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A conservative linearized difference scheme for the nonlinear fractional Schrödinger equation

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Abstract In this paper, we propose a conservative linearized difference scheme for the nonlinear fractional Schrödinger equation. The scheme efficiently avoids the time consuming iteration procedure necessary for the nonlinear scheme and thus is time saving relatively. It is rigorously proved that the scheme is mass conservative and uniquely solvable. Then employing mathematical induction, we further show that the proposed scheme is convergent at the order of $O(\tau^2 + h^2)$ in the l^2 norm with time step τ and mesh size h. Moreover, an extension to coupled nonlinear fractional Schrödinger systems is presented. Finally, numerical tests are carried out to corroborate the theoretical results and investigate the impact of the fractional order α on the collision of two solitons.

Keywords Nonlinear fractional Schrödinger equations · Linearized difference scheme · Conservation · Unique solvability · Convergence

1 Introduction

This paper considers the nonlinear fractional Schrödinger equation (FSE) of the form

$$iu_t - (-\triangle)^{\frac{\alpha}{2}} u + \beta |u|^2 u = 0, \quad a < x < b,\ 0 < t \leqslant T, \tag{1}$$

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with the initial condition

$$u(x,0) = u_0(x), \quad a \leqslant x \leqslant b, \tag{2}$$

and the Dirichlet boundary condition

$$u(a,t) = u(b,t) = 0, \quad 0 \leqslant t \leqslant T, \tag{3}$$

where $i^2 = -1$, $u(x,t)$ is a complex-valued wave function, the parameter β is a real constant describing the strength of the local (or short-range) interactions between particles (positive for repulsive interaction and negative for attractive interaction), and $u_0(x)$ is a given smooth function vanishing at the end points $x = a$ and $x = b$. $-(-\triangle)^{\frac{\alpha}{2}}$ is the fractional Laplace operator which is defined as a pseudo-differential operator with the symbol $-|\xi|^\alpha$:

$$-(-\triangle)^{\frac{\alpha}{2}} u(x,t) = -\mathcal{F}^{-1}(|\xi|^\alpha \hat{u}(\xi,t)),$$

where $\mathcal{F}$ is the Fourier transform. Yang [47] showed that it is indeed equivalent to the Riesz fractional derivative, i.e.,

$$-(-\triangle)^{\frac{\alpha}{2}} u(x,t) = \frac{\partial^\alpha}{\partial |x|^\alpha} u(x,t) := -\frac{1}{2\cos\frac{\alpha\pi}{2}}[{}_{-\infty}D_x^\alpha u(x,t) + {}_xD_{+\infty}^\alpha u(x,t)], \tag{4}$$

where ${}_{-\infty}D_x^\alpha u(x,t)$ and ${}_xD_{+\infty}^\alpha u(x,t)$ are the left- and right-side Riemann-Liouville fractional derivatives [8, 32], respectively. When $\alpha = 2$ the fractional Laplace operator is in accordance with the classical Laplace operator and then this system reduces to the classical cubic nonlinear Schrödinger equation. While for $\alpha = 1$, this system collapses to the relativistic Hartree equation (massless particles case) describing the mean field dynamics of boson stars (see [11] and references therein).

As a natural generalization of the classical (non-fractional) Schrödinger equation, the FSE has been exploited to study fractional quantum phenomena. In last decade many mathematical and numerical studies have been performed in the literature. Along the mathematical front, Naber [30] derived the time FSE involving a Caputo fractional derivative and solved it for a free particle and for a potential well. Laskin [24, 25] extended the Feynman path integral to Lévy one and derived the space FSE. Further he [26] demonstrated the Hermiticity of the fractional Hamilton operator and established the parity conservation law. Some physical applications are discussed by Guo and Xu in [19]. Hu [21] studied the global solution for a class of systems of fractional nonlinear Schrödinger equations. Secchi [33] constructed the ground state solution in $\mathbb{R}^N$. Uzar [38] investigated the fractional Bose-Einstein condensation and compared it with the classical one. In particular, for the problems (1)-(3), Guo et al. [18] studied the existence and uniqueness of the global smooth solution to the period boundary value problem, and arrived at the mass conservation

$$\|u\|_{L^2}^2 = \|u_0\|_{L^2}^2, \tag{5}$$

where $\|\cdot\|_{L^2}$ denotes the L^2 norm.

Along the numerical front, different numerical methods have been developed for time and/or space FSEs. In the time-fractional case, Wei et al. considered an implicit fully discrete local discontinuous Galerkin method for the time FSE [44], and then applied this method to time-fractional coupled Schrödinger systems [45]. Mohebbi [29] proposed a meshless method. In the space-fractional case, Amore et al. [2] developed the collocation method, Atangana [5] considered a difference scheme for the space FSE with the Caputo variable-order fractional derivative (see [4, 6, 7] for the definition and other applications of this fractional derivative). Wang et al. [39, 40] proposed two difference schemes for the coupled fractional Schrödinger equation (CFSE) with the Riesz space fractional derivative. Wang and Huang [41] constructed an energy conservative difference scheme for the nonlinear FSE. Herzallah [20] approximated the time-space FSE by Adomian decomposition method. Ford et al. [17] constructed a difference scheme for the time-space FSE in two dimensions. Bao and Dong [11] proposed a backward Euler and time-splitting sine pseudospectral method, respectively, for computing the ground states and dynamics of the nonlinear relativistic Hartree equation.

