Numer Algor (2015) 68:569–583
DOI 10.1007/s11075-014-9861-9

ORIGINAL PAPER

Convergence for a class of multi-point modified Chebyshev-Halley methods under the relaxed conditions

Xiuhua Wang · Jisheng Kou

Received: 26 November 2013 / Accepted: 4 April 2014 / Published online: 27 May 2014

Abstract In this paper, the semilocal convergence for a class of multi-point modified Chebyshev-Halley methods in Banach spaces is studied. Different from the results in reference Wang and Kou (Numer. Algoritm. **64**, 105–126, 2012), these methods are more general and the convergence conditions are also relaxed. We derive a system of recurrence relations for these methods and based on this, we prove a convergence theorem to show the existence-uniqueness of the solution. A priori error bounds is also given. The R-order of these methods is proved to be $5 + q$ with $\omega-$conditioned third-order Fréchet derivative, where $\omega(\mu)$ is a non-decreasing continuous real function for $\mu > 0$ and satisfies $\omega(0) \geq 0$, $\omega(t\mu) \leq t^q \omega(\mu)$ for $\mu > 0, t \in [0, 1]$ and $q \in [0, 1]$. Finally, we give some numerical results to show our approach.

Keywords Recurrence relations · R-order of convergence · Semilocal convergence · Chebyshev-Halley method · Convergence condition

Mathematics Subject Classifications (2010) 65D10 · 65D99

1 Introduction

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear operator on a non-empty open convex subset Ω of a Banach space X with values in a Banach space Y.

Newton's method [1] is widely applied to find the solution of (1). It converges quadratically under some suitable conditions. Recently, some papers about the third-order methods have been developed since their higher convergence speed. For the

X. Wang · J. Kou (✉)
School of Mathematics and Statistics, Hubei Engineering University, Xiaogan 432000, Hubei, China
e-mail: kjsfine@aliyun.com

classical Chebyshev-Halley methods, see references [2–7]. Though the classical Chebyshev-Halley methods need to compute the second Fréchet derivative, they are useful in some applications. Such as the integral equations [8] and the quadratic equations [9], where for the integral equations, the second Fréchet derivative is easy to compute; for the quadratic equations, the second Fréchet derivative is a constant. Moreover, in some applications where a quick convergence speed is needed, such as the stiff systems, the high-order methods are very useful [10]. So it is interesting to study some high-order methods. In reference [11], we have considered the modified Chebyshev-Halley methods given by

$$\begin{cases} z_n = x_n - \left(I + \frac{1}{2}G(x_n) + \frac{\delta_1}{2}G(x_n)^2\right)\Gamma_n F(x_n), \\ x_{n+1} = z_n - \left[I + G(x_n) + G(x_n)^2 + \frac{1}{2}\Gamma_n F''(u_n)G(x_n)\Gamma_n F(x_n)\right]\Gamma_n F(z_n), \end{cases} \tag{2}$$

where I is the identity operator, $\Gamma_n = F'(x_n)^{-1}$, $G(x_n) = \Gamma_n F''(u_n)\Gamma_n F(x_n)$, $u_n = x_n - \frac{1}{2}\Gamma_n F(x_n)$, δ_1 is a parameter and $\delta_1 \in [-1, 1]$.

By supposing that

(A1) There exists $\Gamma_0 = F'(x_0)^{-1}$ and $\|\Gamma_0\| \le \beta$,

(A2) $\|\Gamma_0 F(x_0)\| \le \eta$,

(A3) $\|F''(x)\| \le M, \quad x \in \Omega$,

(A4) $\|F'''(x)\| \le N, \quad x \in \Omega$,

(A5) $\|F'''(x) - F'''(y)\| \le \omega(\|x - y\|), \quad \forall x, y \in \Omega$, where $\omega(z)$ is a non-decreasing continuous real function for $z > 0$ and satisfy $\omega(0) \ge 0$,

(A6) there exists a non-negative real function $\nu \in C[0, 1]$, with $\nu(t) \le 1$, such that $\omega(tz) \le \nu(t)\omega(z)$, for $t \in [0, 1]$, $z \in (0, +\infty)$,

we have analyzed the semilocal convergence for the methods (2). Numerical results show that the methods (2) can solve some non-linear integral equation of mixed Hammerstein type successfully.

Note that under the conditions (A1)-(A6), we can not study the solution of some equations, for example,

$$f(x) = x^3 \ln(x^2) + 3x^2 - 10x + 1.7 = 0, \tag{3}$$

where $f(x)$ defines in $X = [-1, 1]$, $f(0) = 1.7$. Obviously, $f'''(x)$ can not satisfy the assumption (A4). In reference [12], under the assumptions (A1)-(A3), the convergence for a family of methods are studied and the methods are given by

$$x_{\theta,n+1} = x_{\theta,n} - \left[I + \frac{1}{2}L_F(x_{\theta,n})[I - \theta L_F(x_{\theta,n})]^{-1}\right] F'(x_{\theta,n})^{-1} F(x_{\theta,n}), \tag{4}$$

where $\theta \in [0, 1]$, $L_F(x_n) = F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n)$. This family contains Chebyshev method ($\theta = 0$), Halley method ($\theta = 1/2$) and super-Halley method ($\theta = 1$).

