ORIGINAL PAPER

Numerical simulation of anomalous infiltration in porous media

S. Shen · F. Liu · Q. Liu · V. Anh

Abstract Nonlinear time-fractional diffusion equations have been used to describe the liquid infiltration for both subdiffusion and superdiffusion in porous media. In this paper, some problems of anomalous infiltration with a variable-order time-fractional derivative in porous media are considered. The time-fractional Boussinesq equation is also considered. Two computationally efficient implicit numerical schemes for the diffusion and wave-diffusion equations are proposed. Numerical examples are provided to show that the numerical methods are computationally efficient.

Keywords Anomalous infiltration · Porous media · Subdiffusion and superdiffusion · Time-fractional Boussinesq equation · Time variable order fractional derivative

1 Introduction

Anomalous diffusion is known to describe diffusion processes in fractal media more accurately than normal diffusion. Subdiffusion and superdiffusion are both in the anomalous diffusion regime. Anomalous infiltration, i.e. liquid absorption in porous

S. Shen
School of Mathematical Sciences, Huaqiao University, Quanzhou, Fujian, China
e-mail: shensj12@sina.com

F. Liu (✉) · V. Anh
School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434,
Brisbane Qld. 4001, Australia
e-mail: f.liu@qut.edu.au

Q. Liu
School of Mathematical Sciences, Xiamen University, Xiamen 361005, China

media with a nonlinear dependence of the mean-square value of the wetting front position $\langle x^2 \rangle \sim t^\alpha$, is a subject of many investigations during the last decade [5]. Their analysis leads to a diffusion/wave-diffusion equation for the moisture content u with fractional variable order for the time-derivative operator [5]:

$$\frac{\partial^{\alpha(u(x,t))} u}{\partial t^{\alpha(u(x,t))}} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right). \tag{1}$$

When $0 < \alpha(u(x,t)) < 1$, (1) is a fractional diffusion equation, but for $1 < \alpha(u(x,t)) \le 2$, it is a fractional wave-diffusion equation.

The concept of a variable order operator is of much more recent development, which is a new paradigm in science. Samko and Ross [13, 14] directly generalized the Riemann-Liouvile and Marchaud fractional integration and differentiation to the case of variable order, then showed some properties and an inversion formula. Lorenzo and Hartley [9, 10] suggested that a variable order operator is allowed to vary either as a function of the independent variable of integration or differentiation (t), or as a function of some other (perhaps spatial) variable (x). They also explored the relationship between the mathematical concepts and physical processes. Different authors have proposed different definitions of variable order differential operators, each of which has a specific meaning to suit desired goals. Coimbra [3] took the Laplace transform of Caputo's of the fractional derivative as the starting point to suggest a novel definition for the variable order differential operator. Because of its meaningful physical interpretation, Coimbra's definition is better suited for modeling physical problems. Ingman et al. [6, 7] employed the time dependent variable order operator to model the viscoelastic deformation process. Pedro et al. [11] studied the motion of particles suspended in a viscous fluid with drag force using variable order calculus. Coimbra's paper [3] is also the first paper to actually discuss variable order differential equations.

These pioneering publications focused on mathematical properties of possible candidates for a Variable-Order differintegral operator. In particular, Coimbra et al. have done a number of excellent works [3, 4, 12, 16] in using a Variable-Order differintegral operator.

Since the kernel of the variable order operators has a variable exponent, analytical solutions to the resulting equations are more difficult to obtain, hence the development of numerical techniques to handle these equations has received more attention. Coimbra [3] proposed a consistent (first-order accurate) approximation for the solution of variable order differential equations. Soon et al. [16] employed a second-order Runge-Kutta method consisting of an explicit Euler predictor step followed by an implicit Euler corrector step to numerically integrate the variable order differential equation. Sun et al. [17] introduced a classification of the variable-order fractional diffusion models based on the possible physical origins that motivated the variable-order, and employed the Crank-Nicholson scheme to get the diffusion curve of the variable order differential operator model. Zhuang et al. [20] proposed explicit and implicit Euler approximations for the variable-order fractional advection-diffusion equation with a nonlinear source term. Chen et al. [1] proposed two numerical schemes for a variable-order anomalous subdiffusion equation, one with first order

temporal accuracy and fourth order spatial accuracy, the other with second order temporal accuracy and fourth order spatial accuracy. Shen et al. [15] proposed an approximate scheme for the variable order time fractional diffusion equation, via the technique of Fourier analysis to discuss the stability, convergence and solvability.