For classical Schödinger equations, it is desirable for a numerical scheme to preserve some invariant properties of the original equation because the conservative schemes can perform better than the nonconservative ones [49]. In terms of finite difference, extensive conservative schemes have been constructed and studied in the literature. For the theoretical analysis and numerical comparison of the conservation property for different numerical schemes, we refer to [1, 10, 15, 16, 42, 46, 49] for Schrödinger equations and [23, 27, 35, 36, 43] for coupled Schrödinger equations, or the latest review papers [3, 9, 12] and reference therein. Naturally, it is of interest to investigate conservative difference schemes for FSEs. For example, the difference scheme in [40] is mass conservative for the CFSE and the scheme in [41] is mass and energy conservative. However, both schemes are nonlinear due to the original nonlinearity in the FSEs. This means that at each time step, both schemes require the solution of a nonlinear system and thus it might be very time consuming. In particular, as pointed out in [10], the nonlinear system need be solved numerically to extremely high accuracy, otherwise, the mass and energy conservation could be destroyed. Hence, it is interesting to investigate linearly-implicit conservative schemes. This topic has been considered in [39], where a mass conservative linearized difference scheme is given.

In this paper, we consider a new difference scheme for the FSE (1), which can be regarded a linearization of our previous scheme in [41]. At each time step of the new scheme, only a linear system needs be solved and thus, the computational cost will be significantly reduced. Meanwhile, the mass conservation in the discrete sense can be preserved very well. In addition, for deriving the convergence of the difference solution, it is imperative to show the maximum value of the numerical solution is bounded by some generic constant. In this pursuit, the cut-off technique was adopted in [41]. Here, we propose a mathematical induction method. Another aim of this paper is to numerically investigate the impact of the fractional order α on the collision of two solitons, employing our new scheme.

The remainder of this paper is arranged as follows. In Section 2, the linearized difference scheme is introduced. The conservation property, solvability and convergence are rigorously proved in Section 3. In Section 4, we extend the results to the CFSE. Numerical experiments are performed in Section 5 to confirm our theoretical results and simulate the dynamics. Finally, some conclusions are drawn in Section 6.

Throughout the paper we use C to denote a generic constant whose actual value may change from line to line.

2 Linearized difference scheme

2.1 Notations

For two positive integers N and M, choose the time step $\tau := \frac{T}{N}$ and mesh size $h := \frac{b-a}{M}$. We define a partition of $[a, b] \times [0, T]$ by $\Omega := \Omega_\tau \times \Omega_h$ with the grid $\Omega_\tau = \{t_n = n\tau | n = 0, 1, 2, \dots, N\}$ and $\Omega_h = \{x_j = a + jh | j = 0, 1, 2, \dots, M\}$. Given a grid function $w^n = \{w^n \mid t_n \in \Omega_\tau\}$, denote

$$\delta_t w^{n+\frac{1}{2}} = \frac{w^{n+1} - w^n}{\tau}, \quad w^{n+\frac{1}{2}} = \frac{w^{n+1} + w^n}{2}.$$

Let $\mathcal{V}_h = \{w \mid w = (w_0, w_1, \dots, w_M),\ w_0 = w_M = 0\}$ be the space of grid functions. For any two grid functions $w, v \in \mathcal{V}_h$, define the discrete inner product and the associated l^2 norm as

$$(w, v) = h \sum_{j=1}^{M-1} w_j \bar{v}_j, \quad \|w\|^2 = (w, w),$$

where $\bar{v}$ denote the conjugate of v. We also define

$$\|w\|_\infty = \max_{0 \leqslant j \leqslant M} |w_j|$$

as the discrete maximum norm (or l^∞ norm).

2.2 Derivation of the linearized difference scheme

The Riesz fractional derivative presents some challenges for numerical simulation. For designating efficient and accurate approximating to it, a wide variety of methods including the shifted Grünwald approximation [28], the $L1/L2$ approximation method [47], the matrix transform method [22, 48], the weighted and shifted Grünwald difference method [37] and the finite element method [13, 50] have been developed. Recently, Ortigueira [31] defined the fractional centered difference and Çelik and Duman [14] analyzed the approximation error. This approximation has been successfully applied to solve many problems, e.g., fractional diffusion equations [14], fractional advection-dispersion equations [34] and the FSE [40, 41].

In this paper the fractional centered difference is adopted to approximate the Riesz fractional derivative. For the case $0 < \gamma \leqslant 2$, it is defined as [14]

$$\frac{\partial^\gamma}{\partial |x|^\gamma} u(x, t) = -\frac{1}{h^\gamma} \sum_{l=-(b-x)/h}^{(x-a)/h} \frac{(-1)^l \Gamma(\gamma + 1)}{\Gamma(\gamma/2 - l + 1)\Gamma(\gamma/2 + l + 1)} u(x - lh, t) + O(h^2). \quad (6)$$

Denoting $u_j^n := u(x_j, t_n)$ at the point x_j and at time t_n, and noticing (4) as well as the homogeneous boundary condition (3), we obtain

$$
\begin{aligned}
-(-\triangle)^{\frac{\gamma}{2}} u_j^n &= -\frac{1}{h^\gamma} \sum_{l=-M+j}^{j} c_l^{(\gamma)} u_{j-l}^n + O(h^2) \\
&= -\frac{1}{h^\gamma} \sum_{l=1}^{M-1} c_{j-l}^{(\gamma)} u_l^n + O(h^2), \qquad (7)
\end{aligned}
$$

where the coefficients $c_l^{(\gamma)} := \frac{(-1)^l \Gamma(\gamma+1)}{\Gamma(\gamma/2-l+1)\Gamma(\gamma/2+l+1)}$.

