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# Complexity analysis and numerical implementation of a full-Newton step interior-point algorithm for LCCO

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**Abstract** In this paper, we present a primal-dual interior point algorithm for linearly constrained convex optimization (LCCO). The algorithm uses only full-Newton step to update iterates with an appropriate proximity measure for controlling feasible iterations near the central path during the solution process. The favorable polynomial complexity bound for the algorithm with short-step method is obtained, namely $O(\sqrt{n}\log\frac{n}{\epsilon})$ which is as good as the linear and convex quadratic optimization analogue. Numerical results are reported to show the efficiency of the algorithm.

**Keywords** Linearly constrained convex optimization · Interior point methods · Short-step primal-dual algorithms · Complexity of algorithms

**Mathematical Subject Classifications (2010)** 90C25 · 90C51

## 1 Introduction

In this paper, we consider the following linearly constrained convex optimization (LCCO) problem in its standard form called primal

$$(P) \quad \min f(x) \ \text{ s.t. } Ax = b, \ x \geq 0,$$

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and its Wolfe dual

$$(D) \quad \max \left[ b^T y + f(x) - x^T \nabla f(x) \right] \text{ s.t. } A^T y + z - \nabla f(x) = 0,\ z \geq 0,$$

where $f : \mathbf{R}^n \to \mathbf{R}$ is a convex and continuously differentiable function, and $\nabla f(x)$ is the gradient vector of $f(x)$, $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$.

The LCCO problem has many important applications in mathematical programming and diversified areas of engineering. It includes also the linear optimization (LO) problem and convex quadratic optimization (CQO) problem as special cases. For years, the LCCO has received a considerable attention from researchers and a variant of primal-dual interior point methods (IPMs) have been proposed for its solution since this last has beautiful properties such as polynomial complexity and numerical efficiency [6–8].

In the last decade many primal-dual IPMs for LO have been extended successfully to CQO, semidefinite optimization (SDO), complementarity problems (CP) and conic optimization (CO) problems.

Darvay [4], developed a predictor-corrector algorithm for LCCO where its search direction is based on an algebraic transformation namely the square root function applied to the nonlinear centering equations of the system, which defines the central path. The complexity result for this algorithm is established. We mention that this new technique of algebraic transformations was first introduced by Darvay [3] for LO and later on extended to CQO by Achache [1] and for LCCO by Zhang et al. [9]. And all of them proved that their short-step algorithm matches the best known iteration bound, namely $O\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$.

Recently, Achache and Goutali [2] presented a feasible primal-dual path-following interior point algorithm for CQO based only on full-Newton steps and a suitable proximity measure for controlling feasible iterations produced by the algorithm. They proved that the iteration bound of the short-step algorithm is $O(\sqrt{n} \log \frac{n}{\epsilon})$ which is as good as the CQO and LO analogue.

Motivated by their work, we propose a short-step feasible primal-dual interior point algorithm for LCCO. We adopt the analysis used in [2]. The favorable iteration bound, namely, $O(\sqrt{n} \log \frac{n}{\epsilon})$ for such short-step methods is obtained. Moreover, our analysis is straightforward to CQO analogue. We mention that our analysis is different from the one used in [1, 3, 9]. Finally numerical results are reported to show the efficiency of the algorithm.

The rest of the paper is organized as follows. In Section 2, the generic short step primal-dual interior point algorithm for LCCO is presented. In Section 3, detailed proofs of the complexity result are given. In Section 4, some numerical results are presented. In Section 5, we end the paper with a conclusion.

Throughout the paper, the following notations are used. $\mathbf{R}^n_{++}$ denotes the set of all positive vectors of $\mathbf{R}^n$. Given $x, y \in \mathbf{R}^n$, $x^T y = \sum_{i=1}^n x_i y_i$ is their inner product whereas $xy$ is the vector of their coordinatewise product. The 2-norm and $\infty$-norm of a vector $x$ are denoted by $\|x\|$ and $\|x\|_\infty$, respectively. Let $x, y \in \mathbf{R}^n_{++}$, $\sqrt{x} = \left(\sqrt{x_1}, \ldots, \sqrt{x_n}\right)^T$, $x^{-1} = \left(x_1^{-1}, \ldots, x_n^{-1}\right)^T$ and $\frac{x}{y} = \left(\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n}\right)^T$. Finally, $g(t) = O(f(t)) \Leftrightarrow g(t) \leq cf(t)$ for some positive constant $c$ where $g(t)$ and $f(t)$ are two positive real valued functions. The identity matrix and the vector of ones are denoted by $I$ and $e$, respectively.