The object of this paper is to develop numerical methods for simulation of nonlinear fractional diffusion/wave diffusion equations with variable order time-fractional derivative operator.

2 Nonlinear variable order time-fractional diffusion/wave-diffusion equations

We consider the following nonlinear variable order time-fractional diffusion/wave-diffusion equation:

$$\frac{\partial^{\alpha(u(x,t))}u}{\partial t^{\alpha(u(x,t))}} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) + f(u,x,t), \ (x,t) \in \Omega = [0,L] \times [0,T], \quad (2)$$

with the initial and boundary conditions:

$$\begin{cases} u(x,0) = h(x), \ \dfrac{\partial u(x,0)}{\partial t} = g(x), \ 0 \le x \le L, \\ u(0,t) = \phi(t), u(L,t) = \psi(t), \quad 0 \le t \le T. \end{cases} \quad (3)$$

where $0 < \alpha(u(x,t)) \le 2$ and $\alpha(u(x,t)) \ne 1$. We adopt the Caputo definition of variable-order operator which was introduced in [17, 18]:

$$\frac{\partial^{\alpha(u(x,t))}u}{\partial t^{\alpha(u(x,t))}}$$

$$= \begin{cases} \dfrac{1}{\Gamma(1-\alpha(u(x,t)))} \displaystyle\int_{0^+}^{t} (t-\sigma)^{-\alpha(u(x,t))} \dfrac{\partial u(x,\sigma)}{\partial \sigma} d\sigma, & 0 < \alpha(u(x,t)) < 1, \\ \dfrac{1}{\Gamma(2-\alpha(u(x,t)))} \displaystyle\int_{0^+}^{t} (t-\sigma)^{1-\alpha(u(x,t))} \dfrac{\partial^2 u(x,\sigma)}{\partial \sigma^2} d\sigma, & 1 < \alpha(u(x,t)) \le 2. \end{cases}$$

$$(4)$$

We define the function space $G(\Omega) = \left\{ u(x,t) | \dfrac{\partial^4 u(x,t)}{\partial x^4}, \dfrac{\partial^3 u(x,t)}{\partial t^3} \in C(\Omega) \right\}$. In this paper, we suppose the continuous problem (3)-(4) has a smooth solution $u(x,t) \in G(\Omega)$.

As a special case of (2) , when α is a constant and $D(u) = u$, (2) becomes a time-fractional Boussinesq equation:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial}{\partial x}\left(u(x,t)\frac{\partial u(x,t)}{\partial x}\right) + f(x,t), \quad (5)$$

where $0 < \alpha < 1$ or $1 < \alpha < 2$.

The time-fractional Boussinesq equation can been used to study tidal water table fluctuations.

3 Numerical approximation for the nonlinear variable-order time-fractional diffusion equation

When $0 < \alpha(u(x, t)) < 1$, (2) is a nonlinear variable order time-fractional diffusion equation.

We take an equally spaced mesh of M points for the spatial domain $0 \leq x \leq L$, N constant time steps for the temporal domain $0 \leq t \leq T$, and denote the spatial grid points by

$$x_i = ih, i = 0, 1, \ldots, M,$$

and the temporal grid points by

$$t_n = n\tau, n = 0, 1, \ldots, N,$$

where the grid spacing is simply $h = L/M$ in the spatial domain and $\tau = T/N$ in the temporal domain.

At the grid point (x_i, t_n), (2) becomes

$$\frac{\partial^{\alpha(u(x,t))} u}{\partial t^{\alpha(u(x,t))}}\Big|_{(x_i, t_n)} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right)\Big|_{(x_i, t_n)} + f(u, x, t)\big|_{(x_i, t_n)}, \tag{6}$$

where $f(u, x, t)\big|_{(x_i, t_n)} = f_i^n = f(u(x_i, t_n), x_i, t_n)$.

We denote by u_i^n the numerical approximation to $u(x_i, t_n)$, and D_i^n the numerical approximation to $D(u(x_i, t_n))$.