In this paper, we consider the semilocal convergence for a class of multi-point modified Chebyshev-Halley methods in Banach spaces given by

$$\begin{cases} z_n = x_n - \left[I + \frac{1}{2}G(x_n) + G(x_n)^2 Q(G(x_n))\right]\Gamma_n F(x_n), \\ x_{n+1} = z_n - \left[I + G(x_n) + G(x_n)^2 + \frac{1}{2}\Gamma_n F''(u_n)G(x_n)\Gamma_n F(x_n) + \delta G(x_n)^3\right]\Gamma_n F(z_n), \end{cases} \tag{5}$$

where I is the identity operator, $\Gamma_n = F'(x_n)^{-1}$, $G(x_n) = \Gamma_n F''(u_n)\Gamma_n F(x_n)$, $u_n = x_n - \frac{1}{2}\Gamma_n F(x_n)$, δ is a parameter and $\delta \in [0, 1]$. In the methods (5), Q is an operator which satisfies that there exists a real non-negative and non-decreasing function $\chi(t)$, such that $\|Q(G(x_n))\| \leq \chi(\|G(x_n)\|)$ and $\chi(t)$ is bounded for $t \in (0, s)$, where s will be defined in the latter developments. Obviously, the methods (5) is more general than the methods (2). To relax the conditions considered in reference [11], we study the semilocal convergence of the methods (5) under the conditions (A1)-(A3). Notice that the conditions (A1)-(A3) which have been used in reference [12] are weaker than the conditions (A1)-(A6), since F''' is not required in the former. Applying the recurrence relations, a convergence theorem for methods (5) is proved to show the existence-uniqueness of the solution and a priori error bounds is also given. Since the importance for convergence of iterative methods, in references [2, 3, 7, 10–16], the convergence of some methods are considered.

On the other hand, we give a brief proof to show that the R-order of methods (5) is at least $5 + q$ with $\omega-$conditioned third-order Fréchet derivative, where $\omega(\mu)$ is a non-decreasing continuous real function for $\mu > 0$ and satisfies $\omega(0) \geq 0$, $\omega(t\mu) \leq t^q\omega(\mu)$ for $\mu > 0, t \in [0, 1]$ and $q \in [0, 1]$. Obviously, the R-order of methods (5) is higher than the one of the methods (4) under the same conditions. Finally, some numerical results are given to show our approach.

2 Some preliminary results

Let X and Y be two Banach spaces, and let the nonlinear operator $F : \Omega \subset X \to Y$ be twice Fréchet differentiable in a non-empty open convex subset Ω and the conditions (A1)-(A3) hold, $x_0 \in \Omega$. Define $B(x, r) = \{y \in X : \|y - x\| < r\}$ and $\overline{B(x, r)} = \{y \in X : \|y - x\| \leq r\}$. Furthermore, we define the following functions:

$$p(t) = g_1(t) + \left[1 + t + \frac{3}{2}t^2 + \delta t^3\right] g_2(t), \tag{6}$$

$$h(t) = \frac{1}{1 - tp(t)}, \tag{7}$$

$$\begin{aligned} \varphi(t) = &\ t\left[1 + \frac{3}{2}t + \delta t^2\right] g_2(t) + tg_1(t)\left[1 + t + \frac{3}{2}t^2 + \delta t^3\right] g_2(t) \\ &+ \frac{1}{2}t\left[1 + t + \frac{3}{2}t^2 + \delta t^3\right]^2 g_2(t)^2, \end{aligned} \tag{8}$$

where

$$g_1(t) = 1 + \frac{1}{2}t + t^2\chi(t),$$

$$g_2(t) = \frac{1}{2}t\left[1 + 2t\chi(t) + g_1(t)^2\right].$$

The functions defined above will be used in the later developments, so next we study some of their properties. Let $f(t) = p(t)t - 1$. Since $f(0) = -1 < 0$ and

$f(\frac{1}{2}) > \frac{231}{1024} > 0$, then we can conclude that $f(t) = 0$ has at least a root in $(0, \frac{1}{2})$. Let s be the smallest positive root of $p(t)t - 1 = 0$, then we obtain that $s < \frac{1}{2}$.

Lemma 1 *Let the functions p, h and φ be given in (6–8), s be the smallest positive root of $p(t)t - 1 = 0$; then*
(a) *$p(t)$ and $h(t)$ are increasing and $p(t) > 1$, $h(t) > 1$ for $t \in (0, s)$,*
(b) *For $t \in (0, s)$, $\varphi(t)$ is increasing.*

Define $\eta_0 = \eta$, $\beta_0 = \beta$, $c_0 = M\beta\eta$ and $d_0 = h(c_0)\varphi(c_0)$. Furthermore, we define the following sequences as

$$\eta_{n+1} = d_n\eta_n, \tag{9}$$

$$\beta_{n+1} = h(c_n)\beta_n, \tag{10}$$

$$c_{n+1} = M\beta_{n+1}\eta_{n+1}, \tag{11}$$

$$d_{n+1} = h(c_{n+1})\varphi(c_{n+1}), \tag{12}$$

where $n \geq 0$. Some important properties of the previous sequences are given by the following lemma.

Lemma 2 *If*

$$c_0 < s \quad and \quad h(c_0)d_0 < 1, \tag{13}$$

where s is the smallest positive root of $p(t)t - 1 = 0$, then we have
(a) *$h(c_n) > 1$ and $d_n < 1$ for $n \geq 0$,*
(b) *the sequences $\{\eta_n\}$, $\{c_n\}$ and $\{d_n\}$ are decreasing,*
(c) *$p(c_n)c_n < 1$ and $h(c_n)d_n < 1$ for $n \geq 0$.*

The proof of this lemma can be obtained by induction.