## 2 The generic primal-dual algorithm for LCCO

Throughout the paper, we make the following assumptions on (*P*) and (*D*).

**Assumption 1** The matrix $A$ is of $rank(A) = m$.

**Assumption 2** Interior-Point-Condition (IPC). There exists $\left(x^0, y^0, z^0\right)$ such that

$$Ax^0 = b,\ x^0 > 0,\ A^T y^0 - \nabla f\left(x^0\right) + z^0 = c,\ z^0 > 0.$$

**Assumption 3** $f$ is a convex and twice continuously differentiable function. This implies that the hessian matrix $\nabla^2 f(x)$ of $f$ is positive semidefinite. Finding an optimal solution of (*P*) and (*D*) is equivalent to solving the following system, which represents the Karush-Khun-Tucker optimality conditions

$$\begin{cases} Ax & = b,\ x \geq 0, \\ A^T y + z - \nabla f(x) & = 0,\ z \geq 0, \\ xz & = 0. \end{cases} \tag{1}$$

The basic idea of the primal-dual IPMs is to replace the complementarity equation $xz = 0$ in (1) by the parameterized equation $xz = \mu e$. Thus we consider the system

$$\begin{cases} Ax & = b,\ x > 0, \\ A^T y + z - \nabla f(x) & = c,\ z > 0, \\ xz & = \mu e, \end{cases} \tag{2}$$

with $\mu > 0$.

### 2.1 The central path of LCCO

As $rank(A) = m$ and the IPC hold, then for a fixed $\mu > 0$ the system (2) has a unique solution denoted by $(x(\mu),\ y(\mu),\ z(\mu))$ (see [5]). We call $x(\mu)$ the $\mu-$center of (*P*) and $(y(\mu),\ z(\mu))$ the $\mu-$center of (*D*). The set of $\mu-$center defines a homotopy path, which is called the central path of (*P*) and (*D*). If $\mu$ goes to zero, then the limit of the central path exists and since the limit point satisfies the complementarity condition, the limit yields an optimal solution for both problems (*P*) and (*D*).

### 2.2 The Newton direction and proximity

Now, we proceed to describe a full-Newton step produced by the algorithm for a given $\mu > 0$. Applying Newton's method for (2) for a given feasible point $(x, y, z)$ i.e., the IPC condition holds, we get the following system

$$\begin{pmatrix} A & 0 & 0 \\ -\nabla^2 f(x) & A^T & I \\ Z & 0 & X \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mu e - Xz \end{pmatrix}, \tag{3}$$

where $X$ :=diag$(x)$, $Z$ :=diag$(z)$. Under our assumptions, the linear system has a unique solution $(\Delta x, \Delta y, \Delta z)$. Hence a full-Newton step is defined as

$$x_+ = x + \Delta x,\ y_+ = y + \Delta y \text{ and } z_+ = z + \Delta z.$$

For the analysis of the algorithm, we define also a norm-based proximity measure $\delta(xz;\mu)$ to the central-path as follows

$$\delta(xz;\mu) = \frac{1}{2}\left\| \sqrt{\left(\frac{xz}{\mu}\right)^{-1}} - \sqrt{\frac{xz}{\mu}} \right\|.$$

Thus we have

$$\delta(xz;\mu) = 0 \Leftrightarrow xz = \mu e.$$

Hence, the value of $\delta(xz;\mu)$ can be considered as a measure for the distance from a given pair $(x, y, z)$ to the $\mu$-center $(x(\mu),\ y(\mu),\ z(\mu))$. We use also a threshold value $\beta$ and we suppose that a strictly feasible starting point $(x^0, y^0, z^0)$ such that $\delta(x^0z^0;\mu^0) \le \beta$ for certain $\mu^0$ is known. This defines a $\beta$-neighborhood of the central-path. The details of the generic interior point algorithm is stated in the next sub-section.

### 2.3 Algorithm

Generic Primal-dual interior point algorithm for LCCO (Fig. 1).