The nonlinear spatial derivative on the right hand side of (6) can be approximated by the following expression [5, 8]:

$$
\begin{aligned}
\frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right)\Big|_{(x_i, t_n)} &\approx \frac{D_{i+\frac{1}{2}}^n \frac{u_{i+1}^n - u_i^n}{h} - D_{i-\frac{1}{2}}^n \frac{u_i^n - u_{i-1}^n}{h}}{h} \\
&= \frac{D_{i+\frac{1}{2}}^n (u_{i+1}^n - u_i^n) - D_{i-\frac{1}{2}}^n (u_i^n - u_{i-1}^n)}{h^2} \\
&\approx \frac{\frac{D_{i+1}^n + D_i^n}{2}(u_{i+1}^n - u_i^n) - \frac{D_i^n + D_{i-1}^n}{2}(u_i^n - u_{i-1}^n)}{h^2}.
\end{aligned}
\tag{7}
$$

By Taylor expansion, we have

$$\frac{D_{i+1}^n + D_i^n}{2} = D_{i+\frac{1}{2}}^n + O(h^2), \quad \frac{D_i^n + D_{i-1}^n}{2} = D_{i-\frac{1}{2}}^n + O(h^2),$$

$$u_{i+1}^n - u_i^n = h\left(\frac{\partial u}{\partial x}\right)_{i+\frac{1}{2}}^n + \frac{1}{24}h^3\left(\frac{\partial^3 u}{\partial x^3}\right)_{i+\frac{1}{2}}^n + O(h^5),$$

$$u_i^n - u_{i-1}^n = h\left(\frac{\partial u}{\partial x}\right)_{i-\frac{1}{2}}^n + \frac{1}{24}h^3\left(\frac{\partial^3 u}{\partial x^3}\right)_{i-\frac{1}{2}}^n + O(h^5).$$

Thus,

$$
\begin{aligned}
\frac{D_{i+1}^n + D_i^n}{2}(u_{i+1}^n - u_i^n) &= h\left(D\frac{\partial u}{\partial x}\right)_{i+\frac{1}{2}}^n + \frac{1}{24}h^3\left(D\frac{\partial^3 u}{\partial x^3}\right)_{i+\frac{1}{2}}^n + \left(\frac{\partial u}{\partial x}\right)_{i+\frac{1}{2}}^n \cdot O(h^3) \\
&\quad + \frac{1}{24}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i+\frac{1}{2}}^n \cdot O(h^5) + D_{i+\frac{1}{2}}^n \cdot O(h^5) + O(h^7),
\end{aligned}
\tag{8}
$$

$$\frac{D_i^n + D_{i-1}^n}{2}(u_i^n - u_{i-1}^n) = h\left(D\frac{\partial u}{\partial x}\right)_{i-\frac{1}{2}}^n + \frac{1}{24}h^3\left(D\frac{\partial^3 u}{\partial x^3}\right)_{i-\frac{1}{2}}^n + \left(\frac{\partial u}{\partial x}\right)_{i-\frac{1}{2}}^n \cdot O(h^3)$$

$$+ \frac{1}{24}\left(\frac{\partial^3 u}{\partial x^3}\right)_{i-\frac{1}{2}}^n \cdot O(h^5) + D_{i-\frac{1}{2}}^n \cdot O(h^5) + O(h^7), \tag{9}$$

Combining (8) and (9) we get

$$\frac{D_{i+1}^n + D_i^n}{2}(u_{i+1}^n - u_i^n) - \frac{D_i^n + D_{i-1}^n}{2}(u_i^n - u_{i-1}^n)$$

$$= h\left[\left(D\frac{\partial u}{\partial x}\right)_{i+\frac{1}{2}}^n - \left(D\frac{\partial u}{\partial x}\right)_{i-\frac{1}{2}}^n\right] + \frac{1}{24}h^3\left[\left(D\frac{\partial^3 u}{\partial x^3}\right)_{i+\frac{1}{2}}^n - \left(D\frac{\partial^3 u}{\partial x^3}\right)_{i-\frac{1}{2}}^n\right]$$

$$+ \left[\left(\frac{\partial u}{\partial x}\right)_{i+\frac{1}{2}}^n - \left(\frac{\partial u}{\partial x}\right)_{i-\frac{1}{2}}^n\right] \cdot O(h^3) + O(h^5)$$

$$= h\left[h\frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right)_i^n + O(h^3)\right] + \frac{1}{24}h^3\left[h\frac{\partial}{\partial x}\left(D\frac{\partial^3 u}{\partial x^3}\right)_i^n + O(h^3)\right]$$

$$+ \left[h\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)_i^n + O(h^3)\right] \cdot O(h^3) + O(h^5)$$

$$= h^2\frac{\partial}{\partial x}\left(D\frac{\partial u}{\partial x}\right)_i^n + O(h^4).$$

