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ORIGINAL PAPER

A meshless method based on the moving least squares (MLS) approximation for the numerical solution of two-dimensional nonlinear integral equations of the second kind on non-rectangular domains

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Abstract This paper investigates a numerical method for solving two-dimensional nonlinear Fredholm integral equations of the second kind on non-rectangular domains. The scheme utilizes the shape functions of the moving least squares (MLS) approximation constructed on scattered points as a basis in the discrete collocation method. The MLS methodology is an effective technique for approximating unknown functions which involves a locally weighted least square polynomial fitting. The proposed method is meshless, since it does not need any background mesh or cell structures and so it is independent of the geometry of the domain. The scheme reduces the solution of two-dimensional nonlinear integral equations to the solution of nonlinear systems of algebraic equations. The error analysis of the proposed method is provided. The efficiency and accuracy of the new technique are illustrated by several numerical examples.

Keywords Nonlinear integral equation · Two-dimensional integral equation · Moving least squares (MLS) approximation · Meshless method · Non-rectangular domain · Error analysis

Mathematics Subject Classifications (2010) 47H30 · 65D10 · 41A25 · 41A63

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1 Introduction

Two dimensional integral equations, linear and nonlinear, have significant applications in various fields of science and engineering, such as:

- It is usually required to solve two-dimensional integral equations in the calculation of plasma physics [24].
- The image deblurring problem and its regularization can be modeled by a two-dimensional integral equation [30, 44].
- Two-dimensional integral equations appear in axisymmetric contact problems for bodies with complex rheology [37].
- The formulation of a problem in diffraction theory has led us to consider a two-dimensional integral equation [43].
- A two-dimensional integral equation describes the electrochemical behavior of an inlaid microband electrode (or an array of parallel microbands) for the case of equal diffusion coefficients [38].

  Two-dimensional integral equations are also deduced from some mixed boundary value problems arising in various branches of applied science such as solid and fluid mechanics, electrostatics, heat transfer, diffraction and scattering of waves, etc [8, 15, 16, 35].

These types of integral equations, specially in the nonlinear case, are usually difficult to solve analytically, therefore, it is required to obtain their approximate solutions. A few number of methods have been proposed for the numerical solution of two-dimensional nonlinear integral equations. However, primary works in this area have been done in the last two decades. The projection, iterated projection and Nyström methods [8, 9, 32, 33, 36] are the commonly used approaches for the numerical solutions of two-dimensional nonlinear integral equations. In [28], the iterated discrete Galerkin method was applied to the solution of two-dimensional nonlinear Fredholm integral equations. Han and Wang [26] studied the iterated collocation method to solve two-dimensional nonlinear Volterra integral equations. The numerical solution of two-dimensional nonlinear integral equations, based on the use of spline functions in the Galerkin method, was investigated in [18]. The convergence analysis of the projection and iterated projection methods to solve nonlinear integral equations in several-dimensional spaces is well documented in the literature [7, 9, 50].

The Nyström method was used to solve two-dimensional nonlinear Fredholm integral equations in [7, 27]. Also, the existence of asymptotic error expansion of the Nyström solution for these types of integral equations has been analyzed in [27].

Here, we would like to review some of the most recent works on the square domains. Two-dimensional rationalized Haar (RH) functions are applied to the numerical solution of nonlinear second kind two-dimensional integral equations in [11]. Tari et al. [49] have developed the two-dimensional differential transform (TDDT) method for solving two-dimensional linear and nonlinear Volterra integral equations. Authors of [1] introduced a degenerate method (DM) to obtain the numerical solution of nonlinear two-dimensional Fredholm integral equations with a continuous kernel. In [13], the Gauss product quadrature rules and the collocation method are applied to solve the second- kind nonlinear two-dimensional Fredholm integral equations. A numerical method, based on interpolation by Gaussian radial basis functions (RBFs), for the solution of two-dimensional integral equations is introduced in [2]. Furthermore, a meshless method is presented for solving two-dimensional Fredholm integral equations on non-rectangular domains using RBFs in [5, 6].

To apply the projection or Nyström methods for solving two-dimensional integral equations on non-rectangular regions, we must divide the solution regions into non-overlapping triangular fragments. Also, we need to discretize integral equations based on the approximation and numerical integration over the segments [8, 29]. Therefore, to get rid of the triangulation and mesh refinement, we can use methods based upon scattered data approximations (meshless methods) that approximate a function without any mesh generation on a domain. In recent years, meshless methods have been used in many different areas ranging from artificial intelligence, computer graphics, image processing, neural networks, signal processing and sampling theory, etc [12, 23].

Among meshless methods, the MLS approximation has significant importance applications in different problems of the computational mathematics such as partial differential equations and ordinary differential equations [14, 19, 41, 45, 48]. The MLS consists of a local weighted least square fitting, valid on a small neighborhood of a point and only based on the information provided by its N closet points [40, 46]. The main advantage of using the MLS approximation is that it sets up and solves many small systems, instead of a single, but large, system [23, 52]. Authors of [20] investigated the numerical solution of nonlinear one-dimensional integro-differential equations utilizing the MLS method. Also a mehless method based on the MLS technique is introduced for solving linear one- and two-dimensional Fredholm and Volterra integral equations with error analysis in [39] which applies the MLS expansion for approximating the unknown function.

The main purpose of this article is to present a numerical method based on the MLS approximation for solving two-dimensional nonlinear Fredholm integral equations of the second kind, namely

$$u(x,t) - \lambda \int_D K(x,t,y,s,u(y,s))\mathrm{d}s\mathrm{d}y = f(x,t), \quad (x,t) \in D, \tag{1}$$

where the right hand side function f and the kernel function K are given, $u(x,t)$ is the unknown function to be determined, λ is a constant and $D \subset \mathbb{R}^2$ is a non-rectangular bounded region.

The method utilizes the interpolating extension of the MLS shape functions constructed on the scattered data as a projection operator in the discrete collocation method instead of the commonly extension of the MLS. The scheme approximates its double integrals on the domain D using the composite Gauss-Legendre quadrature formula when the kernel function K is well-behaved and the composite non-uniform Gauss-Legendre quadrature formula when the kernel function K is weakly singular. Therefore, the solution of the nonlinear Fredholm integral equation (1) reduces to the solution of a system of nonlinear algebraic equations. The proposed scheme is a meshless method, which requires no domain elements for interpolation or approximation. Here the distribution of nodes could be selected regularly or randomly in the analyzed domain so the method does not depend to the geometry of the domain. The scheme applies a locally weighted least square fitting unlike the presented method in [5] that utilizes globally RBFs which result in a full and ill-conditioned resultant system for the solution of these types of integral equations. Also, we obtain the error bound and the rate of convergence for the proposed method.

The outline of the paper is as follows: In Section 2, we review some basic formulations and properties of the MLS approximation. In Section 3, we present a computational method for solving (1) utilizing the MLS approximation and also apply this scheme for solving two-dimensional weakly singular integral equations. In Section 4, we provide the error analysis for the method. Numerical examples are given in Section 5. Finally, we conclude the article in Section 6.

