$
    update of μ and ν :
      $\mu := (1 - \theta)\mu;$
      $\nu := (1 - \theta)\nu;$
  end
end

Fig. 1 The algorithm

algorithm.

2.4 The algorithm

A formal description of the new algorithm is given in Fig. 1.

3 Analysis of the algorithm

Let (x, s) be the iterate at the start of an iteration, and assume $\delta(x, s; \mu) \leq \tau$.

3.1 Upper bound for $\delta(v^+)$

As established in Section 2.3, the full-Newton step generates new iterates (x^+, s^+) that satisfy the feasibility condition for (P_{ν^+}), except for possibly the nonnegativity constraints. A crucial element in the analysis is to show that, after the full-Newton step, $\delta(x^+, s^+; \mu^+) \leq \tau$.

Defining

$$d_x := \frac{d\Delta x}{\sqrt{\mu^+}}, \quad d_s := \frac{d^{-1}\Delta s}{\sqrt{\mu^+}}, \quad \text{where} \quad d := \sqrt{\frac{s}{x}}, \tag{6}$$

we have, using the second equation of (4) and (6),

$$\begin{aligned} x^+ s^+ &= xs + (s\Delta x + x\Delta s) + \Delta x \Delta s = (1-\theta)\mu e + \Delta x \Delta s \\ &= (1-\theta)\mu e + \mu^+ d_x d_s = (1-\theta)\mu(e + d_x d_s). \end{aligned} \tag{7}$$

Lemma 4 *The full-Newton step is strictly feasible if and only if* $e + d_x d_s > 0$.

Proof The proof is similar to the proof of Lemma II.48 in [11], and is therefore omitted. □

Corollary 1 *The iterates* (x^+, s^+) *are strictly feasible if* $\|d_x d_s\|_\infty \leq 1$.

Proof By Lemma 4, x^+ and s^+ are strictly feasible if and only if $e + d_x d_s > 0$. Since the last inequality holds if $\|d_x d_s\|_\infty \leq 1$, the corollary follows. □

In the sequel, we use the notation

$$\omega(v) := \frac{1}{2}\left(\|d_x\|^2 + \|d_s\|^2\right). \tag{8}$$

It follows that

$$\|d_x d_s\|_\infty \leq \|d_x d_s\| \leq \|d_x\|\|d_s\| \leq \frac{1}{2}\left(\|d_x\|^2 + \|d_s\|^2\right) = \omega(v). \tag{9}$$

Corollary 2 *If* $\omega(v) < 1$, *then the iterates* (x^+, s^+) *are strictly feasible.*

Proof Due to (9), $\omega(v) < 1$ implies $\|d_x d_s\|_\infty < 1$. By Corollary 1 this implies the desired result. □

Assuming $\omega(v) < 1$, which guarantees strict feasibility of the iterates (x^+, s^+), we proceed by deriving an upper bound for $\delta(x^+, s^+; \mu^+)$. By definition (3), we have

$$\delta(x^+, s^+; \mu^+) = \frac{1}{2}\left\|v^+ - (v^+)^{-1}\right\|, \quad \text{where } v^+ = \sqrt{\frac{x^+ s^+}{\mu^+}}.$$

In what follows, we denote $\delta(x^+, s^+; \mu^+)$ shortly by $\delta(v^+)$.

Lemma 5 *Let* $\omega(v) < 1$. *Then, we have*

$$4\delta(v^+)^2 \leq (n-1)u(0) + \max\left\{u(\omega(v)), u(-\omega(v))\right\}.$$

Proof After dividing both sides in (7) by μ^+ we get

$$(v^+)^2 = \frac{(1-\theta)\mu(e + d_x d_s)}{\mu^+} = e + d_x d_s.$$

Hence, we have

$$\begin{aligned} 4\delta(v^+)^2 &= \left\|\sqrt{e + d_x d_s} - \frac{1}{\sqrt{e + d_x d_s}}\right\|^2 \\ &= \left\|\sqrt{e + d_x d_s}\right\|^2 + \left\|\frac{1}{\sqrt{e + d_x d_s}}\right\|^2 - 2n \\ &= \sum_{i=1}^n \left(1 + d_{xi} d_{si} + \frac{1}{1 + d_{xi} d_{si}} - 2\right) = \sum_{i=1}^n u(d_{xi} d_{si}). \end{aligned}$$

Using Lemma 1 this implies that

$$4\delta(v^+)^2 \leq (n-1)u(0) + \max\left\{u(\omega(v)), u(-\omega(v))\right\},$$

proving the lemma. □

3.2 Upper bound for $\omega(v)$

We start by finding some bounds for the unique solution of the linear system (4).