Lemma 3 *Let the functions p, h and φ be given in (6–8). Let $\alpha \in (0, 1)$, then $p(\alpha t) < p(t)$, $h(\alpha t) < h(t)$, $\varphi(\alpha t) < \alpha^2\varphi(t)$ for $t \in (0, s)$, where s is the smallest positive root of $p(t)t - 1 = 0$.*

3 System of recurrence relations for the methods

For $n = 0$, the existence of Γ_0 implies the existence of u_0, and furthermore, we obtain

$$\|u_0 - x_0\| = \| - \frac{1}{2}\Gamma_0 F(x_0)\| \leq \frac{1}{2}\eta_0. \tag{14}$$

This shows that $u_0 \in B(x_0, R\eta)$, where $R = \frac{p(c_0)}{1-d_0}$. Moreover, we have

$$\|G(x_0)\| \leq \|\Gamma_0\|\|F''(u_0)\|\|\Gamma_0 F(x_0)\| \leq M\beta_0\eta_0 = c_0, \tag{15}$$

and

$$\begin{aligned}\|z_0 - x_0\| &= \left\| -\left[I + \frac{1}{2}G(x_0) + G(x_0)^2 Q(G(x_0)) \right] \Gamma_0 F(x_0) \right\| \\ &\leq \left[1 + \frac{1}{2}c_0 + c_0^2 \chi(c_0) \right] \|\Gamma_0 F(x_0)\| \\ &= g_1(c_0) \|\Gamma_0 F(x_0)\| \leq g_1(c_0)\eta_0. \end{aligned} \tag{16}$$

Furthermore, we obtain

$$\|x_1 - z_0\| \leq \left[1 + c_0 + \frac{3}{2}c_0^2 + \delta c_0^3 \right] \beta_0 \|F(z_0)\|. \tag{17}$$

By Taylor expansion, we have

$$\begin{aligned} F(z_n) &= F(x_n) + F'(x_n)(z_n - x_n) \\ &\quad + \int_0^1 \left[F'(x_n + t(z_n - x_n)) - F'(x_n) \right] (z_n - x_n) dt. \end{aligned} \tag{18}$$

Since

$$z_n - x_n = -\left[I + \frac{1}{2}G(x_n) + G(x_n)^2 Q(G(x_n)) \right] \Gamma_n F(x_n),$$

and

$$G(x_n) = \Gamma_n F''(u_n) \Gamma_n F(x_n), \quad \Gamma_n = F'(x_n)^{-1},$$

we obtain

$$\begin{aligned} F(z_n) &= F(x_n) - F'(x_n) \left[I + \frac{1}{2}G(x_n) + G(x_n)^2 Q(G(x_n)) \right] \Gamma_n F(x_n) \\ &\quad + \int_0^1 \left[F'(x_n + t(z_n - x_n)) - F'(x_n) \right] (z_n - x_n) dt \\ &= -\frac{1}{2} F''(u_n) \left[\Gamma_n F(x_n) \right]^2 - F''(u_n) \Gamma_n F(x_n) G(x_n) Q(G(x_n)) \Gamma_n F(x_n) \\ &\quad + \int_0^1 \left[F'(x_n + t(z_n - x_n)) - F'(x_n) \right] (z_n - x_n) dt. \end{aligned} \tag{19}$$

It follows that

$$\|F(z_0)\| \leq \frac{1}{2} M \|\Gamma_0 F(x_0)\|^2 \left[1 + 2c_0 \chi(c_0) + g_1(c_0)^2 \right], \tag{20}$$

and

$$\begin{aligned} \beta_0 \|F(z_0)\| &\leq \frac{1}{2} c_0 \left[1 + 2c_0 \chi(c_0) + g_1(c_0)^2 \right] \|\Gamma_0 F(x_0)\| \\ &= g_2(c_0) \|\Gamma_0 F(x_0)\| \leq g_2(c_0)\eta_0. \end{aligned} \tag{21}$$

Then we have

$$\|x_1 - x_0\| \leq \|x_1 - z_0\| + \|z_0 - x_0\| \leq p(c_0) \|\Gamma_0 F(x_0)\| \leq p(c_0)\eta_0. \tag{22}$$

This means that $x_1 \in B(x_0, R\eta)$ since the assumption $d_0 < 1/h(a_0) < 1$.
Notice that $a_0 < s$ and $p(a_0) < p(s)$, we have

$$\begin{aligned} \|I - \Gamma_0 F'(x_1)\| &\leq \|\Gamma_0\| \|F'(x_0) - F'(x_1)\| \\ &\leq M\beta_0 \|x_1 - x_0\| \leq c_0 p(c_0) < 1. \end{aligned}$$

By the Banach lemma, we obtain that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\begin{aligned} \|\Gamma_1\| &\leq \frac{\|\Gamma_0\|}{1 - \|\Gamma_0\| \|F'(x_0) - F'(x_1)\|} \\ &\leq \frac{\|\Gamma_0\|}{1 - c_0 p(c_0)} = h(c_0)\|\Gamma_0\| \\ &\leq h(c_0)\beta_0 = \beta_1. \end{aligned} \tag{23}$$

So u_1 is well defined. In order to estimate the bound of $F(x_1)$, we now give the following lemma.

Lemma 4 *Let X and Y be two Banach spaces, Ω be an open set, the nonlinear operator $F : \Omega \subset X \to Y$ be continuously twice Fréchet differentiable. Then we obtain*

$$\begin{aligned} F(x_{n+1}) &= -F''(u_n)\Gamma_n F(x_n)\Gamma_n F(z_n) - F''(u_n)\Gamma_n F(x_n)G(x_n)\Gamma_n F(z_n) \\ &\quad -\frac{1}{2}F''(u_n)G(x_n)\Gamma_n F(x_n)\Gamma_n F(z_n) - \delta F''(u_n)\Gamma_n F(x_n)G(x_n)^2\Gamma_n F(z_n) \\ &\quad + \int_0^1 F''\left(x_n + t(z_n - x_n)\right)(z_n - x_n)dt\,(x_{n+1} - z_n) \\ &\quad + \int_0^1 \left[F'\left(z_n + t(x_{n+1} - z_n)\right) - F'(z_n)\right](x_{n+1} - z_n)dt, \end{aligned} \tag{24}$$

where x_{n+1}, z_n are given by (5), and the definitions of Γ_n, u_n, δ, $G(x_n)$ are same to the ones of (5).

