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Some results on certain generalized circulant matrices

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Abstract In this paper a particular partition on blocks of generalized (h, r)-circulant matrices is determined. We obtain a characterization of generalized (h, r)-circulant matrices and get some results on the values of the permanent and also on the determination of the eigenvalues of r-circulant matrices. At last, a lower bound for the permanent of these matrices is achieved.

Keywords h-circulant matrices · r-circulant matrices · (h, r)-generalized circulant matrices · Permanent · Direct sums

Mathematics Subject Classifications (2010) 15A18 · 65F05 · 65F10

1 Introduction

A matrix $A = [a_{i,j}]$ of type $m \times n$ $(m \leq n)$ is called (h, r)-circulant if it is of the form

$$\begin{cases} a_{1,j} = \alpha_j, \ j = 1, \cdots, n, \\ a_{i,j} = \begin{cases} a_{i-1,j-h}, \ j > h \\ r a_{i-1,j-h+n}, \ j \leq h \end{cases}, \ i = 2, \cdots, m, \ j = 1, \cdots, n, \end{cases}$$

the above equation can be rewritten as

$$\begin{cases} a_{1,j} = \alpha_j, \ j = 1, \cdots, n, \\ a_{i,j} = r^{\theta_1} a_{i-1,(j-h+n) \ mod \ n}, \ i = 2, \cdots, m, \ j = 1, \cdots, n, \end{cases} \tag{1.1}$$

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where $\theta_1 = \begin{cases} 0, & if\ j > h, \\ 1, & if\ j \le h, \end{cases}$ h is a positive integer which satisfies $h < n$, and r is a parameter [1]. Obviously, each row other than the first one, is obtained from the preceding row by shifting the elements cyclically h positions to the right and multiplying the last h elements of the preceding row by r.

When $h = 1$, the matrix A is called r-circulant [1, 2]. When $r = 1$, the matrix A is called h-circulant [3].

Let $P_n(r)$ be the r-circulant matrix of order n with first row $(0, 1, 0, \cdots, 0)$ as follows

$$P_n(r) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ r & 0 & 0 & \ldots & 0 \end{bmatrix};$$

obviously, $P_n^n(r) = rI_n$. If there is not possibility of ambiguity we often drop the subscript n and simply write $P_n(r)$ as $P(r)$.

If $(a_0, a_1, \cdots, a_{n-1})$ is the first row of a r-circulant matrix A of order n, then $A = \sum_{i=0}^{n-1} a_i P^i(r)$.

Lemma 1 *Let m, n, h be positive integers, where $m \le n$, $h < n$ and $mh \equiv 0\ (mod\ n)$, a matrix of type $m \times n$ is (h, r)-circulant if and only if it is satisfied relation*

$$AP_n^h(r) = P_m(r^\theta)A,$$

where $\theta = \frac{mh}{n}$.

Proof Assume A is (h, r)-circulant. Let $a_{1,j} = \alpha_j$ $(j = 1, \cdots, n)$, and $B = (b_{i,j})_{m\times n} = AP_n^h(r)$, it follows

$$b_{1,j} = r^{\theta_1}\alpha_{(j-h+n)\ mod\ n}, \tag{1.2}$$

and

$$b_{i,j} = r^{\theta_1+\theta_2}a_{i-1,[(j-h+n)-h+n]\ mod\ n},\ i = 2, \cdots, m,\ \theta_2 = \begin{cases} 0, & if\ (j-h+n)\ mod\ n > h, \\ 1, & if\ (j-h+n)\ mod\ n \le h. \end{cases} \tag{1.3}$$

Hence

$$\begin{aligned} b_{m,j} =& r^{\theta_1+\theta_2}a_{m-1,(j-2h+2n)\ mod\ n} = \cdots\cdots \\ =& r^{\theta_1+\theta_2+\cdots+\theta_m}a_{m-1-(m-2),[(j-2h+2n)-(m-2)h+(m-2)n]\ mod\ n} \\ =& r^{\theta_1+\theta_2+\cdots+\theta_m}a_{1,(j-mh+mn)\ mod\ n}, \end{aligned} \tag{1.4}$$

where $\theta_i = \begin{cases} 0, & if\ [j-(i-1)h+(i-1)n]\ mod\ n > h, \\ 1, & if\ [j-(i-1)h+(i-1)n]\ mod\ n \le h, \end{cases}$ $i = 1, \cdots, m.$

Now consider $C = (c_{i,j})_{m\times n} = P_m(r^\theta)A$, here $\theta = \frac{mh}{n}$.