## 3 Analysis of Algorithm 2.3

In this section, we show that Algorithm 2.3 solves the LCCO in polynomial time. For its analysis, we introduce the notation

$$v := \sqrt{\frac{xz}{\mu}},\ d := \sqrt{\frac{x}{z}}.$$

**Input**
An accuracy parameter $\epsilon > 0$;
a threshold parameter $0 < \beta < 1$( default $\beta = \frac{1}{\sqrt{2}}$);
a fixed barrier update parameter $0 < \theta < 1$ (default $\theta = \frac{1}{2\sqrt{n}}$);
a feasible point $(x^0, y^0, z^0)$ and $\mu^0$such that $\delta(x^0z^0;\mu^0) \le \beta$;
**begin**
  $x := x^0;\ y := y^0;\ z := z^0;\ \mu := \mu^0;$
  **While** $n\mu \ge \epsilon$ **do**
  **begin**
    Solve system (3) to obtain: $(\Delta x,\ \Delta y, \Delta z)$;
    Update $x = x + \Delta x,\ y = y + \Delta y,\ z = z + \Delta z$;
    $\mu := (1-\theta)\mu$;
  **end**
**end**

**Fig. 1** Algorithm 2.3

The vector $d$ is used to scale the vectors $x$ and $z$ to the same vector $v$

$$\frac{d^{-1}x}{\sqrt{\mu}} = \frac{dz}{\sqrt{\mu}} = v.$$

The scaled search directions are then

$$d_x := \frac{d^{-1}\Delta x}{\sqrt{\mu}}, \ d_z := \frac{d\Delta z}{\sqrt{\mu}}.$$

In addition, we have

$$x\Delta z + z\Delta x = \mu v(d_x + d_z), \tag{4}$$

and

$$\Delta x \Delta z = \mu d_x d_z. \tag{5}$$

Using the notations in (4) and (5), the linear system in (3) and the proximity become

$$\begin{pmatrix} \bar{A} & 0 & 0 \\ -\bar{H} & \bar{A}^T & I \\ I & 0 & I \end{pmatrix} \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v^{-1} - v \end{pmatrix} \tag{6}$$

where $\bar{A} = \sqrt{\mu} AD$ and $\bar{H} = D\nabla^2 f(x) D$ with $D := \text{diag}(d)$ and

$$\delta(v) := \delta(xz; \mu) = \frac{1}{2}\|v^{-1} - v\|.$$

Since $\bar{H}$ is a symmetric positive semidefinite matrix,we have

$$d_x^T d_z = d_x^T(\bar{H}d_x - \bar{A}^T d_y) = d_x^T \bar{H} d_x \geq 0.$$

This means that the directions are not orthogonal in LCCO, in contrast with LO case. Thus makes the analysis different.

The next technical lemma will be used later in the analysis of the algorithm.

**Lemma 3.1** *Let* $(d_x, d_y, d_z)$ *be a solution of* (6) *and if* $\delta := \delta(xz; \mu)$ *and* $\mu > 0$. *Then one has*

$$0 \leq d_x^T d_z \leq 2\delta^2 \tag{7}$$

*and*

$$\|d_x d_z\|_\infty \leq \delta^2, \|d_x d_z\| \leq \sqrt{2}\delta^2. \tag{8}$$

*Proof* Since $0 \leq d_x^T d_z$, the first part of the lemma follows immediately from (6) and the following equality

$$\|d_x\|^2 + \|d_z\|^2 + 2d_x^T d_z = \|d_x + d_z\|^2 = \left\|v^{-1} - v\right\|^2 = 4\delta^2.$$

For the second statement, since

$$d_x d_z = \frac{1}{4}\left((d_x + d_z)^2 - (d_x - d_z)^2\right)$$

and

$$\|d_x + d_z\|^2 = \|d_x - d_z\|^2 + 4d_x^T d_z$$

it follows on one hand that

$$\|d_x - d_z\| \leq \|d_x + d_z\|$$

and on the other hand

$$\begin{aligned}\|d_x d_z\|_\infty &= \frac{1}{4}\left\|(d_x+d_z)^2-(d_x-d_z)^2\right\|_\infty\\ &\le \frac{1}{4}\max\left(\|d_x+d_z\|_\infty^2,\|d_x-d_z\|_\infty^2\right)\\ &\le \frac{1}{4}\max\left(\|d_x+d_z\|^2,\|d_x-d_z\|^2\right)\\ &\le \frac{1}{4}\|d_x+d_z\|^2=\frac{1}{4}\left\|v^{-1}-v\right\|^2=\delta^2\end{aligned}$$