2 The MLS approximation

The moving least squares (MLS) approximation as a general case of Shepard's method [47] has been introduced by Lancaster and Salkauskas [34]. The MLS method is recognized as a meshless method because this approximation is based on a set of scattered points instead of domain elements for interpolation or approximation. Given data values of the function $u(\mathbf{x})$ at certain data sites $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subset D \subset \mathbb{R}^d$. The idea of the MLS method is to approximate $u(\mathbf{x})$ for every point $\mathbf{x} \in D$ in a weighted least square sense. For $\mathbf{x} \in D \subset \mathbb{R}^d$, the value $s_{u,X}(\mathbf{x})$ of the MLS approximation is given by the solution of

$$\min \left\{ \sum_{i=1}^{N} [u(\mathbf{x}_i) - p(\mathbf{x}_i)]^2 w(\mathbf{x}, \mathbf{x}_i) : p \in \Pi_q \left(\mathbb{R}^d\right) \right\}, \tag{2}$$

where $w : D \times D \to [0, \infty]$ is a continuous weight function and $\Pi_q(\mathbb{R}^d)$ is the linear space of polynomials of total degree less than or equal to q in d variables and let $\{p_0(\mathbf{x}), ..., p_Q(\mathbf{x})\}$ be a complete monomial basis of $\Pi_q(\mathbb{R}^d)$. We are mainly interested in local continuous weight function w which gets smaller as its arguments move away from each other. Ideally, w vanishes for arguments $\mathbf{x}, \mathbf{y} \in D$ with $\|\mathbf{x}-\mathbf{y}\|_2$ greater than a certain threshold. Therefore, we can assume that

$$w(\mathbf{x}, \mathbf{y}) = \Phi_\delta(\mathbf{x} - \mathbf{y}) = \phi\left(\frac{\|\mathbf{x} - \mathbf{y}\|_2}{\delta}\right), \quad \delta > 0, \tag{3}$$

where Φ is a radial function, meaning that $\Phi(\mathbf{x}) = \phi(\|\mathbf{x}\|_2)$, $\mathbf{x} \in \mathbb{R}^d$, in witch ϕ is a univariate and nonnegative function, $\phi : [0, \infty) \to \mathbb{R}$, with the property $\phi(r) = 0$ when $r \geq 1$ [53].

In the following, we want to consider a general algorithm of the MLS approximation, but prior to that we present the following definition [17, 52].

Definition 2.1 We call a set of points $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subset \mathbb{R}^d$ q-unisolvent if the only polynomial of total degree at most q, interpolating zero data on X is the zero polynomial.

Suppose that the set $\{\mathbf{x}_1, ..., \mathbf{x}_N\} \subset D$ is q-unisolvent. In this situation, the problem (2) is uniquely solvable and the solution $s_{u,X}(\mathbf{x})$ can be represented as [52]

$$s_{u,X}(\mathbf{x}) = \sum_{i=1}^{N} u\left(\mathbf{x}_i\right) \psi_i(\mathbf{x}) = U^t.\Psi(\mathbf{x}), \tag{4}$$

where

$$\Psi(\mathbf{x}) = [\psi_1(\mathbf{x}), ..., \psi_N(\mathbf{x})]^t, \quad U = [u\left(\mathbf{x}_1\right), ..., u\left(\mathbf{x}_N\right)]^t.$$

We call the basis functions $\psi_i(\mathbf{x})$ as shape functions of the MLS method, corresponding to the nodal point $\mathbf{x}_i$. If $w\left(\mathbf{x}, \mathbf{x}_i\right) \in C^r(D),\ i = 1, ..., N$, then $\psi_i(\mathbf{x}) \in C^r(D)$ and so $s_{u,X}(\mathbf{x}) \in C^r(D)$ [34].

Now, to determine $\Psi(\mathbf{x})$, we define the matrices $\mathbf{P}$ and $\mathbf{W}(\mathbf{x})$ as

$$\mathbf{P}^t = \left[\mathbf{p}^t\left(\mathbf{x}_1\right), \mathbf{p}^t\left(\mathbf{x}_2\right), ..., \mathbf{p}^t\left(\mathbf{x}_N\right)\right]_{(Q+1)\times N}, \quad \mathbf{W}(\mathbf{x}) = \begin{bmatrix} w\left(\mathbf{x}, \mathbf{x}_1\right) & \cdots & 0 \\ \ldots & \ddots & \ldots \\ 0 & \cdots & w\left(\mathbf{x}, \mathbf{x}_N\right) \end{bmatrix}_{N\times N},$$

where $\mathbf{p}^t(\mathbf{x}) = [p_0(\mathbf{x}), ..., p_Q(\mathbf{x})]$. Therefore, we have

$$\Psi^t(\mathbf{x}) = \mathbf{p}^t(\mathbf{x})A^{-1}(\mathbf{x})B(\mathbf{x}), \tag{5}$$

or

$$\psi_i(\mathbf{x}) = \sum_{k=0}^{Q} \mathbf{p}_k(\mathbf{x})[A^{-1}(\mathbf{x})B(\mathbf{x})]_{ki}, \tag{6}$$

where the matrices $A(x)$ and $B(x)$ are defined by

$$A(\mathbf{x}) = \mathbf{P}^t\mathbf{W}\mathbf{P} = B(\mathbf{x})\mathbf{P} = \sum_{j=1}^{N} w(\mathbf{x}, \mathbf{x}_j)\mathbf{p}(\mathbf{x}_j)\mathbf{p}^t(\mathbf{x}_j), \tag{7}$$

$$B(\mathbf{x}) = \mathbf{P}^t\mathbf{W} = \left[\mathbf{w}\left(\mathbf{x}, \mathbf{x}_1\right)\mathbf{p}\left(\mathbf{x}_1\right), w\left(\mathbf{x}, \mathbf{x}_2\right)\mathbf{p}\left(\mathbf{x}_2\right), ..., w\left(\mathbf{x}, \mathbf{x}_N\right)\mathbf{p}\left(\mathbf{x}_n\right)\right]. \tag{8}$$

The matrix A called the moment matrix, is of size $(Q+1)\times(Q+1)$ and under some conditions, is non-singular [54].

The Gaussian and spline weight functions are applied in the present work, respectively as

$$w(\mathbf{x}, \mathbf{x}_j) = \begin{cases} \dfrac{\exp\left[-\left(d_j/\alpha\right)^2\right] - \exp\left[-(\delta/\alpha)^2\right]}{1 - \exp\left[-(\delta/\alpha)^2\right]}, & 0 \le d_j \le \delta, \\ 0, & d_j > \delta, \end{cases} \tag{9}$$

and

$$w\left(\mathbf{x}, \mathbf{x}_j\right) = \begin{cases} 1 - 6\left(d_j/\delta\right)^2 + 8\left(d_j/\delta\right)^3 - 3\left(d_j/\delta\right)^4, & 0 \le d_j \le \delta, \\ 0, & d_j > \delta, \end{cases} \tag{10}$$

where $d_j = \| \mathbf{x} - \mathbf{x}_j \|_2$ (the Euclidean distance between $\mathbf{x}$ and $\mathbf{x}_j$), α is a constant controlling the shape of the weight function $w(\mathbf{x}, \mathbf{x}_j)$ and δ is the size of the support domain.

3 Solution of nonlinear integral equations

In this section, the moving least squares approximation is used to solve two-dimensional nonlinear Fredholm integral equations of the second kind. These equations take the form

$$u(x,t) - \lambda \int_D K(x,t,y,s,u(y,s))\mathrm{d}s\mathrm{d}y = f(x,t), \quad (x,t) \in D, \tag{11}$$

where the right hand side function f and the kernel function K are given, $u(x,t)$ is the unknown function to be determined, λ is a constant and D is an arbitrary bounded two-dimensional domain which is usually non-rectangular.