Let U_j^n be the numerical approximation to $u(x_j, t_n)$ and

$$
\triangle_h^\alpha U_j^n := h^{-\alpha} \sum_{l=1}^{M-1} c_{j-l}^{(\alpha)} U_l^n, \quad 1 \leqslant j \leqslant M-1,\ 0 \leqslant n \leqslant N. \qquad (8)
$$

Then we introduce the linearized difference scheme for the FSE (1)

$$
i\delta_t U_j^{n+\frac{1}{2}} - \triangle_h^\alpha U_j^{n+\frac{1}{2}} + \frac{\beta}{2}\left(3|U_j^n|^2 - |U_j^{n-1}|^2\right) U_j^{n+\frac{1}{2}} = 0, \qquad (9)
$$
$$
1 \leqslant j \leqslant M-1, 0 \leqslant n \leqslant N-1,
$$
$$
U_j^0 = u_0(x_j), \quad 0 \leqslant j \leqslant M, \qquad (10)
$$
$$
U_0^n = U_M^n = 0, \quad 0 \leqslant n \leqslant N. \qquad (11)
$$

This scheme is not selfstarting and the first step values U_j^1 need to be provided by other scheme, such as the one proposed in [40],

$$
i\delta_t U_j^{\frac{1}{2}} - \triangle_h^\alpha U_j^{\frac{1}{2}} + \beta |U_j^{\frac{1}{2}}|^2 U_j^{\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant M-1. \qquad (12)
$$

For scheme (9)-(11), only a linear system is to be solved at each time step. Furthermore, it is worth noting that, when $\alpha = 2$, $\triangle_h^\alpha$ identify with the classical discrete Laplace operator and then the scheme (9)-(11) reduces to the one proposed in [15] for classical nonlinear Schrödinger equations.

3 Theoretical analysis

3.1 Conservation

This subsection is devoted to showing the mass conservation preserved by the scheme (9)-(11).

Lemma 3.1 [41] *For any grid function $U^n \in \mathcal{V}_h$, $0 \leqslant n \leqslant N$, we have*

$$
Im\left(\triangle_h^\alpha U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\right) = 0, \qquad (13)
$$

where "$Im(s)$" means taking the imaginary part of a complex number s.

Theorem 3.1 *The scheme (9)-(11) is conservative in the sense*

$$Q^n \equiv Q^0, \quad 0 \leqslant n \leqslant N, \tag{14}$$

where

$$Q^n := \|U^n\|^2. \tag{15}$$

is the mass in the discrete sense.

Proof The proof is standard and straightfoward. Computing the discrete inner product of (9) with $U^{n+\frac{1}{2}}$ and taking the imaginary part, yield

$$\|U^{n+1}\|^2 = \|U^n\|^2, \quad 0 \leqslant n \leqslant N-1, \tag{16}$$

where (13) was used. Hence, the proof is complete. □

Remark 3.1 From Theorem 3.1, it follows that the numerical solution of (9)-(11) is long-time bounded, i.e., there exists some constant $C > 0$, such that

$$\|U^n\| \leqslant C, \quad 0 \leqslant n \leqslant N. \tag{17}$$

This immediately implies the unconditional stability of scheme (9)-(11).

Remark 3.2 Here the conserved discrete mass Q^n involve only one time level, which is different from that presented in [39] where $Q^n := \frac{1}{2}(\|U^{n+1}\|^2 + \|U^n\|^2)$ and two time levels are included.

3.2 Solvability

We now prove that the difference scheme (9)-(11) is uniquely solvable by means of the Brouwder fixed point theorem and the energy method.

Lemma 3.2 *(Brouwder Fixed Point Theorem) Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a finite dimensional inner product space, $\|\cdot\|$ be the associated norm, and $f : \mathcal{H} \to \mathcal{H}$ be continuous. Assume, moreover, that*

$$\exists \alpha > 0,\ \forall z \in \mathcal{H},\ \|z\| = \alpha,\ Re\langle g(z),\ z\rangle \geqslant 0. \tag{18}$$

Then, there exists a $z^* \in \mathcal{H}$ such that $g(z^*) = 0$ and $\|z^*\| \leqslant \alpha$.

Theorem 3.2 *The solution of finite difference scheme (9)-(11) uniquely exists.*

Proof We discuss the uniqueness and existence of the difference solution in an inductive way. Noticing that $U^0 \in \mathcal{V}_h$ has been determined uniquely from (10)-(11). The assertion for $n = 1$ has been proved in [40]. In scheme (9)-(11), for given U^{n-1}, $U^n \in \mathcal{V}_h$ with $n \leqslant N-1$, we need to prove that there uniquely exists $U^{n+1} \in \mathcal{V}_h$ satisfying difference scheme (9)-(11).

We first prove the uniqueness. Suppose there exist two solutions $U^{(1)}, U^{(2)} \in \mathcal{V}_h$ satisfying scheme (9)-(11), i.e.,

$$i\frac{U_j^{(1)} - U_j^n}{\tau} - \frac{1}{2}\triangle_h^\alpha \left(U_j^{(1)} + U_j^n\right) + \frac{\beta}{4}\left(3|U_j^n|^2 - |U_j^{n-1}|^2\right)\left(U_j^{(1)} + U_j^n\right) = 0, \tag{19}$$

$$i\frac{U_j^{(2)} - U_j^n}{\tau} - \frac{1}{2}\triangle_h^\alpha \left(U_j^{(2)} + U_j^n\right) + \frac{\beta}{4}\left(3|U_j^n|^2 - |U_j^{n-1}|^2\right)\left(U_j^{(2)} + U_j^n\right) = 0. \tag{20}$$

Denoting $\varphi = U^{(1)} - U^{(2)}$ and subtracting (20) from (19), yield

$$i\frac{\varphi_j}{\tau} - \frac{1}{2}\triangle_h^\alpha \varphi_j + \frac{\beta}{4}\left(3|U_j^n|^2 - |U_j^{n-1}|^2\right)\varphi_j = 0, \quad 1 \leqslant j \leqslant M-1. \tag{21}$$

Computing the discrete inner product of (21) with φ, taking the imaginary part and using Lemma 3.1, we obtain $\|\varphi\|^2 = 0$, which implies $\varphi = 0$. Hence $U^{(1)} = U^{(2)}$, i.e., the solution of (9)-(11) is unique.