Proof 1 By Taylor expansion, we obtain

$$\begin{aligned} F(x_{n+1}) &= F(z_n) + F'(z_n)(x_{n+1} - z_n) \\ &\quad + \int_0^1 \left[F'\left(z_n + t(x_{n+1} - z_n)\right) - F'(z_n)\right](x_{n+1} - z_n)dt. \end{aligned} \tag{25}$$

$$F'(z_n) = F'(x_n) + \int_0^1 F''\left(x_n + t(z_n - x_n)\right)(z_n - x_n)dt. \tag{26}$$

Then we have

$$\begin{aligned} F(x_{n+1}) &= F(z_n) + F'(x_n)(x_{n+1} - z_n) \\ &\quad + \int_0^1 F''\left(x_n + t(z_n - x_n)\right)(z_n - x_n)dt\,(x_{n+1} - z_n) \\ &\quad + \int_0^1 \left[F'\left(z_n + t(x_{n+1} - z_n)\right) - F'(z_n)\right](x_{n+1} - z_n)dt. \end{aligned}$$

Since

$$x_{n+1} - z_n = -\left[I + G(x_n) + G(x_n)^2 + \frac{1}{2}\Gamma_n F''(u_n)G(x_n)\Gamma_n F(x_n) + \delta G(x_n)^3\right]\Gamma_n F(z_n),$$

and

$$G(x_n) = \Gamma_n F''(u_n)\Gamma_n F(x_n), \quad \Gamma_n = F'(x_n)^{-1},$$

we obtain

$$\begin{aligned} F(x_{n+1}) &= F(z_n) - F'(x_n)\left[I + G(x_n) + G(x_n)^2\right]\Gamma_n F(z_n) \\ &\quad -F'(x_n)\left[\frac{1}{2}\Gamma_n F''(u_n)G(x_n)\Gamma_n F(x_n) + \delta G(x_n)^3\right]\Gamma_n F(z_n) \\ &\quad + \int_0^1 F''\left(x_n + t(z_n - x_n)\right)(z_n - x_n)dt\,(x_{n+1} - z_n) \\ &\quad + \int_0^1 \left[F'\left(z_n + t(x_{n+1} - z_n)\right) - F'(z_n)\right](x_{n+1} - z_n)dt \\ &= -F''(u_n)\Gamma_n F(x_n)\Gamma_n F(z_n) - F''(u_n)\Gamma_n F(x_n)G(x_n)\Gamma_n F(z_n) \\ &\quad -\frac{1}{2}F''(u_n)G(x_n)\Gamma_n F(x_n)\Gamma_n F(z_n) - \delta F''(u_n)\Gamma_n F(x_n)G(x_n)^2\Gamma_n F(z_n) \\ &\quad + \int_0^1 F''\left(x_n + t(z_n - x_n)\right)(z_n - x_n)dt\,(x_{n+1} - z_n) \\ &\quad + \int_0^1 \left[F'\left(z_n + t(x_{n+1} - z_n)\right) - F'(z_n)\right](x_{n+1} - z_n)dt. \end{aligned}$$

This ends the proof. □

From Lemma 4, we have

$$
\begin{aligned}
\|F(x_1)\| \;\le\; & M\,\|\Gamma_0 F(x_0)\|\left[1+\frac{3}{2}c_0+\delta c_0^2\right]\beta_0\|F(z_0)\| \\
& +M\|z_0-x_0\|\|x_1-z_0\|+\frac{1}{2}M\|x_1-z_0\|^2 \\
\le\; & M\left[1+\frac{3}{2}c_0+\delta c_0^2\right]g_2(c_0)\,\|\Gamma_0 F(x_0)\|^2 \\
& +Mg_1(c_0)\left[1+c_0+\frac{3}{2}c_0^2+\delta c_0^3\right]g_2(c_0)\,\|\Gamma_0 F(x_0)\|^2 \\
& +\frac{1}{2}M\left[1+c_0+\frac{3}{2}c_0^2+\delta c_0^3\right]^2 g_2(c_0)^2\,\|\Gamma_0 F(x_0)\|^2
\end{aligned} \tag{27}
$$

and

$$
\begin{aligned}
\beta_0\|F(x_1)\| \;\le\; & c_0\left[1+\frac{3}{2}c_0+\delta c_0^2\right]g_2(c_0)\,\|\Gamma_0 F(x_0)\| \\
& +c_0 g_1(c_0)\left[1+c_0+\frac{3}{2}c_0^2+\delta c_0^3\right]g_2(c_0)\,\|\Gamma_0 F(x_0)\| \\
& +\frac{1}{2}c_0\left[1+c_0+\frac{3}{2}c_0^2+\delta c_0^3\right]^2 g_2(c_0)^2\,\|\Gamma_0 F(x_0)\| \\
=\; & \varphi(c_0)\,\|\Gamma_0 F(x_0)\| \\
\le\; & \varphi(c_0)\eta_0.
\end{aligned} \tag{28}
$$

From (23) and (28), we have

$$
\begin{aligned}
\|u_1-x_1\| &= \|-\frac{1}{2}\Gamma_1 F(x_1)\| \le \|\Gamma_1 F(x_1)\| \le \|\Gamma_1\|\|F(x_1)\| \\
&\le h(c_0)\varphi(c_0)\,\|\Gamma_0 F(x_0)\| \le h(c_0)\varphi(c_0)\eta_0 \\
&= d_0\eta_0 = \eta_1.
\end{aligned} \tag{29}
$$

Since $p(c_0) > 1$, we obtain

$$
\begin{aligned}
\|u_1-x_0\| &\le \|x_1-x_0\|+\|u_1-x_1\| \\
&< (p(c_0)+d_0)\eta_0 < p(c_0)(1+d_0)\eta_0 < R\eta,
\end{aligned} \tag{30}
$$

which means that $u_1 \in B(x_0, R\eta)$.

