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# Numerical solution of fractional advection-diffusion equation with a nonlinear source term

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**Abstract** In this paper we use the Jacobi collocation method for solving a special kind of the fractional advection-diffusion equation with a nonlinear source term. This equation is the classical advection-diffusion equation in which the space derivatives are replaced by the Riemann-Liouville derivatives of order $0 < \sigma \leq 1$ and $1 < \mu \leq 2$. Also the stability and convergence of the presented method are shown for this equation. Finally some numerical examples are solved using the presented method.

**Keywords** Fractional advection-diffusion equation · Riemann-Liouville derivative · Jacobi polynomials · Operational matrix · Collocation method · Stability analysis and convergence

**Mathematics Subject Classifications (2010)** 35R11 · 26A33 · 33C45 · 44A45 · 65L60

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## 1 Introduction

Fractional calculus has played a significant role in describing many phenomena in engineering, physics, chemistry, economics and other fields. These phenomena can be described very successfully using fractional differential equations. One of the most important fractional differential equation that is used in engineering is the fractional advection-diffusion equation therefore some authors presented numerical and analytical methods for solving the fractional advection-diffusion equation. For example, in [47] Shen et al. derived the fundamental solution for the space-time Riesz-Caputo, fractional advection-diffusion equation with an initial condition. In [24] authors proposed a spectral representation of the fractional Laplacian operator and used the equivalent relationship between fractional Laplacian operator and Riesz fractional derivative and derived analytical solution for the multi-term, time-space, Caputo-Riesz fractional advection-diffusion equation on a finite domain. Wang and Wang [50] developed a fast characteristic finite difference method for solving space fractional transient advection-diffusion equation. Huang et al. [21] presented a finite element method to solve the fractional advection-dispersion equation. Zheng et al. [54] proposed the finite element method for the space fractional advection-diffusion equation with non-homogeneous initial-boundary condition. El-Sayed et al [44] used the Adomian decomposition method for solving an intermediate advection-dispersion equation. Authors of [16] used the homotopy perturbation method for solving the fractional advection-dispersion equation. In [38] authors presented an analytical algorithm to solve the fractional advection-dispersion equation based on homotopy analysis method. Jiang and Lin [23] presented a new method for solving the following advection-dispersion equation in the reproducing kernel space

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = -v(x,t)\frac{\partial^\beta u(x,t)}{\partial x^\beta} + k(x,t)\frac{\partial^\gamma u(x,t)}{\partial x^\gamma} + f(x,t),$$

$$u(x,0) = f(x), \quad u(0,t) = 0; \quad u(L,t) = 0.$$

Ervin and Roop [12] investigated the numerical approximation of the variational solution of the fractional advection-dispersion equation on bounded domains in $\mathbb{R}^d$. Liu et al. [26] proposed an implicit difference method to solve the space-time fractional advection-diffusion equation. In [52] authors used the Melin and Laplace transforms and derived an analytical solution for the time–fractional advection-diffusion equation. In [42] authors used the Sinc-Legendre collocation method for approximating the solution of a class of fractional convection-diffusion equation with variable coefficients. Huang and Liu [22] derived an analytical solution for the space-time fractional advection-dispersion equation based on the Green function and Fourier-Laplace transforms. In [40] Roop, used the finite element method for solving a singularly perturbed fractional convection-diffusion equation. Sousa in [48] derived an explicit finite difference method for solving a fractional advection-diffusion problem. Al-Khaled and Momani [1] presented an approximate solution for a fractional diffusion-wave equation based on the Adomian decomposition method. Authors of [34] used finite difference approximations for solving the fractional advection-diffusion with variable coefficients on a finite domain. Meerschaert and Tadjeran [35] examined some numerical methods such as finite difference method, backward Euler and implicit Euler methods to solve a class of initial-boundary value problems governed by fractional partial differential equations with variable coefficients on a finite domain. Chen et al. [7] derived implicit and explicit difference methods for solving the fractional reaction-subdiffusion equation. Authors of [49] proposed the fractional weighted average finite difference method for the space-fractional advection-dispersion equation. In [10] a fully discrete numerical approximation is used to approximate the solution of a time–dependent nonlinear space-fractional diffusion equation. Yang et al. [53] presented three numerical methods to solve the fractional partial differential equations with Riesz space fractional derivatives, namely the $L1/L2$- approximation method, the standard shifted Grunwald method and the matrix transform method. Baeumer et al. [2] presented numerical solution for the fractional reaction-diffusion equation. Yang et al. [51] presented novel numerical methods for time-space fractional reaction-diffusion equations in two domains. Authors of [37] used the Laplace and Fourier transforms and obtained a general representation of analytical solution of two dimensional space-time Riesz-Caputo fractional diffusion equation in terms of the Mittag-Leffler function. Also they presented a numerical technique for the solution of this equation based on Grunwald-Letnikov approximation. Bueno-Orovio et al. [3] employed Fourier spectral method for solving fractional reaction-diffusion problems. Also they extended this approach to fractional reaction-diffusion problems in rectangular domains of $\mathbb{R}^n$. Also we refer the interested reader to [8, 19, 20, 27–32, 36, 41, 43, 56] for some numerical methods which have been proposed to find the numerical solution of the fractional differential equations.

In this paper we present a Jacobi collocation method for solving fractional advection-diffusion equation with a nonlinear source term. We organize our paper as follows: In Section 2 we present some preliminaries, then introduce fractional advection-diffusion equation. In Section 3 we write this equation in the operational form and apply Jacobi collocation method for approximating its solution. In Section 4 the convergence and the stability of the presented method are shown. Also numerical examples are given in Section 5.

## 2 Preliminaries

In this section we introduce the fractional advection-diffusion equation and describe some preliminaries and notations .

**Definition 1** Suppose $\gamma \geq 0$, $m = \lceil \gamma \rceil$ and $I = [a, b]$. For function $f$ given in the interval $I$, the left and right handed Caputo fractional-order derivatives are defined as [39] :

$$
\begin{aligned}
{}_a^C D_x^\gamma f(x) &= \frac{1}{\Gamma(m-\gamma)} \int_a^x (x-\tau)^{m-\gamma-1} f^{(m)}(\tau) d\tau, \\
{}_x^C D_b^\gamma f(x) &= \frac{(-1)^m}{\Gamma(m-\gamma)} \int_x^b (\tau-x)^{m-\gamma-1} f^{(m)}(\tau) d\tau,
\end{aligned}
$$

and the left and right handed Riemann-Liouville fractional-order derivatives are defined as:

$$ {}_{a}^{R}D_{x}^{\gamma} f(x) = \frac{1}{\Gamma(m-\gamma)} \frac{d^m}{dx^m} \int_{a}^{x} (x-\tau)^{m-\gamma-1} f(\tau) d\tau, $$

$$ {}_{x}^{R}D_{b}^{\gamma} f(x) = \frac{(-1)^m}{\Gamma(m-\gamma)} \frac{d^m}{dx^m} \int_{x}^{b} (\tau-x)^{m-\gamma-1} f(\tau) d\tau. $$

The Riesz fractional-order derivative is defined as [47, 55]:

$$ \frac{\partial^{\gamma} f(x)}{\partial |x|^{\gamma}} = -c \left[ {}_{-\infty}^{R}D_{x}^{\gamma} f(x) + {}_{x}^{R} D_{+\infty}^{\gamma} f(x) \right], $$

where $c = \frac{1}{2\cos(\frac{\gamma\pi}{2})}$. For the function $f$ given in the interval $I$, where $f(a) = f(b) = 0$, we can extend the function to have $f(x) = 0$ to all $x > a$ and $x < b$. Thus we have [55]

$$ \frac{\partial^{\gamma} f(x)}{\partial |x|^{\gamma}} = -c \left[ {}_{a}^{R}D_{x}^{\gamma} f(x) + {}_{x}^{R} D_{b}^{\gamma} f(x) \right], $$

where $c = \frac{1}{2\cos(\frac{\gamma\pi}{2})}$.

In this paper we study the numerical solution of the following fractional advection-diffusion equation with a nonlinear source term :

$$ \begin{array}{l} \frac{\partial u}{\partial t} = \kappa(x,t) R_{\mu} u(x,t) + \rho(x,t) R_{\sigma} u(x,t) + f(u,x,t), \\ (x,t) \in \Omega = [a,b] \times [0,T], \quad 1 < \mu \leq 2, \\ u(x,0) = g(x), \\ u(a,t) = u(b,t) = 0, \end{array} \tag{1} $$

where $0 \leq \kappa(x,t) \leq \bar{\kappa} < \infty$ and $\kappa$, $\rho$, $f$, $g$ are continuous functions and $f$ is a source term which satisfies the Lipschitz condition. Also $R_{\mu}u(x,t)$ is defined by

$$ R_{\mu}u(x,t) = \begin{cases} c_{+}(x,t) {}_{a}^{R}D_{x}^{\mu} u(x,t) + c_{-}(x,t) {}_{x}^{R}D_{b}^{\mu} u(x,t), & 1 < \mu < 2, \\ \frac{d^2}{dx^2} u(x,t), & \mu = 2, \end{cases} $$

where

$$ 0 \leq c_{+}(x,t) \leq c_1, \quad 0 \leq c_{-}(x,t) \leq c_2, $$

where $c_1$, $c_2$ are constants. In this equation if

$$ c_{+}(x,t) = c_{-}(x,t) = \frac{-1}{2\cos(\frac{\pi\mu}{2})}, $$

then $R_{\mu}u(x,t)$ which represents the Riesz fractional derivative is defined by [55]

$$ R_{\mu}u(x,t) = \frac{-1}{2\cos(\frac{\pi\mu}{2})} \left( {}_{a}^{R}D_{x}^{\mu} u(x,t) + {}_{x}^{R}D_{b}^{\mu} u(x,t) \right). $$

In Eq. (1), $R_{\sigma}u(x,t)$ is defined by

$$ R_{\sigma}u(x,t) = d_{+}(x,t) {}_{a}^{R}D_{x}^{\sigma} u(x,t) + d_{-}(x,t) {}_{x}^{R}D_{b}^{\sigma} u(x,t), $$

$$ 0 \leq d_{+}(x,t) \leq d_1 < \infty, \quad 0 \leq d_{-}(x,t) \leq d_2 < \infty, $$

where if $0 \le d_+(x,t) \le d_1 < \infty$, then we have $d_-(x,t) = 0$, $-\infty < -\bar{\rho} \le \rho(x,t) \le 0$ and $0 < \sigma \le 1$. If $0 \le d_-(x,t) \le d_2 < \infty$, then we have $d_+(x,t) = 0$, $0 \le \rho(x,t) \le \bar{\rho} < \infty$ and $0 < \sigma \le 1$. Also if

$$d_+(x,t) = d_-(x,t) = \frac{-1}{2\cos\left(\frac{\pi\sigma}{2}\right)},$$

then $R_\sigma u(x,t)$ is defined by

$$R_\sigma u(x,t) = \frac{-1}{2\cos(\frac{\pi\sigma}{2})}\left({}_a^R D_x^\sigma u(x,t) + {}_x^R D_b^\sigma u(x,t)\right),$$

and $0 \le \rho(x,t) \le \bar{\rho} < \infty$ and $0 < \sigma < 1$.
Table 1 reports research works on the numerical or analytical solutions of (1) in the special cases.

**Lemma 1** *Suppose $\gamma \ge 0$, $m = \lceil\gamma\rceil$ and $I = [a,b]$. Assume that $f$ is such that ${}_a^R D_x^\gamma f$, ${}_x^R D_b^\gamma f$, ${}_a^C D_x^\gamma f$ and ${}_x^C D_b^\gamma f$ exist, then* [39]*:*

$$\frac{1}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}\int_a^x (x-\tau)^{m-\gamma-1} f(\tau)d\tau = \frac{1}{\Gamma(m-\gamma)}\int_a^x (x-\tau)^{m-\gamma-1} f^{(m)}(\tau)d\tau + \sum_{j=0}^{m-1}\frac{f^{(j)}(a)(x-a)^{j-\gamma}}{\Gamma(1+j-\gamma)},$$

$$\frac{(-1)^m}{\Gamma(m-\gamma)}\frac{d^m}{dx^m}\int_x^b (\tau-x)^{m-\gamma-1} f(\tau)d\tau = \frac{1}{\Gamma(m-\gamma)}\int_x^b (\tau-x)^{m-\gamma-1} f^{(m)}(\tau)d\tau + \sum_{j=0}^{m-1}\frac{(-1)^{m-j} f^{(j)}(b)(b-x)^{j-\gamma}}{\Gamma(1+j-\gamma)}.$$

**Lemma 2** *Assume that $\gamma > 0$ and $u$ are such that ${}_a^R D_x^\gamma u$, ${}_x^R D_b^\gamma u$ exist then* [39]*:*

$$\frac{d^k}{dx^k}\left({}_a^R D_x^\gamma u\right) = {}_a^R D_x^{\gamma+k} u, \qquad \frac{d^k}{dx^k}\left({}_x^R D_b^\gamma u\right) = {}_x^R D_b^{\gamma+k} u.$$

**Lemma 3** *The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, are defined as the orthogonal polynomials with respect to the weight function $\omega^{\alpha,\beta}(x) = (x-a)^\alpha(b-x)^\beta$, $(\alpha > -1, \beta > -1)$ on $[a,b]$. These polynomials satisfy the following properties*

**Table 1** The special cases of Eq. (1)

| Reference | $\mu$ | $\sigma$ | $\kappa(x,t)$ | $\rho(x,t)$ | $c_+(x,t)$ | $c_-(x,t)$ | $d_+(x,t)$ | $d_-(x,t)$ | $f(u,x,t)$ |
|---|---|---|---|---|---|---|---|---|---|
| [50] | $1 < \mu \le 2$ | 1 | $\kappa(x,t)$ | $\rho(x,t)$ | 1 | 1 | 1 | 0 | $f(x,t)$ |
| [21] | $1 < \mu \le 2$ | 1 | $\kappa$ | $\rho$ | 1 | 0 | 1 | 0 | 0 |
| [48] | $1 < \mu \le 2$ | 0 | $\kappa$ | 0 | $\frac{-1}{\cos(\frac{\pi\mu}{2})}$ | $\frac{-1}{\cos(\frac{\pi\mu}{2})}$ | 0 | 0 | $f(x,t)$ |
| [49] | $1 < \mu \le 2$ | 1 | $\kappa$ | $\rho$ | 1 | 0 | 1 | 0 | 0 |

[25]:

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} B_k^{(\alpha,\beta,n)}(x-b)^k,$$

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} E_k^{(\alpha,\beta,n)}(x-a)^k,$$

*where*

$$B_k^{(\alpha,\beta,n)} = \frac{(b-a)^{n-k} n!(k+\beta+1)_{n-k}}{(n-k)!(n+k+\alpha+\beta+1)_{n-k} k!},$$

$$E_k^{(\alpha,\beta,n)} = \frac{(a-b)^{n-k} n!(k+\alpha+1)_{n-k}}{(n-k)!(n+k+\alpha+\beta+1)_{n-k} k!},$$