$$C = \{P_m(r^\theta) \times [r^{\theta_1}a_{i-1,(j-h+n)\ mod\ n}]_{m\times n}\}_{m\times n},$$

therefore

$$c_{i,j} = r^{\theta_1} a_{i,(j-h+n)\ mod\ n} = r^{\theta_1+\theta_2} a_{i-1,(j-2h+2n)\ mod\ n},\ i = 2, \cdots, m-1, \quad (1.5)$$

and

$$c_{1,j} = r^{\theta_1} a_{1,(j-h+n)\ mod\ n} = r^{\theta_1} \alpha_{(j-h+n)\ mod\ n}, \quad (1.6)$$

$$c_{m,j} = r^{\theta} a_{1,j} = r^{\theta} \alpha_j. \quad (1.7)$$

Since $n|mh$ and $n > m$, then $j = (j - mh + mn)\ mod\ n$. From the definition of θ_i, there holds $j = j - mh + (\theta_1 + \theta_2 + \cdots, \theta_m)n$, then we have $\theta_1 + \theta_2 + \cdots + \theta_m = \frac{mh}{n}$, that means $b_{m,j} = c_{m,j}$. By (1.2)–(1.7), we can obtain $AP_n^h(r) = P_m(r^{\theta})A$.

If a matrix A of type $m \times n$ is satisfied the relation $AP_n^h(r) = P_m(r^{\theta})A$, let $a_{1,j} = \alpha_j$ $(j = 1, \cdots, n)$, then

$$b_{i,j} = (AP_n^h(r))_{i,j} = r^{\theta_1} a_{i,(j-h+n)\ mod\ n},$$

$$c_{i,j} = (P_m(r^{\theta})A)_{i,j} = \begin{cases} a_{i+1,j}, & i = 1, \cdots, m-1, \\ r^{\theta} a_{1,j}, & i = m. \end{cases}$$

Since $b_{i,j} = c_{i,j}$, then

$$a_{i+1,j} = r^{\theta_1} a_{i,(j-h+n)\ mod\ n}.$$

It follows that A is (h, r)-circulant.

This completes the proof of the lemma. □

Let $A = [a_{i,j}]$ be a matrix of order n. We denote by $d(1, m)$, where $1 \le m \le n$, the diagonal starting in $a_{1,m}$, that is, the sequence of elements $a_{1,m}, a_{2,m+1}, \cdots, a_{n-m+1,n}$ and by $d(m, 1)$ the sequence $a_{m,1}, a_{m+1,2}, \cdots, a_{n,n-m+1}$ ([4]).

Let k and n_i $(i = \overline{1,k})$ be positive integers, A_i $(i = \overline{1,k})$ be square matrices of order n_i, the block diagonal square matrix

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_k \end{bmatrix}$$

of order $n_1 + n_2 + \cdots + n_k$ is called the direct sum of the matrices $A_1, A_2, \cdots A_k$. It is denoted as $A = diag(A_1, A_2, \cdots, A_k)$.

Definition 1 Let h, n be positive integers, where $1 \le h < n$, $k = (n, h)$, $n = km$ and $h = kh^{'}$. A matrix A of order n is said (h, r)-generalized circulant when it is partitioned into k submatrices of type $m \times n$, which are (h, r)-circulant.

In other words a matrix A of order n is (h, r)-generalized circulant when it can be partitioned into (h, r)-circulant submatrices A_j $(1 \leq j \leq k)$ of type $m \times n$ as follows

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix}. \tag{1.8}$$

Since $mh = nh' \equiv 0(\mathrm{mod}\ n)$, then we obtain $A_j P_n^h(r) = P_m(r^\theta)A_j$ $(j = \overline{1, k}$, $\theta = \frac{mh}{n})$ by Lemma 1.

Recall that the permanent of a $n \times n$ matrix $A = [a_{i,j}]$, denoted per A, is defined as

$$\text{per } A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where the sum extends over all permutations σ of S_n .

The computation of the permanent of generic matrices seems to be a very hard task. It has been shown [5] that the computation of the permanent is ♯ P-complete and therefore a polynomial time algorithm is unlikely to exist. Recently, a lot of people have tried to study the permanent of some special matrices such as circulant matrices (see [4, 6–8]), sparse positive matrices (see [9]), Toeplitz matrices (see [10, 11]), Bernoulli matrices (see [12]) etc.

This paper is organized as follows. In next section, we prove a characterization of the (h, r)-generalized circulant matrices. By using this result we are able to prove that the matrix $A = \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_n^{jh}(r)$ is similar to the matrix $B = \text{diag}\left\{\sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_m^j(r^\theta), \cdots, \sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_m^j(r^\theta)\right\}$, the direct sum of k matrices coinciding with $\sum_{j=0}^{\lfloor \frac{n}{h} \rfloor} a_j P_m^j(r^\theta)$. This implies new results on the values of the permanent and also on the determination of the eigenvalues of r-circulant matrices.