Besides, we have

$$
M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \le h(c_0)d_0c_0 = c_1. \tag{31}
$$

Using induction, we can obtain the following items:

(1) There exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \le h(c_{n-1})\|\Gamma_{n-1}\|$,

(2) $\|\Gamma_n F(x_n)\| \le h(c_{n-1})\varphi(c_{n-1})\|\Gamma_{n-1}F(x_{n-1})\|$,

(3) $M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \le c_n$,

(4) $\|z_n - x_n\| \le g_1(c_n)\|\Gamma_n F(x_n)\|$,
(5) $\|x_{n+1} - x_n\| \le p(c_n)\|\Gamma_n F(x_n)\|$,
where $n \ge 0$.

Moreover, we can get the following lemma.

Lemma 5 *Let the assumptions of Lemma 2 and the conditions (A1)-(A3) hold; then we have*

$$\|u_n - x_0\| \le R\eta, \quad \|z_n - x_0\| \le R\eta, \quad \|x_{n+1} - x_0\| \le R\eta, \tag{32}$$

where $R = \frac{p(c_0)}{1-d_0}$.

To get the proof of Lemma 5, we now give the following lemma.

Lemma 6 *Under the assumptions of Lemma 2, let* $\gamma = h(c_0)d_0$ *and* $\lambda = 1/h(c_0)$*; then we have*

$$\prod_{i=0}^{n} d_i \le \lambda^{n+1}\gamma^{\frac{3^{n+1}-1}{2}}, \tag{33}$$

$$\eta_n \le \eta\lambda^n\gamma^{\frac{3^n-1}{2}}, \quad n \ge 0, \tag{34}$$

$$\sum_{i=n}^{n+m} \eta_i \le \eta\lambda^n\gamma^{\frac{3^n-1}{2}} \frac{1-\lambda^{m+1}\gamma^{\frac{3^n(3^m+1)}{2}}}{1-\lambda\gamma^{3^n}}, \quad n \ge 0, \ m \ge 1. \tag{35}$$

Proof 2 Since $c_1 = \gamma c_0$, by Lemma 3, we have

$$d_1 = h(\gamma c_0)\varphi(\gamma c_0) < \gamma^2 d_0 = \gamma^{3^1-1}d_0 = \lambda\gamma^{3^1}.$$

Suppose $d_k \le \lambda\gamma^{3^k}$, $k \ge 1$. Then by Lemma 2, we have $c_{k+1} < c_k$ and $h(c_k)d_k < 1$. Then

$$d_{k+1} < h(c_k)\varphi\left(h(c_k)d_k c_k\right) < h(c_k)^2 d_k^3 < \lambda\gamma^{3^{k+1}}.$$

Therefore it holds that $d_n \le \lambda\gamma^{3^n}$, $n \ge 0$. Moreover, we have

$$\prod_{i=0}^{n} d_i \le \prod_{i=0}^{n} \lambda\gamma^{3^i} = \lambda^{n+1}\gamma^{\sum_{i=0}^{n} 3^i} = \lambda^{n+1}\gamma^{\frac{3^{n+1}-1}{2}}, n \ge 0.$$

From (9) and (33), we have

$$\eta_n = d_{n-1}\eta_{n-1} = d_{n-1}d_{n-2}\eta_{n-2} = \cdots = \eta\left(\prod_{i=0}^{n-1} d_i\right) \le \eta\lambda^n\gamma^{\frac{3^n-1}{2}}, n \ge 0.$$

Let

$$\rho = \sum_{i=n}^{n+m} \lambda^i\gamma^{\frac{3^i}{2}},$$

where $n \geq 0, m \geq 1$. Since

$$\rho \leq \lambda^n \gamma^{\frac{3^n}{2}} + \gamma^{3^n} \left(\sum_{i=n+1}^{n+m} \lambda^i \gamma^{\frac{3^i-1}{2}} \right) = \lambda^n \gamma^{\frac{3^n}{2}} + \lambda \gamma^{3^n} \left(\rho - \lambda^{n+m} \gamma^{\frac{3^{n+m}}{2}} \right),$$

we have

$$\rho \leq \lambda^n \gamma^{\frac{3^n}{2}} \frac{1 - \lambda^{m+1} \gamma^{\frac{3^n(3^m+1)}{2}}}{1 - \lambda \gamma^{3^n}}.$$

Moreover, we obtain

$$\sum_{i=n}^{n+m} \eta_i \leq \eta \left(\sum_{i=n}^{n+m} \lambda^i \gamma^{\frac{3^i-1}{2}} \right) \leq \eta \lambda^n \gamma^{\frac{3^n-1}{2}} \frac{1 - \lambda^{m+1} \gamma^{\frac{3^n(3^m+1)}{2}}}{1 - \lambda \gamma^{3^n}}.$$

Next we give a brief proof of Lemma 5.

Proof 3 From (14), we know that $\|u_0 - x_0\| < R\eta$. For $n \geq 1$, by (5) and Lemma 6, we obtain

$$\begin{aligned} \|u_n - x_0\| &\leq \|u_n - x_n\| + \|x_n - x_0\| \\ &\leq \|u_n - x_n\| + \sum_{i=0}^{n-1} \|x_{i+1} - x_i\| \\ &\leq \eta_n + p(c_0) \sum_{i=0}^{n-1} \eta_i \leq p(c_0) \sum_{i=0}^{n} \eta \lambda^i \gamma^{\frac{3^i-1}{2}} \\ &\leq p(c_0) \eta \frac{1 - \lambda^{n+1} \gamma^{\frac{3^n+1}{2}}}{1 - d_0} < R\eta. \end{aligned}$$