Hence

$$\frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right)\Big|_{(x_i,t_n)}$$

$$= \frac{\frac{D_{i+1}^n + D_i^n}{2}(u_{i+1}^n - u_i^n) - \frac{D_i^n + D_{i-1}^n}{2}(u_i^n - u_{i-1}^n)}{h^2} + O(h^2). \tag{10}$$

When $0 < \alpha(u(x,t)) < 1$, the variable order of the time-derivative operator on the left hand side of (6) can be approximated by the following expression [15]:

$$\frac{\partial^{\alpha(u(x,t))}u}{\partial t^{\alpha(u(x,t))}}\Big|_{(x_i,t_n)} = \frac{\tau^{-\alpha_i^n}}{\Gamma(2-\alpha_i^n)}\sum_{k=0}^{n-1}a_{n-k-1,i}^n(u_i^{k+1} - u_i^k) + O(\tau), \tag{11}$$

where $\alpha_i^n = \alpha(u(x_i, t_n))$ and $a_{j,i}^n = (j+1)^{1-\alpha_i^n} - j^{1-\alpha_i^n}$ $(j = 0, 1, \cdots, n-1)$.

From (10) and (11), we obtain the following implicit approximation scheme to (2) for $0 < \alpha(u(x,t)) < 1$:

$$\frac{\tau^{-\alpha_i^n}}{\Gamma(2-\alpha_i^n)}\sum_{k=0}^{n-1}a_{n-k-1,i}^n(u_i^{k+1} - u_i^k)$$

$$= \frac{\frac{D_{i+1}^n + D_i^n}{2}(u_{i+1}^n - u_i^n) - \frac{D_i^n + D_{i-1}^n}{2}(u_i^n - u_{i-1}^n)}{h^2} + f_i^n, \tag{12}$$

$$i = 1, 2, \cdots, M-1; \quad n = 1, 2, \cdots, N,$$

with initial and boundary conditions

$$\begin{cases} u_i^0 = h(x_i), \\ u_0^n = \phi(t_n), \ u_M^n = \psi(t_n). \end{cases} \tag{13}$$

The accuracy of the scheme (12) to approximate (2) is $O(h^2 + \tau)$.

4 Numerical approximation for the nonlinear variable-order time-fractional wave-diffusion equation

When $1 < \alpha(u(x,t)) \leq 2$, (2) is a nonlinear variable-order time-fractional wave-diffusion equation. The variable order of the time-derivative operator on the right hand side of (2) can be approximated by the following expression [2, 19]:

$$\begin{aligned} &\frac{\partial^{\alpha(u(x,t))} u}{\partial t^{\alpha(u(x,t))}} \Big|_{(x_i,t_n)} \\ &\approx \frac{1}{\tau \Gamma(2-\alpha_i^n)} \left[b_{0,i}^n \frac{\nabla_t u_i^n}{\tau} - \sum_{k=1}^{n-1} (b_{n-k-1,i}^n - b_{n-k,i}^n) \frac{\nabla_t u_i^k}{\tau} - b_{n-1,i}^n g_i \right], \end{aligned} \tag{14}$$

where $\nabla_t u_i^n = u_i^n - u_i^{n-1}$, $b_{j,i}^n = \frac{\tau^{2-\alpha_i^n}}{2-\alpha_i^n} \left[(j+1)^{2-\alpha_i^n} - j^{2-\alpha_i^n} \right]$ and $g_i = g(x_i)$.

When α is a constant, the accuracy of the approximation (14) is $O(\tau^{3-\alpha})$, which was established in [2, 19]. Hence, the accuracy of (14) is at least $O(\tau)$ for $1 < \alpha(u(x,t)) \leq 2$.