In Section 3, we consider the problem of studying the r-circulant matrix $A = a_0 I + a_i P^i(r) + a_j P^j(r)$, and determine a lower bound for the values of the permanent of these matrices.

2 Characterization

Consider a generalized (h, r)-circulant matrix A of order $n = km$, where h, n, k, m are positive integers and $(n, h) = k$, let A_j $(1 \leq j \leq k)$ be the submatrix of A of type $m \times n$ formed by the rows of A

$$1 + (j-1)m, 2 + (j-1)m, \cdots, jm.$$

Theorem 1 *A matrix A of order n is generalized (h, r)-circulant, where $(n, h) = k$ and $n = km$, if and only if it is satisfied the relation*

$$AP^h(r) = P'(r^\theta)A \tag{2.1}$$

where $P^{'}(r^{\theta})$ is direct sum of k matrices coinciding with $P_m(r^{\theta})$ and $\theta = \frac{h}{k}$.

Proof Assume that a matrix A of order n is satisfied (2.1). The matrix $AP^h(r)$ is obtained by multiplying elements of last h columns of A by r and shifting each column h positions to the right cyclicaly. Taking into account the partitioned form of A, we have

$$AP^h(r) = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} P^h(r) = \begin{bmatrix} A_1P^h(r) \\ A_2P^h(r) \\ \vdots \\ A_kP^h(r) \end{bmatrix}. \quad (2.2)$$

Hence $(AP^h(r))_j = A_jP^h(r)$, for $1 \le j \le k$.

Now consider the product $P^{'}(r^{\theta})A$. From the definition of $P^{'}(r^{\theta})$ and the partitioned of A we have

$$\begin{bmatrix} P_m(r^{\theta}) & & & \\ & P_m(r^{\theta}) & & \\ & & \ddots & \\ & & & P_m(r^{\theta}) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_k \end{bmatrix} = \begin{bmatrix} P_m(r^{\theta})A_1 \\ P_m(r^{\theta})A_2 \\ \vdots \\ P_m(r^{\theta})A_k \end{bmatrix} \quad (2.3)$$

From the equalities (2.1), (2.2) and (2.3) it follows $A_jP^h(r) = P_m(r^{\theta})A_j$ $(1 \le j \le k)$, that is to say, the submatrices A_j $(1 \le j \le k)$ are (h, r)-circulant matrices of type $m \times n$. By Definition 1, A is generalized (h, r)-circulant.

Conversely, assume that A is generalized (h, r)-circulant, then each A_j $(1 \le j \le k)$ is (h, r)-circulant. That means $A_jP^h(r) = P_m(r^{\theta})A_j$ $(1 \le j \le k)$. Since $(AP^h(r))_j = A_jP^h(r)$ and $(P^{'}(r^{\theta})A)_j = P^{'}(r^{\theta})A_j$, then $(AP^h(r))_j = (P^{'}(r^{\theta})A)_j$ $(1 \le j \le k)$. It follows that $AP^h(r) = P^{'}(r^{\theta})A$. □

When $k = 1$ a matrix A which satisfies (2.1) turns out to be a (h, r)-circulant matrix; thus (h, r)-generalized circulant matrix turns out to be a generalization of the notion of (h, r)-circulant matrix.

Definition 2 A matrix $Q(r) = [q_{i,j}]$ of order n with $q_{1,1} = 1$ and $q_{1,j} = 0$ $(j = \overline{2, n})$ is said (h, r)-regular, when it can be partitioned into (h, r)-circulant submatrices $Q_j(r)$ of type $m \times n$ $(1 \le j \le k,\ n = km)$, such that every submatrix, distinct from the first, is obtained from the preceding by shifting every column one position to the right.

The definition implies that also $q_{1+(i-1)m,i} = 1$ $(i = \overline{1, k})$.

It is easy to get that $Q^{-1}(r) = \left[Q(\frac{1}{r})\right]^T$, and for an arbitrary matrix A of order n, there holds per $(Q(r)AQ^{-1}(r)) =$ per (A).

Notice that a (h, r)-regular matrix of order n is uniquely determined.

Theorem 2 *Let $A = a_0I + a_1P^h(r) + \cdots + a_tP^{th}(r)$ be a matrix of order n, where $1 < h < n$, $(n, h) = k, n = km$, $t = \left\lfloor \frac{n}{h} \right\rfloor$ and a_i $(1 \le i \le t)$ be real numbers;*

moreover let $Q(r)$ be the (h, r)-regular generalized permutation matrix of order n. Then A is similar to the direct sum of k matrices coinciding with $\sum_{i=0}^{t} a_i P_m^i(r^\theta)$, here $\theta = \frac{h}{k}$.