Throughout this paper we assume that the following conditions on f and K hold:

(I) $f \in V(D)$, where $V(D)$ is the framework of some complete function space on D, such as $C(D)$.

(II) $K(x,t,y,s,u)$ satisfies the uniform Lipschitz condition for any $(y_i, s_i) \in D$ and $u_i \in (-\infty, \infty)$, $i = 1, 2$, namely

$$|K(x,t,y_1,s_1,u_1) - K(x,t,y_2,s_2,u_2)| \le \hat{K}(x,t)\left[\|(y_1,s_1) - (y_2,s_2)\|_2 + |u_1 - u_2|\right], \tag{12}$$

where $\hat{K}(x,t)$ is a nonnegative bounded function on D and its upper bound is taken as

$$k = \sup_{(x,t)\in D} \hat{K}(x,t). \tag{13}$$

We rewrite (11) in abstract form as

$$u = Tu, \tag{14}$$

where the operator $T : V(D) \to V(D)$, is defined as

$$Tu(x,t) = \lambda \int_D K(x,t,y,s,u(y,s))\mathrm{d}s\mathrm{d}y + f(x,t). \tag{15}$$

Note that solving (11) is equivalent to finding the fixed points of T. Under Assumptions (I) and (II), since the operator T is a contraction operator if $\lambda k < 1$, by the Banach contraction mapping principle, (11) or (14) has a unique solution $u_0(x,t) \in V(D)$ [7, 51].

To apply the method, at first N nodal scattered points are selected to initiate the MLS method. The distribution of nodes could be selected regularly or randomly on the whole of the domain D, such as $X = \{(x_1, t_1), ..., (x_N, t_N)\}$. Therefore, to solve (11), we estimate the unknown function $u(x,t)$ by the MLS approximation as

$$u(x,t) \approx \sum_{j=1}^{N} u_j \psi_j(x,t), \tag{16}$$

where $\psi_j(x,t)$ are the shape functions of the MLS method corresponding to the nodal point (x_j, t_j) and the coefficients $\{u_1, ..., u_N\}$ are found by solving the following system. Then instead of $u(x,t)$ in (11), we replace the expansion (16) and install the collocation points (x_i, t_i), $i = 1, 2, ..., N$. Therefore (11) becomes

$$\sum_{j=1}^{N} u_j \psi_j(x_i, t_i) - \lambda \int_D K\left(x_i, t_i, y, s, \sum_{j=1}^{N} u_j \psi_j(y, s)\right) \mathrm{d}s\mathrm{d}y = f(x_i, t_i). \quad (17)$$

We consider the m_N-point numerical integration scheme over D relative to the coefficients $\{(y_p, s_p)\}$ and weights $\{w_p\}$ for solving integrals in (17), i.e.,

$$\int_D K(x, t, y, s, u(y, s))\mathrm{d}s\mathrm{d}y \approx \sum_{p=1}^{m_N} w_p K\left(x, t, y_p, s_p, u\left(y_p, s_p\right)\right). \quad (18)$$

Applying the numerical integration rule (18) in (17) yields

$$\sum_{j=1}^{N} \hat{u}_j \psi_j(x_i, t_i) - \lambda \sum_{p=1}^{m_N} K\left(x_i, t_i, y_p, s_p, \sum_{j=1}^{N} \hat{u}_j \psi_j\left(y_p, s_p\right)\right) w_p = f(x_i, t_i), \quad i = 1, 2, ..., N. \quad (19)$$

Finding unknowns $\hat{u} = [\hat{u}_1, \hat{u}_2, ..., \hat{u}_N]$ by solving the nonlinear system of algebraic equations (19) yields the following approximate solution at any point $(x, t) \in D$ by

$$\hat{u}_N(x, t) = \sum_{j=1}^{N} \hat{u}_j \psi_j(x, t), \quad (x, t) \in D. \quad (20)$$

Now, we proceed by discussing on the selection of numerical integration formula (18) based upon the classification of the domain D. Let D be a normal domain with a smooth boundary, so we can assume that

$$D = \{(y, s) \in \mathbb{R}^2 : a \leq y \leq b \; and \; \alpha_1(y) \leq s \leq \alpha_2(y)\}, \quad (21)$$

where $a, b \in \mathbb{R}$ and α_1, α_2 are sufficiently smooth functions of y. Without lose of generality, we can assume that $a = 0$ and $b = 1$.

If the kernel function K is a well-behaved function (that is, it is several times continuously differentiable), then the reduction formula for the double integrals gives

$$\int_D K(x, t, y, s, u(y, s))\mathrm{d}y\mathrm{d}s = \int_0^1 \int_{\alpha_1(y)}^{\alpha_2(y)} K(x, t, y, s, u(y, s))\mathrm{d}y\mathrm{d}s = \int_0^1 \bar{K}(x, t, y)\mathrm{d}y. \quad (22)$$

Similar to [6], the integral $\int_0^1 \bar{K}(x, t, y)\mathrm{d}y$ can be approximated by the composite m_N-point Gauss-Legendre quadrature rule using M subintervals relative to the coefficients $\{y_k\}$ and weights $\{w_k\}$ in interval $[-1, 1]$. Thus, in the y direction we can write

$$\int_0^1 \bar{K}(x, t, y)\mathrm{d}y \approx \frac{\Delta y}{2} \sum_{q=1}^{M} \sum_{k=1}^{m_N} w_k \bar{K}\left(x, t, \theta_k^q\right), \quad (23)$$

where $\Delta y = \frac{1}{M}$ and $\theta_k^q = \frac{\Delta y}{2} y_k + \left(q - \frac{1}{2}\right) \Delta y$. For each node θ_k^q, the approximate evaluation of the integral $\bar{K}\left(x, t, \theta_k^q\right)$ is then carried out by the composite m_N-point

Gauss-Legendre quadrature rule using M subintervals relative to the coefficients $\{s_p\}$ and weights $\{w_p\}$ in interval $[-1, 1]$

$$\bar{K}\left(x, t, \theta_k^q\right) = \int_{\alpha_1(\theta_k^q)}^{\alpha_2(\theta_k^q)} K\left(x, t, \theta_k^q, s, u\left(\theta_k^q, s\right)\right) \mathrm{d}s \approx \frac{\Delta s\left(\theta_k^q\right)}{2} \sum_{r=1}^{M} \sum_{p=1}^{m_N} w_p K\left(x, t, \theta_k^q, \eta_p^r, u\left(\theta_k^q, \eta_p^r\right)\right), \tag{24}$$

where $\Delta s\left(\theta_k^q\right) = \frac{\alpha_2(\theta_k^q)-\alpha_1(\theta_k^q)}{M}$ and $\eta_p^r = \frac{\Delta s}{2} s_p + \left(r - \frac{1}{2}\right) \Delta s$.