*and*

$$(c)_0 = 1, \quad (c)_k = \prod_{i=0}^{k}(c+i-1), \quad k = 1, 2, 3, ...,$$

## 3 The collocation method for solving Eq. (1) based on the Jacobi polynomials

Let $\mathbb{P}_N$ be the space of all polynomials of degree at most $N$ and $\mathbb{P}_N^0 = \{P \in \mathbb{P}_N| \quad P(a) = P(b) = 0\}$. In this section we apply the Jacobi collocation method for solving (1), this means that we look for a mapping $u_N \in \mathbb{P}_N^0$, such that for any $t \in [0, T]$ we have:

$$\frac{\partial}{\partial t} u_N(x_k^{\alpha,\beta}, t) = \frac{\kappa(x_k^{\alpha,\beta}, t)}{\Gamma(2-\mu)} \left\{ c_+(x_k^{\alpha,\beta}, t) \left[ \frac{d^2}{d\zeta^2} \int_a^\zeta (\zeta-\eta)^{1-\mu} u_N(\eta, t) d\eta \right]_{\zeta = x_k^{\alpha,\beta}} + c_-(x_k^{\alpha,\beta}, t) \left[ \frac{d^2}{d\zeta^2} \int_\zeta^b (\eta-\zeta)^{1-\mu} u_N(\eta, t) d\eta \right]_{\zeta = x_k^{\alpha,\beta}} \right\} + \frac{\rho(x_k^{\alpha,\beta}, t)}{\Gamma(1-\sigma)} \left\{ d_+(x_k^{\alpha,\beta}, t) \left[ \frac{d}{d\zeta} \int_a^\zeta (\zeta-\eta)^{-\sigma} u_N(\eta, t) d\eta \right]_{\zeta = x_k^{\alpha,\beta}} - d_-(x_k^{\alpha,\beta}, t) \left[ \frac{d}{d\zeta} \int_\zeta^b (\eta-\zeta)^{-\sigma} u_N(\eta, t) d\eta \right]_{\zeta = x_k^{\alpha,\beta}} \right\} + f(u_N(x_k^{\alpha,\beta}, t), x_k^{\alpha,\beta}, t), \quad k = 1, 2, ..., N-1, \tag{2}$$

and

$$u_N(x_k^{\alpha,\beta}, 0) = g(x_k^{\alpha,\beta}), \quad k = 0, 1, .., N, \tag{3}$$

where $x_k^{\alpha,\beta}$ $(0 \leq k \leq N)$ are the Gauss-Jacobi points (The roots of $(N+1)$-th Jacobi polynomials). Suppose

$$u_N(x, t) = \sum_{j=0}^{N} a_j(t) P_j^{(\alpha,\beta)}(x). \tag{4}$$

In this section for simplicity we use $P_j(x)$ instead of $P_j^{(\alpha,\beta)}(x)$, therefore we can write

$$u_N(x, t) = R(x) A^T(t), \tag{5}$$

where

$$R(x) = [P_0(x), P_1(x), ..., P_N(x)], \tag{6}$$

and

$$A(t) = [a_0(t), a_1(t), ..., a_N(t)]. \tag{7}$$

Therefore we have the following system of ordinary differential equations

$$\begin{cases} R(a)A^T(t) = 0, \\ R(x_k^{\alpha,\beta})\frac{d}{dt}A^T(t) = \frac{\kappa(x_k^{\alpha,\beta},t)}{\Gamma(2-\mu)}\left\{c_+(x_k^{\alpha,\beta},t)I_1(x_k^{\alpha,\beta})A^T(t) + c_-(x_k^{\alpha,\beta},t)I_2(x_k^{\alpha,\beta})A^T(t)\right\} + \\ \frac{\rho(x_k^{\alpha,\beta},t)}{\Gamma(1-\sigma)}\left\{d_+(x_k^{\alpha,\beta},t)I_3(x_k^{\alpha,\beta})A^T(t) - d_-(x_k^{\alpha,\beta},t)I_4(x_k^{\alpha,\beta})A^T(t)\right\} + f(R(x_k^{\alpha,\beta})A^T(t), x_k^{\alpha,\beta}, t), \\ k = 1, 2, ..., N-1, \\ R(b)A^T(t) = 0, \end{cases} \tag{8}$$

with the initial condition

$$[R(x_0^{\alpha,\beta})A^T(t), R(x_1^{\alpha,\beta})A^T(t), R(x_2^{\alpha,\beta})A^T(t), ..., R(x_N^{\alpha,\beta})A^T(t)]^T = G, \quad k = 0, 1, .., N, \tag{9}$$

where

$$I_1(x) = \left[\frac{d^2}{d\zeta^2}\int_a^\zeta (\zeta-\eta)^{1-\mu}R(\eta)d\eta\right]_{\zeta=x}, \quad I_2(x) = \left[\frac{d^2}{d\zeta^2}\int_\zeta^b (\eta-\zeta)^{1-\mu}R(\eta)d\eta\right]_{\zeta=x},$$

$$I_3(x) = \left[\frac{d}{d\zeta}\int_a^\zeta (\zeta-\eta)^{-\sigma}R(\eta)d\eta\right]_{\zeta=x}, \quad I_4(x) = \left[\frac{d}{d\zeta}\int_\zeta^b (\eta-\zeta)^{-\sigma}R(\eta)d\eta\right]_{\zeta=x},$$

$$G = [g(x_0^{\alpha,\beta}), g(x_1^{\alpha,\beta}), ..., g(x_N^{\alpha,\beta})]^T.$$

In the current paper we use the classic 4-stage Runge–Kutta method which is a fourth-order formula [4]. For this purpose we first evaluate

$$\left[\frac{d^m}{d\zeta^m}\int_a^\zeta (\zeta-\eta)^{m-\mu-1}R(\eta)d\eta\right]_{\zeta=x_k^{\alpha,\beta}}, \quad \left[\frac{d^m}{d\zeta^m}\int_\zeta^b (\eta-\zeta)^{m-\mu-1}R(\eta)d\eta\right]_{\zeta=x_k^{\alpha,\beta}},$$

where $1 \le k \le N-1, \quad m = 1,\ 2$. Using Lemma 1 we can write:

$$\frac{1}{\Gamma(m-\mu)}\left[\frac{d^m}{d\zeta^m}\int_a^\zeta (\zeta-\eta)^{m-\mu-1}R(\eta)d\eta\right]_{\zeta=x_k^{\alpha,\beta}} = \frac{1}{\Gamma(m-\mu)}\int_a^{x_k^{\alpha,\beta}} \frac{R^{(m)}(\eta)}{\left(x_k^{\alpha,\beta}-\eta\right)^{\mu-m+1}}d\eta + \sum_{j=0}^{m-1}\frac{R^{(j)}(a)\left(x_k^{\alpha,\beta}-a\right)^{j-\mu}}{\Gamma(1+j-\mu)}, \tag{10}$$

and

$$\frac{(-1)^m}{\Gamma(m-\mu)}\left[\frac{d^m}{d\zeta^m}\int_\zeta^b (\eta-\zeta)^{m-\mu-1}R(\eta)d\eta\right]_{\zeta=x_k^{\alpha,\beta}} = \frac{1}{\Gamma(m-\mu)}\int_{x_k^{\alpha,\beta}}^b \frac{R^{(m)}(\eta)}{\left(\eta-x_k^{\alpha,\beta}\right)^{\mu-m+1}}d\eta + \sum_{j=0}^{m-1}\frac{(-1)^{m-j}R^{(j)}(b)\left(b-x_k^{\alpha,\beta}\right)^{j-\mu}}{\Gamma(1+j-\mu)}. \tag{11}$$

On the other hand the relations between the matrix $R(x)$ and its derivatives, $R^{(k)}(x)$, are:

$$R^{(k)}(x) = R(x)Z^k, \quad k = 1, 2, ..., \tag{12}$$

where $Z$ is the operational matrix of derivative and it can be seen that [13]:

$$Z = \begin{bmatrix} 0 & z_{1,0} & z_{2,0} & \cdots & z_{n,0} \\ 0 & 0 & z_{2,1} & \cdots & z_{n,1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & z_{n,n-1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

where

$$z_{i,i-1} = \frac{P_i^{(i)}(1)}{(i-1)!B_{i-1}^{(\alpha,\beta,i-1)}} = \frac{\alpha+\beta+2i}{2(\alpha+\beta+i)},$$

and

$$z_{i,j} = \frac{P_i^{(j+1)}(1) - j!\sum\limits_{k=i-1}^{j+1} B_j^{(\alpha,\beta,k)} z_{i,k}}{j!B_j^{(\alpha,\beta,j)}}, \quad j = 0, 1, 2, ..., i-2.$$

Now using Eqs. (10) and (12) yield:

$$\left[\frac{d^m}{d\zeta^m}\int\limits_a^\zeta (\zeta-\eta)^{m-\mu-1}R(\eta)d\eta\right]_{\zeta=x_k^{\alpha,\beta}} = \Gamma(m-\mu)\left(F_3(x_k^{\alpha,\beta})E^T Z^m + \sum_{j=0}^{m-1}\frac{\left(x_k^{\alpha,\beta}-a\right)^{j-\mu}F_1(a)E^T Z^j}{\Gamma(1+j-\mu)}\right), \tag{13}$$

and employing (11) and (12) we conclude:

$$\left[\frac{d^m}{d\zeta^m}\int\limits_\zeta^b (\eta-\zeta)^{m-\mu-1}R(\eta)d\eta\right]_{\zeta=x_k^{\alpha,\beta}} = \Gamma(m-\mu)\left(F_4(x_k^{\alpha,\beta})B^T Z^m + \sum_{j=0}^{m-1}\frac{(-1)^{m-j}\left(b-x_k^{\alpha,\beta}\right)^{j-\mu}F_2(b)B^T Z^j}{\Gamma(1+j-\mu)}\right), \tag{14}$$

where

$$E = \begin{bmatrix} E_0^{(\alpha,\beta,0)} & 0 & 0 & \cdots & 0 \\ E_0^{(\alpha,\beta,1)} & E_1^{(\alpha,\beta,1)} & 0 & \cdots & 0 \\ E_0^{(\alpha,\beta,2)} & E_1^{(\alpha,\beta,2)} & E_2^{(\alpha,\beta,2)} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ E_0^{(\alpha,\beta,3)} & B_1^{(\alpha,\beta,n)} & \cdots & \cdots & E_n^{(\alpha,\beta,n)} \end{bmatrix},$$

$$B = \begin{bmatrix} B_0^{(\alpha,\beta,0)} & 0 & 0 & \cdots & 0 \\ B_0^{(\alpha,\beta,1)} & B_1^{(\alpha,\beta,1)} & 0 & \cdots & 0 \\ B_0^{(\alpha,\beta,2)} & B_1^{(\alpha,\beta,2)} & B_2^{(\alpha,\beta,2)} & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ B_0^{(\alpha,\beta,3)} & B_1^{(\alpha,\beta,n)} & \cdots & \cdots & B_n^{(\alpha,\beta,n)} \end{bmatrix},$$

$$F_1(x) = \left[\,1,\ (x-a),\ (x-a)^2,\ \cdots,\ (x-a)^n\,\right]^T,$$

and

$$F_2(x) = \Big[\ 1,\ (x-b),\ (x-b)^2,\ \cdots,\ (x-b)^n\ \Big]^T,$$

$$F_3(x_k^{\alpha,\beta}) = \frac{1}{\Gamma(m-\mu)}\left[\ \int_a^{x_k^{\alpha,\beta}} \frac{1}{\left(x_k^{\alpha,\beta}-\eta\right)^{\mu-m+1}}d\eta,\ \int_a^{x_k^{\alpha,\beta}} \frac{(\eta-a)}{\left(x_k^{\alpha,\beta}-\eta\right)^{\mu-m+1}}d\eta,\ \cdots\ \int_a^{x_k^{\alpha,\beta}} \frac{(\eta-a)^n}{\left(x_k^{\alpha,\beta}-\eta\right)^{\mu-m+1}}d\eta\ \right],$$

and

$$F_4(x_k^{\alpha,\beta}) = \frac{1}{\Gamma(m-\mu)}\left[\ \int_{x_k^{\alpha,\beta}}^{b} \frac{1}{\left(\eta-x_k^{\alpha,\beta}\right)^{\mu-m+1}}d\eta,\ \int_{x_k^{\alpha,\beta}}^{b} \frac{(\eta-b)}{\left(\eta-x_k^{\alpha,\beta}\right)^{\mu-m+1}}d\eta,\ \cdots\ \int_{x_k^{\alpha,\beta}}^{b} \frac{(\eta-b)^n}{\left(\eta-x_k^{\alpha,\beta}\right)^{\mu-m+1}}d\eta\ \right].$$

## 4 The convergence and stability analysis

In this section we present the convergence and stability analysis of the collocation method for Eq. (1). Let us present some definitions and lemmas before we prove the main theorems of this section.

**Definition 2** Suppose $I = (a, b)$ and $L^2_{\omega^{\alpha,\beta}}(I)$ is the space of square integrable functions in $I$. Now we can define the following inner product and norm on $L^2_{\omega^{\alpha,\beta}}(I)$:

$$(u, v)_{\omega^{\alpha,\beta}} = \int_a^b \omega^{\alpha,\beta}(x)u(x)v(x)dx, \quad \forall u, v \in L^2_{\omega^{\alpha,\beta}}(I),$$

$$\|u\|_{\omega^{\alpha,\beta}} = \left(\int_a^b \omega^{\alpha,\beta}(x)(u(x))^2dx\right)^{\frac{1}{2}}, \quad \forall u \in L^2_{\omega^{\alpha,\beta}}(I).$$

Suppose $I = (a, b)$, therefore we define [5]:

$$H^k_{\omega^{\alpha,\beta}}(I) = \left\{u|\ \partial^l_x u \in L^2_{\omega^{\alpha,\beta}}(I),\ \ 0 \le l \le k\right\},$$

where $\partial^l_x u = \frac{\partial^l u}{\partial x^l}$. $H^k_{\omega^{\alpha,\beta}}(I)$ is a Hilbert space with respect to the inner product :

$$(u, v)_{k,\omega^{\alpha,\beta}} = \sum_{m=0}^{k}\left(\partial^m_x u, \partial^m_x v\right)_{\omega^{\alpha,\beta}},$$

which induces the norm:

$$\|u\|_{k,\omega^{\alpha,\beta}} = \left(\sum_{j=0}^{k}\left\|\partial^j_x u\right\|^2_{\omega^{\alpha,\beta}}\right)^{\frac{1}{2}},$$

also it is easy to see that :

$$\left\|\partial^m_x u\right\|_{\omega^{\alpha,\beta}} \le \|u\|_{k,\omega^{\alpha,\beta}}, \quad 0 \le m \le k. \tag{15}$$

We define [45]:

$$H^k_{\omega^{\alpha,\beta},*}(I) = \left\{u|\ \partial^j_x u \in L^2_{\omega^{\alpha+j,\beta+j}}(I),\ \ 0 \le j \le k,\quad k \in \mathbb{N}\right\},$$

where $\partial^j_x u = \frac{\partial^j u}{\partial x^j}$. $H^k_{\omega^{\alpha,\beta},*}(I)$ is a Hilbert space with respect to the inner product

$$(u, v)_{k,\omega^{\alpha,\beta},*} = \sum_{j=0}^{k}\left(\partial^j_x u, \partial^j_x v\right)_{\omega^{\alpha+j,\beta+j}},$$

which induces the norm:

$$\|u\|_{k,\omega^{\alpha,\beta},*} = \left(\sum_{j=0}^{k} \left\|\partial_x^j u\right\|^2_{\omega^{\alpha+j,\beta+j}}\right)^{\frac{1}{2}}.$$

**Definition 3** Suppose $\mathbb{P}_N$ is the space of all polynomials of degree at most $N$. $\Pi_{N,\omega^{\alpha,\beta}} : L^2_{\omega^{\alpha,\beta}}(I) \to \mathbb{P}_N$ is an orthogonal projection if and only if for any $u \in L^2_{\omega^{\alpha,\beta}}(I)$, we have [5]:

$$\left(\Pi_{N,\omega^{\alpha,\beta}}(u(x)) - u(x), v(x)\right)_{\omega^{\alpha,\beta}} = 0, \quad \forall v \in \mathbb{P}_N.$$

**Lemma 4** *For any $u \in H^p_{\omega^{\alpha,\beta}}(I)$, $p \geq 1$, there exists a constant $C_1$ independent of $N$, such that* [5]*:*

$$\left\|u - \Pi_{N,\omega^{\alpha,\beta}}(u)\right\|_{\omega^{\alpha,\beta}} \leq C_1 N^{-p} \|u\|_{p,\omega^{\alpha,\beta}}.$$

**Lemma 5** *Suppose $u \in \mathbb{P}_N$, therefore there exists a constant $C_1$ independent of $N$ such that* [45]*:*

$$\left\|\partial_x^k u\right\|_{\omega^{\alpha+k,\beta+k}} \leq C_1 N^k \|u\|_{\omega^{\alpha,\beta}}, \quad \forall k \in \mathbb{N}.$$

**Definition 4** It is known that the general form of Gauss quadrature rules is given by:

$$\int_a^b f(x)dw(x) = \sum_{j=1}^{n} w_j f(x_j) + \sum_{k=1}^{m} v_k f(x_k) + R[f],$$

where the weighs $\{w_j\}_{j=1}^n$ and $\{v_k\}_{k=1}^m$ and the nodes $\{x_j\}_{j=1}^n$ are unknowns and the nodes $\{z_k\}_{k=1}^m$ are predetermined. $w$ is also a positive measure on $[a,b]$. In the special cases if $m = 1$ and $z_k = a$ or $m = 2$ and $z_1 = a$ and $z_2 = b$ the resultant quadrature rules are called the Gauss-Radau and Gauss-Lobatto quadrature rules, respectively. For more introduction about Gauss quadrature weights and nodes we refer the interested reader to [5, 14, 15, 17].