Proof By Theorem 1, it satisfies the relation

$$Q(r)P^h(r) = P^{'}(r^\theta)Q(r),$$

where $P^{'}(r^\theta)$ is the direct sum of k matrices coinciding with $P_m(r^\theta)$.

Then $P^h(r) = Q^{-1}(r)P^{'}(r^\theta)Q(r)$, it follows

$$A = a_0 I + a_1 Q^{-1}(r)P^{'}(r^\theta)Q(r) + \cdots + a_t Q^{-1}(r)\left[P^{'}(r^\theta)\right]^t Q(r)$$

then

$$Q(r)AQ^{-1}(r) = a_0 I + a_1 P^{'}(r^\theta) + \cdots + a_t \left[P^{'}(r^\theta)\right]^t = \bigoplus \sum_{i=0}^{t} a_i P_m^i(r^\theta).$$

This completes the proof of the theorem. □

In the case of $h = 1$, an immediate consequence is that the r-circulant matrix $A = a_0 + a_1 P^h(r) + \cdots + a_s P^{sh}(r)$ where $s \leq \left\lfloor \frac{n}{h} \right\rfloor$, is similar to the r-circulant matrix $B = a_0 I + a_1 P(r) + \cdots + P^s(r)$. Another consequence is the following

Corollary 1 *Let $A = \sum_{i=0}^{s} a_i P^{ih}(r)$ be a r-circulant matrix of n, where $1 < h < n$, (n,h)=k, n=km, $t = \left\lfloor \frac{n}{h} \right\rfloor$, $\theta = \frac{h}{k}$ and $0 \leq s \leq t$. Then we have*

$$Per\left(\sum_{i=0}^{s} a_i P^{ih}(r)\right) = \left(Per\left(\sum_{i=0}^{s} a_i P_m^i(r^\theta)\right)\right)^k.$$

Proof It is well known that for an arbitrary matrix A of n and a (h, r)-regular generalized permutation matrix $Q(r)$ of order n, the relation $\mathrm{Per}(A) = \mathrm{Per}(QAQ^{-1})$ always holds. Then by Theorem 2, it follows the result. □

Example As an example of (6,2)-regular generalized permutation matrix of order 9, we consider the following matrix

$$Q(2) = \begin{bmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&2&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&2&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&2&0&0&0 \end{bmatrix}.$$

Then consider the 9×9 matrix $A = I + P^6$, we obtain

$$Q(2)AQ^{-1}(2) = \begin{bmatrix} 1 & 1 & 0 & & & & & & \\ 0 & 1 & 1 & & & & & & \\ 4 & 0 & 1 & & & & & & \\ & & & 1 & 1 & 0 & & & \\ & & & 0 & 1 & 1 & & & \\ & & & 4 & 0 & 1 & & & \\ & & & & & & 1 & 1 & 0 \\ & & & & & & 0 & 1 & 1 \\ & & & & & & 4 & 0 & 1 \end{bmatrix}.$$

□

Recall that if A is a r-circulant matrix with first row $[a_0, a_1, \cdots, a_{n-1}]$, the polynomial $p(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$ is said the Hall polynomial of the matrix A.

Denote by $\omega_k = \rho e^{\frac{2\pi i}{n}k}$ $(k = \overline{0, n-1})$, where

$$\rho = \begin{cases} \sqrt[n]{r}, & r > 0 \\ \sqrt[n]{|r|}\left(cos\frac{\pi}{n} + isin\frac{\pi}{n}\right), & r < 0 \end{cases}.$$

Then the eigenvalues of r-circulant matrix A are $p(\omega_0), p(\omega_1), \cdots, p(\omega_{n-1})$.

Denote by $\varpi_k = \varrho e^{\frac{2\pi i}{m}k}$ $(k = \overline{0, m-1})$, where

$$\varrho = \begin{cases} \sqrt[m]{r}, & r > 0 \\ \sqrt[m]{|r|}\left(cos\frac{\pi}{m} + isin\frac{\pi}{m}\right), & r < 0 \end{cases}.$$

By Theorem 2, we can get the following

Corollary 2 *Let the r-circulant matrix $A = a_0 I + a_1 P^h(r) + \cdots + a_s P^{sh}(r)$, and $q(\lambda)$ be the Hall polynomial of the r-circulant matrix $B = a_0 I + a_1 P_m(r^\theta) + \cdots + a_s P_m^s(r^\theta)$, where $1 \le h < n$, $k = (n, h)$, $n = km$, $\theta = \frac{h}{k}$, and $1 \le s \le \left\lfloor \frac{n}{h} \right\rfloor$.*

Then the sets of eigenvalues of A and B coincide for $k = 1$. In the case of $k > 1$, the set of eigenvalues of A is the union of k sets coinciding with $\{q(\varpi_0), q(\varpi_1), \cdots, q(\varpi_{m-1})\}$.