Next, we prove the existence. For a fixed n, substituting $U_j^{n+1} = 2U_j^{n+\frac{1}{2}} - U_j^n$ into (9) yields

$$U_j^{n+\frac{1}{2}} = U_j^n - i\frac{\tau}{2}\left[\triangle_h^\alpha U_j^{n+\frac{1}{2}} - \frac{\beta}{2}\left(3|U_j^n|^2 - |U_j^{n-1}|^2\right)U_j^{n+\frac{1}{2}}\right], \quad 1 \leqslant j \leqslant M-1. \tag{22}$$

Consider the mapping $\mathfrak{F} : \mathcal{V}_h \to \mathcal{V}_h$ defined as

$$\mathfrak{F}(w)_j = w_j - U_j^n + i\frac{\tau}{2}\left[\triangle_h^\alpha w_j - \frac{\beta}{2}\left(3|U_j^n|^2 - |U_j^{n-1}|^2\right)w_j\right], \quad 1 \leqslant j \leqslant M-1, \tag{23}$$

which is obviously continuous. Noticing (13), computing the discrete inner product of (23) with w and taking the real part, we obtain

$$\begin{aligned}
&Re\big(\mathfrak{F}(w), w\big)\\
&= \|w\|^2 - Re(U^n, w) - \frac{\tau}{2}Im\Big(\big(\triangle_h^\alpha w, w\big) - \frac{\beta}{2}h\sum_{j=1}^{M-1}\left(3|U_j^n|^2 - |U_j^{n-1}|^2\right)|w_j|^2\Big)\\
&= \|w\|^2 - Re(U^n, w)\\
&\geqslant \|w\|^2 - \|w\| \cdot \|U^n\|\\
&= \|w\|\big(\|w\| - \|U^n\|\big),
\end{aligned} \tag{24}$$

where "$Re(s)$" means taking the real part of a complex number s. It follows from (24) that, if we let $\|w\| = \|U^n\|$, there is $Re\big(\mathfrak{F}(w)\big) \geqslant 0$. Then using Lemma 3.2 we know that there exists a $w^* \in \mathcal{V}_h$ such that $\mathfrak{F}(w^*) = 0$ and $\|w^*\| \leqslant \|U^n\|$. Thus, the existence is proved. □

3.3 Convergence

In this subsection, we analyze the convergence of the scheme (9)-(11). The key ingredient in the analysis is to show the maximum value of the difference solution, i.e. $\|U^n\|_\infty$, is bounded by some constant. The authors in [41] adopted the cut-off technique (see [10]) to show the convergence of the nonlinear CN scheme. Here, we propose a mathematical induction method. The inverse inequality is needful in our analysis and we first introduce it in the following lemma.

Lemma 3.3 [46] *For any grid function $U^n \in \mathcal{V}_h,\ 0 \leqslant n \leqslant N$, the inequality*

$$\|U^n\|_\infty^2 \leqslant \frac{1}{h}\|U^n\|^2 \tag{25}$$

holds.

Let $R_j^{n+\frac{1}{2}}$ be the local truncation error of scheme (9). Then

$$i\delta_t u_j^{n+\frac{1}{2}} - \triangle_h^\alpha u_j^{n+\frac{1}{2}} + \frac{\beta}{2}(3|u_j^n|^2 - |u_j^{n-1}|^2)u_j^{n+\frac{1}{2}} = R_j^{n+\frac{1}{2}},$$
$$1 \leqslant j \leqslant M-1,\ 1 \leqslant n \leqslant N-1. \tag{26}$$

From (7) and Taylor's expansion, we have

$$|R_j^{n+\frac{1}{2}}| \leqslant C_R(\tau^2 + h^2), \quad 1 \leqslant j \leqslant M-1,\ 0 \leqslant n \leqslant N-1, \tag{27}$$

which gives

$$\|R^{n+\frac{1}{2}}\|^2 \leqslant (b-a)\big(C_R(\tau^2+h^2)\big)^2, \quad 0 \leqslant n \leqslant N-1. \tag{28}$$

Define the error function $e^n \in \mathcal{V}_h$ for $0 \leqslant n \leqslant N$ as

$$e_j^n = u_j^n - U_j^n, \quad 1 \leqslant j \leqslant M-1.$$

Then we have the following result.

Theorem 3.3 *Suppose that the original problem (1)-(3) has a smooth solution. Assume $\tau \leqslant Ch$, then there exist $\tau_0 > 0$ and $h_0 > 0$ sufficiently small such that, when $0 < \tau \leqslant \tau_0$ and $0 < h \leqslant h_0$, we have*

$$\|e^n\| \leqslant \sqrt{2(b-a)(n\tau)}C_R \exp\big(2(2+C_{M_0})n\tau\big)(\tau^2+h^2), \quad \|U^n\|_\infty \leqslant 1 + M_0, \quad 0 \leqslant n \leqslant N, \tag{29}$$

where $M_0 = \max_{0\leqslant t\leqslant T}\|u(\cdot,t)\|_{L^\infty}$, $C_{M_0} = 12\beta^2 M_0^2(1+M_0)^2$.