The approximation for the nonlinear spatial derivative on the right hand side of (2) is the same as (10).

From (10) and (14), we obtain the following implicit approximation scheme to (2) for $1 < \alpha(u(x,t)) \leq 2$:

$$\begin{aligned} &\frac{1}{\tau \Gamma(2-\alpha_i^n)} \left[b_{0,i}^n \frac{\nabla_t u_i^n}{\tau} - \sum_{k=1}^{n-1} (b_{n-k-1,i}^n - b_{n-k,i}^n) \frac{\nabla_t u_i^k}{\tau} - b_{n-1,i}^n g_i \right] \\ &= \frac{\frac{D_{i+1}^n + D_i^n}{2}(u_{i+1}^n - u_i^n) - \frac{D_i^n + D_{i-1}^n}{2}(u_i^n - u_{i-1}^n)}{h^2} + f_i^n, \\ &i = 1, 2, \cdots, M-1; \ n = 1, 2, \cdots, N, \end{aligned} \tag{15}$$

with initial and boundary conditions

$$\begin{cases} u_i^0 = h(x_i), \ \dfrac{\partial u(x,t)}{\partial t}\Big|_i^0 = g(x_i), \\ u_0^n = \phi(t_n), \ u_M^n = \psi(t_n). \end{cases} \tag{16}$$

The accuracy of the scheme (15) to approximate (2) is at least $O(h^2 + \tau)$.

5 Numerical approximation for the time-fractional Boussinesq equation

From (12), we obtain the following implicit approximation scheme to the time-fractional Boussinesq equation for $0 < \alpha < 1$:

$$
\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} a_{n-k-1}(u_i^{k+1} - u_i^k)
$$
$$
= \frac{\dfrac{u_{i+1}^n + u_i^n}{2}(u_{i+1}^n - u_i^n) - \dfrac{u_i^n + u_{i-1}^n}{2}(u_i^n - u_{i-1}^n)}{h^2} + f_i^n, \tag{17}
$$
$$
i = 1, 2, \cdots, M-1; \quad n = 1, 2, \cdots, N,
$$

with initial and boundary conditions

$$
\begin{cases} u_i^0 = h(x_i), \\ u_0^n = \phi(t_n), \ u_M^n = \psi(t_n), \end{cases} \tag{18}
$$

where $a_j = (j+1)^{1-\alpha} - j^{1-\alpha}$ $(j = 0, 1, \cdots, n-1)$ and $f_i^n = f(x_i, t_n)$.

From (15), we obtain the following implicit approximation scheme to the time-fractional Boussinesq equation for $1 < \alpha \leq 2$:

$$
\frac{1}{\tau\Gamma(2-\alpha)} \left[b_0 \frac{\nabla_t u_i^n}{\tau} - \sum_{k=1}^{n-1} (b_{n-k-1} - b_{n-k}) \frac{\nabla_t u_i^k}{\tau} - b_{n-1} g_i \right]
$$
$$
= \frac{\dfrac{u_{i+1}^n + u_i^n}{2}(u_{i+1}^n - u_i^n) - \dfrac{u_i^n + u_{i-1}^n}{2}(u_i^n - u_{i-1}^n)}{h^2} + f_i^n, \tag{19}
$$
$$
i = 1, 2, \cdots, M-1; \quad n = 1, 2, \cdots, N,
$$

with initial and boundary conditions

$$
\begin{cases} u_i^0 = h(x_i), \ \left.\dfrac{\partial u(x,t)}{\partial t}\right|_i^0 = g(x_i), \\ u_0^n = \phi(t_n), \ u_M^n = \psi(t_n), \end{cases} \tag{20}
$$

where $b_j = \dfrac{\tau^{2-\alpha}}{2-\alpha} \left[(j+1)^{2-\alpha} - j^{2-\alpha} \right]$ $(j = 0, 1, \cdots, n-1)$.

6 Numerical results

In this section, we carry out a number of numerical examples to verify the theoretical results. Here the Jacobi iteration technique is used for solving the implicit difference schemes (17), (19) and (15).