A consequence is that, when $k > 1$, each eigenvalue of A has multiplicity at leat k.

3 Sparse r-circulant matrices

In this section, we consider the r-circulant matrix of order n

$$A = a_0 I + a_i P^i(r) + a_j P^j(r),$$

where i, j, n are positive integers and $1 \le i < j \le n$.

Lemma 2 *Let $Q(r)$ be the (h, r)-regular matrix of order n, where $1 < h < n$, $(n, h) = k > 1$, $n = km$ and $h = kh'$. Then the nonzero element in the n-th column* *of $Q(r)$ is in one of the last m rows; and the nonzero element in the n-th row of $Q^{-1}(r)$ is in one of the last m columns.*

Proof For the last m rows of $Q(r)$, let $i = (k-1)m + q \quad (1 \leq q \leq m)$, then $\sigma(i) = k + (q-1)h$ (σ is defined in following (3.2)). The nonzero element of $Q(r)$ which is in position (i, n) means $n|\sigma(i)$. For the m integers: $1 + h^{'}, 1 + 2h^{'}, \cdots, 1 + (m-1)h^{'}$, by the pigeonhole principle, there exists $q \in \{1, 2, \cdots, m\}$ such that $1 + (q-1)h^{'} \equiv 0(\text{mod } m)$,which is equivalent to $k + (q-1)h \equiv 0(\text{mod } n)$. Since there is one and only one nonzero element in each row and column of $Q(r)$, Then the nonzero element in the n-th column of $Q(r)$ must be in one of the last m rows.

Similarly, we can proof another result of lemma. □

Lemma 3 *Let $Q(r)$ be the (h, r)-regular generalized permutation matrix of order n, where $1 < h < n$, $(n, h) = k > 1$, $n = km$ and $h = kh^{'}$. The matrix $Q(r)P(r)Q^{-1}(r)$ may be partitioned into the following superdiagonal $k \times k$ block form:*

$$B = Q(r) \cdot P(r) \cdot Q^{-1}(r) = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ \Delta_m^s & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{3.1}$$

where $\Delta_m = diag(\underbrace{r^{-\alpha}, \cdots, r^{-\alpha}}_{number:\, m-s}, \underbrace{r^{\beta}, \cdots, r^{\beta}}_{number:\, s}) \times P_m(1)$, s is the inverse of $h^{'}$ modulo m, and $\alpha = \left\lfloor \frac{sh-k+1}{n} \right\rfloor$, $\beta = \left\lfloor \frac{k-1-sh+mh}{n} \right\rfloor + 1$.

Proof Notice that the (h, r)-regular matrix $Q(r) = [q_{i,j}]$ of order n can be written in the form

$$q_{i,j} = \begin{cases} r^{\left\lfloor \frac{\sigma(i)}{n} \right\rfloor}, & \text{if } \sigma(i) \equiv j (\text{mod } n) \text{ and } n \nmid \sigma(i) \\ r^{\left\lfloor \frac{\sigma(i)}{n} \right\rfloor - 1}, & \text{if } \sigma(i) \equiv j (\text{mod } n) \text{ and } n | \sigma(i) \\ 0, & \text{otherwise} \end{cases}.$$

where σ is a permutation of n elements be represented by the following array:

$$\begin{pmatrix} 1 & \cdots & m & m+1 & \cdots & 2m & \cdots & (k-1)m+1 & \cdots & km \\ 1 & \cdots & 1+(m-1)h & 2 & \cdots & 2+(m-1)h & \cdots & k & \cdots & k+(m-1)h \end{pmatrix}. \tag{3.2}$$

Similarly, the circulant matrix $P(1)$ represents a permutation π.

Firstly, let us consider the simple case of $r = 1$, then

$$\sigma\pi\sigma^{-1} = \begin{pmatrix} 1 & 2 & \cdots & (k-1)m & (k-1)m+1 & \cdots & km \\ m+1 & m+2 & \cdots & km & s+1 & \cdots & s \end{pmatrix}, \tag{3.3}$$

where the integers are taken modulo n and $k + 1 \equiv 1 + sh(\text{mod } n)$; then s is the inverse of $h^{'}$ modulo m. As a consequence, the corresponding permutation matrix may be partitioned into blocks of order m in the following form:

$$\begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ (P_m(1))^s & 0 & 0 & \ldots & 0 \end{bmatrix},$$

it is just the result in [4].

Now let us consider the case of $r \neq 1$.