**Lemma 6** *Suppose $I = (a,b)$ and $x_j^{\alpha,\beta}$, $(0 \leq j \leq N)$, are the Gauss-Jacobi or Gauss-Radau or Gauss-Lobatto quadrature nodes and $w_j$, $(0 \leq j \leq N)$, are the Gauss-Jacobi or Gauss-Radau or Gauss-Lobatto quadrature weights* [5] *. The Jacobi interpolation is denoted by $I_N^{\alpha,\beta}(u)$. For any $u \in H^k_{\omega^{\alpha,\beta}}(I)$, $k \geq 1$, there exists a constant $C_2$ independent of $N$, such that* [5, 18, 45]*:*

$$\left\|u - I_N^{\alpha,\beta}(u)\right\|_{\omega^{\alpha,\beta}} \leq C_2 N^{-k} \|u\|_{k,\omega^{\alpha,\beta}}, \tag{16}$$

*and for any $u \in H^m_{\omega^{\alpha,\beta},*}(I)$, $m > k$, there exist constants $C_3$ and $C_4$ independent of $N$, such that:*

$$\left\|\partial_x^k \left(u - \Pi_{N,\omega^{\alpha,\beta}}(u)\right)\right\|_{\omega^{\alpha+k,\beta+k}} \leq C_3 N^{k-m} \left\|\partial_x^m u\right\|_{\omega^{\alpha+m,\beta+m}}, \tag{17}$$

$$\left\|\partial_x^k \left(u - I_N^{\alpha,\beta}(u)\right)\right\|_{\omega^{\alpha+k,\beta+k}} \leq C_4 N^{k-m} \left\|\partial_x^m u\right\|_{\omega^{\alpha+m,\beta+m}}. \tag{18}$$

*Also for any $u \in H^m_{\omega^{\alpha,\beta}}(I)$, $1 \leq k \leq m$, there exist constants $C_5$ and $C_6$ independent of $N$, such that:*

$$\left\|u - \Pi_{N,\omega^{\alpha,\beta}}(u)\right\|_{k,\omega^{\alpha,\beta}} \leq C_5 N^{2k-\frac{1}{2}-m} \|u\|_{m,\omega^{\alpha,\beta}},$$

$$\left\|u - I_N^{\alpha,\beta}(u)\right\|_{k,\omega^{\alpha,\beta}} \leq C_6 N^{2k-\frac{1}{2}-m} \|u\|_{m,\omega^{\alpha,\beta}}.$$

For any $u, v$ continuous on $[a,b]$, we set

$$(u,v)_{N,\omega^{\alpha,\beta}} = \sum_{i=0}^{N} u(x_i^{\alpha,\beta}) v(x_i^{\alpha,\beta}) w_i^{\alpha,\beta},$$

where $x_i^{\alpha,\beta}$, $(0 \leq i \leq N)$, are the Gauss-Jacobi or Gauss-Radau or Gauss-Lobatto quadrature nodes and $w_i^{\alpha,\beta}$, $(0 \leq i \leq N)$, are the Gauss-Jacobi or Gauss-Radau or Gauss-Lobatto quadrature weights. Therefore we define

$$\|u\|_{N,\omega^{\alpha,\beta}} = \sum_{i=0}^{N} \omega_i^{\alpha,\beta} u^2(x_i^{\alpha,\beta}).$$

The Gauss integration formulas imply that:

$$(u, v)_{N,\omega^{\alpha,\beta}} = (u, v)_{\omega^{\alpha,\beta}}, \quad if \quad uv \in \mathbb{P}_{2N+\delta},$$

where $\delta = 1, 0, -1$ for Gauss, Gauss-Radau or Gauss-Lobatto integration rules, respectively.

**Lemma 7** *For the Gauss and Gauss-Radau integration rules we have* [45] *:*

$$\left|(u, \varphi)_{\omega^{\alpha,\beta}} - (u, \varphi)_{N,\omega^{\alpha,\beta}}\right| \leq \left\|u - I_N^{\alpha,\beta}(u)\right\|_{\omega^{\alpha,\beta}} \|\varphi\|_{\omega^{\alpha,\beta}}, \quad \forall \varphi \in \mathbb{P}_N.$$

**Definition 5** Let $s > 0$, we define the seminorms [33]:

$$|u|_{H_l^s(I)} := \left\|{}_a^R D_x^s u\right\|_{\omega^{0,0}}, \quad |u|_{H_r^s(I)} := \left\|{}_x^R D_b^s u\right\|_{\omega^{0,0}},$$

and the norms:

$$\|u\|_{H_l^s(I)} := \left(\|u\|_{\omega^{0,0}}^2 + |u|_{H_l^s(I)}^2\right)^{\frac{1}{2}}, \quad \|u\|_{H_r^s(I)} := \left(\|u\|_{\omega^{0,0}}^2 + |u|_{H_r^s(I)}^2\right)^{\frac{1}{2}}.$$

Suppose

$$C_0^\infty(I) := \left\{u | \, u \in C^\infty(I) \; with \; compact \; support \; in \; (a, b)\right\}.$$

Let us define $H_l^s(I)$ as the closure of $C_0^\infty(I)$, with respect to norm $\|.\|_{H_l^s(I)}$ and $H_r^s(I)$ as the closure of $C_0^\infty(I)$, with respect to norm $\|.\|_{H_r^s(I)}$.

**Lemma 8** *Let $s > 0$, the spaces $H_l^s(I)$ and $H_r^s(I)$ are equal in the sense that their seminorms as well as norms are equivalent* [33].

**Lemma 9** *If $q \leq s$, then we have* [33]*:*

$$H_r^s(I) \subset H_r^q(I), \quad H_l^s(I) \subset H_l^q(I).$$

**Lemma 10** *For any $u \in H_l^s(I)$, $0 < s < \mu$, there exists a constant $C_1$ such that* [33]*:*

$$|u|_{H_l^s(I)} \leq C_1 |u|_{H_l^\mu(I)},$$

*and for any $u \in H_r^s(I)$, $0 < s < \mu$, there exists a constant $C_2$ such that:*

$$|u|_{H_r^s(I)} \leq C_2 |u|_{H_r^\mu(I)}.$$

**Definition 6** Let $\alpha > 0$ and $k \in \mathbb{N}$, we define the seminorms:

$$|u|_{H_l^{\alpha,k}(I)} := \left\|{}_a^R D_x^\alpha u\right\|_{k,\omega^{0,0}}, \quad |u|_{H_r^{\alpha,k}(I)} := \left\|{}_x^R D_b^\alpha u\right\|_{k,\omega^{0,0}},$$

and the norms:

$$\|u\|_{H_l^{\alpha,k}(I)} := \left(\|u\|_{k,\omega^{0,0}}^2 + |u|_{H_l^{\alpha,k}(I)}^2\right)^{\frac{1}{2}}, \quad \|u\|_{H_r^{\alpha,k}(I)} := \left(\|u\|_{k,\omega^{0,0}}^2 + |u|_{H_r^{\alpha,k}(I)}^2\right)^{\frac{1}{2}}.$$

Let us define $H_l^{\alpha,k}(I)$ as the closure of $C_0^\infty(I)$, with respect to norm $\|.\|_{H_l^{\alpha,k}(I)}$ and $H_r^{\alpha,k}(I)$, the closure of $C_0^\infty(I)$, with respect to norm $\|.\|_{H_r^{\alpha,k}(I)}$.

**Lemma 11** *Let $\alpha > 0$, the spaces $H_l^{\alpha,k}(I)$ and $H_r^{\alpha,k}(I)$ are equal in the sense that their seminorms as well as norms are equivalent.*

*Proof* It can be proved using Lemmas 2 and 8. □

**Lemma 12** *For any $u \in H_l^{s,k}(I),\ 0 < s < \mu,\ k \geq 0$ there exists a constant $C_1$ such that:*

$$|u|_{H_l^{s,k}(I)} \leq C_1 |u|_{H_l^{\mu,k}(I)}.$$

*Also for any $u \in H_r^{s,k}(I),\ 0 < s < \mu,\ k \geq 0$ there exists a constant $C_2$ such that:*

$$|u|_{H_r^{s,k}(I)} \leq C_2 |u|_{H_r^{\mu,k}(I)}.$$

*Proof* This lemma can be easily proved using Lemmas 2 and 10. □

**Definition 7** We define

$$L^\infty(I) = \{u\ |u\ is\ measurable\ on\ I\ and\ \|u\|_{\infty,I} < \infty\},$$

where

$$\|u\|_{\infty,I} := ess \sup_{x\in I} |u(x)|\,.$$

**Lemma 13** *(Sobolev inequality) Suppose $u \in H^1_{\omega^{\alpha,\beta}}(I)$, therefore we have [5]:*

$$\|u\|_\infty \leq \left(\frac{1}{b-a}+2\right)^{\frac{1}{2}} \|u\|^{\frac{1}{2}}_{\omega^{\alpha,\beta}}\ \|u\|^{\frac{1}{2}}_{1,\omega^{\alpha,\beta}}\,.$$

**Lemma 14** *Suppose*

$$I(u,x) := \tfrac{1}{\Gamma(m-\gamma)}\int_a^x (x-\eta)^{m-\gamma-1}u(\eta)d\eta,\quad and\quad I^*(u,x) := \tfrac{(-1)^m}{\Gamma(m-\gamma)}\int_x^b (\eta-x)^{m-\gamma-1}u(\eta)d\eta,$$

*where $m-1 < \gamma < m$. Also suppose $u \in L^2_{\omega^{\alpha,\beta}}(I)$, therefore we conclude $I : L^2_{\omega^{\alpha,\beta}}(I) \to L^2_{\omega^{\alpha,\beta}}(I)$ and $I^* : L^2_{\omega^{\alpha,\beta}}(I) \to L^2_{\omega^{\alpha,\beta}}(I)$ are bounded operators, this means that there exist constants $C$ and $C'$ such that [11]:*

$$\|I(u)\|_{\omega^{\alpha,\beta}} \leq C\|u\|_{\omega^{\alpha,\beta}},\qquad and\qquad \left\|I^*(u)\right\|_{\omega^{\alpha,\beta}} \leq C'\|u\|_{\omega^{\alpha,\beta}}.$$

**Lemma 15** *If $u \in \mathbb{P}_N$, then [18]:*

$$\|u\|_{\omega^{\alpha,\beta}} \leq \|u\|_{N,\omega^{\alpha,\beta}} \leq \sqrt{2+\frac{1}{N}}\|u\|_{\omega^{\alpha,\beta}}.$$

For the Sobolev space $X$, let

$$H^m_{\omega^{\alpha,\beta}}(J;X) := \left\{u\,|\ \|u(x,.)\|_X \in H^m_{\omega^{\alpha,\beta}}(J)\right\},\quad m \geq 0,$$

endowed with the norm:

$$\|u\|_{m,\omega^{\alpha,\beta},X} := \|\,\|u(x,.)\|_X\|_{m,\omega^{\alpha,\beta}}.$$

### 4.1 The stability and convergence of the collocation method for Eq. (1)

*Remark 1* For simplicity, we use $x_k, \omega_k, \Pi_N$ and $I_N$ instead of $x_k^{0,0}, \omega_k^{0,0}, \Pi_{N,\omega^{0,0}}$ and $I_N^{0,0}$, respectively.

**Theorem 1** (The stability theorem) *Suppose $u_N \in \mathbb{P}_N^0$ is the approximate solution of (2) obtained by the method presented in Section 3, also suppose $\{x_k\}_{k=0}^N$ and $\{\omega_k\}_{k=0}^N$ are the Gauss-Legendre quadrature nodes and weights, respectively and $I = (a, b)$. Assume that $f \in H^1_{\omega^{0,0}}(I)$, $\kappa$, $\rho$, $c_+$, $c_-$, $d_+$, $d_- \in H^1_{\omega^{1,1},*}(I)$, $g \in H^3_{\omega^{0,0}}(I)$ and $f$ and $\partial_x f$ satisfy in the Lipschitz condition. Also suppose $\kappa$, $\rho$, $c_+$, $c_-$, $d_+$, $d_-$, $g$ and $u_N$ in (2) and (3) have the errors $\tilde{\kappa}$, $\tilde{\rho}$, $\tilde{c}_+$, $\tilde{c}_-$, $\tilde{d}_+$, $\tilde{d}_-$, $\tilde{g}$ and $\tilde{u} \in \mathbb{P}_N^0$, respectively. Therefore we have*

$$\|\tilde{u}\|_{N,\omega^{0,0}} \leq C\|\tilde{u}(0)\|_{3,\omega^{0,0}} + C'\|\tilde{u}(0)\|_{N,\omega^{0,0}},$$

*where $C$ and $C'$ are constants.*

*Proof* We know that $u_N$ and $u_N + \tilde{u}$ satisfy in the following equations:

$$\frac{\partial u_N}{\partial t}(x_k) = \kappa(x_k, t)\left(c_+(x_k, t)\,{}_a^R D_x^{\mu} u_N(x, t)\Big|_{x=x_k} + c_-(x_k, t)\,{}_x^R D_b^{\mu} u_N(x, t)\Big|_{x=x_k}\right) +$$
$$\rho(x_k, t)\left(d_+(x_k, t)\,{}_a^R D_x^{\sigma} u_N(x, t)\Big|_{x=x_k} + d_-(x_k, t)\,{}_x^R D_b^{\sigma} u_N(x, t)\Big|_{x=x_k}\right) + f\left(u_N(x_k), x_k, t\right), \quad (19)$$
$$k = 1, 2, ..., N-1,$$

$$u_N(x_k, 0) = g(x_k), \quad k = 0, 1, 2, ..., N, \quad (20)$$

and

$$\frac{\partial(u_N+\tilde{u})}{\partial t}(x_k) = (\kappa + \tilde{\kappa})(x_k, t)\left((c_+ + \tilde{c}_+)(x_k, t)\,{}_a^R D_x^{\mu}(u_N + \tilde{u})(x, t)\Big|_{x=x_k} +\right.$$
$$\left.(c_- + \tilde{c}_-)(x_k, t)\,{}_x^R D_b^{\mu}(u_N + \tilde{u})(x, t)\Big|_{x=x_k}\right) + (\rho + \tilde{\rho})(x_k, t)\left(\left(d_+ + \tilde{d}_+\right)(x_k, t)\right.$$
$${}_a^R D_x^{\sigma}(u_N + \tilde{u})(x, t)\Big|_{x=x_k} + \left.\left(d_- + \tilde{d}_-\right)(x_k, t)\,{}_x^R D_b^{\sigma}(u_N + \tilde{u})(x, t)\Big|_{x=x_k}\right) + \quad (21)$$
$$f\left((u_N + \tilde{u})(x_k), x_k, t\right), \quad k = 1, 2, ..., N-1,$$

$$\tilde{u}(x_k, 0) + u_N(x_k, 0) = g(x_k) + \tilde{g}(x_k), \quad k = 0, 1, 2, ..., N. \quad (22)$$