Table 1 The error of the numerical solution from the exact solution when $t = 1$, $h = 1/16$, $\tau = 1/256$

Space (x_i)	Numerical solution	Exact solution	Error
0.125	0.38441101	0.38268343	0.00172758
0.250	0.71092937	0.70710677	0.00382260
0.375	0.92910177	0.92387952	0.00522224
0.500	1.00571372	1.00000000	0.00571372
0.625	0.92910179	0.92387955	0.00522224
0.750	0.71092940	0.70710681	0.00382260
0.875	0.38441106	0.38268348	0.00172758

Example 1 We consider the following time-fractional Boussinesq equation for $0 < \alpha < 1$:

$$
\begin{cases}
\dfrac{\partial^{0.7} u(x,t)}{\partial t^{0.7}} = \dfrac{\partial}{\partial x}\left\{ u(x,t)\left[\dfrac{\partial u(x,t)}{\partial x}\right]\right\} + f(x,t), & (x,t) \in \Omega = [0,1] \times [0,1], \\
u(x,0) = 0, & 0 \le x \le 1, \\
u(0,t) = u(1,t) = 0, & 0 \le t \le 1,
\end{cases}
\tag{21}
$$

where $f(x,t) = \sin(\pi x)\dfrac{t^{0.3}}{\Gamma(1.3)} - t^2\pi^2\cos(2\pi x)$.

The exact solution is $u(x,t) = t\sin(\pi x)$. Here, we use the implicit numerical scheme (17).

A comparison of the numerical solution and the exact solution when $t = 1$ for Example 1 is provided in Table 1. Here, we take $h = 1/16$ and $\tau = 1/256$. It shows that the numerical solution is in good agreement with the exact solution.

Table 2 shows that, when the number of spatial subintervals / time steps is doubled (i.e., step sizes are halved), the error behavior versus grid size reduction at $t = 1$ for Example 1 is observed, as expected with the order $O(h^2 + \tau)$ of the method, where the convergence order is calculated by the following formula:

$$
\text{Cvge. Order} = \log_{\frac{h_1}{h_2}} \frac{e_1}{e_2}.
$$

Table 2 Maximum error behavior versus grid size reduction at $t = 1$ when $h^2 = \tau$

h	τ	Maximum error	Cvge. Order of space	Cvge. Order of time
1/4	1/16	0.09855078		
1/6	1/36	0.04187286	2.111	1.055
1/8	1/64	0.02319386	2.053	1.027
1/12	1/144	0.01019669	2.027	1.013
1/16	1/256	0.00571372	2.013	1.007

Table 3 The error of the numerical solution from the exact solution when $t = 1$, $h = 1/16$, $\tau = 1/256$

Space (x_i)	Numerical solution	Exact solution	Error
0.125	0.44131098	0.43750000	0.00381098
0.250	0.75334297	0.75000000	0.00334297
0.375	0.94055349	0.93750000	0.00305349
0.500	1.00295655	1.00000000	0.00295655
0.625	0.94055349	0.93750000	0.00305349
0.750	0.75334297	0.75000000	0.00334297
0.875	0.44131098	0.43750000	0.00381098

Example 2 We consider the following time-fractional Boussinesq equation for $1 < \alpha \leq 2$:

$$\begin{cases} \dfrac{\partial^{1.8} u(x,t)}{\partial t^{1.8}} = \dfrac{\partial}{\partial x}\left\{ u(x,t)\left[\dfrac{\partial u(x,t)}{\partial x}\right]\right\} + f(x,t), & (x,t) \in \Omega = [0,1] \times [0,1], \\ u(x,0) = 0, \quad \dfrac{\partial u(x,0)}{\partial t} = 0, & 0 \leq x \leq 1, \\ u(0,t) = u(1,t) = 0, & 0 \leq t \leq 1, \end{cases}$$

$$(22)$$

where $f(x,t) = 8(x - x^2)\dfrac{t^{0.2}}{\Gamma(1.2)} - 16t^4(1 - 6x + 6x^2)$.

The exact solution is $u(x,t) = 4t^2(x - x^2)$. Here, we use the implicit numerical scheme (19).

A comparison of the numerical solution and the exact solution when $t = 1$ for Example 2 is provided in Table 3. Here, we take $h = 1/16$ and $\tau = 1/256$. It shows that the numerical solution is in good agreement with the exact solution.