It is evident that two matrices $Q(r)P(r)Q^{-1}(r)$ and $Q(1)P(1)Q^{-1}(1)$ have the same sparse structure and only different in nonzero elements.

For the first $(k-1)m$ rows of the matrix $Q(r)P(r)Q^{-1}(r)$, let $i = (t-1)m + q$, where $1 \leq t \leq k-1$, $1 \leq q \leq m$ and t, q are integers. Using (3.2), $\sigma(i) = t + (q-1)h$. Then by Lemma 2, we can get $n \nmid \sigma(i)$ for $1 \leq i \leq (k-1)m$.

From the structure of (h, r)-regular matrices, the nonzero element in the i-th row of $Q(r)$ is r^{μ}, where $\mu = \left\lfloor \frac{t+(q-1)h}{n} \right\rfloor$.

By (3.3), $j = \sigma\pi\sigma^{-1}(i) = tm + q$, then the nonzero element in the j-th column of $Q^{-1}(r)$ is $r^{-\nu}$, where

$$\nu = \begin{cases} \left\lfloor \frac{t+1+(q-1)h}{n} \right\rfloor, & \text{if } n \nmid (t+1+(q-1)h) \\ \left\lfloor \frac{t+1+(q-1)h}{n} \right\rfloor - 1, & \text{if } n | (t+1+(q-1)h) \end{cases}.$$

If $n \nmid (t+1+(q-1)h)$, then $\left\lfloor \frac{t+(q-1)h}{n} \right\rfloor = \left\lfloor \frac{t+1+(q-1)h}{n} \right\rfloor$; if $n|(t+1+(q-1)h)$, then $\left\lfloor \frac{t+(q-1)h}{n} \right\rfloor = \left\lfloor \frac{t+1+(q-1)h}{n} \right\rfloor - 1$, it follows $\mu = \nu$ and $B(i, \sigma(i)) = r^{\mu} \times 1 \times r^{-\nu} = 1$.

Then we can conclude that all the nonzero elements in the first $(k-1)m$ rows of the matrix $Q(r)P(r)Q^{-1}(r)$ are 1.

Now we turn to consider the nonzero elements in the last m rows of the matrix $Q(r)P(r)Q^{-1}(r)$.

Let $i = (k-1)m + q$ $(1 \leq q \leq m)$. By (3.2), $\sigma(i) = k + (q-1)h$, then the nonzero element in the i-th row of $Q(r)$ is r^{μ}, where

$$\mu = \begin{cases} \left\lfloor \frac{k+(q-1)h}{n} \right\rfloor, & \text{if } n \nmid (k+(q-1)h) \\ \left\lfloor \frac{k+(q-1)h}{n} \right\rfloor - 1, & \text{if } n | (k+(q-1)h) \end{cases}.$$

By (3.3), $j = \sigma\pi\sigma^{-1}(i) = s + q$. Then using Lemma 2 again, we can get $n \nmid (1 + (q-1+s)h)$, it follows $n \nmid (1 + (q-1+s-m)h)$.

Notice that for the matrix $P(r)$, its element in position $(n, 1)$ is r, and other nonzero elements are 1. Then the nonzero element in the j-th column of $Q^{-1}(r)$ is $r^{-\nu}$, where

$$\nu = \begin{cases} \left\lfloor \frac{1+(q-1+s)h}{n} \right\rfloor, & \text{if } q \le m-s \\ \left\lfloor \frac{1+(q-1+s-m)h}{n} \right\rfloor, & \text{if } q > m-s \end{cases}.$$

By $sh^{'} \equiv 1 \pmod{m}$, we can get

$$(1 + (q-1+s)h) - (k + (q-1)h) \equiv 1 (\bmod\, n), \tag{3.4}$$

and

$$(k + (q-1)h) - (1 + (q-1+s-m)h) \equiv n-1 (\bmod\, n). \tag{3.5}$$

Therefore, for $q \le m-s$, if $n \nmid (k+(q-1)h)$, then $B(i, \sigma(i)) = r^{\mu} \times 1 \times r^{-\nu} = r^{-(\nu-\mu)} = r^{-\left(\left\lfloor \frac{(1+(q-1+s)h)}{n} \right\rfloor - \left\lfloor \frac{k+(q-1)h}{n} \right\rfloor\right)}$; if $n|(k+(q-1)h)$, then $B(i, \sigma(i)) = r^{\mu} \times r \times r^{-\nu} = r^{-\left(\left\lfloor \frac{(1+(q-1+s)h)}{n} \right\rfloor - 1 - \left\lfloor \frac{k+(q-1)h}{n} \right\rfloor + 1\right)}$.

By (3.4), we can get $B(i, \sigma(i)) = r^{-\left\lfloor \frac{sh-k+1}{n} \right\rfloor}$.