Suppose $h(x) = (x - x_0)^2(x - x_N)^2(x - a)^2(x - b)^2$. Now subtracting (21) from (19) and multiplying both sides of the new equation by $w_k h(x_k)\tilde{u}(x_k)$, $k = 0, 1, ..., N$ and summing up these equations we can write

$$\left(h\frac{\partial\tilde{u}}{\partial t}, \tilde{u}\right)_{\omega^{0,0}} = \left(h\tilde{\kappa}\tilde{c}_+\,{}_a^R D_x^{\mu}\tilde{u}, \tilde{u}\right)_{N,\omega^{0,0}} + \left(h\tilde{\kappa}\tilde{c}_-\,{}_x^R D_b^{\mu}\tilde{u}, \tilde{u}\right)_{N,\omega^{0,0}} + \left(h\tilde{\rho}\tilde{d}_+\,{}_a^R D_x^{\sigma}\tilde{u}, \tilde{u}\right)_{N,\omega^{0,0}} +$$
$$\left(h\tilde{\rho}\tilde{d}_-\,{}_x^R D_b^{\sigma}\tilde{u}, \tilde{u}\right)_{N,\omega^{0,0}} + \left(h\tilde{f}, \tilde{u}\right)_{N,\omega^{0,0}} + \left(\left(\frac{\partial\tilde{u}}{\partial t}h, \tilde{u}\right)_{\omega^{0,0}} - \left(\frac{\partial\tilde{u}}{\partial t}h, \tilde{u}\right)_{N,\omega^{0,0}}\right) \pm \left(h\tilde{f}, \tilde{u}\right)_{\omega^{0,0}}, \quad (23)$$

where $\tilde{f}(\tilde{u}) = f(u_N + \tilde{u}) - f(u_N)$. It is easy to see that

$$\left(\frac{\partial\tilde{u}}{\partial t}h, \tilde{u}\right)_{\omega^{0,0}} \leq \left|\left(h\tilde{\kappa}\tilde{c}_+\,{}_a^R D_x^{\mu}\tilde{u}, \tilde{u}\right)_{N,\omega^{0,0}}\right| + \left|\left(h\tilde{\kappa}\tilde{c}_-\,{}_x^R D_b^{\mu}\tilde{u}, \tilde{u}\right)_{N,\omega^{0,0}}\right| + \left|\left(h\tilde{\rho}\tilde{d}_+\,{}_a^R D_x^{\sigma}\tilde{u}, \tilde{u}\right)_{N,\omega^{0,0}}\right| +$$
$$\left|\left(h\tilde{\rho}\tilde{d}_-\,{}_x^R D_b^{\sigma}\tilde{u}, \tilde{u}\right)_{N,\omega^{0,0}}\right| + \left|\left(h\tilde{f}, \tilde{u}\right)_{N,\omega^{0,0}} - \left(h\tilde{f}, \tilde{u}\right)_{\omega^{0,0}}\right| + \left|\left(h\tilde{f}, \tilde{u}\right)_{\omega^{0,0}}\right| +$$
$$\left|\left(\left(\frac{\partial\tilde{u}}{\partial t}h, \tilde{u}\right)_{\omega^{0,0}} - \left(\frac{\partial\tilde{u}}{\partial t}h, \tilde{u}\right)_{N,\omega^{0,0}}\right)\right|. \quad (24)$$

Now we have

$$\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u},\tilde{u}\right)_{N,\omega^{0,0}}=\left(I_N\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right),\tilde{u}\right)_{\omega^{0,0}}, \tag{25}$$

therefore we can write

$$\left|\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u},\tilde{u}\right)_{N,\omega^{0,0}}\right|\le\left\|I_N\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\right\|_{\omega^{0,0}}\|\tilde{u}\|_{\omega^{0,0}}. \tag{26}$$

Employing Lemma 6 yields

$$\begin{aligned}&\left\|I_N\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\right\|_{\omega^{0,0}}-\left\|h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{0,0}}\le\left\|I_N\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)-\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\right\|_{\omega^{0,0}}\le\\&C_1N^{-1}\left\|\partial_x\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\right\|_{\omega^{1,1}}\le C_1N^{-1}\left(\left\|\partial_x\left(\tilde{\kappa}\tilde{c}_{+}\right)h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}+\left\|\partial_x\left(h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\tilde{\kappa}\tilde{c}_{+}\right\|_{\omega^{1,1}}\right),\end{aligned} \tag{27}$$

therefore we have

$$\begin{aligned}&\left\|I_N\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\right\|_{\omega^{0,0}}\le C_1N^{-1}\left(\left\|\partial_x\left(\tilde{\kappa}\tilde{c}_{+}\right)h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}+\left\|\partial_x\left(h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\tilde{\kappa}\tilde{c}_{+}\right\|_{\omega^{1,1}}\right)+\\&\gamma_1\left\|h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{0,0}}\le C_1N^{-1}\left(\left\|\partial_x\left(\tilde{\kappa}\tilde{c}_{+}\right)h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}+\left\|\partial_x\left(h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\right\|_{\omega^{1,1}}\|\tilde{\kappa}\tilde{c}_{+}\|_{\omega^{1,1}}\right)+\\&\gamma_1\left\|h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{0,0}},\end{aligned} \tag{28}$$

where $\gamma_1=\|\tilde{\kappa}\tilde{c}_{+}\|_{\omega^{0,0}}$. But we know that

$$\left\|\partial_x\left(h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\right\|_{\omega^{1,1}}\le\left\|h'\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}+\left\|h\partial_x\left({}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\right\|_{\omega^{1,1}}. \tag{29}$$

Using Lemmas 1 and 14, respectively we conclude

$$\begin{aligned}&\left\|h'\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}\le\left\|\frac{h'\tilde{u}'(a)(x-a)^{1-\mu}}{\Gamma(2-\mu)}\right\|_{\omega^{1,1}}+\left\|h'\,{}_{a}^{C}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}\le\left\|\frac{h'(x-a)^{1-\mu}}{\Gamma(2-\mu)}\right\|_{\omega^{1,1}}\left|\tilde{u}'(a)\right|+\\&\gamma_2\left\|{}_{a}^{C}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}\le\gamma_3\left|\tilde{u}'(a)\right|+C_2\left\|\partial_x^2\tilde{u}\right\|_{\omega^{1,1}}\le\gamma_3\left|\tilde{u}'(a)\right|+C_3\left\|\partial_x^2\tilde{u}\right\|_{\omega^{0,0}},\end{aligned} \tag{30}$$

and following (30) we have

$$\left\|h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{0,0}}\le\gamma_4\left|\tilde{u}'(a)\right|+C_4\left\|\partial_x^2\tilde{u}\right\|_{\omega^{0,0}}, \tag{31}$$

where $\gamma_2=\|h'\|_{\omega^{1,1}}$, $\gamma_3=\left\|\frac{h'(x-a)^{1-\mu}}{\Gamma(2-\mu)}\right\|_{\omega^{1,1}}$, $\gamma_4=\|h\|_{\omega^{0,0}}$. Using Lemmas 1, 2 and 14, respectively we obtain

$$\begin{aligned}&\left\|h\partial_x\left({}_{a}^{R}D_{x}^{\mu}\tilde{u}\right)\right\|_{\omega^{1,1}}=\left\|h\,{}_{a}^{R}D_{x}^{\mu+1}\tilde{u}\right\|_{\omega^{1,1}}\le\left\|h\left(\frac{\tilde{u}'(a)(x-a)^{-\mu}}{\Gamma(2-\mu)}\right)\right\|_{\omega^{1,1}}+\left\|h\left(\frac{\tilde{u}''(a)(x-a)^{-\mu+1}}{\Gamma(2-\mu)}\right)\right\|_{\omega^{1,1}}+\\&\left\|h\,{}_{a}^{C}D_{x}^{\mu+1}\tilde{u}\right\|_{\omega^{1,1}}\le\gamma_5\left|\tilde{u}'(a)\right|+\gamma_6\left|\tilde{u}''(a)\right|+C_5\left\|\partial_x^3\tilde{u}\right\|_{\omega^{0,0}},\end{aligned} \tag{32}$$

where $\gamma_5=\left\|\frac{h(x-a)^{-\mu}}{\Gamma(2-\mu)}\right\|_{\omega^{1,1}}$ and $\gamma_6=\left\|\frac{h(x-a)^{-\mu+1}}{\Gamma(2-\mu)}\right\|_{\omega^{1,1}}$. Now employing Lemmas 1 and 14, one can write

$$\left\|\partial_x\left(\tilde{\kappa}\tilde{c}_{+}\right)h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}\le\gamma_7\left\|h\,{}_{a}^{R}D_{x}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}\le\gamma_7\gamma_8\left|\tilde{u}'(a)\right|+C_6\left\|\partial_x^2\tilde{u}\right\|_{\omega^{0,0}}, \tag{33}$$

where $\gamma_7=\|\partial_x\left(\tilde{\kappa}\tilde{c}_{+}\right)\|_{\omega^{1,1}}$ and $\gamma_8=\left\|\frac{h(x-a)^{1-\mu}}{\Gamma(2-\mu)}\right\|_{\omega^{1,1}}$. Therefore substituting (28)-(33) in (26), respectively and using (15) we can write:

$$\begin{aligned}&\left|\left(h\tilde{\kappa}\tilde{c}_{+}\,{}_{a}^{R}D_{x}^{\mu}\tilde{u},\tilde{u}\right)_{N,\omega^{0,0}}\right|\le C_1N^{-1}\|\tilde{u}\|_{\omega^{0,0}}\left(C_7\left|\tilde{u}'(a)\right|+C_8\left|\tilde{u}''(a)\right|+C_9\left\|\partial_x^2\tilde{u}\right\|_{\omega^{0,0}}+C_{10}\left\|\partial_x^3\tilde{u}\right\|_{\omega^{0,0}}\right)+\\&\|\tilde{u}\|_{\omega^{0,0}}\left(\gamma_8\left|\tilde{u}'(a)\right|+C_5\left\|\partial_x^2\tilde{u}\right\|_{\omega^{0,0}}\right)\le\varsigma_0\left|\tilde{u}'(a)\right|^2+\varsigma_1\left|\tilde{u}''(a)\right|^2+\varsigma_2\left\|\partial_x^2\tilde{u}\right\|_{\omega^{0,0}}^2+\varsigma_3\left\|\partial_x^3\tilde{u}\right\|_{\omega^{0,0}}^2+\\&\varsigma_4\|\tilde{u}\|_{\omega^{0,0}}^2\le\varsigma_5\left|\tilde{u}'(a)\right|^2+\varsigma_6\left|\tilde{u}''(a)\right|^2+\varsigma_7\|\tilde{u}\|_{3,\omega^{0,0}}^2+\varsigma_4\|\tilde{u}\|_{\omega^{0,0}}^2.\end{aligned} \tag{34}$$

We know that

$$\left(h\tilde{\kappa}\tilde{c}_{-}\,{}_{x}^{R}D_{b}^{\mu}\tilde{u},\tilde{u}\right)_{N,\omega^{0,0}}=\left(I_{N}\left(h\tilde{\kappa}\tilde{c}_{-}\,{}_{x}^{R}D_{b}^{\mu}\tilde{u}\right),\tilde{u}\right)_{\omega^{0,0}}. \tag{35}$$

Now using Lemmas 6, 1, 2 and 14 and Eq. (15) and follow the process that we obtained (25)–(34) we can write

$$\left|\left(h\tilde{\kappa}\tilde{c}_{-}\,{}_{x}^{R}D_{b}^{\mu}\tilde{u},\tilde{u}\right)_{N,\omega^{0,0}}\right|\leq\left\|I_{N}\left(h\tilde{\kappa}\tilde{c}_{-}\,{}_{x}^{R}D_{b}^{\mu}\tilde{u}\right)\right\|_{\omega^{0,0}}\|\tilde{u}\|_{\omega^{0,0}}\leq C_{11}N^{-1}\left(\left\|\partial_{x}\left(\tilde{\kappa}\tilde{c}_{-}\right)h\,{}_{x}^{R}D_{b}^{\mu}\tilde{u}\right\|_{\omega^{1,1}}+\right.$$

$$\left.\left\|\partial_{x}\left(h\,{}_{x}^{R}D_{b}^{\mu}\tilde{u}\right)\right\|_{\omega^{1,1}}\|\tilde{\kappa}\tilde{c}_{-}\|_{\omega^{1,1}}\right)+\gamma_{8}\left\|h\,{}_{x}^{R}D_{b}^{\mu}\tilde{u}\right\|_{\omega^{0,0}},\leq C_{11}N^{-1}\|\tilde{u}\|_{\omega^{0,0}}\left(C_{12}\left|\tilde{u}'(b)\right|+C_{13}\left|\tilde{u}''(b)\right|+\right.$$

$$\left.C_{14}\left\|\partial_{x}^{2}\tilde{u}\right\|_{\omega^{0,0}}+C_{15}\left\|\partial_{x}^{3}\tilde{u}\right\|_{\omega^{0,0}}\right)+\|\tilde{u}\|_{\omega^{0,0}}\left(\gamma_{9}\left|\tilde{u}'(b)\right|+C_{16}\left\|\partial_{x}^{2}\tilde{u}\right\|_{\omega^{0,0}}\right)\leq\varsigma_{8}\|\tilde{u}\|_{\omega^{0,0}}^{2}+\varsigma_{9}\left|\tilde{u}'(b)\right|^{2}+$$

$$\varsigma_{10}\left|\tilde{u}''(b)\right|^{2}+\varsigma_{11}\|\tilde{u}\|_{3,\omega^{0,0}}^{2}, \tag{36}$$

where $\gamma_8 = \|\tilde{\kappa}\tilde{c}_{-}\|_{\omega^{0,0}}$ and $\gamma_9 = \left\|\frac{h(b-x)^{1-\mu}}{\Gamma(2-\mu)}\right\|_{\omega^{1,1}}$. Therefore one can write

$$\left|\left(h\tilde{\kappa}\tilde{c}_{-}\,{}_{x}^{R}D_{b}^{\mu}\tilde{u},\tilde{u}\right)_{N,\omega^{0,0}}\right|\leq\varsigma_{8}\|\tilde{u}\|_{\omega^{0,0}}^{2}+\varsigma_{9}\left|\tilde{u}'(b)\right|^{2}+\varsigma_{10}\left|\tilde{u}''(b)\right|^{2}+\varsigma_{11}\|\tilde{u}\|_{3,\omega^{0,0}}^{2}. \tag{37}$$