Proof We will prove this theorem by the method of mathematical induction. For $n = 0$, combining (2) with (10) straightforwardly implies the validity of (29). From the error analysis shown in [40], when τ and h sufficiently small, we get

$$\begin{aligned}\|e_j^1\| &\leqslant \sqrt{b-a}C_R\tau(\tau^2+h^2)\\ &\leqslant \sqrt{b-a}C_R\sqrt{\tau}(\tau^2+h^2)\\ &\leqslant \sqrt{2(b-a)}C_R\sqrt{\tau}\exp\big(2(2+C_{M_0})\tau\big)\left(\tau^2+h^2\right). \end{aligned} \quad (30)$$

Under the assumption $\tau \leqslant Ch$, noticing (25), we have

$$\|e^1\|_\infty \leqslant h^{-\frac{1}{2}}\|e^1\| \leqslant C_1 h^{\frac{3}{2}}, \quad (31)$$

where $C_1 := \sqrt{2(b-a)T}C_R\exp\big(2(2+C_{M_0})T\big)(1+C^2)$. Hence

$$\|U^1\|_\infty \leqslant \|u^1\|_\infty + \|e^1\|_\infty \leqslant M_0 + C_1 h^{\frac{3}{2}}. \quad (32)$$

Let $h_0 = C_1^{-\frac{2}{3}}$. When $0 < h \leqslant h_0$, we obtain

$$\|U^1\|_\infty \leqslant M_0 + 1. \quad (33)$$

It follows from (30) and (33) that (29) holds for $n = 1$. Now we assume that (29) is valid for all $0 \leqslant n \leqslant m-1 \leqslant N-1$, we then need to show that it is still valid for $n = m$.

Subtracting (26) from (9) yields

$$\begin{aligned} i\delta_t e_j^{n+\frac{1}{2}} - \triangle_h^\alpha e_j^{n+\frac{1}{2}} + \frac{\beta}{2}(3|U_j^n|^2 - |U_j^{n-1}|^2)e_j^{n+\frac{1}{2}} = G_j^n + R_j^{n+\frac{1}{2}},\\ 1 \leqslant j \leqslant M-1,\ 1 \leqslant n \leqslant m-1, \end{aligned} \quad (34)$$

where

$$\begin{aligned} G_j^n &= \frac{\beta}{2}\big[(3|U_j^n|^2 - |U_j^{n-1}|^2) - (3|u_j^n|^2 - |u_j^{n-1}|^2)\big]u_j^{n+\frac{1}{2}}\\ &= \frac{\beta}{2}\big[3(|U_j^n|^2 - |u_j^n|^2) - (|U_j^{n-1}|^2 - |u_j^{n-1}|^2)\big]u_j^{n+\frac{1}{2}}. \end{aligned} \quad (35)$$

Since (29) is valid for $n \leqslant m-1$, we have

$$|G_j^n| \leqslant |\beta|M_0(1+M_0)\big(3|e_j^n| + |e_j^{n-1}|\big), \quad 1 \leqslant j \leqslant M-1,\ 1 \leqslant n \leqslant m-1, \quad (36)$$

which deduces

$$\|G^n\|^2 \leqslant C_{M_0}(\|e^n\|^2 + \|e^{n-1}\|^2), \quad 1 \leqslant n \leqslant m-1. \quad (37)$$

Computing the discrete inner product of (34) with $e^{n+\frac{1}{2}}$ and taking the imagine part, using the triangular and Cauchy inequalities, noticing (13), (27) and (37), we have for $1 \leqslant n \leqslant m-1$,

$$\begin{aligned} \|e^{n+1}\|^2 - \|e^n\|^2 &= \tau Im(G^n + R^{n+\frac{1}{2}}, e^{n+1} + e^n)\\ &\leqslant \tau\big(\|e^{n+1}\|^2 + \|e^n\|^2 + \|G^n\|^2 + \|R^{n+\frac{1}{2}}\|^2\big)\\ &\leqslant \tau\Big(\|e^{n+1}\|^2 + \|e^n\|^2 + C_{M_0}\big(\|e^n\|^2 + \|e^{n-1}\|^2\big) + (b-a)\big(C_R(\tau^2+h^2)\big)^2\Big). \end{aligned}$$

When $\tau \leqslant \frac{1}{2}$, we obtain

$$\|e^{n+1}\|^2 - \|e^n\|^2 \leqslant 2(2+C_{M_0})\tau\|e^n\|^2 + 2C_{M_0}\tau\|e^{n-1}\|^2 + 2\tau(b-a)\big(C_R(\tau^2+h^2)\big)^2.$$

Using the above inequality, noticing (29), we get

$$\begin{aligned}\|e^m\|^2 &\leqslant \big(1+2(2+C_{M_0})\tau\big)\|e^{m-1}\|^2+2C_{M_0}\tau\|e^{m-2}\|^2+2\tau(b-a)\big(C_R(\tau^2+h^2)\big)^2\\ &\leqslant 2(b-a)\big(C_R(\tau^2+h^2)\big)^2\Big[\big(1+2(2+C_{M_0})\tau\big)(m-1)\tau\exp\big(4(2+C_{M_0})(m-1)\tau\big)\\ &\quad +2C_{M_0}\tau(m-2)\tau\exp\big(4(2+C_{M_0})(m-2)\tau\big)+\tau\Big]\\ &\leqslant 2(b-a)\big(C_R(\tau^2+h^2)\big)^2\Big[\big(1+4(2+C_{M_0})\tau\big)(m-1)\tau\exp\big(4(2+C_{M_0})(m-1)\tau\big)+\tau\Big]\\ &\leqslant 2(b-a)\big(C_R(\tau^2+h^2)\big)^2\Big[(m-1)\tau\exp\big(4(2+C_{M_0})m\tau\big)+\tau\Big]\\ &\leqslant 2(b-a)\big(C_R(\tau^2+h^2)\big)^2\Big[m\tau\exp\big(4(2+C_{M_0})m\tau\big)\Big],\end{aligned}$$

which immediately implies

$$\|e^m\| \leqslant \sqrt{2(b-a)(m\tau)}C_R\exp\big(2(2+C_{M_0})m\tau\big)(\tau^2+h^2). \tag{38}$$