From (16), we know that $\|z_0 - x_0\| \leq g_1(c_0)\eta < p(c_0)\eta < R\eta$. For $n \geq 1$, by (4), (5) and Lemma 6, we obtain

$$\begin{aligned} \|z_n - x_0\| &\leq \|z_n - x_n\| + \|x_n - x_0\| \\ &\leq \|z_n - x_n\| + \sum_{i=0}^{n-1} \|x_{i+1} - x_i\| \\ &\leq g_1(c_n)\eta_n + p(c_0) \sum_{i=0}^{n-1} \eta_i \leq p(c_0) \sum_{i=0}^{n} \eta \lambda^i \gamma^{\frac{3^i-1}{2}} < R\eta. \end{aligned}$$

Similarly, for $n \geq 0$, By (5) and Lemma 6, we have

$$\|x_{n+1} - x_0\| \leq \sum_{i=0}^{n} \|x_{i+1} - x_i\| \leq p(c_0) \sum_{i=0}^{n} \eta_i < R\eta.$$

This ends the proof. □

Lemma 7 *Let $R = \frac{p(c_0)}{1-d_0}$, If $h(c_0)d_0 < 1$ and $c_0 < s$, where s is the smallest positive root of $p(t)t - 1 = 0$, then we have $R < \frac{1}{c_0}$.*

4 Semilocal convergence

In this section, we prove the following theorem which shows the existence and uniqueness of the solution and gives a priori error bounds.

Theorem 1 *Let X and Y be two Banach spaces, the nonlinear operator $F : \Omega \subset X \to Y$ be twice Fréchet differentiable in a non-empty open convex subset Ω. Assume that $x_0 \in \Omega$ and all conditions (A1)-(A3) hold. Let $c_0 = M\beta\eta$ and $d_0 = h(c_0)\varphi(c_0)$ satisfy $c_0 < s$ and $h(c_0)d_0 < 1$, where s is the smallest positive root of $p(t)t - 1 = 0$ and p, h, φ are defined by (6–8). Let $\overline{B(x_0, R\eta)} \subseteq \Omega$ where $R = \frac{p(c_0)}{1-d_0}$, then starting from x_0, the sequence $\{x_n\}$ generated by the method (5) converges to a solution x^* of $F(x) = 0$ with x_n, x^* belong to $\overline{B(x_0, R\eta)}$ and x^* is the unique solution of $F(x) = 0$ in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$.*

Furthermore, a priori error estimate is given by

$$\|x_n - x^*\| \leq p(c_0)\eta\lambda^n\gamma^{\frac{3^n-1}{2}} \frac{1}{1-\lambda\gamma^{3^n}}, \tag{36}$$

where $\gamma = h(c_0)d_0$ and $\lambda = 1/h(c_0)$.

Proof 4 From Lemma 5, we can obtain that the sequence $\{x_n\}$ is well-defined in $\overline{B(x_0, R\eta)}$. Now we prove that $\{x_n\}$ is a Cauchy sequence. Since

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \leq p(c_0) \sum_{i=n}^{n+m-1} \eta_i \\ &\leq p(c_0)\eta\lambda^n\gamma^{\frac{3^n-1}{2}} \frac{1-\lambda^m\gamma^{\frac{3^n(3^{m-1}+1)}{2}}}{1-\lambda\gamma^{3^n}}. \end{aligned} \tag{37}$$

We have that there exists a x^* such that $\lim_{n\to\infty} x_n = x^*$.

Letting $n = 0, m \to \infty$ in (37), we have

$$\| x^* - x_0 \| \leq R\eta, \tag{38}$$

which means that $x^* \in \overline{B(x_0, R\eta)}$.

From Lemma 4, we have

$$\begin{aligned} \|F(x_{n+1})\| \leq\ & M\left[1 + \frac{3}{2}c_0 + \delta c_0^2\right] g_2(c_0)\eta_n^2 \\ &+ Mg_1(c_0)\left[1 + c_0 + \frac{3}{2}c_0^2 + \delta c_0^3\right] g_2(c_0)\eta_n^2 \\ &+ \frac{1}{2}M\left[1 + c_0 + \frac{3}{2}c_0^2 + \delta c_0^3\right]^2 g_2(c_0)^2\eta_n^2 \end{aligned} \tag{39}$$

Let $n \to \infty$ in (39), then we obtain that $\|F(x_n)\| \to 0$ since $\eta_n \to 0$. By the continuity of $F(x)$ in Ω, we get that $F(x^*) = 0$.

Next we prove the uniqueness of x^* in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$. By Lemma 7, we obtain

$$\frac{2}{M\beta} - R\eta = \left(\frac{2}{c_0} - R\right)\eta > \frac{1}{c_0}\eta > R\eta,$$

and then $\overline{B(x_0, R\eta)} \subseteq B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$, thus $x^* \in B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$.

Assume that $x^{**} \in B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$ and x^{**} satisfies $F(x^{**}) = 0$, then we have that

$$0 = F(x^{**}) - F(x^*) = \int_0^1 F'((1-t)x^* + tx^{**})dt(x^{**} - x^*). \tag{40}$$

Notice that

$$\begin{aligned} &\|\Gamma_0\| \left\| \int_0^1 [F'((1-t)x^* + tx^{**}) - F'(x_0)]dt \right\| \\ \leq\ & M\beta \int_0^1 [(1-t)\|x^* - x_0\| + t\|x^{**} - x_0\|]dt \\ <\ & \frac{M\beta}{2}\left[R\eta + \frac{2}{M\beta} - R\eta\right] = 1, \end{aligned} \tag{41}$$

then by the Banach lemma, we have that $\int_0^1 F'((1-t)x^* + tx^{**})dt$ is invertible and hence $x^{**} = x^*$.

Finally, letting $m \to \infty$ in (37), we obtain (36).

Next we consider two examples, where the conditions of theorem 1 are satisfied, but the assumption (A4) can not be satisfied.

Example 4.1

$$f(x) = x^3 \ln(x^2) + 3x^2 - 10x + 1.7 = 0,$$

where $f(x)$ defines in $X = [-1, 1]$, $f(0) = 1.7$.