Table 4 shows that, when the number of spatial subintervals / time steps is doubled (i.e., step sizes are halved), the error behavior versus grid size reduction at $t = 1$ for Example 2 is observed, as expected with the order $O(h^2 + \tau)$ of the method.

Table 4 Maximum error behavior versus grid size reduction at $t = 1$ when $h^2 = \tau$

h	τ	Maximum error	Cvge. Order of space	Cvge. Order of time
1/4	1/16	0.07360206		
1/6	1/36	0.03196249	2.057	1.029
1/8	1/64	0.01796630	2.002	1.001
1/12	1/144	0.00784773	2.043	1.021
1/16	1/256	0.00408902	2.266	1.133

Example 3 We consider the following nonlinear variable order time-fractional diffusion/wave-diffusion equation:

$$
\begin{cases}
\dfrac{\partial^{\alpha(u(x,t))} u}{\partial t^{\alpha(u(x,t))}} = \dfrac{\partial}{\partial x}\left[D(u)\left(\dfrac{\partial u}{\partial x}\right)\right] + f(u,x,t), \ (x,t) \in \Omega = [0,2] \times [0,1], \\
\quad u(x,0) = \cos\left(\dfrac{\pi}{4}x\right), \ \dfrac{\partial u(x,0)}{\partial t} = 0, \qquad\qquad 0 \le x \le 2, \\
\qquad u(0,t) = 1, \ \ u(2,t) = 0, \qquad\qquad\qquad 0 \le t \le 1,
\end{cases}
\tag{23}
$$

where

$$
f(u,x,t) = 2x(2-x)\frac{t^{2-\alpha(u(x,t))}}{\Gamma(3-\alpha(u(x,t)))}
$$
$$
-e^u\left\{ 4(1-x)^2 t^4 - \left[\pi(1-x)\sin\left(\frac{\pi}{4}x\right) + 2\right]t^2 + \frac{\pi^2}{16}\left[\sin^2\left(\frac{\pi}{4}x\right) - \cos\left(\frac{\pi}{4}x\right)\right]\right\},
$$
$$
D(u) = e^u, \text{ and}
$$

$$
\alpha(u(x,t)) = \begin{cases}
\dfrac{2 + \sin(xt)}{4}, & \text{when } u < 1.4; \\
1.5 + 0.4\sin(0.5\pi xt), & \text{when } u \ge 1.4.
\end{cases}
$$

The exact solution is $u(x,t) = \cos\left(\dfrac{\pi}{4}x\right) + t^2 x(2-x)$.

When $\alpha(u(x,t)) = \dfrac{2 + \sin(xt)}{4}$, we use the implicit numerical scheme (12), and when $\alpha(u(x,t)) = 1.5 + 0.4\sin(0.5\pi xt)$, we use the implicit numerical scheme (15).

A comparison of the numerical solution and the exact solution when $t = 1$ for Example 3 is provided in Table 5. Here, we take $h = 1/16$ and $\tau = 1/256$. It shows that the numerical solution is in good agreement with the exact solution.

Table 6 shows that, when the number of spatial subintervals / time steps is doubled (i.e., step sizes are halved), the error behavior versus grid size reduction at $t = 1$ for Example 3 is observed, as expected with the order $O(h^2 + \tau)$ of the method.

Table 5 The error of the numerical solution from the exact solution when $t = 1$, $h = 1/16$, $\tau = 1/256$

Space (x_i)	Numerical solution	Exact solution	Error
0.25	1.41851485	1.41828528	0.00022957
0.50	1.67448931	1.67387954	0.00060977
0.75	1.76968925	1.76896962	0.00071963
1.00	1.70756071	1.70710679	0.00045392
1.25	1.49299964	1.49307025	0.00007060
1.50	1.13210223	1.13268345	0.00058122
1.75	0.63190752	0.63259035	0.00068282

Table 6 Maximum error behavior versus grid size reduction at $t = 1$ when $h^2 = \tau$

h	τ	Maximum error	Cvge. Order of space	Cvge. Order of time
1/2	1/4	0.05182303		
1/4	1/16	0.01300180	1.995	0.997
1/8	1/64	0.00301043	2.111	1.055
1/16	1/256	0.00072752	2.049	1.024

7 Conclusions

This paper developed an implicit numerical approach for numerical simulation of nonlinear variable-order time-fractional diffusion/wave-diffusion equations. These equations are useful to describe liquid infiltration for both subdiffusion and superdiffusion in porous media. As a special case, a time-fractional Boussinesq equation has also been considered. Three numerical examples demonstrated the accuracy and efficiency of the implicit numerical approach.