Again under the assumption $\tau \leqslant Ch$, combining the above inequality with (25) gives

$$\|U^m\|_\infty \leqslant \|u^m\|_\infty+\|e^m\|_\infty \leqslant M_0+h^{-\frac{1}{2}}\|e^m\| \leqslant M_0+C_1h^{\frac{3}{2}}, \tag{39}$$

and consequently, when $0 < h \leqslant h_0$, we have

$$\|U^m\|_\infty \leqslant M_0+1. \tag{40}$$

This together with (38) implies (29) for $n = m$ and thus completes the proof by the method of mathematical induction. □

4 Extension

The ideas for designing linearized difference scheme for the FSE (1)-(3) in the previous sections can be easily extended to the CFSE.

We consider the following CFSE

$$\begin{aligned}&iu_t-(-\triangle)^{\frac{\alpha}{2}}u+\rho\left(|u|^2+\beta|v|^2\right)u=0, \quad a<x<b,\ 0<t\leqslant T,\\ &iv_t-(-\triangle)^{\frac{\alpha}{2}}v+\rho\left(|v|^2+\beta|u|^2\right)v=0, \quad a<x<b,\ 0<t\leqslant T,\\ &u(x,0)=u_0(x),\ v(x,t)=v_0(x), \quad a\leqslant x\leqslant b,\\ &u(a,t)=u(b,t)=0,\ v(a,t)=v(b,t)=0, \quad 0\leqslant t\leqslant T,\end{aligned} \tag{41}$$

where the parameters ρ and β are some real constants. This equation has been studied mathematically in [21] and numerically in [39, 40]. When $\rho = 0$, this system is decoupled and becomes the FSE of free particles. When $\beta = 0$, it reduces to the single FSE. Moreover, the CFSE (41) conserves the mass, i.e.,

$$\|u\|^2_{L^2}=\|u_0\|^2_{L^2}, \quad \|v\|^2_{L^2}=\|v_0\|^2_{L^2}. \tag{42}$$

Table 1 The errors $e(h,\tau)$ and the order for $\alpha = 2$ with $\tau = 0.1h$

τ	h	$e(h,\tau)$	order
0.02	0.2	2.2322e-01	-
0.01	0.1	5.4744e-02	2.0277
0.005	0.05	1.3611e-02	2.0079
0.0025	0.025	3.3981e-03	2.0020
0.00125	0.0125	8.4922e-04	2.0005

The conservative linearized difference scheme for the CFSE (41) reads

$$i\delta_t U_j^{n+\frac{1}{2}} - \Delta_h^\alpha U_j^{n+\frac{1}{2}} + \frac{\rho}{2}\big(3|U_j^n|^2 - |U_j^{n-1}|^2 + \beta(3|V_j^n|^2 - |V_j^{n-1}|^2)\big)U_j^{n+\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant M-1,\ 1 \leqslant n \leqslant N-1, \tag{43}$$

$$i\delta_t V_j^{n+\frac{1}{2}} - \Delta_h^\alpha V_j^{n+\frac{1}{2}} + \frac{\rho}{2}\big(3|V_j^n|^2 - |V_j^{n-1}|^2 + \beta(3|U_j^n|^2 - |U_j^{n-1}|^2)\big)V_j^{n+\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant M-1,\ 1 \leqslant n \leqslant N-1, \tag{44}$$

$$U_j^0 = u_0(x_j), \quad V_j^0 = v_0(x_j), \quad 0 \leqslant j \leqslant M, \tag{45}$$

$$U_0^n = U_M^n = 0, \quad V_0^n = V_M^n = 0, \quad 0 \leqslant n \leqslant N. \tag{46}$$

Following the analysis analogous to that performed for scheme (9)-(11) in the above section, it can be easily shown that the scheme (43)-(46) is mass conservative, uniquely solvable and convergent at the order of $O(\tau^2 + h^2)$ in the l^2 norm. Here we omit the details due to space limitations.

5 Numerical experiments

In this section, some numerical experiments are performed. In the first example, we consider the FSE and pay particular attention to verifying the numerical accuracy and mass conservation. The CFSE is presented in the second example by which we aim to simulate the collision of two soliton waves and investigate the effect of fractional order α.

Table 2 The errors $e(h,\tau)$ and the order for $1 < \alpha < 2$ with $\tau = 0.1h$

α	$h = 0.2$	$h = 0.1$	order
1.4	1.9558e-01	4.9293e-02	1.9883
1.6	2.3322e-01	5.4692e-02	2.0923
1.8	2.2895e-01	5.3138e-02	2.1072
1.99	2.2023e-01	5.1589e-02	2.0939

Table 3 The value of Q^n at different time with $\tau = h = 0.05$

	$\alpha = 1.4$	$\alpha = 1.7$	$\alpha = 2$
$T = 0$	2.000000000000002	2.000000000000002	2.000000000000002
$T = 2$	2.000000000000008	2.000000000000005	2.000000000000002
$T = 4$	2.000000000000010	2.000000000000001	2.000000000000006
$T = 6$	2.000000000000011	2.000000000000001	2.000000000000000
$T = 8$	2.000000000000008	2.000000000000002	1.999999999999990
$T = 10$	2.000000000000007	2.000000000000005	1.999999999999990

Example 1 We consider the problem [40]

$$iu_t - (-\triangle)^{\frac{\alpha}{2}}u + \beta|u|^2u = 0, \tag{47}$$

with the initial value

$$u(x, 0) = \mathrm{sech}(x) \cdot \exp(2ix). \tag{48}$$

Here we take $\beta = 2$. When $\alpha = 2$, the problem reduces to the classical cubic nonlinear Schrödinger equation and the exact solution is given by

$$u(x, t) = \mathrm{sech}(x - 4t) \cdot \exp(i(2x - 3t)).$$

For practical computations, as in [40, 41], the whole space problems are usually truncated into a large bounded interval $[a, b]$ and set $u(a, t) = u(b, t) = 0$. In this example, we choose $a = -20$ and $b = 20$.