Lemma 6 (Lemma 3.3 in [9]) *If HLCP is $P_*(\kappa)$, then for any $a, \tilde{b} \in \mathbb{R}^n$ and any $z = (x^T, s^T)^T \in \mathbb{R}^{2n}_{++}$ the linear system*

$$Qu + Rv = \tilde{b}, \quad su + xv = a, \tag{10}$$

has a unique solution $w := (u^T, v^T)^T$ and the following inequality is satisfied:

$$\|w\|_z \leq \sqrt{1+2\kappa}\|\tilde{a}\|_2 + (1+\sqrt{2+4\kappa})\,\zeta(z,\tilde{b}),$$

where

$$\tilde{a} = (xs)^{-\frac{1}{2}}a,\, D = X^{-\frac{1}{2}}S^{\frac{1}{2}}, \quad \|w\|_z^2 = \|(u^T, v^T)^T\|_z^2 = \|Du\|^2 + \|D^{-1}v\|^2,$$

and

$$\zeta(z,\tilde{b})^2 = \min\{\|(\tilde{u}^T, \tilde{v}^T)^T\|_z^2 : Q\tilde{u} + R\tilde{v} = \tilde{b}\} = \tilde{b}^T(QD^{-2}Q^T + RD^2R^T)^{-1}\tilde{b}.$$

Comparing system (10) with the system (4) and considering $(u, v) = (\Delta x, \Delta s)$, $\tilde{b} = \theta\nu r_q^0$ and $a = (1-\theta)\mu e - xs$ in (10), we get

$$\begin{aligned}\|D\Delta x\|^2 + \|D^{-1}\Delta s\|^2 &\leq \Big(\sqrt{1+2\kappa}\|(xs)^{-\frac{1}{2}}((1-\theta)\mu e - xs)\| \\ &\qquad +(1+\sqrt{2(1+2\kappa)})\zeta(z,\theta\nu r_q^0)\Big)^2 \\ &\leq \Big(\sqrt{\mu(1+2\kappa)}\left((1-\theta)\|v^{-1}-v\| + \theta\|v\|\right) \\ &\qquad +(1+\sqrt{2(1+2\kappa)})\theta\nu\zeta(z,r_q^0)\Big)^2,\end{aligned} \tag{11}$$

where the last inequality follows by the following inequality

$$\begin{aligned}\left\|(xs)^{-\frac{1}{2}}((1-\theta)\mu e - xs)\right\| &= \left\|\frac{(1-\theta)\mu e - xs}{\sqrt{xs}}\right\| = \left\|\frac{(1-\theta)\mu e - \mu v^2}{\sqrt{\mu}v}\right\| \\ &= \sqrt{\mu}\|(1-\theta)v^{-1} - v\| \leq \sqrt{\mu}\left((1-\theta)\|v^{-1}-v\| + \theta\|v\|\right).\end{aligned}$$

If $\delta := \delta(v)$ is given, then $\|v\|$ is maximal if $v \geq e$ and all elements of v are equal to [12]

$$\frac{\delta}{\sqrt{n}} + \sqrt{1+\frac{\delta^2}{n}},$$

which implies that

$$\|v\| = \sqrt{n}\left(\frac{\delta}{\sqrt{n}} + \sqrt{1+\frac{\delta^2}{n}}\right). \tag{12}$$

By using (3) and (12) we get

$$(1-\theta)\|v^{-1}-v\| + \theta\|v\| \leq 2(1-\theta)\delta + \theta\sqrt{n}\left(\frac{\delta}{\sqrt{n}} + \sqrt{1+\frac{\delta^2}{n}}\right). \tag{13}$$

Since

$$r_q^0 = q - Qx^0 - Rs^0 = Q(x^* - x^0) + R(s^* - s^0),$$

by definition of $\zeta(z, r_q^0)$ and Lemma 2, we obtain

$$\begin{aligned}
\zeta(z, r_q^0)^2 &\leq \|D(x^*-x^0)\|^2 + \|D^{-1}(s^*-s^0)\|^2 \\
&\leq \rho_p^2\|De\|^2 + \rho_d^2\|D^{-1}e\|^2 = \rho_p^2\left\|\sqrt{\frac{s}{x}}\right\|^2 + \rho_d^2\left\|\sqrt{\frac{x}{s}}\right\|^2 \\
&\leq \frac{\rho_p^2}{\mu}\left\|\frac{s}{v}\right\|_1^2 + \frac{\rho_d^2}{\mu}\left\|\frac{x}{v}\right\|_1^2 \\
&\leq \frac{1}{\mu(-\delta+\sqrt{1+\delta^2})}\left(\rho_p^2\|s\|_1^2 + \rho_d^2\|x\|_1^2\right).
\end{aligned} \tag{14}$$