Now, using the composite m_N-point Gauss-Legendre quadrature formula (24), we obtain

$$\int_D K(x, t, y, s, u(y, s))\mathrm{d}s\mathrm{d}y \approx \frac{1}{2M} \sum_{q=1}^{M} \sum_{k=1}^{m_N} w_k \frac{\Delta s\left(\theta_k^q\right)}{2} \sum_{r=1}^{M} \sum_{p=1}^{m_N} w_p K\left(x, t, \theta_k^q, \eta_p^r, u\left(\theta_k^q, \eta_p^r\right)\right). \tag{25}$$

Utilizing this numerical integration rule for numerically solving the integrals in (17), we obtain the following nonlinear system of algebraic equations

$$\sum_{j=1}^{N} \hat{u}_j \psi_j\left(x_i, t_i\right) - \lambda \frac{1}{2M} \sum_{q=1}^{M} \sum_{k=1}^{m_N} w_k \frac{\Delta s\left(\theta_k^q\right)}{2} \sum_{r=1}^{M} \sum_{p=1}^{m_N} w_p K\left(x_i, t_i, \theta_k^q, \eta_p^r, \sum_{j=1}^{N} \hat{u}_j \psi_j\left(\theta_k^q, \eta_p^r\right)\right) = f\left(x_i, t_i\right). \tag{26}$$

Therefore the solution of the nonlinear integral equation reduces to the solution of the nonlinear system of algebraic equations (26) and the values of $u(x, t)$ at any point $(x, t) \in D$ can be approximated by

$$\hat{u}_N(x, t) = \sum_{j=1}^{N} \hat{u}_j \psi_j(x, t), \quad (x, t) \in D. \tag{27}$$

Remark 1 Suppose that the domain D is described as

$$D = \{(y, s) \in \mathbb{R}^2 : c \leq s \leq d \text{ and } \beta_1(s) \leq y \leq \beta_2(s)\}, \tag{28}$$

where $c, d \in \mathbb{R}$, β_1 and β_2 are sufficiently smooth functions of s. In this case, the computations are similarly performed, but the variables are commuted.

Next, let D be a normal domain with a piecewise smooth boundary, so we can assume

$$D = D_1 \cup D_2 \cup ... \cup D_L, \tag{29}$$

where D_l's are domains of the form (21), i.e.

$$D_l = \{(y, s) \in \mathbb{R}^2 : a_l \leq y \leq b_l \text{ and } \alpha_{1,l}(y) \leq s \leq \alpha_{2,l}(y)\}, \quad l = 1, 2, ..., L, \tag{30}$$

where $a_l, b_l \in \mathbb{R}$ and $\alpha_{1,l}, \alpha_{2,l}$ are sufficiently smooth functions of y.

We present the following theorem regarding the composite m_N-point Gauss-Legendre quadrature rule on the domain (29) with error analysis.

Theorem 3.1 *Suppose that f is defined on $D \subseteq [0,1]\times[0,1]$, where D is a domain with a piecewise smooth boundary. Assume*

$$\left|\frac{\partial^{2m_N} f}{\partial y^{2m_N}}\right| < C_1 < \infty, \qquad \left|\frac{\partial^{2m_N} f}{\partial s^{2m_N}}\right| < C_2 < \infty, \tag{31}$$

for all $(y,s) \in D$ and $\alpha_{1,l}, \alpha_{2,l} \in C^{2m_N}[a_l, b_l]$. Then, for any given integer M, we have

$$\int_D f(y,s)\mathrm{d}y\mathrm{d}s = \sum_{l=1}^{L} \frac{(b_l-a_l)}{2M} \sum_{q=1}^{M}\sum_{k=1}^{m_N} w_k \frac{\Delta s^l\left(\theta_k^{q,l}\right)}{2} \sum_{r=1}^{M}\sum_{p=1}^{m_N} w_p f\left(\theta_k^{q,l}, \eta_p^{r,l}\right) + O\left(\frac{1}{M^{2m_N}}\right), \tag{32}$$

where

$$\Delta s^l\left(\theta_k^{q,l}\right) = \frac{\alpha_{2,l}\left(\theta_k^{q,l}\right) - \alpha_{1,l}\left(\theta_k^{q,l}\right)}{M}, \qquad \eta_p^{r,l} = \frac{\Delta s^l}{2} s_p + \left(r - \frac{1}{2}\right)\Delta s,$$

with $\theta_k^{q,l} = \frac{(b_l-a_l)}{2} y_k + \left(q - \frac{1}{2}\right)(b_l - a_l)$.

Proof From the definition of D in (29), we have

$$\begin{aligned}\int_D f(y,s)\mathrm{d}y\mathrm{d}s &= \sum_{l=1}^{L}\int_{D_l} f(y,s)\mathrm{d}s\mathrm{d}y = \sum_{l=1}^{L}\int_{a_l}^{b_l}\int_{\alpha_{1,l}(y)}^{\alpha_{2,l}(y)} f(y,s)\mathrm{d}s\mathrm{d}y \\ &= \sum_{l=1}^{L}\int_{a_l}^{b_l} F_l(y)\mathrm{d}y,\end{aligned} \tag{33}$$

and $\alpha_{1,l}, \alpha_{2,l}$ are m_N times continuously differentiable functions on $[a_l, b_l]$ into $[0,1]$. Now, we apply the composite m_N-point Gauss-Legendre quadrature rule with M uniform subdivisions relative to the coefficients $\{y_k\}$ and weights $\{w_k\}$ in interval $[-1,1]$ to each integral [42]. Thus

$$\int_0^1 F_l(y)\mathrm{d}y = \frac{(b_l-a_l)}{2M}\sum_{q=1}^{M}\sum_{k=1}^{m_N} w_k F_l\left(\theta_k^{q,l}\right) + \frac{C_{m_N}^l}{M^{2m_N}} F_l^{2m_N}(c), \quad \text{for some } c \in (0,1), \tag{34}$$

where $\theta_k^{q,l} = \frac{(b_l-a_l)}{2} y_k + \left(q - \frac{1}{2}\right)(b_l - a_l)$ and $C_{m_N}^l$ is a constant independent of F_l. On the other hand, we can write

$$F_l\left(\theta_k^{q,l}\right) = \frac{\Delta s^l\left(\theta_k^{q,l}\right)}{2}\sum_{r=1}^{M}\sum_{p=1}^{m_N} w_p f\left(\theta_k^{q,l}, \eta_p^{r,l}\right) + C_{m_N}^l\left(\Delta s^l\left(\theta_k^{q,l}\right)\right)^{2m_N}\frac{\partial^{2m_N} f\left(\theta_k^{q,l}, c_{k,q}^l\right)}{\partial s^{2m_N}}, \tag{35}$$

where

$$\Delta s^l\left(\theta_k^{q,l}\right)=\frac{\alpha_{2,l}\left(\theta_k^{q,l}\right)-\alpha_{1,l}\left(\theta_k^{q}\right)}{M},\qquad \eta_p^{r,l}=\frac{\Delta s^l}{2}s_p+\left(r-\frac{1}{2}\right)\Delta s,$$

and for some $c_{k,q}^l\in\left(\alpha_{1,l}\left(\theta_k^{q,l}\right),\alpha_{2,l}\left(\theta_k^{q,l}\right)\right)$. Therefore, we obtain

$$\int_D f(y,s)\mathrm{d}y\mathrm{d}s=\sum_{l=1}^{L}\frac{(b_l-a_l)}{2M}\sum_{q=1}^{M}\sum_{k=1}^{m_N}w_k\frac{\Delta s^l\left(\theta_k^{q,l}\right)}{2}\sum_{r=1}^{M}\sum_{p=1}^{m_N}w_pf\left(\theta_k^{q,l},\eta_p^{r,l}\right)+E, \tag{36}$$

where

$$E=\sum_{l=1}^{L}\left\{\frac{1}{2M}\sum_{q=1}^{M}\sum_{k=1}^{m_N}w_kC_{m_N}^l\left(\Delta s^l\left(\theta_k^{q,l}\right)\right)^{2m_N}\frac{\partial^{2m_N}f\left(\theta_k^{q,l},c_{k,q}^l\right)}{\partial s^{2m_N}}+\frac{C_{m_N}^l}{M^{2m_N}}F_l^{2m_N}(c)\right\}. \tag{37}$$