Hence

$$\|d_x d_z\|_\infty \le \delta^2.$$

For the last part of the second statement, we have

$$\begin{aligned}\|d_x d_z\|^2 &= e^T(d_x d_z)^2\\ &= \frac{1}{16}e^T((d_x+d_z)^2-(d_x-d_z)^2)^2,\\ &= \frac{1}{16}\left\|(d_x+d_z)^2-(d_x-d_z)^2\right\|^2\\ &\le \frac{1}{16}\left(\left\|(d_x+d_z)^2\right\|^2+\left\|(d_x-d_z)^2\right\|^2\right)\\ &\le \frac{1}{16}\left(\|d_x+d_z\|^4+\|d_x-d_z\|^4\right)\\ &\le \frac{1}{8}\|d_x+d_z\|^4=\frac{1}{8}\left\|v^{-1}-v\right\|^4=2\delta^4.\end{aligned}$$

This implies that

$$\|d_x d_z\| \le \sqrt{2}\delta^2.$$

This completes the proof. □

In the next lemmas, we show under the condition $\delta(xz;\mu) < 1$ that the full-Newton step is strictly feasible.

**Lemma 3.2** *Let $(x, z)$ be a strictly feasible primal-dual point. Hence $x_+ > 0$ and $z_+ > 0$ if and only if $e + d_x d_z > 0$.*

*Proof* Let $x_+ = x + \Delta x$ and $z_+ = z + \Delta z$ we introduce a step length $\alpha \in [0, 1]$ and we define

$$x(\alpha) = x + \alpha\Delta x \text{ and } z(\alpha) = z + \alpha\Delta z.$$

So $x^0 = x(0) = x,\ x(1) = x_+$ and one can introduce similar notations for $z$ hence $x^0 z^0 = xz > 0$. We have

$$x(\alpha)z(\alpha) = (x + \alpha\Delta x)(z + \alpha\Delta z\ ) = xz + \alpha\,(x\Delta z + z\Delta x) + \alpha^2\Delta x\Delta z.$$

Now by using again (4), we get

$$x(\alpha)z(\alpha) = xz + \alpha\,(\mu e - xz) + \alpha^2\Delta x\Delta z.$$

Suppose that $e + d_x d_z > 0$, we deduce that $\mu e + \Delta x \Delta z > 0$ which is equivalent to $\Delta x \Delta z > -\mu e$. Substitution gives

$$\begin{aligned} x(\alpha)z(\alpha) &> xz + \alpha(\mu e - xz) - \alpha^2 \mu e \\ &= (1-\alpha)\, xz + \left(\alpha - \alpha^2\right) \mu e \\ &= (1-\alpha)\, xz + \alpha(1-\alpha)\, \mu e. \end{aligned}$$

Since $xz$ and $\mu e$ are positive, it follows that $x(\alpha)z(\alpha) > 0$ for $\alpha \in [0, 1]$. Hence, none of the entries of $x(\alpha)$ and $z(\alpha)$ vanish for $\alpha \in [0, 1]$, and by continuity the vectors $x\,(1)$ and $z\,(1)$ must be positive. This completes the proof. □

For convenience, we may write

$$v_+^2 = \frac{x_+ z_+}{\mu}$$

and it is easy to have

$$v_+^2 = e + d_x d_z.$$

**Lemma 3.3** *If* $\delta := \delta(xz; \mu) < 1$. *Then the primal-dual full-Newton step is strictly feasible.*

*Proof* By Lemma 3.2, $x_+ > 0$ and $z_+ > 0$ are strictly feasible if $e + d_x d_z > 0$. So $e + d_x d_z > 0$ holds if $1 + (d_x d_z)_i > 0$ for all $i$. Since

$$\begin{aligned} 1 + (d_x d_z)_i &\geq 1 - |(d_x d_z)_i| \text{ for all } i \\ &\geq 1 - \|d_x d_z\|_\infty, \end{aligned}$$

it follows by (8) in Lemma 3.1, that:

$$1 - \|d_x d_z\|_\infty \geq 1 - \delta^2.$$

Thus $e + d_x d_z > 0$ holds if $\delta < 1$. This completes the proof. □

The local quadratic convergence of the full-Newton step to the target $(x(\mu), y(\mu), z(\mu))$ is proved in the following lemma.