Using $D\Delta x = \sqrt{\mu^+}d_x$, $D^{-1}\Delta s = \sqrt{\mu^+}d_s$, and substituting two bounds (14) and (13) into (11) and using the definition of $\omega(v)$, we obtain

$$\omega(v) \leq \left(\frac{\sqrt{1+2\kappa}\left((2-\theta)\delta + \theta\sqrt{n+\delta^2}\right)}{\sqrt{2(1-\theta)}} + \frac{(1+\sqrt{2(1+2\kappa)})\theta}{\mu^0\sqrt{2(1-\theta)}}\sqrt{\frac{\rho_p^2\|s\|_1^2 + \rho_d^2\|x\|_1^2}{-\delta+\sqrt{1+\delta^2}}}\right)^2.$$

Lemma 7 (Lemma 4.5 in [6]) *Let (x, s) be feasible for the perturbed problem (P_ν) and $(x^0, s^0) = (\rho_p e, \rho_d e)$. Then for any solution (x^*, s^*) of (P), we have*

$$\nu\left(x^T s^0 + s^T x^0\right) \leq (1+4\kappa)\left(\nu^2 (x^0)^T s^0 + \nu(1-\nu)\left((s^0)^T x^* + (x^0)^T s^*\right) + x^T s\right).$$

Since $x^0 = \rho_p e$, $s^0 = \rho_d e$, $\|x^*\|_\infty \leq \rho_p$ and $\|s^*\|_\infty \leq \rho_d$, we have

$$(s^0)^T x^* + (x^0)^T s^* \leq 2n\rho_p\rho_d, \quad (x^0)^T s^0 = n\rho_p\rho_d.$$

Hence, by Lemma 7, $x^T s = \mu e^T v^2 = \mu \|v\|^2$ and (12), we obtain

$$\begin{aligned} x^T s^0 + s^T x^0 &\leq (1+4\kappa)\left(2n\rho_p\rho_d + \frac{x^T s}{\nu}\right) = (1+4\kappa)\left(2n\rho_p\rho_d + \mu^0\|v\|^2\right) \\ &\leq (1+4\kappa)\left(2 + \left(\frac{\delta}{\sqrt{n}} + \sqrt{1+\frac{\delta^2}{n}}\right)^2\right) n\rho_p\rho_d. \end{aligned}$$

This implies that

$$\|x\|_1 \leq (1+4\kappa)\left(2 + \left(\frac{\delta}{\sqrt{n}} + \sqrt{1+\frac{\delta^2}{n}}\right)^2\right) n\rho_p,$$

and

$$\|s\|_1 \leq (1+4\kappa)\left(2 + \left(\frac{\delta}{\sqrt{n}} + \sqrt{1+\frac{\delta^2}{n}}\right)^2\right) n\rho_d.$$

Substitution the bounds of $\|x\|_1$ and $\|s\|_1$ yields

$$\begin{aligned} \omega(v) \leq \Bigg(&\frac{\sqrt{1+2\kappa}\left((2-\theta)\delta + \theta\sqrt{n+\delta^2}\right)}{\sqrt{2(1-\theta)}} \\ &+ \frac{(1+4\kappa)\left(1+\sqrt{2(1+2\kappa)}\right) n\theta \left(2 + \left(\frac{\delta}{\sqrt{n}} + \sqrt{1+\frac{\delta^2}{n}}\right)^2\right)}{\sqrt{(1-\theta)(-\delta+\sqrt{1+\delta^2})}} \Bigg)^2. \end{aligned} \tag{15}$$

3.3 Values for θ and τ

Our aim is to find a positive number τ such that if $\delta := \delta(v) \leq \tau$, then $\delta(v^+) \leq \tau$. By Lemma 5, this will hold if $\omega := \omega(v) < 1$ and

$$\frac{1}{2}\sqrt{(n-1)u(0) + \max\{u(\omega), u(-\omega)\}} \leq \tau. \tag{16}$$