Since

$$F_l^{2m_N}(y)=\int_{\alpha_{1,l}(y)}^{\alpha_{2,l}(y)}\frac{\partial^{2m_N}f(y,s)}{\partial y^{2m_N}}\mathrm{d}y$$
$$+\text{ a combination of lower partial derivatives of } f \text{ with respect to } y,$$

from the assumption (31), we obtain $\left|F_l^{2m_N}(y)\right|<C_1$. Moreover,

$$\left|\frac{\partial^{2m_N}f\left(\theta_k^{q,l},s\right)}{\partial s^{2m_N}}\right|<C_2\quad \textit{and}\quad \left|\Delta s^l\left(\theta_k^{q,l}\right)\right|\leq\frac{1}{M},$$

which provide a bound for absolute value of the residual E as

$$|E|<\frac{\left\{\frac{1}{2M}C_2+C_1\right\}\sum_{l=1}^{L}C_{m_N}^l}{M^{2m_N}}=\frac{C}{M^{2m_N}}. \tag{38}$$

This completes the proof. □

Therefore, using the composite m_N-point Gauss-Legendre quadrature formula (32) yields

$$\int_D K(x,t,y,s,u(y,s))\mathrm{d}s\mathrm{d}y\approx\sum_{l=1}^{L}\frac{(b_l-a_l)}{2M}\sum_{q=1}^{M}\sum_{k=1}^{m_N}$$
$$w_k\frac{\Delta s^l\left(\theta_k^{q,l}\right)}{2}\sum_{r=1}^{M}\sum_{p=1}^{m_N}w_pK\left(x,t,\theta_k^{q,l},\eta_p^{r,l},u\left(\theta_k^{q,l},\eta_p^{r,l}\right)\right).$$

Thus we obtain the final system

$$\sum_{j=1}^{N}\hat{u}_j\psi_j(x_i,t_i)-\lambda\sum_{l=1}^{L}\frac{(b_l-a_l)}{2M}\sum_{q=1}^{M}\sum_{k=1}^{m_N}w_k\frac{\Delta s^l\left(\theta_k^{q,l}\right)}{2}\sum_{r=1}^{M}\sum_{p=1}^{m_N}$$
$$w_pK\left(x_i,t_i,\theta_k^{q,l},\eta_p^{r,l},\sum_{j=1}^{N}\hat{u}_j\psi_j\left(\theta_k^{q,l},\eta_p^{r,l}\right)\right)=f(x_i,t_i). \tag{39}$$

Remark 2 Monte Carlo integration techniques are the preferred methods for approximating multidimensional integrals when the region of integral is not in the above classification. The method is based upon the interpretation of the integral as a statistic mean value. Suppose $f(y, s)$ is a continuous function on the domain $D \subset \mathbb{R}^2$, thus

$$\int_D f(x, t)\mathrm{d}x\mathrm{d}t \approx \frac{\mathcal{S}(D)}{m_N} \sum_{p=1}^{m_N} f(y_p, s_p), \tag{40}$$

where $\mathcal{S}(D)$ denotes the area of the domain D and the points $\{(y_p, s_p)\}_{p=1}^{m_N}$ are randomly selected on the domain D. Note that the approximate rate of the m_N-point Monte Carlo method is of $O(1/\sqrt{m_N})$ as $m_N \to \infty$, which is independent of dimension (for more details see Chapter 9 of [42]). Now, utilizing the Monte Carlo method as the quadrature rule (40), we have

$$\int_D K(x, t, y, s, u(y, s))\mathrm{d}s\mathrm{d}y \approx \frac{\mathcal{S}(D)}{m_N} \sum_{p=1}^{m_N} K(x, t, y_p, s_p, u(y_p, s_p)), \tag{41}$$

then, the nonlinear system (19) is converted to

$$\sum_{j=1}^{N} \hat{u}_j \phi_j(x_i, t_i) - \frac{\lambda \mathcal{S}(D)}{m_N} \sum_{p=1}^{m_N} K\left(x_i, t_i, y_p, s_p, \sum_{j=1}^{N} \hat{u}_j \phi_j(y_p, s_p)\right) = f(x_i, t_i). \tag{42}$$

3.1 Notes on weakly singular integral equations

We now solve two-dimensional nonlinear Fredholm integral equations with weakly singular kernels given in the form

$$u(x) - \lambda \int_D H(x, t, y, s)L(x, t, y, s, u(y, s))\mathrm{d}s\mathrm{d}y = f(x, t), \quad (x, t) \in D, \tag{43}$$

where the kernel function

$$K(x, t, y, s, u(y, s)) = H(x, t, y, s)L(x, t, y, s, u(y, s)),$$

and the right hand side function $f(x, t)$ are given, $u(x, t)$ is the unknown function to be determined, λ is a constant and D is a bounded non-rectangular two-dimensional domain. We assume that H is continuous for all $(x, t), (y, s) \in D$, $(x, t) \neq (y, s)$, and there exist positive constants M and $\alpha \in (0, m]$ such that for all $(x, t), (y, s) \in D$, $(x, t) \neq (y, s)$, we have

$$|H(x, t, y, s)| \leq M \left\{(x - y)^2 + (t - s)^2\right\}^{\frac{\alpha - m}{2}}, \tag{44}$$

and L is a given well-behaved function (that is, it is several times continuously differentiable). This equation occasionally arises in the problem of determining the cross-sectional distribution of current in an infinitely long rectangular conducting bar which carries an alternating current [25].

In this setting most such discontinuous functions H have an infinite singularity and the most important examples are

$$H(x,t,y,s)=\begin{cases}\ln\sqrt{(x-y)^2+(t-s)^2},\\ \left\{(x-y)^2+(t-s)^2\right\}^{\frac{\alpha}{2}},\end{cases} \tag{45}$$

for some $-1<\alpha<0$ and variants of them [32].

As before, we need N nodal scattered points to initiate the MLS method. These nodes are given or selected arbitrary on the whole of the domain D, such as $X=\{(x_1,t_1),...,(x_N,t_N)\}$. Therefore, to solve (43), we estimate the unknown function $u(x,t)$ by the MLS approximation. We replace the expansion (16) with $u(x,t)$ and install the collocation points (x_i,t_i), $i=1,2,...,N$ in (43). Thus we obtain

$$\bar{u}_N(x_i,t_i)-\lambda\int_D H(x_i,t_i,y,s)L(x_i,t_i,y,s,\bar{u}_N(y,s))\mathrm{d}s\mathrm{d}y=f(x_i,t_i), \tag{46}$$

where

$$\bar{u}_N(x,t)=\sum_{j=1}^{N}u_j\psi_j(x,t). \tag{47}$$

Let D be a non-rectangular normal domain with a smooth boundary as (21). The singular integrals in (46) cannot be computed with common numerical integration and thus the special numerical integration rule is required. Therefore we present the double m_N-point Gauss-Legendre integration rule with M non-uniform subdivisions over the domain D with its error analysis in the following theorem from [22, 31].