Also we have

$$\left|\left(h\tilde{\rho}\tilde{d}_{+}\,{}_{a}^{R}D_{x}^{\sigma}\tilde{u},\tilde{u}\right)_{N,\omega^{0,0}}\right|\leq\left\|I_{N}\left(h\tilde{\rho}\tilde{d}_{+}\,{}_{a}^{R}D_{x}^{\sigma}\tilde{u}\right)\right\|_{\omega^{0,0}}\|\tilde{u}\|_{\omega^{0,0}}\leq C_{17}N^{-1}\left(\left\|\partial_{x}\left(\tilde{\rho}\tilde{d}_{+}\right)h\,{}_{a}^{R}D_{x}^{\sigma}\tilde{u}\right\|_{\omega^{1,1}}+\right.$$

$$\left.\left\|\partial_{x}\left(h\,{}_{a}^{R}D_{x}^{\sigma}\tilde{u}\right)\right\|_{\omega^{1,1}}\left\|\tilde{\rho}\tilde{d}_{+}\right\|_{\omega^{1,1}}\right)+\gamma_{10}\left\|h\,{}_{a}^{R}D_{x}^{\sigma}\tilde{u}\right\|_{\omega^{0,0}},\leq C_{17}N^{-1}\|\tilde{u}\|_{\omega^{0,0}}\left(C_{18}\left|\tilde{u}'(a)\right|+C_{19}\left\|\partial_{x}^{2}\tilde{u}\right\|_{\omega^{0,0}}\right)$$

$$+C_{20}\|\partial_{x}\tilde{u}\|_{\omega^{0,0}}\|\tilde{u}\|_{\omega^{0,0}}\leq\varsigma_{12}\|\tilde{u}\|_{\omega^{0,0}}^{2}+\varsigma_{13}\left|\tilde{u}'(a)\right|^{2}+\varsigma_{14}\left|\tilde{u}''(a)\right|^{2}+\varsigma_{15}\|\tilde{u}\|_{3,\omega^{0,0}}^{2}, \tag{38}$$

and

$$\left|\left(h\tilde{\rho}\tilde{d}_{-}\,{}_{x}^{R}D_{b}^{\sigma}\tilde{u},\tilde{u}\right)_{N,\omega^{0,0}}\right|\leq\left\|I_{N}\left(h\tilde{\rho}\tilde{d}_{-}\,{}_{x}^{R}D_{b}^{\sigma}\tilde{u}\right)\right\|_{\omega^{0,0}}\|\tilde{u}\|_{\omega^{0,0}}\leq C_{21}N^{-1}\left(\left\|\partial_{x}\left(\tilde{\rho}\tilde{d}_{-}\right)h\,{}_{x}^{R}D_{b}^{\sigma}\tilde{u}\right\|_{\omega^{1,1}}+\right.$$

$$\left.\left\|\partial_{x}\left(h\,{}_{x}^{R}D_{b}^{\sigma}\tilde{u}\right)\right\|_{\omega^{1,1}}\left\|\tilde{\rho}\tilde{d}_{-}\right\|_{\omega^{1,1}}\right)+\gamma_{11}\left\|h\,{}_{x}^{R}D_{b}^{\sigma}\tilde{u}\right\|_{\omega^{0,0}},\leq C_{21}N^{-1}\|\tilde{u}\|_{\omega^{0,0}}\left(C_{22}\left|\tilde{u}'(b)\right|+\right.$$

$$\left.C_{23}\left\|\partial_{x}^{2}\tilde{u}\right\|_{\omega^{0,0}}\right)+C_{24}\|\partial_{x}\tilde{u}\|_{\omega^{0,0}}\|\tilde{u}\|_{\omega^{0,0}}\leq\varsigma_{16}\|\tilde{u}\|_{\omega^{0,0}}^{2}+\varsigma_{17}\left|\tilde{u}'(a)\right|^{2}+\varsigma_{18}\left|\tilde{u}''(a)\right|^{2}+\varsigma_{19}\|\tilde{u}\|_{3,\omega^{0,0}}^{2}, \tag{39}$$

where $\gamma_{10} = \left\|\tilde{\rho}\tilde{d}_{+}\right\|_{\omega^{0,0}}$ and $\gamma_{11} = \left\|\tilde{\rho}\tilde{d}_{-}\right\|_{\omega^{0,0}}$. Employing Lemmas 7 and 6, respectively we obtain

$$\left|\left(h\tilde{f},\tilde{u}\right)_{N,\omega^{0,0}}-\left(h\tilde{f},\tilde{u}\right)_{\omega^{0,0}}\right|\leq\left\|I_{N}\left(h\tilde{f}\right)-h\tilde{f}\right\|_{\omega^{0,0}}\|\tilde{u}\|_{\omega^{0,0}}\leq\nu_{1}N^{-1}\left\|\partial_{x}\left(h\tilde{f}\right)\right\|_{\omega^{1,1}}\leq\nu_{1}N^{-1}$$

$$\left(\left\|h\partial_{x}\tilde{f}\right\|_{\omega^{1,1}}+\left\|h'\tilde{f}\right\|_{\omega^{1,1}}\right)\|\tilde{u}\|_{\omega^{0,0}}\leq N^{-1}\left(\nu_{2}\left\|\partial_{x}\tilde{f}\right\|_{\omega^{1,1}}+\nu_{3}\left\|\tilde{f}\right\|_{\omega^{1,1}}\right)\|\tilde{u}\|_{\omega^{0,0}}, \tag{40}$$

where $\nu_2 = \nu_1\|h\|_{\omega^{1,1}}$ and $\nu_3 = \nu_1\left\|h'\right\|_{\omega^{1,1}}$. $f$ and $\partial_x f$, satisfy in the Lipschitz condition therefore one can write

$$\left\|\partial_{x}\tilde{f}\right\|_{\omega^{1,1}}=\|\partial_{x}\left(f(u_{N}+\tilde{u})-f(u_{N})\right)\|_{\omega^{1,1}}\leq\nu_{4}\|\tilde{u}\|_{\omega^{1,1}}\leq\nu_{5}\|\tilde{u}\|_{\omega^{0,0}}, \tag{41}$$

and

$$\left\|\tilde{f}\right\|_{\omega^{1,1}}=\|f(u_{N}+\tilde{u})-f(u_{N})\|_{\omega^{1,1}}\leq\nu_{6}\|\tilde{u}\|_{\omega^{1,1}}\leq\nu_{7}\|\tilde{u}\|_{\omega^{0,0}}. \tag{42}$$

Substituting (41) and (42) in (40) gives

$$\left|\left(h\tilde{f},\tilde{u}\right)_{N,\omega^{0,0}}-\left(h\tilde{f},\tilde{u}\right)_{\omega^{0,0}}\right|\leq\nu_{8}N^{-1}\|\tilde{u}\|_{\omega^{0,0}}^{2}. \tag{43}$$

Also we have

$$\left|\left(h\tilde{f},\tilde{u}\right)_{\omega^{0,0}}\right| \leq \left\|h\tilde{f}\right\|_{\omega^{0,0}}\|\tilde{u}\|_{\omega^{0,0}} \leq \gamma_8\left\|\tilde{f}\right\|_{\omega^{0,0}}\|\tilde{u}\|_{\omega^{0,0}} \leq \nu_9\,\|\tilde{u}\|^2_{\omega^{0,0}}\,. \tag{44}$$

Using Lemmas 7 and 6, respectively one can write

$$\left|\left(\tfrac{\partial\tilde{u}}{\partial t}h,\tilde{u}\right)_{\omega^{0,0}} - \left(\tfrac{\partial\tilde{u}}{\partial t}h,\tilde{u}\right)_{N,\omega^{0,0}}\right| = \left|\left(\tfrac{\partial\tilde{u}}{\partial t}\tilde{u},h\right)_{\omega^{0,0}} - \left(\tfrac{\partial\tilde{u}}{\partial t}\tilde{u},h\right)_{N,\omega^{0,0}}\right| \leq \vartheta_1 N^{-3}\left\|\tfrac{\partial\tilde{u}}{\partial t}\tilde{u}\right\|_{3,\omega^{1,1}}\|h\|_{\omega^{0,0}} \leq$$
$$\tfrac{\vartheta_2}{2}N^{-3}\tfrac{d}{dt}\,\|\tilde{u}\|^2_{3,\omega^{0,0}}\,. \tag{45}$$

Substituting (34), (37), (38), (39), (43), (44) and (45) in (24) we have

$$\left(\tfrac{\partial\tilde{u}}{\partial t}h,\tilde{u}\right)_{\omega^{0,0}} \leq \vartheta_3\left|\tilde{u}'(a)\right|^2 + \vartheta_4\left|\tilde{u}''(a)\right|^2 + \vartheta_5\left|\tilde{u}'(b)\right|^2 + \vartheta_6\left|\tilde{u}''(b)\right|^2 + \vartheta_7\,\|\tilde{u}\|^2_{3,\omega^{0,0}} +$$
$$\vartheta_8\,\|\tilde{u}\|^2_{\omega^{0,0}} + \vartheta_9 N^{-3}\tfrac{d}{dt}\,\|\tilde{u}\|^2_{3,\omega^{0,0}}\,. \tag{46}$$

Using Lemma 16 we conclude there exists a constant $\xi \in (a,b)$, such that $h(\xi) \neq 0$ and

$$\left(\frac{\partial\tilde{u}}{\partial t}h,\tilde{u}\right)_{\omega^{0,0}} = h(\xi)\left(\frac{\partial\tilde{u}}{\partial t},\tilde{u}\right)_{\omega^{0,0}} = \frac{h(\xi)}{2}\frac{d}{dt}\|\tilde{u}\|^2_{\omega^{0,0}}, \tag{47}$$

also using Lemma 13 and Eq. (15) we have

$$\left|\tilde{u}'(a)\right|^2 \leq \left\|\tilde{u}'\right\|^2_\infty \leq \tau_0\,\|\tilde{u}\|^2_{3,\omega^{0,0}}\,, \tag{48}$$

$$\left|\tilde{u}''(a)\right|^2 \leq \left\|\tilde{u}''\right\|^2_\infty \leq \tau_1\,\|\tilde{u}\|^2_{3,\omega^{0,0}}\,, \tag{49}$$

$$\left|\tilde{u}'(b)\right|^2 \leq \left\|\tilde{u}'\right\|^2_\infty \leq \tau_2\,\|\tilde{u}\|^2_{3,\omega^{0,0}}\,, \tag{50}$$

and

$$\left|\tilde{u}''(b)\right|^2 \leq \left\|\tilde{u}''\right\|^2_\infty \leq \tau_3\,\|\tilde{u}\|^2_{3,\omega^{0,0}}\,. \tag{51}$$

Now substituting (47)–(51) in (46) yields

$$\frac{d}{dt}\,\|\tilde{u}\|^2_{\omega^{0,0}} + \vartheta_{10}\frac{d}{dt}\,\|\tilde{u}\|^2_{3,\omega^{0,0}} \leq \vartheta_{10}\,\|\tilde{u}\|^2_{3,\omega^{0,0}} + \vartheta_{11}\,\|\tilde{u}\|^2_{\omega^{0,0}} + \left(\vartheta_{12}N^{-3} + \vartheta_{10}\right)\frac{d}{dt}\,\|\tilde{u}\|^2_{3,\omega^{0,0}}\,, \tag{52}$$

therefore we have

$$\vartheta_{10}\frac{d}{dt}\,\|\tilde{u}\|^2_{3,\omega^{0,0}} \leq \vartheta_{10}\,\|\tilde{u}\|^2_{3,\omega^{0,0}} + \vartheta_{13}\,\|\tilde{u}\|^2_{\omega^{0,0}} + \left(\vartheta_{12}N^{-3} + \vartheta_{10}\right)\frac{d}{dt}\,\|\tilde{u}\|^2_{3,\omega^{0,0}} - \frac{d}{dt}\,\|\tilde{u}\|^2_{\omega^{0,0}}\,. \tag{53}$$

Employing Gronwall lemma [5] one can write

$$\vartheta_{14}\,\|\tilde{u}\|^2_{3,\omega^{0,0}} \leq e^t\,\|\tilde{u}(0)\|^2_{3,\omega^{0,0}} + \int_0^t\left(\left(\vartheta_{16}N^{-3} + \vartheta_{14}\right)\tfrac{d}{ds}\,\|\tilde{u}\|^2_{3,\omega^{0,0}} - \tfrac{d}{ds}\,\|\tilde{u}\|^2_{\omega^{0,0}}\right)e^{(t-s)}ds+$$
$$\vartheta_{13}\int_0^t\|\tilde{u}\|^2_{\omega^{0,0}}e^{(t-s)}ds, \tag{54}$$

by integrating by parts from $\int_0^t\left(\left(\vartheta_{16}N^{-3} + \vartheta_{14}\right)\tfrac{d}{ds}\,\|\tilde{u}\|^2_{3,\omega^{0,0}} - \tfrac{d}{ds}\,\|\tilde{u}\|^2_{\omega^{0,0}}\right)e^{(t-s)}ds$ we conclude

$$\vartheta_{14}\,\|\tilde{u}\|^2_{3,\omega^{0,0}} + \|\tilde{u}\|^2_{\omega^{0,0}} \leq e^t\left(\|\tilde{u}(0)\|^2_{3,\omega^{0,0}} + \|\tilde{u}(0)\|^2_{\omega^{0,0}}\right) + \left(\vartheta_{16}N^{-3} + \vartheta_{14}\right)\left(\|\tilde{u}\|^2_{3,\omega^{1,1}}\right.$$
$$\left. - \|\tilde{u}(0)\|^2_{3,\omega^{0,0}}\,e^t + \int_0^t\|\tilde{u}\|^2_{3,\omega^{0,0}}\,e^{(t-s)}ds\right), \tag{55}$$

and employing the Gronwall lemma [5] two times, we can write

$$\|\tilde{u}\|^2_{\omega^{0,0}} \leq \vartheta_{17}\left(\vartheta_{14}\,\|\tilde{u}(0)\|^2_{3,\omega^{0,0}} + \|\tilde{u}(0)\|^2_{\omega^{0,0}}\right)\,. \tag{56}$$

Finally using Lemma 15 we have

$$\|\tilde{u}\|_{N,\omega^{0,0}} \leq C\|\tilde{u}(0)\|_{3,\omega^{0,0}} + C'\|\tilde{u}(0)\|_{N,\omega^{0,0}}. \tag{57}$$

□

**Theorem 2** (The convergence theorem) *Suppose $u$ is the exact solution of (1) and $u_N$ is the solution of (2) which is obtained by the method presented in Section 2, also suppose $\{x_k\}_{k=0}^N$ and $\{\omega_k\}_{k=0}^N$ are the Gauss-Legendre quadrature nodes and weights, respectively and $I = (a, b)$. Assume that $u \in H^1_{\omega^{0,0}}\left([0, T], H_l^{\mu,m}(I)\right)$, $1 < \mu < 2$, and $\partial_x^i f \in C(I)$, $i = 0, 1, ..., m$ and $\rho$, $\kappa$, $c_+$, $c_-$, $d_+$ and $d_- \in H^m_{\omega^{0,0}}(I)$ and $g \in H^l_{\omega^{0,0}}(I)$, $m > 2$ and $l > 2m - \frac{3}{2}$. Then*

$$\|u(t) - u_N(t)\|_{\omega^{0,0}} \to 0, \quad \forall t \in [0, T],$$

*as $N \to \infty$.*

*Proof* Suppose $h(x) = (x - x_0)(x - x_N)(x - a)^{(m+1)}(x - b)^{(m+1)}$. For simplicity suppose $P_j(x)$ $(j = 0, 1, ..., N)$ are the Legendre polynomials of degree $j$, $(j = 0, 1, ..., N)$. By multiplying both sides of (2) in $w_k h(x_k)P_j(x_k)$, $(k = 0, 1, ..., N)$, and summing up these equations we have

$$\sum_{k=0}^{N} \tfrac{\partial}{\partial t} u_N(x_k, t) w_k h(x_k) P_j(x_k) = \sum_{k=0}^{N} \left\{\kappa(x_k, t)c_+(x_k, t)\, {}_a^R D_x^{\mu} u_N(x, t)\big|_{x=x_k} w_k h(x_k)P_j(x_k)\right\} +$$
$$\sum_{k=0}^{N} \left\{\kappa(x_k, t)c_-(x_k, t)\, {}_x^R D_b^{\mu} u_N(x, t)\big|_{x=x_k} w_k h(x_k)P_j(x_k)\right\} + \sum_{k=0}^{N} \{\rho(x_k, t)d_+(x_k, t)$$
$${}_a^R D_x^{\sigma} u_N(x, t)\big|_{x=x_k}\ w_k h(x_k)P_j(x_k)\} + \sum_{k=0}^{N} \{\rho(x_k, t)d_-(x_k, t)\ {}_x^R D_b^{\sigma} u_N(x, t)\big|_{x=x_k}$$
$$w_k h(x_k)P_j(x_k)\} + \sum_{k=0}^{N} \{f(u_N(x_k, t), x_k, t)\ \ w_k h(x_k)P_j(x_k)\}. \tag{58}$$