**Lemma 3.4** *If* $\delta(xz; \mu) < 1$. *Then*

$$\delta_+ := \delta(x_+ z_+; \mu) \leq \frac{\delta^2}{\sqrt{2(1-\delta^2)}}.$$

*If* $\delta \leq \frac{1}{2}$, *then* $\delta_+ \leq \delta^2$, *which means the quadratic convergence of the full-Newton step.*

*Proof* We have

$$\begin{aligned} 4\delta_+^2 &= \left\| v_+^{-1} - v_+ \right\|^2 \\ &= \left\| v_+^{-1} \left( e - v_+^2 \right) \right\|^2. \end{aligned}$$

But $v_+^2 = e + d_x d_z$ and $v_+^{-1} = \frac{e}{\sqrt{e+d_x d_z}}$, then it follows that

$$4\delta_+^2 = \left\| \frac{d_x d_z}{\sqrt{e + d_x d_z}} \right\|^2 \leq \frac{\|d_x d_z\|^2}{1 - \|d_x d_z\|_\infty}.$$

In view of Lemma 3.1, the result follows. □

In the following lemma, we investigate the effect on the proximity measure of a full-Newton step followed by an update of the barrier parameter $\mu$.

**Lemma 3.5** *If* $\delta(xz; \mu) \leq \frac{1}{\sqrt{2}}$ *and* $\mu_+ = (1-\theta)\,\mu$ *where* $0 < \theta < 1$. *Then*

$$\delta^2(x_+z_+; \mu_+) \leq (1-\theta)\delta_+^2 + \frac{\theta^2(n+1)}{4(1-\theta)} + \frac{\theta}{2}.$$

*Furthermore, if* $\delta \leq \frac{1}{\sqrt{2}}$, $\theta = \frac{1}{2\sqrt{n}}$ *and* $n \geq 2$, *then we have*

$$\delta(x_+z_+; \mu_+) \leq \frac{1}{\sqrt{2}}.$$

*Proof* Let $v_+ = \sqrt{\frac{x_+z_+}{\mu}}$ and $\mu_+ = (1-\theta)\,\mu$. Then

$$\begin{aligned}
4\delta^2(x_+z_+; \mu_+) &= \left\| \left(\sqrt{\frac{\mu_+}{x_+z_+}}\right) - \left(\sqrt{\frac{x_+z_+}{\mu_+}}\right) \right\|^2 \\
&= \left\| \sqrt{1-\theta}\, v_+^{-1} - \frac{1}{\sqrt{1-\theta}} v_+ \right\|^2 \\
&= \left\| \sqrt{1-\theta}\,(v_+^{-1} - v_+) - \frac{\theta}{\sqrt{1-\theta}} v_+ \right\|^2 \\
&= (1-\theta) \left\| v_+^{-1} - v_+ \right\|^2 + \frac{\theta^2}{1-\theta} \|v_+\|^2 - 2\theta (v_+^{-1} - v_+)^T v_+ \\
&= 4(1-\theta)\delta_+^2 + \frac{\theta^2}{1-\theta} \|v_+\|^2 - 2\theta (v_+^{-1} - v_+)^T v_+ \\
&= 4(1-\theta)\delta_+^2 + \frac{\theta^2}{1-\theta} \|v_+\|^2 - 2\theta n + 2\theta \|v_+\|^2,
\end{aligned}$$

since $(v_+^{-1})^T v_+ = n$ and $v_+^T v_+ = \|v_+\|^2$. As

$$x_+^T z_+ = \mu \left(n + d_x^T d_z\right),$$

then by Lemma 3.1 (7) and if $\delta \leq \frac{1}{\sqrt{2}}$, it follows

$$\|v_+\|^2 = \frac{1}{\mu} x_+^T z_+ \leq (n+1).$$

Consequently,

$$\delta^2(x_+z_+;\mu_+) \leq (1-\theta)\delta_+^2 + \frac{\theta^2(n+1)}{4(1-\theta)} + \frac{\theta}{2}.$$