Theorem 3.2 *Suppose that f is defined on $D\subseteq[0,1]\times[0,1]$ and satisfies*

$$\left|\frac{\partial^{2m_N}f}{\partial y^{2m_N}}\right|<C_1y^{-\epsilon-2m_N},\qquad \left|\frac{\partial^{2m_N}f}{\partial s^{2m_N}}\right|<C_2y^{-\epsilon}, \tag{48}$$

for all $(y,s)\in D$ and $\alpha_1,\alpha_2\in C^{2m_N}[0,1]$. Then, for any given integer M, we have

$$\int_D f(y,s)\mathrm{d}y\mathrm{d}s=\sum_{q=1}^{M}\frac{\Delta y_q}{2}\sum_{k=1}^{m_N}w_k\frac{\Delta s\left(\theta_k^q\right)}{2}\sum_{r=1}^{M_{p,r}}\sum_{p=1}^{m_N}w_pf\left(\theta_k^q,\eta_p^r\left(\theta_k^q\right)\right)+O\left(\frac{1}{M^{2m_N}}\right), \tag{49}$$

where

$$\theta_k^q=\frac{\Delta x_q}{2}v_k+\bar{x}_q,\ \ \Delta x_q=x_q-x_{q-1}\ \textit{and}\ \bar{x}_q=\frac{x_q+x_{q-1}}{2},$$

with

$$x_q=\left(\frac{q}{M}\right)^s,\quad s=\frac{2q_N+1}{1-\epsilon},\quad M_{p,r}=1+\left[M\left(\alpha_2\left(\theta_k^q\right)-\alpha_1\left(\theta_k^q\right)\right)\right],$$

$$\Delta s\left(\theta_k^q\right)=\frac{\alpha_2\left(\theta_k^q\right)-\alpha_1\left(\theta_k^q\right)}{M_{p,r}}\ \textit{and}\ \eta_p^r\left(\theta_k^q\right)=\frac{\Delta s\left(\theta_k^q\right)}{2}s_p+\alpha_1\left(\theta_k^q\right)+\left(r-\frac{1}{2}\right)\Delta s\left(\theta_k^q\right).$$

In the following we assume that $H(x,t,y,s)=\ln\sqrt{(x-y)^2+(t-s)^2}$, thus we consider the logarithm-like singular integrals in (46), i.e.

$$I_i=\int_D \ln\sqrt{(x_i-y)^2+(t_i-s)^2}L(x_i,t_i,y,s,\bar{u}_N(y,s))\mathrm{d}y\mathrm{d}s,\ \ i=1,...,N. \tag{50}$$

It is clear that the integrals I_i can not approximate utilizing the quadrature rule (49), since the singularity occurs at the point (x_i,t_i). To overcome this problem, we split the integrals I_i in two separate integrals as

$$\begin{aligned}I_i &= \int_0^{x_i}\int_{\alpha_1(y)}^{\alpha_2(y)} \ln\sqrt{(x_i-y)^2+(t_i-s)^2}L(x_i,t_i,y,s,\bar{u}_N(y,s))\mathrm{d}y\mathrm{d}s\\ &+\int_{x_i}^{1}\int_{\alpha_1(y)}^{\alpha_2(y)} \ln\sqrt{(x_i-y)^2+(t_i-s)^2}L(x_i,t_i,y,s,\bar{u}_N(y,s))\mathrm{d}y\mathrm{d}s,\end{aligned}$$

and so, we find that a change of variables for these integrals is most helpful. Let

$$y=x_i(1-\gamma)\quad\text{and}\quad y=\xi(1-x_i)+x_i.$$

With this change of variables, we have

$$\begin{aligned}I_i=x_i\int_0^1\int_{\alpha_1(y)}^{\alpha_2(y)} K_1(x_i,t_i,\gamma,s,\bar{u}_N(\gamma,s))\mathrm{d}\gamma\mathrm{d}s+(1-x_i)\int_0^1\int_{\alpha_1(y)}^{\alpha_2(y)}\\ K_2(x_i,t_i,\xi,s,\bar{u}_N(\xi,s))\mathrm{d}\xi\mathrm{d}s,\end{aligned}$$

where

$$K_1(x_i,t_i,\gamma,s,\bar{u}_N(\gamma,s))=\ln\sqrt{x_i^2\gamma^2+(t_i-s)^2}L(x_i,t_i,x_i(1-\gamma),s,\bar{u}_N(x_i(1-\gamma),s)),$$

and

$$\begin{aligned}K_2(x_i,t_i,\xi,s,\bar{u}_N(\xi,s))=\ln\sqrt{(1-x_i)^2\xi^2+(t_i-s)^2}L(x_i,t_i,\xi(1-x_i)\\ +x_i,s,\bar{u}_N(\xi(1-x_i)+x_i,s)).\end{aligned}$$

Note that K_1 and K_2 have the singularities in at points $\gamma=0,\ s=t_i$ and $\xi=0,\ s=t_i$, respectively, so these functions satisfy the assumptions (48) for any positive integer m_N and for any small positive number ϵ [22, 31]. Therefore, using the quadrature rule (49), we obtain

$$\begin{aligned}I_i &\approx x_i\sum_{q=1}^{M}\frac{\Delta y_q}{2}\sum_{k=1}^{m_N}w_k\frac{\Delta s\left(\theta_k^q\right)}{2}\sum_{r=1}^{M_{p,r}}\sum_{p=1}^{m_N}w_pK_1\left(x_i,t_i,\theta_k^q,\eta_p^r\left(\theta_k^q\right),\hat{u}_N\left(\theta_k^q,\eta_p^r\left(\theta_k^q\right)\right)\right)\\ &+(1-x_i)\sum_{q=1}^{M}\frac{\Delta y_q}{2}\sum_{k=1}^{m_N}w_k\frac{\Delta s\left(\theta_k^q\right)}{2}\sum_{r=1}^{M_{p,r}}\sum_{p=1}^{m_N}w_pK_2\left(x_i,t_i,\theta_k^q,\eta_p^r\left(\theta_k^q\right),\hat{u}_N\left(\theta_k^q,\eta_p^r\left(\theta_k^q\right)\right)\right).\end{aligned}$$

Utilizing the above numerical integration scheme in the system (46) gets the final nonlinear system of algebraic equations for the unknowns $\hat{u}=[\hat{u}_1,\hat{u}_2,...,\hat{u}_N]$ which the solution of this system eventually leads to the numerical solution

$$\hat{u}_N(x,t)=\sum_{j=1}^{N}\hat{u}_j\psi_j(x,t). \tag{51}$$

It should be noted that the argument discussed here is similarly applied to other weakly singular integral equations.

4 Error analysis

This section includes the error estimate and the rate of convergence of the presented method. This analysis is based on the approximation of fixed points of the nonlinear compact integral operators for two-dimensional cases [7, 50]. The error analysis for linear integral equations based on the error expansion of the Nyström solution is also provided in [39].