We first testify the numerical accuracy of the scheme (9)-(11). In order to quantify the numerical accuracy, we compute the l^2 norm errors $e(h, \tau) = \|u - u_h\|$ between the numerical solution u_h and the exact solution u at $T = 1$. Then the convergence rates are calculated as $\log_2(e(h, \tau)/e(h/2, \tau/2))$. When $1 < \alpha < 2$, we derive the numerical "exact" solution u by the scheme proposed in [40] with a very fine mesh and a small time step, e.g., $h = 0.025$ and $\tau = 0.0001$. Table 1 shows the errors for $\alpha = 2$ and Table 2 displays the similar results for $1 < \alpha < 2$, which together demonstrate that the proposed scheme is second order accurate in both space and time, and hence the theoretical results in Theorem 3.3 are confirmed.

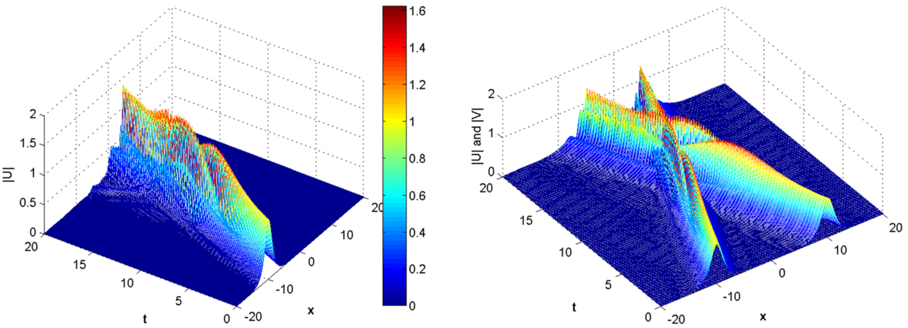

Fig. 1 Evolution of $|U|$ (*left*) and collision of two solitons (*right*) for $\alpha = 1.3$

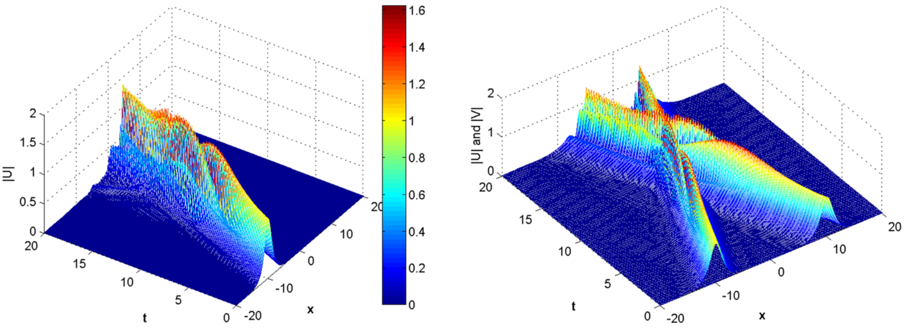

Fig. 2 Evolution of $|U|$ (*left*) and collision of two solitons (*right*) for $\alpha = 1.5$

Then we test the discrete mass conservation law. Table 3 gives the values of mass Q^n at different time for $\alpha = 1.4, 1.7, 2$, respectively, with $\tau = h = 0.05$. It is observed that the scheme (9)-(11) preserves the mass conservation very well and is suitable for long-term simulation. More precisely, the scheme conserves the discrete mass over time with the machine precision, while the nonlinear schemes developed in [40, 41] only with 8 significant digits due to the iteration (see Table 2 in [40] and Table 3 in [41]). In order to improve the conservation accuracy of the non-linear schemes, a smaller iteration tolerance must be imposed and accordingly, the computational cost will increase rapidly.

Example 2 Consider the coupled system [40]

$$
\begin{aligned}
&iu_t - (-\triangle)^{\frac{\alpha}{2}} u + \rho(|u|^2 + \beta|v|^2)u = 0,\\
&iv_t - (-\triangle)^{\frac{\alpha}{2}} v + \rho(|v|^2 + \beta|u|^2)v = 0,
\end{aligned} \tag{49}
$$

subject to the initial conditions

$$
\begin{aligned}
&u(x,0) = \text{sech}(x+10)\cdot\exp(i\upsilon_1 x),\\
&v(x,0) = \text{sech}(x-10)\cdot\exp(-i\upsilon_2 x),
\end{aligned} \tag{50}
$$

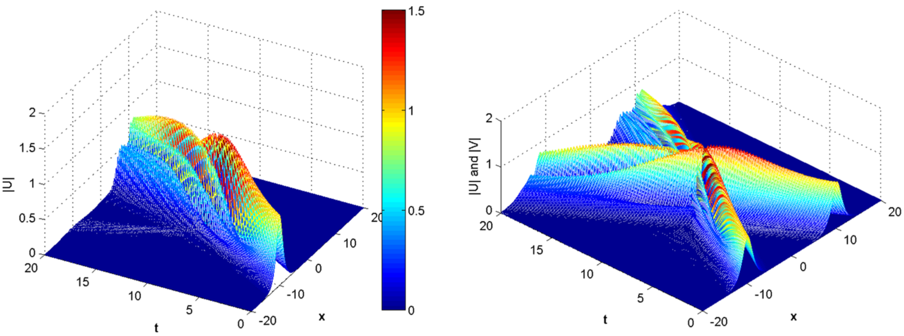

Fig. 3 Evolution of $|U|$ (*left*) and collision of two solitons (*right*) for $\alpha = 1.7$