Assuming $\delta(v) \leq \tau$, we therefore need to find τ such that the above inequalities hold, with θ as large as possible. We choose

$$\theta = \frac{1}{27n(1+2\kappa)^2}, \quad \tau = \frac{1}{6(1+2\kappa)}. \tag{17}$$

Using $\delta \leq \tau$, we obtain from (15) that

$$\left(\frac{\sqrt{1+2\kappa}\left((2-\theta)\delta+\theta\sqrt{n+\delta^2}\right)}{\sqrt{2(1-\theta)}}\right.$$
$$\left.+\frac{(1+4\kappa)\left(1+\sqrt{2(1+2\kappa)}\right)n\theta\left(2+\left(\frac{\delta}{\sqrt{n}}+\sqrt{1+\frac{\delta^2}{n}}\right)^2\right)}{\sqrt{(1-\theta)(-\delta+\sqrt{1+\delta^2})}}\right)^2$$
$$\leq\left(\frac{\sqrt{1+2\kappa}\left((2-\theta)\tau+\theta\sqrt{n+\tau^2}\right)}{\sqrt{2(1-\theta)}}\right.$$
$$\left.+\frac{(1+4\kappa)\left(1+\sqrt{2(1+2\kappa)}\right)n\theta\left(2+\left(\frac{\tau}{\sqrt{n}}+\sqrt{1+\frac{\tau^2}{n}}\right)^2\right)}{\sqrt{(1-\theta)(-\tau+\sqrt{1+\tau^2})}}\right)^2.$$

We define the function $h(n,\tau)$ as follows

$$h(n,\tau)=\left(\frac{\sqrt{1+2\kappa}\left((2-\theta)\tau+\theta\sqrt{n+\tau^2}\right)}{\sqrt{2(1-\theta)}}\right.$$
$$\left.+\frac{(1+4\kappa)\left(1+\sqrt{2(1+2\kappa)}\right)n\theta\left(2+\left(\frac{\tau}{\sqrt{n}}+\sqrt{1+\frac{\tau^2}{n}}\right)^2\right)}{\sqrt{(1-\theta)(-\tau+\sqrt{1+\tau^2})}}\right)^2,$$

where $\theta=\frac{1}{27n(1+2\kappa)^2}$. This function, which provides an upper bound for ω, decreases when n increases. In order to have (16), when defining

$$g_+(\omega):=\frac{1}{2}\sqrt{(n-1)u(0)+u(\omega)},\quad g_-(\omega):=\frac{1}{2}\sqrt{(n-1)u(0)+u(-\omega)},$$

we need to show that $g_+(\omega)\leq\tau$ and $g_-(\omega)\leq\tau$ holds for all ω such that $\omega\leq h(n,\tau)$. Note that if n increases, then $h(n,\tau)$ converges to the positive value

$$h(\infty,\tau)=\left(\frac{\sqrt{2}}{6(1+2\kappa)}+\frac{\sqrt{6}(1+4\kappa)(\sqrt{2(1+2\kappa)}-1)}{9(1+2\kappa)^{\frac{3}{2}}\sqrt{\sqrt{1+(6+12\kappa)^2}-1}}\right)^2\leq 0.2779.$$

We have

$$(n-1)u(0)=(n-1)\times 0=0.$$

On the other hand, we have

$$u(\omega) = 1 + \omega + \frac{1}{1+\omega} - 2 = \frac{\omega^2}{1+\omega}, \quad u(-\omega) = \frac{\omega^2}{1-\omega}.$$

It follows that

$$\frac{1}{2}\sqrt{(n-1)u(0) + \max\{u(\omega), u(-\omega)\}} = \frac{\omega}{2\sqrt{1-\omega}}.$$

One easily verifies that the last expression is less than or equal to $\tau = \frac{1}{6(1+2\kappa)}$ if and only if

$$\omega \leq 2\tau(\sqrt{1+\tau^2} - \tau) = \frac{\sqrt{36(1+2\kappa)^2+1} - 1}{18(1+2\kappa)^2} \leq 0.2824.$$

Since $h(\infty, \tau) \leq 0.2779$, which is strictly less than 0.2824, it follows that (16) is strictly satisfied if n goes to infinity.

Based on the above analysis, we may state the following result without further proof.