At first, the error estimate of the MLS method is presented in terms of the fill distance parameter $h_{X,D}$. This discussion follows mostly from [23, 52]. Here, we restrict ourselves to the domains satisfying an interior cone condition defined as follows [52]:

Definition 4.1 A set $D \subset \mathbb{R}^d$ is said to satisfy an interior cone condition if there exists an angle $\theta \in (0, \pi/2)$ and a radius $r > 0$ such that for every $\mathbf{x} \in D$ a unit vector $\xi(\mathbf{x})$ exists such that the cone

$$C(\mathbf{x}, \xi(\mathbf{x}), \theta, r) = \{\mathbf{x} + \lambda \mathbf{y} : \ \mathbf{y} \in \mathbb{R}^d, \ \|\mathbf{y}\|_2 = 1, \ \mathbf{y}^t \xi(\mathbf{x}) \geq \cos\theta, \ \lambda \in [0, r]\}, \quad (52)$$

is contained in D.

We present some definitions from [17, 52] that are important to measure the quality of data points and to estimate the rate of convergence in the MLS and other meshless methods.

Definition 4.2 The fill distance of a set of points $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\} \subseteq D$ for a bounded domain D is defined by

$$h_{X,D} = \sup_{x \in D} \min_{0 \leq j \leq N} \|\mathbf{x} - \mathbf{x}_j\|_2.$$

Definition 4.3 The separation distance of $X = \{\mathbf{x}_1, ..., \mathbf{x}_N\}$ is defined by

$$q_X = \frac{1}{2} \min_{i \neq j} \|\mathbf{x}_i - \mathbf{x}_j\|_2.$$

The set X is said to be quasi-uniform with respect to a constant $c > 0$ if

$$q_X \leq h_{X,D} \leq c q_X.$$

Finally, in the MLS approximation, for every sample point $\mathbf{x} \in D$, the following theorem was proven by Wendland [52, 53].

Theorem 4.1 *Suppose $D \subseteq \mathbb{R}^d$ is compact and satisfies an interior cone condition with angle $\theta \in (0, \pi/2)$ and radius $r > 0$. Then there exists a constant $C > 0$ that can be computed explicitly such that for all $u \in C^{q+1}(D^*)$, where D^* is defined as the closure of $\cup_{\mathbf{x} \in D} B(\mathbf{x}, 2\tau h_0)$, and all quasi-uniform $X \subset D$ with $h_{X,D} \leq h_0$, where*

$h_0 = r/\tau$ *with* $\tau = \frac{16(1+\sin\theta)^2 q^2}{3\sin^2\theta}$, *the approximation error is bounded as follows*

$$\|u - s_{u,X}\|_{L^\infty(D)} \leq C h_{X,D}^{q+1} |u|_{C^{q+1}(D^*)}. \tag{53}$$

The semi-norm on the right-hand side is defined by $|u|_{C^{q+1}(D^*)} = \max_{|\alpha|=q+1} \|D^\alpha u\|_{L^\infty(D^*)}$.

Note that, in [4] Armentano and Duran proved error estimates in L^∞, for the function and its derivatives in the one-dimensional case. Also, in [3] Armentano obtained the error estimates in L^∞ and L^2 norms for one and higher dimensions.

Now, we introduce a sequence of numerical integral operators T_N on $C(D)$ by

$$T_N u(x,t) = \lambda \sum_{p=1}^{m_N} w_p K(x,t,y_p,s_p,u(y_p,s_p)) + f(x,t), \quad N \geq 1. \tag{54}$$

The required hypotheses on T and $T_N,\ N \geq 1$ are listed and labeled in the following [7, 10]:

Hypothesis H1. T and $T_N,\ N \geq 1$, are completely continuous nonlinear operators on $C(D)$ into $C(D)$.

Hypothesis H2. $T_N,\ N \geq 1$ is a collectively compact family on $C(D)$, i.e., for every bounded set $B \subset C(D)$, the closure of the set $\cup_{N=1}^{\infty} T_N(B)$ is compact in $C(D)$.

Hypothesis H3. T_N is pointwise convergent to T on $C(D)$, i.e., $T_N u \to Tu,\ u \in C(D)$.

Hypothesis H4. At each point of $C(D)$, $\{T_N\}$ is an equicontinuous family.

Hypothesis H5. T and $T_N,\ N \geq 1$ are twice Frechet differentiable on $B(u_0, r) = \{u : \|u - u_0\| \leq r,\ r > 0\}$ and moreover

$$\|T_N''\| \leq \alpha < \infty, \quad N \geq 1, \quad u \in B(u_0, r). \tag{55}$$

Remark 3 Note that since we apply the composite m_N-point Gauss-Legendre quadrature rule for approximating integrals when the kernel function K is an m_N times continuously differentiable function

$$T_N u(x,t) = \lambda \frac{1}{2M} \sum_{q=1}^{M} \sum_{k=1}^{m_N} w_k \frac{\Delta s\left(\theta_k^q\right)}{2} \sum_{r=1}^{M} \sum_{p=1}^{m_N} w_p K\left(x,t,\theta_k^q,\eta_p^r,u\left(\theta_k^q,\eta_p^r\right)\right) + f(x,t),$$

and when the kernel function K is a weakly singular function

$$\begin{aligned} T_N u(x,t) &= \lambda x_i \sum_{q=1}^{M} \frac{\Delta y_q}{2} \sum_{k=1}^{m_N} w_k \frac{\Delta s\left(\theta_k^q\right)}{2} \sum_{r=1}^{M_{p,r}} \sum_{p=1}^{m_N} w_p K_1\left(x_i,t_i,\theta_k^q,\eta_p^r\left(\theta_k^q\right),u\left(\theta_k^q,\eta_p^r\left(\theta_k^q\right)\right)\right) \\ &\quad + \lambda(1-x_i) \sum_{q=1}^{M} \frac{\Delta y_q}{2} \sum_{k=1}^{m_N} w_k \frac{\Delta s\left(\theta_k^q\right)}{2} \sum_{r=1}^{M_{p,r}} \sum_{p=1}^{m_N} \\ &\qquad w_p K_2\left(x_i,t_i,\theta_k^q,\eta_p^r\left(\theta_k^q\right),u\left(\theta_k^q,\eta_p^r\left(\theta_k^q\right)\right)\right) + f(x,t). \end{aligned}$$

Now, from Theorems 3.1 and 3.2, we conclude that

$$\|Tu - T_N u\|_{L^\infty(D)} = O(1/M^{2m_N}), \quad u \in C(D).$$

We define the collocation projection operator $P_N : C(D) \to C_N(D)$ as

$$P_N u(x,t) = \sum_{j=1}^{N} c_j \psi_j(x,t), \quad N \geq 1, \tag{56}$$

where $C_N(D) = \text{span}\{\psi_1, ..., \psi_N\} \subset C(D)$ and the vector $C = [c_1, ..., c_N]^t$ is determined by solving the linear system

$$P_N(x_i, t_i) = u(x_i, t_i), \quad i = 1, ..., N. \tag{57}$$

Let $\ell_j \in C_N(D)$ be elements that satisfy the interpolation conditions

$$\ell_j(x_i, t_i) = \delta_{ij}, \quad i, j = 1, 2, ..., N.$$

There is a unique such functions ℓ_j are called Lagrange basis functions, and the set $\{\ell_1, ..., \ell_N\}$ is a new basis for $C_N(D)$ [8]. With this new basis, we can write

$$P_N u(x,t) = \sum_{j=1}^{N} u(x_j, t_j)\ell_j(x,t), \quad N \geq 1. \tag{58}$$

In addition, in the collocation projection operator, we need to assume that [8, 10]

$$\sum_{j=1}^{N} |\ell_j(x,t)| \leq \gamma_1 < \infty. \tag{59}$$

Remark 4 Note that, $s_{u,X} \neq P_N u$ because for some $(x_i, t_i) \in X$, we have $s_{u,X}(x_i, t_i) \neq u(x_i, t_i)$.