References

1. Chen, C., Liu, F., Anh, V., Turner, I.: Numerical schemes with high spatial accuracy for a variable-order anomalous subdiffusion equation, SIAM. J. Sci. Commun. **32**(4), 1740–1760 (2010)
2. Chen, J., Liu, F., Anh, V., Shen, S., Liu, Q., Liao, C.: The analytical solution and numerical solution of the fractional diffusion-wave equation with damping. Appl. Math. Comput. **219**, 1737–1748 (2012)
3. Coimbra, C.F.M.: Mechanics with variable-order differential operators. Annalen der Physik **12**(11-12), 692–703 (2003)
4. Diaz, G., Coimbra, C.F.M.: Nonlinear dynamics and control of a variable order oscillator with application to the van der Pol equation. Nonlinear Dyn. **56**(1-2), 145–157 (2009)
5. Gerasimov, D.N., Kondratieva, V.A., Sinkevich, O.A.: An anomalous non-self-similar infiltration and fractional diffusion equation. Physica D: Nonlinear Phenom. **239**(16), 1593–1597 (2010)
6. Ingman, D., Suzdalnitsky, J., Zeifman, M.: Constitutive dynamic-order model for nonlinear contact phenomena. J. Appl. Mech. **67**(2), 383–390 (2000)
7. Ingman, D., Suzdalnitsky, J.: Control of damping oscilations by fractional differential operator with time-dependent order. Comput. Methods Appl. Mech. Eng. **193**(52), 5585–5595 (2004)
8. Li, C., Deng, W., Wu, Y.: Numerical analysis and physical simulations for the time fractional radial diffusion equation. Comput Math. Appl. **62**(3), 1024–1037 (2011)
9. Lorenzo, C.F., Hartley, T.T.: Initialization, conceptualization, and application in the generalized (fractional) calculus. Crit. Rev. Biomed. Eng. **35**(6), 447–553 (2007)
10. Lorenzo, C.F., Hartley, T.T.: Variable order and distributed order fractional operators. Nonlinear Dyn. **29**(1-4), 57–98 (2002)
11. Pedro, H.T.C., Kobayashi, M.H., Pereira, J.M.C., Coimbra, C.F.M.: Variable order modeling of diffusive-convective effects on the oscillatory flow past a sphere. J. Vib. Control. **14**(9-10), 1659–1672 (2008)
12. Ramirez, L.E.S., Coimbra, C.F.M.: A variable order constitutive relation for viscoelasticity. Ann. Phys **16**(7-8), 543–552 (2007)
13. Samko, S.G., Ross, B.: Intergation and differentiation to a variable fractional order. Integr. Transforms and Spec. Funct. **1**(4), 277–300 (1993)

14. Samko, S.G.: Fractional integration and differentiation of variable order. Anal. Math. **21**(3), 213–236 (1995)
15. Shen, S., Liu, F., Chen, J., Turner, I., Anh, V.: Numerical techniques for the variable order time fractional diffusion equation. Appl. Math. Comput. **218**(22), 10861–10870 (2012)
16. Soon, C.M., Coimbra, C.F.M., Kobayashi, M.H.: The variable viscoelasticity oscillator. Annalen der Physik **14**(6), 378–389 (2005)
17. Sun, H., Chen, W., Chen, Y.: Variable-order fractional differential operators in anomalous diffusion modeling. Physica A **388**, 4586–4592 (2009)
18. Sun, H., Chen, W., Sheng, H., Chen, Y.: On mean square displacement behaviors of anomalous diffusions with variable and random orders. Phys. Lett. A **374**(7), 906–910 (2010)
19. Sun, Z., Wu, X.: A fully discrete difference scheme for a diffusion-wave system. Appl. Numer. Math. **56**(2), 193–209 (2006)
20. Zhuang, P., Liu, F., Anh, V., Turner, I.: Numerical methods for the variable-order fractonal advection-diffusion equation with a nonlinear source term, SIAM. J. Numer. Anal. **47**(3), 1760–1781 (2009)