Here, we take $Q(G(x_n)) = 0$, $\delta = 1$ in the methods (5). Let $\Omega = B(0, 1)$, $x_0 = 0$, we obtain

$$\lim_{x\to 0} x^3 \ln(x^2) = 0, \quad \lim_{x\to 0} x^2 \ln(x^2) = 0, \quad \lim_{x\to 0} x \ln(x^2) = 0.$$

Note that $f'''(x)$ can not satisfy the assumption (A4). But we have

$$|1/f'(0)| = 0.1, \quad |f(0)/f'(0)| = 0.17, \quad \sup_{x\in X} |f''(x)| = 16.$$

Since $c_0 = 0.272$,

$$p(c_0)c_0 = 0.427\ldots < 1,$$

then we get $c_0 < s$. Furthermore,

$$h(c_0)d_0 = 0.875\ldots < 1,$$

Then the conditions of Theorem 1 are satisfied. The solution x^* belongs to $\overline{B(x_0, R\eta)} = \overline{B(0, 0.535\ldots)} \subseteq \Omega$ and x^* is the unique solution of $f(x) = 0$ in $B(0, 0.714\ldots) \bigcap \Omega$.

Example 4.2 Consider a nonlinear integral equation

$$x(s) = 1 + \frac{9}{8}\int_0^1 G(s,t)x(t)^{5/2}dt, \quad s \in [0, 1],$$

where $x \in C[0, 1]$, $t \in [0, 1]$, $G(s, t)$ is the Green function defined by

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t. \end{cases}$$

To find the solution of this equation, we need to solve the equation $F(x) = 0$, where $F : \Omega \subseteq C[0, 1] \to C[0, 1]$,

$$[F(x)](s) = x(s) - 1 - \frac{9}{8}\int_0^1 G(s,t)x(t)^{5/2}dt, \quad s \in [0, 1].$$

Here, we take $\Omega = B(0, 2)$. The Fréchet derivatives of F are given by

$$F'(x)y(s) = y(s) - \frac{45}{16}\int_0^1 G(s,t)x(t)^{3/2}y(t)dt, \quad y \in \Omega,$$

$$F''(x)yz(s) = -\frac{135}{32}\int_0^1 G(s,t)x(t)^{1/2}y(t)z(t)dt, \quad y,\ z \in \Omega,$$

$$F'''(x)yzv(s) = -\frac{135}{64}\int_0^1 G(s,t)x(t)^{-1/2}y(t)z(t)v(t)dt, \quad y,\ z,\ v \in \Omega.$$

Obviously, F''' can not satisfy the condition (A4). We take $Q(G(x_n)) = 0, \delta = 1$ in the methods (5) and choose $x_0(t) = 1$ as the initial approximate solution. Then we obtain that

$$\|F(x_0)\| = \frac{9}{64}, \quad \|I - F'(x_0)\| = \frac{45}{128},$$

$$\|\Gamma_0\| = \|F'(x_0)^{-1}\| \leq \frac{1}{1 - \|I - F'(x_0)\|} = \frac{128}{83} \equiv \beta,$$

$$\|\Gamma_0 F(x_0)\| \leq \frac{18}{83} \equiv \eta, \quad \|F''(x)\| \leq \frac{135\sqrt{2}}{256} \equiv M.$$

Here, the max norm is used. Since $c_0 = 0.249\ldots$,

$$p(c_0)c_0 = 0.376\ldots < 1,$$

then $c_0 < s$. Moreover,

$$h(c_0)d_0 = 0.583\ldots < 1,$$

Then the conditions of theorem 1 are satisfied. The solution x^* belongs to $\overline{B(x_0, R\eta)} = \overline{B(1, 0.514\ldots)} \subseteq \Omega$ and x^* is the unique solution of $F(x) = 0$ in $B(1, 1.224\ldots) \bigcap \Omega$.

5 Higher *R*-order convergence analysis

(B) $\|F'''(x) - F'''(y)\| \leq \omega(\|x-y\|),\ \forall x, y \in \Omega$, where $\omega(\mu)$ is a non-decreasing continuous real function for $\mu > 0$ and satisfies $\omega(0) \geq 0$, $\omega(t\mu) \leq t^q \omega(\mu)$ for $\mu > 0$, $t \in [0, 1]$ and $q \in [0, 1]$.

Define the function ψ as

$$\begin{aligned}\psi(t,u,v) = &\left[\frac{1}{4}tu + \frac{1}{q+1}\frac{1}{2^{(q+1)}}\left(1+\frac{3}{2}t\right)v + \left(\frac{1}{2}t + t^2\chi(t)\right)u\right]\phi(t,u)\\ &+\left[\delta t^3 + t^3\chi(t) + \left(\frac{3}{2}+\delta t\right)t^3 + \frac{t^2}{2}\left(\frac{3}{2}+\delta t\right)u\right]\phi(t,u)\\ &+\frac{1}{(q+1)(q+2)}\left(1+t+\frac{3}{2}t^2+\delta t^3\right)v\phi(t,u)\\ &+t^3\left(\frac{1}{2}+t\chi(t)\right)\left(1+\frac{3}{2}t+\delta t^2\right)\phi(t,u)\\ &+\frac{t^2}{2}\left(\frac{1}{2}+t\chi(t)\right)^2\left(1+t+\frac{3}{2}t^2+\delta t^3\right)u\phi(t,u)\\ &+\frac{t}{2}\left(1+t+\frac{3}{2}t^2+\delta t^3\right)^2\phi(t,u)^2, \end{aligned} \tag{42}$$

where

$$\phi(t,u) = t^2\chi(t) + \frac{5}{12}u + t^2\left(\frac{1}{2}+t\chi(t)\right) + \frac{t}{2}\left(\frac{1}{2}+t\chi(t)\right)u + \frac{t^3}{2}\left(\frac{1}{2}+t\chi(t)\right)^2.$$