Theorem 1 *If θ and τ are given by* (17)*, then there exists a number N such that for all $n \geq N$ the algorithm is well defined, in the sense that the property $\delta(x, s; \mu) \leq \tau$ is maintained during the iterations.*

3.4 Complexity analysis

We have found that if n is large enough and at the start of an iteration the iterates satisfy $\delta(x, s, \mu) \leq \tau$, and τ and θ are as defined in (17), then after the full-Newton step, the new iterates satisfy $\delta(x^+, s^+; \mu^+) \leq \tau$. This establishes the algorithm to be well-defined.

In each iteration, both the value of $x^T s$ and the norm of the residual vector are reduced by the factor $1 - \theta$. Hence, the total number of main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\{(x^0)^T s^0, \|r_q^0\|\}}{\epsilon}.$$

Since $\theta = \frac{1}{27n(1+2\kappa)^2}$, this yields the following result.

Theorem 2 *Let* (P) *be feasible and $\rho_p > 0$ and $\rho_d > 0$ such that $\|x^*\|_\infty \leq \rho_p$ and $\|s^*\|_\infty \leq \rho_d$ for some solution (x^*, s^*) of* (P)*. If n is large enough, then after at most*

$$27n(1+2\kappa)^2 \log \frac{\max\{(x^0)^T s^0, \|r_q^0\|\}}{\epsilon}$$

iterations the algorithm finds an ϵ-solution of (P).

Remark 1 It is worth noting that this result improves the iteration bound in ([6], Theorem 4.1) with factor 2.2.

Table 1 The number of iterations and CPU

Problem	$\|x^*\|_\infty$	$\|s^*\|_\infty$	Algor. [6]			New algor.	
			Ou. Iter.	T. Iter.	Time	Iter.	Time
$M_{2,3}$	25.4602	6.4640	1066	2123	0.541832	1442	0.803425
$M_{2,5}$	27.1710	7.0309	1833	3953	1.017019	2478	1.415518
$M_{2,7}$	27.7903	7.2359	2617	4687	1.575742	3536	2.312912
$M_{2,10}$	28.2184	7.3775	3814	7489	2.695927	5152	3.399544
$M_{2,15}$	28.5321	7.4813	5847	11763	2.936233	7896	6.973262
$M_{2,20}$	28.6831	7.5312	7986	14461	3.950233	10784	8.803626
$M_{1,3}$	33.6392	0.0000	1066	2935	0.716764	1442	0.840011
$M_{1,5}$	33.6803	0.0000	1833	4812	1.233801	2478	1.396211
$M_{1,7}$	33.6978	0.0000	2617	6649	2.014513	3536	2.044211
$M_{1,10}$	33.7109	0.0000	3814	9220	2.822196	5152	3.373976
$M_{1,15}$	33.7210	0.0000	5847	13347	4.837527	7896	5.960868
$M_{1,20}$	33.7261	0.0000	7913	15722	4.537763	10686	9.758106

4 Numerical results

The algorithm is tested on a number of LCP problems from the literature [2, 8]. We have written MATLAB codes for proposed algorithm and the algorithm presented in [6]. The numerical experiments are implemented by using MATLAB version 7.8.0.347 (R2009a) on a PC with 2 GB RAM under Windows XP. In our experiments, we choose $x = \rho_p e = 50e$, $s = \rho_d e = 40e$ and $\mu = \rho_p \rho_d$ as the starting data. We set $\epsilon = 10^{-4}$, and we take the set of parameters $\tau = 5 \times 10^{-6}$ and $\theta = \frac{1}{20n}$ for the algorithm in [6] and $\theta = \frac{1}{27n}$ for proposed algorithm. The algorithms were terminated when the duality gap satisfied $x^T s \le \epsilon = 10^{-4}$. The numbers of iterations (Iter.) and time (in second) (CPU) required for the two algorithms on two set of problems of various sizes with the corresponding matrices and $q = -e$, as given below, are noted in Table 1:

$$M_{1,n} = \begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \dots & 2 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}, \quad M_{2,n} = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & & \vdots \\ 2 & 6 & 10 & \cdots & 4(n-1)+1 \end{bmatrix}.$$

The executing time and the number of iterations for the two set of problems are given in Table 1, which shows that the number of iterations of the algorithm depends on the size of the corresponding matrix. The executing time of the algorithm increases as the size of the corresponding matrix is increased. However, based on the numerical results obtained, as shown in Table 1, the CPU times cost by the new algorithm are larger than by the algorithm given in [6], even when the new algorithm needs less iterations.

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