For $q > m-s$, by (3.5), similarly, we can obtain $B(i, \sigma(i)) = r^{\left\lfloor \frac{k-1-sh+mh}{n} \right\rfloor + 1}$.

At last, set $\alpha = \left\lfloor \frac{sh-k+1}{n} \right\rfloor$, $\beta = \left\lfloor \frac{k-1-sh+mh}{n} \right\rfloor + 1$, we complete the proof of the lemma. □

Example Let $Q(3)$ be the (6, 3)-regular matrix of order 8, and $P(3)$ be the 3-circulant matrix of order 8 with first row $(0, 1, 0, \cdots, 0)$. They are of the following forms

$$Q(3) = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&3&0&0&0 \\ 0&0&9&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&3&0&0 \\ 0&0&0&9&0&0&0&0 \end{bmatrix}, \quad Q^{-1}(3) = \left[Q\left(\frac{1}{3}\right)\right]^T = \begin{bmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&\frac{1}{9}&0&0&0&0 \\ 0&0&0&0&0&0&0&\frac{1}{9} \\ 0&0&\frac{1}{3}&0&0&0&0&0 \\ 0&0&0&0&0&0&\frac{1}{3}&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{bmatrix},$$

and

$$P(3) = \begin{bmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 3&0&0&0&0&0&0&0 \end{bmatrix},$$

then

$$Q(3)\cdot P(3)\cdot Q^{-1}(3)=\begin{bmatrix}0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&\frac{1}{9}&0&0&0&0\\3&0&0&0&0&0&0&0\\0&3&0&0&0&0&0&0\\0&0&3&0&0&0&0&0\end{bmatrix}.$$

□

Now, let us consider the $n \times n$ sparse r-circulant matrix

$$A = a_0 I + a_i P^i(r) + a_j P^j(r),$$

where i, j, n are positive integers such that $1 \le i < j \le n$.

Theorem 3 *If the positive integers i, j, n have a non-trivial common factor, say h, n=hm, $i = hi'$, $j = hj'$ and $\theta = 1$, then A is similar to the direct sum of h matrices coinciding with $a_0 I_m + a_i P_m^{i'}(r^\theta) + a_j P_m^{j'}(r^\theta)$.*

It is the special case of Theorem 2.

Now assume that i, j, n have not a common factor. In particular, as first case, assume that n and i are coprime.

Theorem 4 *Let $A = a_0 I + a_i P^i(r) + a_j P^j(r)$ be a r-circulant matrix of order n, where $1 \le i < j \le n-1$. $(n, i) = 1$, $j = iq + t$, $0 \le t < i$. Then A is similar to $a_0 I + a_i P(r^\theta) + a_j r^{-at} P^{q+st}(r^\theta)$, where s is the inverse of i modulo n, $a = \frac{si-1}{n}$, and $\theta = i$.*

Proof Let $Q(r)$ be the (i, r)-regular matrix of order n. Then $P^i(r) = Q^{-1}(r)P(r^\theta)Q(r)$ and

$$A = a_0 I + a_i Q^{-1}(r)P(r^\theta)Q(r) + a_j \left(Q^{-1}(r)P(r^\theta)Q(r)\right)^q P^t(r),$$

it follows $Q(r)AQ^{-1}(r) = a_0 I + a_i P(r^\theta) + a_j P^q(r^\theta)(Q(r)P(r)Q^{-1}(r))^t$.

Since $P^s(r^\theta) = (Q(r)P^i(r)Q^{-1}(r))^s = r^a Q(r)P(r)Q^{-1}(r)$, then

$$Q(r)AQ^{-1}(r) = a_0 I + a_i P(r^\theta) + a_j r^{-at} P^{q+st}(r^\theta).$$

□

Theorem 5 *Let $A = a_0 I + a_i P^i(r) + a_j P^j(r)$ be a r-circulant matrix of order n, where $1 \le i < j < n$, $(n, i) = k > 1$, $n = km$, $i = ki'$, $j = iq + t, 0 < t < i$ and i, j, n have not a non-trivial common factor.*

Then A is similar to a $k \times k$ block matrix whose elements on the main diagonal coincide with $a_0 I + a_i P_m(r^\theta)$ (here $\theta = i'$), while other elements are 0, but on the diagonals $d(1, t+1)$ and $d(n-t, 1)$, where they coincide with $a_j P_m^q(r^\theta)$ and

$a_j P_m^q(r^\theta) \times \Delta_m^s$, respectively, where s is the inverse of i' modulo m, Δ_m is in the form of which in (3.1).