Suppose

$$A_j(t) = \sum_{k=0}^{N} \tfrac{\partial}{\partial t} u_N(x_k, t) w_k h(x_k) P_j(x_k),$$
$$B_j(t) = \sum_{k=0}^{N} \left\{\kappa(x_k, t)c_+(x_k, t)\, {}_a^R D_x^{\mu} u_N(x, t)\big|_{x=x_k} w_k h(x_k)P_j(x_k)\right\},$$
$$C_j(t) = \sum_{k=0}^{N} \left\{\kappa(x_k, t)c_-(x_k, t)\, {}_x^R D_b^{\mu} u_N(x, t)\big|_{x=x_k} w_k h(x_k)P_j(x_k)\right\},$$
$$D_j(t) = \sum_{k=0}^{N} \left\{\rho(x_k, t)d_+(x_k, t)\, {}_a^R D_x^{\sigma} u_N(x, t)\big|_{x=x_k} w_k h(x_k)P_j(x_k)\right\},$$
$$E_j(t) = \sum_{k=0}^{N} \left\{\rho(x_k, t)d_-(x_k, t)\, {}_x^R D_b^{\sigma} u_N(x, t)\big|_{x=x_k} w_k h(x_k)P_j(x_k)\right\},$$
$$F_j(t) = \sum_{k=0}^{N} \{f(u_N(x_k, t), x_k, t)\ \ w_k h(x_k)P_j(x_k)\},$$

therefore (58) can be written in the form:

$$\int_a^b \left(\tfrac{\partial}{\partial t} u_N(x, t)h(x)P_j(x)\omega^{0,0}(x)\right) dx = B_j(t) + C_j(t) + D_j(t) + E_j(t) + F_j(t) +$$
$$\int_a^b \left(\tfrac{\partial}{\partial t} u_N(x, t)h(x)P_j(x)\omega^{0,0}(x)\right) dx - A_j(t). \tag{59}$$

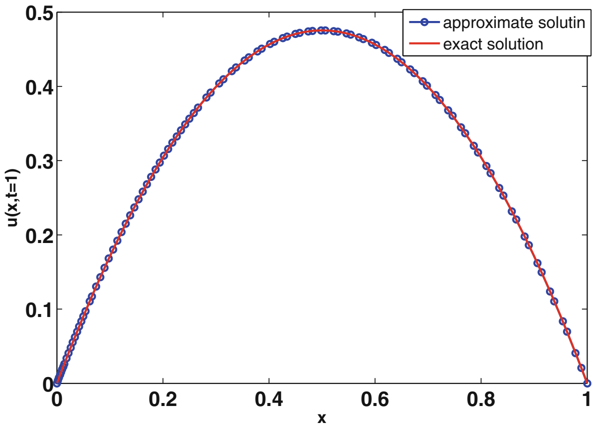


**Fig. 1** The comparison of the exact and approximate solutions for $N = 3$ and $(\alpha, \beta) = (-0.1, 0)$, $\tau = \frac{1}{60}$ at the time $t = 1.0$ (Example 1)

Multiplying both sides of (59) in $P_j(x)$ and summing up from $j = 0$ to $j = N$ yield

$$\frac{\partial}{\partial t}u_N(x,t)h(x) = \sum_{j=0}^{N} B_j(t)P_j(x) + \sum_{j=0}^{N} C_j(t)P_j(x) + \sum_{j=0}^{N} D_j(t)P_j(x) + \sum_{j=0}^{N} E_j(t)P_j(x)$$
$$+\sum_{j=0}^{N}\left(\int_a^b\left(\frac{\partial}{\partial t}u_N(x,t)h(x)P_j(x)\right)dx - A_j(t)\right)P_j(x) + \sum_{j=0}^{N} F_j(t)P_j(x) + \frac{\partial}{\partial t}u_N(x,t)h(x) - \tag{60}$$
$$\sum_{j=0}^{N}\left(\int_a^b\left(\frac{\partial}{\partial t}u_N(x,t)h(x)P_j(x)\right)dx\right)P_j(x).$$

Also we can write

$$\sum_{j=0}^{N} B_j(t)P_j(x) = L_1(x,t) + L_2(x,t) + h(x)\kappa(x,t)c_+(x,t)\,{}_a^R D_x^\mu u_N(x,t), \tag{61}$$

where

$$L_1(x,t) = \sum_{j=0}^{N}\left\{\left(h\kappa c_+\,{}_a^R D_x^\mu u_N, P_j\right)_{N,\omega^{0,0}} - \left(h\kappa c_+\,{}_a^R D_x^\mu u_N, P_j\right)_{\omega^{0,0}}\right\}P_j(x),$$
$$L_2(x,t) = \Pi_{N,\omega^{0,0}}\left(h(x)\kappa(x,t)\ c_+(x,t)\,{}_a^R D_x^\mu u_N(x,t)\right) - h(x)\kappa(x,t)c_+(x,t)\,{}_a^R D_x^\mu u_N(x,t).$$

It is easy to see that $h\,{}_a^R D_x^\mu u_N \in H^m_{\omega^{0,0}}(I)$, $m > 2$ [9]. Now using Lemmas 7 and 6 and Eq. (15), respectively we can write:

$$\|L_1(x,t)\|_{\omega^{0,0}} \le 2N^2 C_1 \left\| I_N\left(c_+\, h\kappa\ {}_a D_x^\mu u_N\right) - \left(c_+\, h\kappa\ {}_a D_x^\mu u_N\right)\right\|_{\omega^{0,0}}$$
$$\le 2N^{2-m}\tau_0 C_1' \left\|h\ {}_a D_x^\mu u_N\right\|_{m,\omega^{0,0}},$$

where $\tau_0 = \max_{0\le t\le T}\|\kappa c_+\|_{m,\omega^{0,0}}$ and $C_1$ and $C_1'$ are constants. Suppose

$$e_N(x,t) = u(x,t) - u_N(x,t),$$

therefore using (15) it is easy to see that

$$\|L_1(x,t)\|_{\omega^{0,0}} \le 2N^{2-m}\lambda_0 C_1''\left\|{}_a D_x^\mu u\right\|_{m,\omega^{0,0}} + 2N^{2-m}\lambda_0 C_1''\left\|{}_a D_x^\mu e_N\right\|_{m,\omega^{0,0}}, \tag{62}$$

where $\lambda_0 = \tau_0\|h\|_{m,\omega^{0,0}}$ and $C_1''$ is a constant. Noting to Lemma 4 and Eq. (15) we conclude

$$\|L_2(x,t)\|_{\omega^{0,0}} \le \tau_0 N^{-m} C_2 \left\|h\ {}_aD_x^{\mu} u_N\right\|_{m,\omega^{0,0}} \le \lambda_0 N^{-m} C_2' \left\|{}_aD_x^{\mu} u\right\|_{m,\omega^{0,0}} + \lambda_0 C_2' N^{-m} \left\|{}_aD_x^{\mu} e_N\right\|_{m,\omega^{0,0}}, \tag{63}$$

where $C_2$ and $C_2'$ are constants. Similar to (61)–(63) we can write

$$\sum_{j=0}^{N} C_j(t)P_j(x) = L_3(x,t) + L_4(x,t) + h(x)\kappa(x,t)c_-(x,t){}_x^RD_b^{\mu} u_N(x,t),$$

$$\sum_{j=0}^{N} D_j(t)P_j(x) = L_5(x,t) + L_6(x,t) + h(x)\rho(x,t)d_+(x,t){}_a^RD_x^{\sigma} u_N(x,t),$$

$$\sum_{j=0}^{N} E_j(t)P_j(x) = L_7(x,t) + L_8(x,t) + h(x)\rho(x,t)d_-(x,t){}_x^RD_b^{\sigma} u_N(x,t),$$

where

$$L_3(x,t) = \sum_{j=0}^{N} \left\{\left(h\kappa c_-\ {}_x^RD_b^{\mu} u_N, P_j\right)_{N,\omega^{0,0}} - \left(h\kappa c_-\ {}_x^RD_b^{\mu} u_N, P_j\right)_{\omega^{0,0}}\right\} P_j(x),$$

$$L_4(x,t) = \Pi_{N,\omega^{0,0}}\left(h(x)\kappa(x,t)\ c_-(x,t){}_x^RD_b^{\mu} u_N(x,t)\right) - h(x)\kappa(x,t)c_-(x,t){}_x^RD_b^{\mu} u_N(x,t),$$

$$L_5(x,t) = \sum_{j=0}^{N} \left\{\left(h\rho d_+\ {}_a^RD_x^{\sigma} u_N, P_j\right)_{N,\omega^{0,0}} - \left(h\rho d_+\ {}_a^RD_x^{\sigma} u_N, P_j\right)_{\omega^{0,0}}\right\} P_j(x),$$

$$L_6(x,t) = \Pi_{N,\omega^{0,0}}\left(h(x)\rho(x,t)\ d_+(x,t){}_a^RD_x^{\sigma} u_N(x,t)\right) - h(x)\rho(x,t)d_+(x,t){}_a^RD_x^{\sigma} u_N(x,t),$$

$$L_7(x,t) = \sum_{j=0}^{N} \left\{\left(h\rho d_-\ {}_x^RD_b^{\sigma} u_N, P_j\right)_{N,\omega^{0,0}} - \left(h\rho d_-\ {}_x^RD_b^{\sigma} u_N, P_j\right)_{\omega^{0,0}}\right\} P_j(x),$$

$$L_8(x,t) = \Pi_{N,\omega^{0,0}}\left(h(x)\rho(x,t)\ d_-(x,t){}_x^RD_b^{\sigma} u_N(x,t)\right) - h(x)\rho(x,t)d_-(x,t){}_x^RD_b^{\sigma} u_N(x,t),$$

and

$$\|L_3(x,t)\|_{\omega^{0,0}} \le N^{2-m} C_3\tau_1 \left\|h\ {}_xD_b^{\mu} u_N\right\|_{m,\omega^{0,0}} \le N^{2-m} C_3'\lambda_1 \left\|{}_xD_b^{\mu} u\right\|_{m,\omega^{0,0}} + N^{2-m} C_3'\lambda_1 \left\|{}_xD_b^{\mu} e_N\right\|_{m,\omega^{0,0}}, \tag{64}$$

$$\|L_4(x,t)\|_{\omega^{0,0}} \le N^{-m} C_4\tau_1 \left\|h\ {}_xD_b^{\mu} u_N\right\|_{m,\omega^{0,0}} \le N^{-m} C_4'\lambda_1 \left\|{}_xD_b^{\mu} u\right\|_{m,\omega^{0,0}} + N^{-m} C_4'\lambda_1 \left\|{}_xD_b^{\mu} e_N\right\|_{m,\omega^{0,0}}. \tag{65}$$

$$\|L_5(x,t)\|_{\omega^{0,0}} \le N^{2-m} C_5\tau_2 \left\|h\ {}_aD_x^{\sigma} u_N\right\|_{m,\omega^{0,0}} \le C_5'\lambda_2 N^{2-m} \left\|{}_aD_x^{\sigma} u\right\|_{m,\omega^{0,0}} + C_5'\lambda_2 N^{2-m} \left\|{}_aD_x^{\sigma} e_N\right\|_{m,\omega^{0,0}}, \tag{66}$$

$$\|L_6(x,t)\|_{\omega^{0,0}} \le N^{-m} C_6\tau_2 \left\|h\ {}_aD_x^{\sigma} u_N\right\|_{m,\omega^{0,0}} \le N^{-m} C_6'\lambda_2 \left\|{}_aD_x^{\sigma} u\right\|_{m,\omega^{0,0}} + N^{-m} C_6'\lambda_2 \left\|{}_aD_x^{\sigma} e_N\right\|_{m,\omega^{0,0}}, \tag{67}$$

$$\|L_7(x,t)\|_{\omega^{0,0}} \le N^{2-m} C_7\tau_3 \left\|h\ {}_xD_b^{\sigma} u_N\right\|_{m,\omega^{0,0}} \le C_7'\lambda_3 N^{2-m} \left\|{}_xD_b^{\sigma} u\right\|_{m,\omega^{0,0}} + C_7'\lambda_3 N^{2-m} \left\|{}_xD_b^{\sigma} e_N\right\|_{m,\omega^{0,0}}, \tag{68}$$

$$\|L_8(x,t)\|_{\omega^{0,0}} \le N^{-m} C_8\tau_3 \left\|h\ {}_xD_b^{\sigma} u_N\right\|_{m,\omega^{0,0}} \le C_8'\lambda_3 N^{-m} \left\|{}_xD_b^{\sigma} u\right\|_{m,\omega^{0,0}} + C_8'\lambda_3 N^{-m} \left\|{}_xD_b^{\sigma} e_N\right\|_{m,\omega^{0,0}}, \tag{69}$$

where $\lambda_1 = \tau_1 \|h\|_{m,\omega^{0,0}}$, $\tau_1 = \max_{0\le t\le T} \|\kappa c_-\|_{m,\omega^{0,0}}$, $\lambda_2 = \tau_2 \|h\|_{m,\omega^{0,0}}$, $\tau_2 = \max_{0\le t\le T} \|\rho d_+\|_{m,\omega^{0,0}}$, $\lambda_3 = \tau_3 \|h\|_{m,\omega^{0,0}}$, $\tau_3 = C \max_{0\le t\le T} \|\rho d_-\|_{m,\omega^{0,0}}$. Suppose

$$L_9(x,t) = \sum_{j=0}^{N} \left\{ \left(hf(u_N), P_j\right)_{N,\omega^{0,0}} - \left(hf(u_N), P_j\right)_{\omega^{0,0}} \right\} P_j(x),$$

$$L_{10}(x,t) = \left(\Pi_{N,\omega^{0,0}} \left(f(u_N(x,t),x,t)h(x)\right) - f(u_N(x,t),x,t)h(x)\right),$$

therefore we can write

$$\sum_{j=0}^{N} F_j(t)P_j(x) = L_9(x,t) + L_{10}(x,t) + f(u_N(x,t),x,t)h(x). \tag{70}$$

Using Lemmas 7, 6, 4 and Eq. (15), respectively and noting that $f$ satisfies in the Lipschitz condition, give

$$\begin{aligned} &\|L_9(x,t)\|_{\omega^{0,0}} \le N^{2-m}\delta_1 \|f(u_N)\|_{m,\omega^{0,0}} \le N^{2-m}\delta_1 \|f(u_N)-f(u)\|_{m,\omega^{0,0}} + \\ &N^{2-m}\delta_1 \|f(u)\|_{m,\omega^{0,0}} \le \delta_2 N^{2-m} \|u-u_N\|_{m,\omega^{0,0}} + N^{2-m}\delta_1' \|f(u)\|_{m,\omega^{0,0}} \le C_9 N^{2-m} \\ &\|e_N\|_{m,\omega^{0,0}} + N^{2-m} C_9 \|f(u)\|_{m,\omega^{0,0}}, \end{aligned} \tag{71}$$

and

$$\begin{aligned} &\|L_{10}(x,t)\|_{\omega^{0,0}} \le N^{-m}\delta_3 \|f(u_N)\|_{m,\omega^{0,0}} \le N^{-m} C_{10} \|e_N\|_{m,\omega^{0,0}} + N^{-m} \\ &C_{10} \|f(u)\|_{m,\omega^{0,0}}. \end{aligned} \tag{72}$$