Let the function ψ be defined by (42), $\alpha \in (0,1)$, then $\psi(\alpha t, \alpha^2 u, \alpha^{(2+p)} v) < \alpha^{(4+p)}\psi(t,u,v)$ for $t \in (0,s)$, where s is the smallest positive root of $p(t)t - 1 = 0$, the function p is given in (6).
Define the following sequences as

$$\widetilde{\eta}_{n+1} = \widetilde{d}_n\widetilde{\eta}_n,\quad \widetilde{\beta}_{n+1} = h(\widetilde{c}_n)\widetilde{\beta}_n, \tag{43}$$

$$\widetilde{c}_{n+1} = M\widetilde{\beta}_{n+1}\widetilde{\eta}_{n+1},\quad \widetilde{b}_{n+1} = N\widetilde{\beta}_{n+1}\widetilde{\eta}^2_{n+1},\quad \widetilde{a}_{n+1} = \widetilde{\beta}_{n+1}\widetilde{\eta}^2_{n+1}w(\widetilde{\eta}_{n+1}), \tag{44}$$

$$\widetilde{d}_{n+1} = h(\widetilde{c}_{n+1})\psi(\widetilde{c}_{n+1}, \widetilde{b}_{n+1}, \widetilde{a}_{n+1}), \tag{45}$$

where $n \geq 0$. Here, we choose $\widetilde{\eta}_0 = \eta$, $\widetilde{\beta}_0 = \beta$, $\widetilde{c}_0 = M\beta\eta$, $\widetilde{b}_0 = N\beta\eta^2$, $\widetilde{a}_0 = \beta\eta^2\omega(\eta)$ and $\widetilde{d}_0 = h(\widetilde{c}_0)\psi(\widetilde{c}_0, \widetilde{b}_0, \widetilde{a}_0)$. From the definitions of $\widetilde{c}_{n+1}, \widetilde{b}_{n+1}, \widetilde{a}_{n+1}$ and Eq (43), we can obtain

$$\widetilde{c}_{n+1} = h(\widetilde{c}_n)\widetilde{d}_n\widetilde{c}_n,\quad \widetilde{b}_{n+1} = h(\widetilde{c}_n)\widetilde{d}^2_n\widetilde{b}_n,\quad \widetilde{a}_{n+1} \leq h(\widetilde{c}_n)\widetilde{d}^{2+q}_n\widetilde{a}_n. \tag{46}$$

Similar to the derivation in Section 3 and Section 4, we can establish the semilocal convergence of methods (5) under the conditions (A1)-(A4), (B). Moreover, we can get a priori error estimate

$$\|x_n - x^*\| \leq \frac{p(\widetilde{c}_0)\eta}{\widetilde{\gamma}^{1/(4+q)}(1-\widetilde{d}_0)}\left(\widetilde{\gamma}^{1/(4+q)}\right)^{(5+q)^n}, \tag{47}$$

where $\widetilde{\gamma} = h(\widetilde{c}_0)\widetilde{d}_0$ and $\widetilde{\lambda} = 1/h(\widetilde{c}_0)$. This error estimate shows that under the conditions (A1)-(A4), (B), the methods (5) has, at least, R-order $5 + q$.

Acknowledgement

This work is supported by the Scientific and Technical Research Project of Hubei Provincial Department of Education (No. D20132701).

References

1. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equation in Several Variables. Academic Press, New York (1970)
2. Candela, V., Marquina, A.: Recurrence relations for rational cubic methods I: The Halley method. Comput. **44**, 169–184 (1990)
3. Gutiérrez, J.M., Hernández, M.A.: Recurrence relations for the super-Halley method, Comput. Math. Applic. **36**, 1-8 (1998)
4. Argyros, I.K., Chen, D.: Results on the Chebyshev method in banach spaces. Proyecciones **12**(2), 119–128 (1993)
5. Hernández, M.A., Salanova, M.A.: Modification of the Kantorovich assumptions for semilocal convergence of the Chebyshev method. J. Comput. Appl. Math. **126**, 131–143 (2000)
6. Hernández, M.A.: Chebyshev's approximation algorithms and applications. Comput.Math. with Appl. **41**, 433-445 (2001)
7. Candela, V., Marquina, A.: Recurrence relations for rational cubic methods II: The Chebyshev method. Comput. **45**, 355-367 (1990)
8. Amat, S., Busquier, S., Gutiérrez, J.M.: Geometric constructions of iterative functions to solve nonlinear equations. J. Comput. Appl. Math **157**, 197–205 (2003)
9. Argyros, I.K.: Quadratic equations and applications to Chandrasekhar's and related equations. Bull. Austral. Math. Soc **32**, 275–292 (1985)
10. Hernández, M.A.: Second-derivative-free variant of the Chebyshev method for nonlinear equations. J. Optim. Theory Appl. **104**(3), 501–515 (2000)
11. Wang, X., Kou, J.: Semilocal convergence and R-order for modified Chebyshev-Halley methods. Numer. Algoritm. **64**, 105–126 (2012)
12. Gutiérrez, J.M., Hernández, M.A.: Third-order iterative methods for operators with bounded second derivative. J. Comput. Appl. Math. **82**, 171–183 (1997)
13. Proinov, P.D.: New general convergence theory for iterative processes and its applications to Newton-Kantorovich type theorems. J. Complexity **26**, 3–42 (2010)
14. Argyros, I.K.: Improved generalized differentiability conditions for Newton-like methods. J.Complexity **26**, 316–333 (2010)
15. Gutiérrez, J.M., Magreñán, Á.A., Romero, N.: On the semilocal convergence of Newton-Kantorovich method under center-Lipschitz conditions. Appl. Math. Comput. **221**, 79–88 (2013)
16. Amat, S., Magreñán, Á.A., Romero, N.: On a two-step relaxed Newton-type method. Appl. Math. Comput. **219**(24), 11341–11347 (2013)