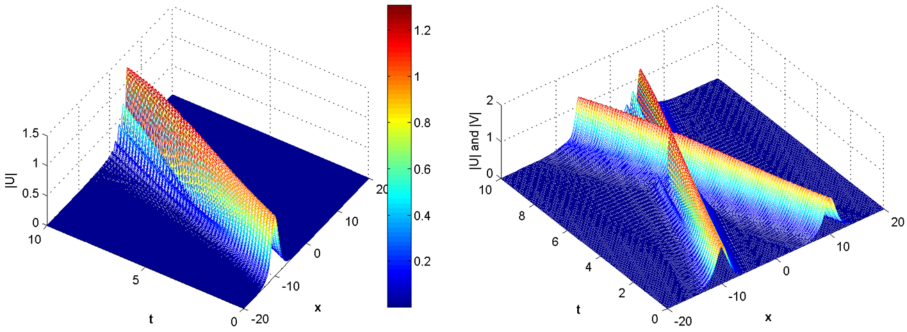

Fig. 4 Evolution of $|U|$ (*left*) and collision of two solitons (*right*) for $\alpha = 1.9$

where υ_j $(j = 1, 2)$ are velocities. As in Example 5.1, we solve (49)-(50) on $[-20, 20]$ with homogeneous Dirichlet boundary conditions. Here we choose $\upsilon_1 = \upsilon_2 = 3, \tau = h = 0.1$.

Now we investigate the impact on the collision of two solitons brought by fractional order α. In order to illustrate the fact clearly, we specify the elastic collisions. Choosing $\rho = \beta = 1$, when $\alpha = 2$, the system is the Manakovs equations which is completely integrable and the collision is elastic, i.e., the waves retain their shape and velocity after interaction (see Fig. 5). We denote t_c the time when the two solitons completely collide. Figures 1-5 show the evolution of the modulus of the wave function (left) and of the collision of two solitons (right) for $\alpha = 1.3, 1.5, 1.7, 1.9, 2$, respectively.

From Figs. 1-5, we can draw the following conclusions: (i) The order α will greatly affect the height and width of the soliton. The smaller α becomes, the more severely the shape of the soliton changes. This feature is consistent with the observation in [40, 41]. (ii) The time t_c varies with the fractional order α. More precisely, t_c will increase subsequently when α becomes small. These phenomena are greatly different from that in the non-fractional case and, essentially, features the nonlocal character of the fractional Laplace operator. In addition, these special properties can be used

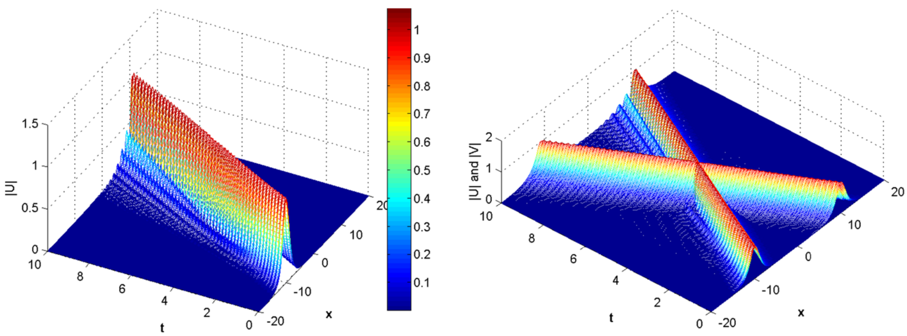

Fig. 5 of $|U|$ (*left*) and collision of two solitons (*right*) for $\alpha = 2$

Table 4 The value of Q_1^n at different time with $\tau = h = 0.1$

	$\alpha = 1.4$	$\alpha = 1.7$	$\alpha = 2$
$T = 0$	1.999999995451730	1.999999995451730	1.999999995451730
$T = 2$	1.999999995451729	1.999999995451728	1.999999995451731
$T = 4$	1.999999995451733	1.999999995451732	1.999999995451733
$T = 6$	1.999999995451735	1.999999995451734	1.999999995451732
$T = 8$	1.999999995451735	1.999999995451730	1.999999995451731
$T = 10$	1.999999995451735	1.999999995451733	1.999999995451733

in physics to modify the shape of wave and collision time without change of the nonlinearity and dispersion effects.

Table 4 lists the values of mass $Q_1^n := \|U^n\|^2$ at different time for $\alpha = 1.4, 1.7, 2$, respectively. We choose $\rho = 1$ and $\beta = 2$ here. The values of $Q_2^n := \|V^n\|^2$ are equal to Q_1^n and not shown here for brevity. It is observed that the scheme (43)-(46) preserves the mass conservation very well.

6 Conclusions

A linearized difference scheme has been given for solving the nonlinear fractional Schrödinger equation. For the proposed scheme, at each time step, only a linear system is to be solved. Thus it is significantly cheaper than the nonlinear one in the view of computation time, and meanwhile, preserves the mass conservation in the discrete sense very well. In addition, we proved rigorously that the scheme is uniquely solvable and second order convergent. We further extended the methods to solve the coupled fractional Schrödinger equation. Finally, numerical tests were performed and the accuracy and discrete conservation law were confirmed. Based on the numerical simulation for the collision of two solitons, we observed that the fractional order α dramatically affect the wave shape and collision time.

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