For the last statement the proof goes as follows. If $\delta \leq \frac{1}{\sqrt{2}}$, then $\delta_+^2 \leq \frac{1}{4}$ and this yields

$$\delta^2(x_+z_+;\mu_+) \leq \frac{\theta^2(n+1)}{4(1-\theta)} + \frac{(1-\theta)}{4} + \frac{\theta}{2}.$$

Letting $\theta = \frac{1}{2\sqrt{n}}$ so $\theta^2 = \frac{1}{4n}$ , it follows that

$$\delta^2(x_+z_+;\mu_+) \leq \frac{\frac{1}{4n}(n+1)}{4(1-\theta)} + \frac{(1-\theta)}{4} + \frac{\theta}{2}.$$

As $\frac{n+1}{4n} \leq \frac{3}{8}$ for all $n \geq 2$, we get

$$\delta^2(x_+z_+;\mu_+) \leq \frac{3}{32(1-\theta)} + \frac{(1-\theta)}{4} + \frac{\theta}{2}.$$

If $n \geq 2$, then $0 \leq \theta \leq \frac{1}{2\sqrt{2}}$. The function

$$f(\theta) = \frac{3}{32(1-\theta)} + \frac{(1-\theta)}{4} + \frac{\theta}{2}$$

is continuous and monotonic increasing on $0 \leq \theta \leq \frac{1}{2\sqrt{2}}$, consequently

$$f(\theta) \leq f\left(\frac{1}{2\sqrt{2}}\right) = 0.48341 < \frac{1}{2}, \forall\theta \in \left[0, \frac{1}{2\sqrt{2}}\right].$$

Hence $\delta(x_+z_+;\mu_+) \leq \frac{1}{\sqrt{2}}$. This completes the proof. □

We deduce from Lemma 3.5, that for the defaults $\theta = \frac{1}{2\sqrt{n}}$, and $\beta = \frac{1}{\sqrt{2}}$, the conditions $x > 0$, $z > 0$ and $\delta^2(x_+z_+;\mu_+) \leq \frac{1}{\sqrt{2}}$ are maintained during the solution process. Hence the algorithm is well defined.

Next lemma gives an upper bound of the duality gap after a full-Newton step.

**Lemma 3.6** *If* $\delta := \delta(xz;\mu) \leq \frac{1}{\sqrt{2}}$. *Then after a full-Newton step the duality gap satisfies:*

$$x_+^T z_+ \leq \mu(n+1). \tag{9}$$

*Proof* It follows straightforwardly from the proof in Lemma 3.5. □

The following lemma gives an upper bound for the total number of iterations produced by Algorithm 2.3.

**Lemma 3.7** *Let $x^{k+1}$ and $z^{k+1}$be the $(k+1)-th$ iteration produced by Algorithm 2.3 with $\mu := \mu_k$. Then $(x^{k+1})^T z^{k+1} \leq \epsilon$ if*

$$k \geq \left[\frac{1}{\theta} \log \frac{2\mu_0 n}{\epsilon}\right].$$

*Proof* It follows from Lemma 3.6 (9), that

$$\left(x^{k+1}\right)^T z^{k+1} \leq \mu_k(n+1)$$

with

$$\mu_k = (1-\theta)\mu_{k-1} = (1-\theta)^k \mu_0.$$

Then it follows that:

$$\left(x^{k+1}\right)^T z^{k+1} \leq (1-\theta)^k \mu_0(n+1).$$

Since $n+1 \leq 2n$ for all $n \geq 1$, then we have

$$\left(x^{k+1}\right)^T z^{k+1} \leq (1-\theta)^k 2\mu_0 n.$$

Thus the inequality $(x^k)^T z^k \leq \epsilon$ is satisfied if $(1-\theta)^k 2\mu_0 n \leq \epsilon$.
Taking logarithms, we obtain

$$k \log(1-\theta) \leq \log \epsilon - \log 2\mu_0 n$$

and using $-\log(1-\theta) \geq \theta$ for $0 < \theta < 1$, then $(1-\theta)^k 2\mu_0 n \leq \epsilon$ holds if

$$k\theta \geq \log \frac{2n\mu_0}{\epsilon}.$$

This completes the proof. □

**Theorem 3.1** *Let $\theta = \frac{1}{2\sqrt{n}}$ and $\mu^0 = \frac{1}{2}$. Then Algorithm 2.3 requires at most*

$$O\left(\sqrt{n} \log \frac{n}{\epsilon}\right)$$

*iterations.*

*Proof* By taking $\theta = \frac{1}{2\sqrt{n}}$ and $\mu^0 = \frac{1}{2}$ in Lemma 3.7, the proof is straightforward. □