We know that $u_N$ is a polynomial of degree at most $N$, therefore $hu_N \in H^m_{\omega^{0,0}}(I)$ and employing Lemmas 7, 6, 4 and Eq. (15), respectively we get

$$\begin{aligned} &\left\| \sum_{j=0}^{N} \left( \int_a^b \tfrac{\partial}{\partial t} u_N(x,t)h(x)P_j(x)dx - A_j(t) \right) P_j(x) \right\|_{\omega^{0,0}} \le C_{11} N^{2-m} \left\| \tfrac{\partial}{\partial t} u_N \right\|_{m,\omega^{0,0}} \\ &\le C_{11}' N^{2-m} \left\| \tfrac{\partial}{\partial t} e_N \right\|_{m,\omega^{0,0}} + N^{2-m} C_{11}' \left\| \tfrac{\partial}{\partial t} u \right\|_{m,\omega^{0,0}}, \end{aligned} \tag{73}$$

and

$$\begin{aligned} &\left\| \tfrac{\partial}{\partial t} u_N(x,t)h(x) - \sum_{j=0}^{N} \left( \int_a^b \tfrac{\partial}{\partial t} u_N(x,t)h(x)P_j(x)dx \right) P_j(x) \right\|_{\omega^{0,0}} \le C_{12} N^{-m} \left\| \tfrac{\partial}{\partial t} e_N \right\|_{m,\omega^{0,0}} + \\ &N^{-m} C_{12} \left\| \tfrac{\partial}{\partial t} u \right\|_{m,\omega^{0,0}}. \end{aligned} \tag{74}$$

By multiplying both sides of (1) in $h(x)$ and subtracting from (60) we can write:

$$\begin{aligned} &h(x)\tfrac{\partial}{\partial t} e_N(x,t) = L_1(x,t) + L_2(x,t) + L_3(x,t) + L_4(x,t) + L_5(x,t) + L_6(x,t) + \\ &L_7(x,t) + L_8(x,t) + L_9(x,t) + L_{10}(x,t) + h(x)\kappa(x,t)c_+(x,t)\,{}_a^R D_x^{\mu} e_N(x,t) + \kappa(x,t) \\ &h(x)c_-(x,t)\,{}_x^R D_b^{\mu} e_N(x,t) + h(x)\rho(x,t)d_+(x,t)\,{}_a^R D_x^{\sigma} e_N(x,t) + \rho(x,t)d_-(x,t) \\ &{}_x^R D_b^{\sigma} e_N(x,t) + \left(f(u_N(x,t),x,t) - f(u(x,t),x,t)\right)h(x), \end{aligned} \tag{75}$$

therefore we conclude

$$\begin{aligned}
&\tfrac{d}{dt}\|e_N\|_{\omega^{0,0}} \le N^{-m}\vartheta_0\left\|{}_aD_x^{\mu}u\right\|_{m,\omega^{0,0}} + N^{-m}\vartheta_1\left\|{}_xD_b^{\mu}u\right\|_{m,\omega^{0,0}} + N^{-m}\vartheta_2\left\|{}_aD_x^{\sigma}u\right\|_{m,\omega^{0,0}}+\\
&N^{-m}\vartheta_2\left\|{}_xD_b^{\sigma}u\right\|_{m,\omega^{0,0}} + \left(C_9N^{2-m}+C_{10}N^{-m}\right)\delta_3\tfrac{d}{dt}\|u\|_{m,\omega^{0,0}} + \vartheta_4\left\|{}_aD_x^{\mu}e_N\right\|_{m,\omega^{0,0}}+\\
&\vartheta_5\left\|{}_xD_b^{\mu}e_N\right\|_{m,\omega^{0,0}} + \vartheta_6\left\|{}_aD_x^{\sigma}e_N\right\|_{m,\omega^{0,0}} + \vartheta_7\left\|{}_xD_b^{\sigma}e_N\right\|_{m,\omega^{0,0}} + \vartheta_8N^{-m}\|f(u)\|_{m,\omega^{0,0}}+\\
&\left(C_9N^{2-m}+C_{10}N^{-m}\right)\|e_N\|_{m,\omega^{0,0}}+\left(C_{11}N^{2-m}+C_{12}N^{-m}\right)\tfrac{d}{dt}\|e_N\|_{m,\omega^{0,0}},
\end{aligned} \tag{76}$$

where

$$\begin{aligned}
&\vartheta_0 = \lambda_0\left(2N^2C_1+C_2\right)r,\ \ \vartheta_1 = \lambda_1\left(2N^2C_3+C_4\right)r,\\
&\vartheta_2 = \lambda_2\left(2N^2C_5+C_6\right)r,\ \ \vartheta_3 = \lambda_3\left(2N^2C_7+C_8\right)r,\\
&\vartheta_4 = \lambda_0\left(2N^{2-m}C_1+N^{-m}C_2+1\right)r,\ \ \vartheta_5 = \lambda_1\left(2N^{2-m}C_3+C_4N^{-m}+1\right)r,\\
&\vartheta_6 = \lambda_2\left(2N^{2-m}C_5+N^{-m}C_6+1\right)r,\ \ \vartheta_7 = \lambda_3\left(2N^{2-m}C_7+C_8N^{-m}+1\right)r,\\
&\vartheta_8 = \left(N^2+1\right)\delta_1 r,\ \ r=\delta_3\|h\|_{\omega^{0,0}}^{-1},
\end{aligned}$$

where $\delta_1$ and $\delta_2$ are constants. Using Lemmas 12 and 11 we can write

$$\begin{aligned}
&\tfrac{d}{dt}\|e_N\|_{\omega^{0,0}} \le \left(N^{-m}\left(\vartheta_0+\vartheta_1+\vartheta_2+\vartheta_3\right)+\left(C_9N^{2-m}+C_{10}N^{-m}\right)D\right)\|u\|_{H_l^{\mu,m}(I)}+\\
&\left(\left(\vartheta_4+\vartheta_5+\vartheta_6+\vartheta_7\right)+\left(C_9N^{2-m}+C_{10}N^{-m}\right)\right)\|e_N\|_{H_l^{\mu,m}(I)}+\left(C_{11}N^{2-m}+C_{12}N^{-m}\right)\\
&\tfrac{d}{dt}\|e_N\|_{H_l^{\mu,m}(I)}+\left(C_9N^{2-m}+C_{10}N^{-m}\right)\delta_2\tfrac{d}{dt}\|u\|_{m,\omega^{0,0}}+\vartheta_8N^{-m}\|f(u)\|_{m,\omega^{0,0}}.
\end{aligned} \tag{77}$$

Suppose

$$\begin{aligned}
&\lambda_1 = \left(N^{-m}\left(\vartheta_0+\vartheta_1+\vartheta_2+\vartheta_3\right)+\left(C_9N^{2-m}+C_{10}N^{-m}\right)\delta_1\right),\\
&\lambda_2 = \left(\left(\vartheta_4+\vartheta_5+\vartheta_6+\vartheta_7\right)+\left(C_9N^{2-m}+C_{10}N^{-m}\right)\right),\\
&\lambda_3 = \left(C_{11}N^{2-m}+C_{12}N^{-m}\right),\ \ \lambda_4 = \left(C_9N^{2-m}+C_{10}N^{-m}\right)\delta_2,\ \ \lambda_5 = \vartheta_8N^{-m},
\end{aligned}$$

therefore we can write

$$\begin{aligned}
&\tfrac{d}{dt}\|e_N\|_{\omega^{0,0}}+\lambda_2\tfrac{d}{dt}\|e_N\|_{H_l^{\mu,m}(I)} \le \lambda_1\|u\|_{H_l^{\mu,m}(I)}+\lambda_2\|e_N\|_{H_l^{\mu,m}(I)}+\left(\lambda_3+\lambda_2\right)\tfrac{d}{dt}\|e_N\|_{H_l^{\mu,m}(I)}+\\
&\lambda_4\tfrac{d}{dt}\|u\|_{m,\omega^{0,0}}+\lambda_5\|f(u)\|_{m,\omega^{0,0}}.
\end{aligned}$$

Using the Gronwall lemma [5] arrives at

$$\begin{aligned}
&\|e_N\|_{L^2_\omega(I)}+\lambda_2\|e_N\|_{H_l^{\mu,m}(I)} \le \|e_N(0)\|_{L^2_\omega(I)}+\lambda_2\|e_N(0)\|_{H_l^{\mu,m}(I)}+\lambda_1\int_0^t\|u\|_{H_l^{\mu,m}(I)}\mathrm{e}^{(t-s)}ds+\\
&\left(\lambda_3+\lambda_2\right)\int_0^t\tfrac{d}{ds}\|e_N\|_{H_l^{\mu,m}(I)}\mathrm{e}^{(t-s)}ds+\lambda_4\int_0^t\tfrac{d}{dt}\|u\|_{m,\omega^{0,0}}e^{(t-s)}ds+\lambda_5\int_0^t\|f(u)\|_{m,\omega^{0,0}}e^{(t-s)}ds.
\end{aligned}$$

Integrating by parts from $\int_0^t \frac{d}{ds}\|e_N\|_{H_l^{\mu,m}(I)} \mathrm{e}^{(t-s)}ds$, yields

$$\|e_N\|_{\omega^{0,0}} + \lambda_2\|e_N\|_{H_l^{\mu,m}(I)} \leq \|e_N(0)\|_{L^2_\omega(I)} + \lambda_2\|e_N(0)\|_{H_l^{\mu,m}(I)} + \lambda_1\int_0^t \|u\|_{H_l^{\mu,m}(I)} \mathrm{e}^{t-s}ds + (\lambda_3+\lambda_2)\left(\|e_N\|_{H_l^{\mu,m}(I)} - \|e_N(0)\|_{H_l^{\mu,m}(I)}\mathrm{e}^t + \int_0^t \|e_N\|_{H_l^{\mu,m}(I)}\ \mathrm{e}^{t-s}ds\right) + \lambda_4\int_0^t \frac{d}{dt}\|u\|_{m,\omega^{0,0}}e^{(t-s)}ds + \lambda_5\int_0^t \|f(u)\|_{m,\omega^{0,0}}e^{(t-s)}ds. \tag{78}$$

Now one can use (78) and Gronwall lemma [5] two times to conclude:

$$\|e_N\|_{L^2_\omega(I)} + \lambda_2\|e_N\|_{H_l^{\mu,m}(I)} \leq \gamma_2\left(\|e_N(0)\|_{L^2_\omega(I)} + \lambda_2\|e_N(0)\|_{H_l^{\mu,m}(I)}\right) + \lambda_4\int_0^t\left(\int_0^\upsilon\left(\int_0^\tau \frac{d}{dt}\|u\|_{m,\omega^{0,0}}e^{(t-s)}ds\right)e^{\gamma_1'(\upsilon-\tau)}d\tau\right)e^{\gamma_3'(t-\upsilon)}d\upsilon + \lambda_5\int_0^t\left(\int_0^\upsilon\left(\int_0^\tau \|f(u)\|_{m,\omega^{0,0}}e^{(t-s)}ds\right)e^{\gamma_1''(\upsilon-\tau)}d\tau\right)e^{\gamma_3''(t-\upsilon)}d\upsilon\ \lambda_1\int_0^t\left(\int_0^\upsilon\left(\int_0^\tau \|u\|_{H_l^{\mu,m}(I)}\mathrm{e}^{(\tau-s)}ds\right)\mathrm{e}^{\gamma_1(\upsilon-\tau)}d\tau\right)\mathrm{e}^{\gamma_3(t-\upsilon)}d\upsilon. \tag{79}$$

From (3) we can write:

$$e_N(x,0) = I_N(g(x)) - g(x).$$

Using Lemmas 13 and 6, respectively we can conclude:

$$\left\|\frac{1}{\Gamma(m-\mu)}\frac{d^2}{dx^2}\int_a^x (x-\eta)^{1-\mu}e_N(\eta,0)d\eta\right\|_{m,\omega^{0,0}} \leq \left\|\frac{1}{\Gamma(m-\mu)}\frac{d^2}{dx^2}\int_a^x (x-\eta)^{1-\mu} \|e_N(\eta,0)\|_{\infty,(a,x)}d\eta\right\|_{m,\omega^{0,0}} \leq C_{12}\left\|\frac{1}{\Gamma(m-\mu)}\frac{d^2}{dx^2}\int_a^x (x-\eta)^{1-\mu}\|e_N(\eta,0)\|_{1,\omega^{0,0}}d\eta\right\|_{m,\omega^{0,0}} \leq C_{13}N^{\frac{3}{2}-l}. \tag{80}$$

Also employing Lemma 6, yields

$$\|e_N(0)\|_{m,\omega^{0,0}} \leq C_{14}N^{2m-\frac{1}{2}-l}\|g\|_{l,\omega^{0,0}}. \tag{81}$$

Using (80) and (81) we have:

$$\|e_N(0)\|_{H_l^{\mu,m}(I)} \leq \gamma_3 N^{2m-\frac{1}{2}-l}\|g\|_{l,\omega^{0,0}} + \gamma_4 N^{\frac{3}{2}-l}. \tag{82}$$

Therefore from (79), (82) and Lemma 6 we conclude:

$$\|e_N\|_{\omega^{0,0}} \leq \gamma_5 N^{2-m} + \gamma_6 N^{2m-\frac{1}{2}-l}\|g\|_{l,\omega^{0,0}} + \gamma_7 N^{\frac{3}{2}-l} + \gamma_8 N^{-l}, \tag{83}$$

where $\gamma_5, \gamma_6,\ \gamma_7$ and $\gamma_8$ are constants independent of $N$. Therefore the proof is completed. □

**Table 2** The maximum of absolute error for $N = 3$. (Example 1)

| $\tau\diagdown E(\alpha,\beta)$ | $E(-0.2,-0.2)$ | $E(-0.1,0)$ | $E(0,0)$ | $E(0.1,0)$ | $E(0.2,0.2)$ |
|---|---|---|---|---|---|
| $\frac{1}{30}$ | $1.2922(-6)$ | $1.1938(-6)$ | $1.1484(-6)$ | $1.1788(-6)$ | $1.1197(-6)$ |
| $\frac{1}{40}$ | $3.7285(-7)$ | $3.53658(-7)$ | $3.6320(-7)$ | $3.4907(-7)$ | $3.5449(-7)$ |
| $\frac{1}{60}$ | $7.3599(-8)$ | $7.4482(-8)$ | $7.1744(-8)$ | $6.8996(-8)$ | $7.0070(-8)$ |
| $\frac{1}{100}$ | $9.085(-9)$ | $9.6514(-9)$ | $9.2991(-9)$ | $7.1748(-9)$ | $9.0850(-9)$ |

**Table 3** The maximum of absolute error for $N = 4$ and $\gamma = 1.9$ (Example 2)

| $\tau \diagdown E(\alpha, \beta)$ | $E(-0.5, 0)$ | $E(0, 0)$ | $E(0, 0.5)$ | $E(0.5, 0.5)$ |
|---|---|---|---|---|
| $\frac{1}{20}$ | 4.3451(−3) | 3.3650(−6) | 3.7088(−6) | 2.5397(−6) |
| $\frac{1}{50}$ | 1.2298(−7) | 5.4399(−8) | 5.8558(−8) | 4.9580(−8) |
| $\frac{1}{100}$ | 4.4365(−9) | 3.0039(−9) | 3.0731(−9) | 3.0267(−9) |
| $\frac{1}{200}$ | 2.0135(−10) | 1.8786(−10) | 8.7576(−10) | 1.8836(−10) |

*Remark 2* In Theorem 2 the order of convergence is $\max\left\{2 - m, 2m - \frac{1}{2} - l, \frac{3}{2} - l, -l\right\}$.

## 5 Numerical results

In this section $u_N^{\alpha,\beta}$ represents the numerical solution of equations using the technique presented in the current article.

*Example 1* Consider the following equation [49]:

$$\frac{\partial u}{\partial t} = -4.5\frac{\partial u}{\partial x} + \Gamma(0.2)x^{0.8}\,{}_a^R D_x^{\gamma} u + 5e^{-t}\left(x^2 + 3.5\right), \tag{84}$$

$$\begin{aligned} &(x, t) \in \Omega = [0, 1] \times [0, T], \\ &u(0, t) = u(1, t) = 0, \quad 0 \le t \le T, \\ &u(x, 0) = 5x(1 - x), \quad 0 \le x \le 1, \end{aligned}$$

where $\gamma = 1.8$. The exact solution of this equation is $u(x, t) = 5e^{-t}x(1 - x)$.