Proof Denote by $Q(r)$ the (i, r)-regular matrix of order n. By Theorem 1

$$P^i(r) = Q^{-1}(r)P'(r^\theta)Q(r),$$

where $P'(r^\theta)$ is direct sum of k matrices coinciding with $P_m(r^\theta)$. By Lemma 3 we have that

$$\left(Q(r)P(r)Q^{-1}(r)\right)^t = \begin{bmatrix} 0 & \cdots & 0 & I_m & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & I_m \\ \Delta_m^s & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \Delta_m^s & 0 & \cdots & 0 \end{bmatrix},$$

where in the first row the matrix I_m is in position $(1, t+1)$ and in the first column Δ_m^s is in position $(n-t, 1)$. As $\left(P'(r^\theta)\right)^q$ is direct sum of k matrices coinciding with $P_m^q(r^\theta)$, it follows that $\left(P'(r^\theta)\right)^q \cdot \left(Q(r) \cdot P(r) \cdot Q^{-1}(r)\right)^t$ is a block matrix of order k having the same structure as the preceding one, but in which I_m and Δ_m^s are replaced by $P_m^q(r^\theta)$ and $P_m^q(r^\theta) \times \Delta_m^s$, respectively.

Then

$$Q(r)AQ^{-1}(r) = \begin{bmatrix} a_0 I_m + a_i P_m(r^\theta) & \cdots & 0 & a_j P_m^q(r^\theta) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_0 I_m + a_i P_m(r^\theta) & 0 & \cdots & a_j P_m^q(r^\theta) \\ a_j P_m^q(r^\theta) \times \Delta_m^s & \cdots & 0 & a_0 I_m + a_i P_m(r^\theta) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_j P_m^q(r^\theta) \times \Delta_m^s & 0 & \cdots & a_0 I_m + a_i P_m(r^\theta) \end{bmatrix},$$

and the result holds. □

Now we consider a lower bound for the sparse r-circulant matrices.

Lemma 4 *Let $A = a_0 I + a_i P^i(r)$ be a r-circulant matrix of order n, where a_0, a_i are real numbers, $(n, i) = k$ and $n = km,\ i = ki'$. Then per $(A) = (a_0^m + a_i^m r^\theta)^k$, here $\theta = i'$.*

Proof Let $Q(r)$ be the (i, r)-regular matrix of order n. By Theorem 1 we have that $P^i(r) = Q^{-1}(r)P'(r^\theta)Q(r)$, where $P'(r^\theta)$ is a direct sum of k matrices coinciding with $P_m(r^\theta)$. Then $Q(r)AQ^{-1}(r) = \bigoplus(a_0 I_m + a_i P_m(r^\theta))$ and per $(A) =$ (per $(a_0 I_m + a_i P_m(r^\theta)))^k$. Obviously, per $(a_0 I_m + a_i P_m(r^\theta)) = a_0^m + a_i^m r^\theta$. It follows per $(A) = (a_0^m + a_i^m r^\theta)^k$. □

Now let us consider the case that i, j, n have not a non-trivial common factor.

Theorem 6 *Let* $A = a_0 I + a_i P^i(r) + a_j P^j(r)$ *be a* r*-circulant matrix of order* n*, where* a_0, a_i, a_j *are real numbers,* $1 \leq i < j \leq n-1$*,* $(n, i) = k$*,* $n = km$*,* $i = ki'$ *,* $j = kq + t$*,* $0 \leq t < k$ *. Then*

$$per\ (A) \geq (a_0^m + a_i^m r^\theta)^k + a_j^n r^{\theta qk} \times r^{s^2\alpha + s^2\beta - ms\alpha},$$

where $s, \alpha, \beta, \Delta_m, \theta$ *are in the forms of which in Theorem 5.*

Proof By Theorem 5, A is similar to a $k \times k$ block matrix, which coincide with $a_0 I + a_i P_m(r^\theta)$ on the main diagonal, while other elements are 0, but on the diagonals $d(1, t+1)$ and $d(n-t, 1)$, where they coincide with $a_j P_m^q(r^\theta)$ and $a_j P_m^q(r^\theta) \times \Delta_m^s$, respectively. Then

$$\begin{aligned}\text{per } (A) &= \text{per } (Q(r) A Q^{-1}(r)) \geq (\text{per } (a_0 I + a_i P_m(r^\theta)))^k + a_j^n \text{per } (P_m^{qk}(r^\theta)\Delta_m^{st}) \\ &\geq (a_0^m + a_i^m r^\theta)^k + a_j^n r^{\theta qk} \times r^{s^2 t\alpha + s^2 t\beta - mst\alpha} \\ &= (a_0^m + a_i^m r^\theta)^k + a_j^n r^{\theta qk + s^2 t(\alpha+\beta) - mst\alpha}.\end{aligned}$$

This completes the proof of the theorem. □

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