## 4 Numerical results

In this section, we give some numerical results on two LCCO problems. Different values of the parameter barrier $\mu$ and the update barrier $\theta$ are presented to show their influence in reducing the number of iterations produced by our algorithm. The initial primal-dual point $(x^0, y^0, z^0)$ is chosen such that the interior point condition holds and the proximity $\delta(x^0 z^0; \mu^0)$ do not exceed the threshold $\beta$. The tolerance, the theoretical update barrier and the threshold used in the implementation are

**Table 1** Numerical results for Problem 1

| $\mu_0\backslash\theta$ | theoretical $\theta = \frac{1}{2\sqrt{n}}$ | relaxed $\theta = \frac{1}{\sqrt{n}}$ |
|---|---|---|
| 0.5 theoretical | 29 | 14 |
| 0.05 relaxed | 23 | 11 |
| 0.005 relaxed | 17 | 9 |
| 0.0005 relaxed | 12 | 6 |

$\epsilon = 10^{-6}$, $\theta = \frac{1}{2\sqrt{n}}$ and $\beta = \frac{1}{\sqrt{2}}$, respectively.

**Problem 1** The LCCO problem with its data is given by

$$f(x) = \sum_{i=1}^{8} x_i \ln x_i \;\; \text{s.t.} \;\; Ax = b, \; x \geq 0$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \;\; b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The initial primal-dual point is:

$$\begin{aligned} x^0 &= (0.4, 0.6, 0.4, 0.6, 0.6, 0.4, 0.6, 0.4)^T. \\ y^0 &= (0, 0, 0, 0)^T. \\ z^0 &= (0.083709, 0.48917, 0.083709, 0.48917, 0.48917, \\ &\quad 0.083709, 0.48719, 10.083709)^T. \end{aligned}$$

An exact optimal primal-dual solution of problem 1 is:

$$\begin{aligned} x^{opt} &= (0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5)^T. \\ y^{opt} &= (0.320702,\; 0.320702,\; 0.320702,\; 0.320702)^T. \\ z^{opt} &= (0, 0, 0, 0, 0, 0, 0, 0)^T. \end{aligned}$$

The numerical results of this problem are stated in Table 1.

**Problem 2** The convex quadratic LCCO problem is given by

$$f(x) = \frac{1}{2} x^T Q x + c^T x \;\; \text{s.t.} \;\; Ax = b, \; x \geq 0$$

where

$$Q = \begin{pmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \;\; A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 5 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ and } c = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}.$$

**Table 2** Numerical results for Problem 2

| $\mu_0\backslash\theta$ | theoretical $\theta = \frac{1}{2\sqrt{n}}$ | relaxed $\theta = \frac{1}{\sqrt{n}}$ |
|---|---|---|
| 0.5 theoretical | 20 | 9 |
| 0.05 relaxed | 16 | 8 |
| 0.005 relaxed | 12 | 6 |
| 0.0005 relaxed | 8 | 4 |

The initial starting point is:

$$x^0 = (0.6, 0.3, 1.1, 1.8)^T.$$
$$y^0 = (-0.5, -0.5)^T.$$
$$z^0 = (3.8, 5, \ 1, 0.5)^T.$$

An exact optimal solution is:

$$x^{opt} = (0, 0, 2, 3)^T.$$
$$y^{opt} = (0, 0)^T.$$
$$z^{opt} = (1, 2, 0, 0)^T.$$

The numerical results of this problem are stated in Table 2.

The tables show that the lowest iterations number produced by the algorithm, is obtained by the relaxed update barrier parameter $\theta = \frac{1}{\sqrt{n}}$ and the relaxed barrier $\mu_0 = 0.0005$.

## 5 Conclusion

In this paper, we have presented a feasible short-step primal-dual interior point algorithm for solving LCCO. At each iteration, we use only full-Newton step. The favorable iteration bound with short-steps method is deserved, namely $O(\sqrt{n}\log\frac{n}{\epsilon})$. Moreover, the resulting analysis is straightforward to CQO analogue. Few numerical results are reported to show the efficiency of the proposed algorithm.

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