In this example $\tau$ is the time step size and we choose $(\alpha, \beta)$ and $\tau$ according to Table 2. In this table we define $E(\alpha, \beta) = \max\limits_{\substack{0\le x\le 1\\ 0\le t\le 1}} \left|u(x, t) - u_N^{\alpha,\beta}(x, t)\right|$. Also in Fig. 1 we plot the approximate solution of this equation that has been obtained by the presented method for $N = 3$ and $(\alpha, \beta) = (-0.1, 0)$, $\tau = \frac{1}{60}$ at the time $t = 1.0$ .

*Example 2* Consider the following equation [6]

$$\begin{aligned} &\tfrac{\partial u}{\partial t} = \tfrac{\partial^{\gamma} u}{\partial |x|^{\gamma}} + f(x, t), \quad (x, t) \in (0, 1) \times (0, T], \quad \gamma \in (1, 2], \\ &u(x, 0) = x^2(1 - x)^2, \quad 0 < x < 1, \\ &u(0, t) = 0, \quad 0 < t \le T, \quad u(1, t) = 0, \quad 0 < t \le T, \end{aligned} \tag{85}$$

**Table 4** The maximum of absolute error for $N = 5$ and $\gamma = 1.5$ (Example 2)

| $\tau \diagdown E(\alpha, \beta)$ | $E(-0.1, -0.1)$ | $E(-0.1, 0)$ | $E(0, 0)$ | $E(0.1, 0.1)$ |
|---|---|---|---|---|
| $\frac{1}{20}$ | 2.3469(−6) | 7.1146(−7) | 4.5674(−7) | 2.2847(−3) |
| $\frac{1}{50}$ | 5.1047(−9) | 5.1987(−9) | 5.2202(−8) | 5.2448(−9) |
| $\frac{1}{100}$ | 3.3093(−10) | 3.2984(−10) | 3.2910(−10) | 3.2843(−10) |

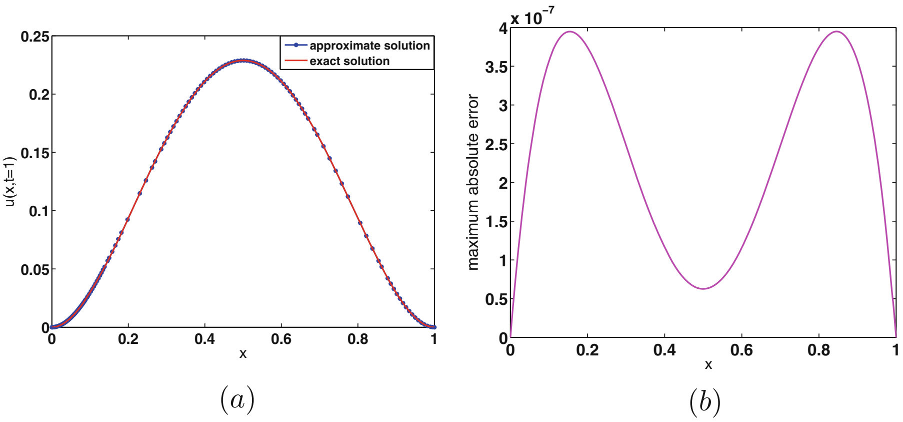


**Fig. 2** **a** The comparison of the exact and approximate solutions for $N = 4$ and $(\alpha, \beta) = (0, 0)$, $\gamma = 1.9$, $\tau = \frac{1}{20}$ at time $t = 1.0$ and **b** The maximum absolute error for $N = 5$, $(\alpha, \beta) = (0.1, 0.1)$, $\gamma = 1.5$ and $\tau = \frac{1}{20}$ at time $t = 1.0$ (Example 2)

where

$$f(x,t) = (1+t)^{-1+\gamma}(-1+x)^2x^2\gamma + \frac{x^{-\gamma}}{\Gamma(5-\gamma)}\left(\left(\frac{1+t}{1-x}\right)^{\gamma}(-1+x)^2x^{\gamma}\left(12x^2 - 6x\gamma\right.\right.$$
$$\left.\left.(-1+\gamma)\gamma\right) + (1+t)^{\gamma}x^2\left(12(-1+x)^2 + (-7+6x)\gamma + \gamma^2\right)\right)\sec\left(\frac{\pi\gamma}{2}\right).$$

The exact solution of the above equation is $u(x,t) = (t+1)^{\gamma}x^2(1-x)^2$. In Tables 3 and 4 we show the maximum absolute error for Example 2 when $\gamma = 1.9$, $N = 4$ and $\gamma = 1.5$, $N = 5$, respectively and in these tables $\tau$ represents the time step size and we define $E(\alpha,\beta) = \max\limits_{\substack{0\le x\le 1\\ 0\le t\le T}} \left|u(x,t) - u_N^{\alpha,\beta}(x,t)\right|$. Also in Fig. 2a we graph the exact and approximate solutions of Example 2 for $N = 4$, $\gamma = 1.9$, $\tau = \frac{1}{20}$ and $(\alpha,\beta) = (0,0)$ and in Fig. 2b we plot the absolute error for Example 2 when $N = 5$ and $\gamma = 1.5$, $\tau = \frac{1}{20}$ and $(\alpha,\beta) = (0.1, 0.1)$.

*Example 3* Consider the following equation :

$$\frac{\partial u}{\partial t} = \Gamma(1.2)x^{1.8}\,{}_a^R D_x^{\gamma} u - u^3 + u^2 + e^{-t}\left(-x^9 + 3x^8 - 3x^7 + 2x^5 - x^4 + 6x^3 - 3x^2\right), \qquad (86)$$

$$(x,t) \in \Omega = [0,1] \times [0,T],$$
$$u(0,t) = u(1,t) = 0, \quad 0 \le t \le T,$$
$$u(x,0) = x^2 - x^3, \quad 0 \le x \le 1,$$

where $\gamma = 1.8$. The exact solution of this equation is $u(x,t) = 5e^{-t}(x^2 - x^3)$. In this example we choose $(\alpha,\beta)$ and $\tau$ according to Table 5. Note that in this table $\tau$ represents the time step size and

**Table 5** The maximum of absolute error for $N = 6$. (Example 3)

| $\tau \diagdown E(\alpha,\beta)$ | $E(-0.3,-0.3)$ | $E(-0.3,0)$ | $E(0,0)$ | $E(0.3,0)$ | $E(0.3,0.3)$ |
|---|---|---|---|---|---|
| $\frac{1}{40}$ | $1.6666(-5)$ | $8.5046(-6)$ | $1.8098(-5)$ | $1.2534(-5)$ | $1.4785(-5)$ |
| $\frac{1}{100}$ | $1.5413(-7)$ | $1.4208(-7)$ | $1.4782(-7)$ | $1.5331(-7)$ | $1.4307(-7)$ |
| $\frac{1}{200}$ | $9.9964(-8)$ | $9.2050(-8)$ | $3.8520(-8)$ | $4.9839(-8)$ | $9.5283(-8)$ |
| $\frac{1}{300}$ | $2.0047(-9)$ | $1.8378(-9)$ | $5.9116(-9)$ | $3.2837(-9)$ | $4.7393(-9)$ |

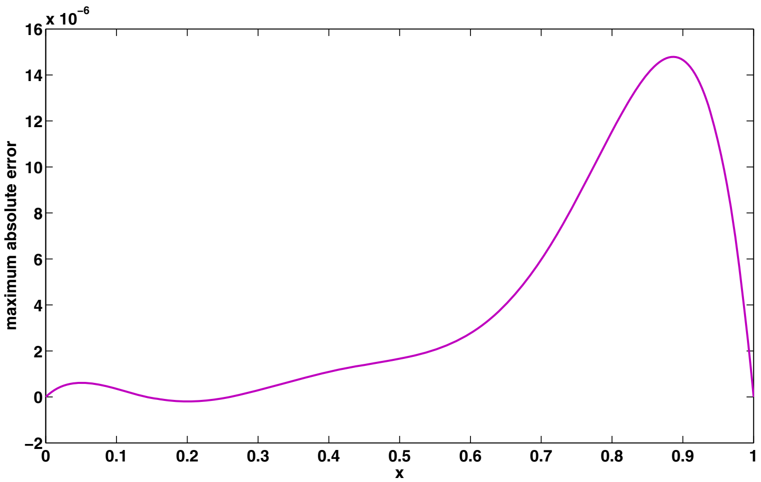


**Fig. 3** The maximum absolute error for $N = 6$, $(\alpha, \beta) = (0.3, 0.3)$ and $\tau = \frac{1}{40}$ at time $t = 1.0$ (Example 3)

$E(\alpha, \beta) = \max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq 1}} \left| u(x,t) - u_N^{\alpha,\beta}(x,t) \right|$. Also in Fig 3 we plot the absolute error for Example 3 when $N = 6$, $\tau = \frac{1}{40}$ and $(\alpha, \beta) = (0.3, 0.3)$.

*Example 4* Consider the following fractional advection-diffusion equation with the Riesz fractional derivative [46]:

$$\begin{aligned} &\tfrac{\partial u(x,t)}{\partial t} = A \tfrac{\partial^{\sigma} u(x,t)}{\partial |x|^{\sigma}} + B \tfrac{\partial^{\gamma} u(x,t)}{\partial |x|^{\gamma}} + f(x,t), \\ &u(x,0) = 0, \quad 0 < x < 1, \quad 0 \leq t \leq T, \\ &u(0,t) = u(1,t) = 0, \quad 0 < x < 1, \quad 0 \leq t \leq T, \end{aligned} \tag{87}$$

where

**Table 6** Error of solving (87) using the presented method for $N = 7$, $\tau = 0.002$ at $t = 2.0$

| $x \diagdown (\alpha, \beta)$ | $(-0.5. -0.5)$ | $(-0.5.0.5)$ | $(0, 0)$ | $(0.5. -0.5)$ | $(0.5.0.5)$ |
|---|---|---|---|---|---|
| 0.1 | 1.6428(−7) | 2.8451(−7) | 0.9412(−7) | 5.5611(−7) | 4.018(−7) |
| 0.2 | 4.3028(−7) | 3.0620(−7) | 1.9270(−7) | 7.0349(−7) | 5.9037(−7) |
| 0.3 | 5.6309(−7) | 5.2031(−7) | 2.4914(−7) | 7.0565(−7) | 6.4620(−7) |
| 0.4 | 8.6271(−7) | 6.4729(−7) | 5.4825(−7) | 7.9463(−7) | 8.2421(−7) |
| 0.5 | 9.2639(−7) | 8.6203(−7) | 6.9436(−7) | 9.1382(−7) | 8.9414(−7) |
| 0.6 | 1.9702(−6) | 9.5395(−7) | 7.0156(−7) | 1.0415(−6) | 2.0731(−6) |
| 0.7 | 3.0517(−6) | 2.3742(−6) | 9.1281(−7) | 3.5399(−6) | 3.2307(−6) |
| 0.8 | 6.0183(−6) | 5.2767(−6) | 2.8926(−6) | 4.1631(−6) | 5.8310(−6) |
| 0.9 | 9.4919(−6) | 7.0582(−6) | 5.0387(−6) | 6.0619(−6) | 8.3261(−6) |

**Table 7** Error of solving (87) using the presented method for $N = 5$, $\tau = 0.0002$ at $t = 2.0$

| $x \diagdown (\alpha, \beta)$ | $(-0.5. -0.5)$ | $(-0.5.0.5)$ | $(0, 0)$ | $(0.5. -0.5)$ | $(0.5.0.5)$ |
|---|---|---|---|---|---|
| 0.1 | 5.9310(−11) | 2.4493(−11) | 3.3502(−11) | 1.4299(−11) | 3.0155(−11) |
| 0.2 | 5.0610(−11) | 3.0195(−11) | 4.9153(−11) | 3.5518(−11) | 4.6024(−11) |
| 0.3 | 7.6913(−11) | 3.9015(−11) | 6.0449(−11) | 3.9149(−11) | 4.9345(−11) |
| 0.4 | 8.0183(−11) | 5.1661(−11) | 6.9422(−11) | 5.0153(−11) | 6.4462(−11) |
| 0.5 | 9.1491(−11) | 8.0531(−11) | 7.1466(−11) | 7.2068(−11) | 6.9250(−11) |
| 0.6 | 1.9361(−10) | 9.0617(−11) | 9.3104(−11) | 8.5347(−11) | 9.6771(−11) |
| 0.7 | 2.4551(−10) | 1.8106(−10) | 1.7154(−10) | 9.0614(−11) | 2.5305(−10) |
| 0.8 | 5.3550(−10) | 2.0928(−10) | 4.6992(−10) | 1.8273(−10) | 3.6502(−10) |
| 0.9 | 6.2405(−10) | 5.1609(−10) | 7.1591(−10) | 3.5304(−10) | 6.0316(−10) |

$$f(x,t) = \frac{At^{\gamma}e^{\sigma t}}{2\cos(\frac{\pi\sigma}{2})}\left\{\frac{2}{\Gamma(3-\sigma)}\left[x^{2-\sigma}+(1-x)^{2-\sigma}\right] - \frac{12}{\Gamma(4-\sigma)}\left[x^{3-\sigma}+(1-x)^{3-\sigma}\right] + \frac{24}{\Gamma(5-\sigma)}\left[x^{4-\sigma}+(1-x)^{4-\sigma}\right]\right\} + \frac{Bt^{\gamma}e^{\sigma t}}{2\cos(\frac{\pi\gamma}{2})}\left\{\frac{2}{\Gamma(3-\gamma)}\left[x^{2-\gamma}+(1-x)^{2-\gamma}\right] - \frac{12}{\Gamma(4-\gamma)}\left[x^{3-\gamma}+(1-x)^{3-\gamma}\right] + \frac{24}{\Gamma(5-\gamma)}\left[x^{4-\gamma}+(1-x)^{4-\gamma}\right]\right\} + t^{\gamma-1}e^{\sigma t}\left(\gamma+\sigma t\right)x^2(1-x)^2.$$

The above problem has the exact solution $u(x,t) = t^{\gamma}e^{\sigma t}x^2(1-x)^2$. In this example we take $A = 2$, $B = 2$, $\sigma = 0.4$ and $\gamma = 1.7$. In Tables 6 and 7 we show the results of solving (87) using the presented method in the special points of $x$ at $t = 2.0$. In Table 6, we show the error of the presented method for solving this example when $N = 7$, $\tau = 0.002$ and in Table 7, we show the error of presented method for solving this example when $N = 5$, $\tau = 0.0002$ and in these tables $\tau$ represents the time step. In [46], for solving (87) the authors presented two methods, explicit difference approximations (EDA) and implicit difference approximations (IDA). The results of solving this equation using these methods in the special points of $x$ are shown in Table 8. Comparison between the presented method and the methods of [46] shows the efficiency of the new method.

**Table 8** Error of solving (87) using the EDA and IDA methods at $t = 2.0$

| $x$ | EFDA method with $\tau = 0.0002$ and $h = 0.02$ $(N = 50)$ | IDA method with $\tau = 0.002$ and $h = 0.01$ $(N = 100)$ |
|---|---|---|
| 0.1 | 0.000123 | 0.0000839 |
| 0.2 | 0.000823 | 0.0009065 |
| 0.3 | 0.001839 | 0.0017731 |
| 0.4 | 0.002563 | 0.0023931 |
| 0.5 | 0.002822 | 0.0026157 |
| 0.6 | 0.002563 | 0.0023931 |
| 0.7 | 0.001840 | 0.0017731 |
| 0.8 | 0.000824 | 0.0009065 |
| 0.9 | 0.000123 | 0.0000839 |

## 6 Conclusion

In this paper we applied the collocation method to solve a special kind of fractional advection-diffusion equation with a nonlinear source term. For this purpose after rewriting the given fractional partial differential equation in the operational form, we replaced the Gauss-Jacobi points in the equation. Then by solving the resultant system of ordinary differential equations using the four-stage Runge-Kutta method, the approximate solution of this equation was obtained. Moreover in two main theorems we studied the stability and convergence of the presented method. Finally some examples were solved using the presented method and the numerical simulations were reported. These results show the applicability and efficiency of the new method.

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