For ease reference, we present the following lemma:

Lemma 4.1 *Having in mind the assumptions of Theorem 4.1, suppose that P_N is the collocation projection operator for the shape functions of the MLS approximation corresponding to the nodal points $X = \{(x_1, t_1), ..., (x_N, t_N)\} \subset D$. If $u \in C^{q+1}(D^*)$ then $P_N u$ converges to u as $N \to \infty$ and moreover*

$$\|P_N u - u\|_{L^\infty(D)} \leq (1 + \gamma_1) C h_{X,D}^{q+1} |u|_{C^{q+1}(D^*)}. \tag{60}$$

Proof We consider the inequality

$$\|P_N u - u\|_{L^\infty(D)} \leq \|P_N u - s_{u,X}\|_{L^\infty(D)} + \|s_{u,X} - u\|_{L^\infty(D)}. \tag{61}$$

From the properties of projection operators [8], since $s_{u,X} \in C_N(D)$, then $P_N(s_{u,X}) = s_{u,X}$. Thus

$$\begin{aligned} \|P_N u - u\|_{L^\infty(D)} &\leq \|P_N u - P_N s_{u,X}\|_{L^\infty(D)} + \|s_{u,X} - u\|_{L^\infty(D)} \\ &\leq \|P_N\| \|u - s_{u,X}\|_{L^\infty(D)} + \|s_{u,X} - u\|_{L^\infty(D)} \\ &= (1 + \|P_N\|) \|s_{u,X} - u\|_{L^\infty(D)}. \end{aligned} \tag{62}$$

The definition of the operator P_N in (58) yields

$$\|P_N\| = \max_{(x,t)\in D} \sum_{j=1}^{N} |\ell_j(x,t)|. \tag{63}$$

Moreover, from the assumption (59), we have $\|P_N\| \leq \gamma_1$. Therefore (62) becomes

$$\|P_N u - u\|_{L^\infty(D)} \leq (1+\gamma_1)\|s_{u,X} - u\|_{L^\infty(D)}. \tag{64}$$

Using Theorem 4.1, for every $u \in C^{q+1}(D^*)$, we obtain

$$\|P_N u - u\|_{L^\infty(D)} \leq (1+\gamma_1) C h_{X,D}^{q+1} |u|_{C^{q+1}(D^*)}. \tag{65}$$

Since $h_{X,D} \to 0$ as $N \to \infty$ (justified by the quasi-uniform condition on X), so $P_N u \to u$. $\square$

Now, utilizing the operators (54) and (56), we can rewrite (26) in the operator form

$$\hat{u}_N = P_N T_N \hat{u}_N. \tag{66}$$

Define the iterated solution by

$$\bar{u}_N = T_N(\hat{u}_N), \tag{67}$$

then it is easily seen that

$$P_N \bar{u}_N = \hat{u}_N, \tag{68}$$

and so

$$\bar{u}_N = T_N P_N \bar{u}_N. \tag{69}$$

Remark 5 Assuming that $\{T_N\}$ satisfies H1-H5, it is shown in [10] that $\{T_N P_N\}$ also satisfies H1-H5.

Now, we give the following theorem from [10] about iterated collocation method that is used to obtain the error analysis of the presented method.

Theorem 4.2 *Suppose H1-H4. Let u_0 be a solution of (11) or a fixed point of T, and assume that 1 is not an eigenvalue of $T'(u_0)$, where $T'(u_0)$ denotes the Frechet derivative of T at u_0. If H5 is satisfied on $B(u_0, r) \subseteq C(D)$, then u_0 is an isolated solution of (11), of nonzero index. Moreover, there are $\varepsilon, \bar{M} > 0$ such that for every $N > \bar{M}$, $T_N P_N$ has a unique fixed point $\bar{u}_N$ in $B(u_0, \varepsilon)$. Also, there is a constant $\gamma_2 > 0$ such that*

$$\|\bar{u}_N - u_0\|_{L^\infty(D)} \leq \gamma_2 \|T u_0 - T_N P_N u_0\|_{L^\infty(D)}, \quad N \geq \bar{M}. \tag{70}$$

This gives a bound on the rate of convergence of the iterated solution $\bar{u}_N$ to u_0.

Here, we complete the error analysis by the following theorem:

Theorem 4.3 *In the situation of Theorem 4.2 and Lemma 4.1 assume that $u_0 \in C^{q+1}(D^*)$ be the exact solution of (11). Furthermore, suppose that the kernel function K is an m_N times continuously differentiable function and the domain D satisfies* *the hypotheses of Theorem 3.1 (or the kernel function K is a weakly singular function and the domain D satisfies the hypotheses of Theorem 3.2), then there are $\hat{\varepsilon}, \hat{M} > 0$ such that for every $N > \hat{M}$, the proposed method has a unique solution $\hat{u}_N \in B(u_0, \hat{\varepsilon})$. Also, we have*

$$\|\hat{u}_N - u_0\|_{L^\infty(D)} \to 0, \quad as \quad N \to \infty, \tag{71}$$

and

$$\|\hat{u}_N - u_0\|_{L^\infty(D)} \leq (1+\gamma_1\gamma_3)(1+\gamma_1)Ch_{X,D}^{q+1}|u_0|_{C^{q+1}(D^*)} + \gamma_1\gamma_2\frac{C_0}{M^{2m_N}}, \quad N \geq \hat{M}, \tag{72}$$

where $\gamma_1, \gamma_2, \gamma_3, C_0$ and C are constants.

Proof From Theorem 4.2, there exists $\varepsilon, \bar{M} > 0$ such that for every $N \geq \bar{M}$, (69) has a unique iterated solution $\bar{u}_N \in B(u_0, \varepsilon)$ and furthermore

$$\begin{aligned}\|\bar{u}_N - u_0\|_{L^\infty(D)} &\leq \gamma_2\|Tu_0 - T_N P_N u_0\|_{L^\infty(D)} \\ &\leq \gamma_2[\|Tu_0 - T_N u_0\|_{L^\infty(D)} + \|T_N(u_0 - P_N u_0)\|_{L^\infty(D)}].\end{aligned} \tag{73}$$

Using the hypothesis H1, we can assume that $\|T_N\| \leq \gamma_3$, so that

$$\|\bar{u}_N - u_0\|_{L^\infty(D)} \leq \gamma_2[\|Tu_0 - T_N u_0\|_{L^\infty(D)} + \gamma_3\|u_0 - P_N u_0\|_{L^\infty(D)}]. \tag{74}$$

Let $\hat{u}_N = P_N\bar{u}_N$ then $\hat{u}$ is a fixed point of (66) and consequently, a solution for the presented method. Consider the decomposition

$$u_0 - \hat{u}_N = u_0 - P_N\bar{u}_N = (u_0 - P_N u_0) + P_N(u_0 - \bar{u}_N), \tag{75}$$

which yields

$$\|\hat{u}_N - u_0\|_{L^\infty(D)} \leq \|u_0 - P_N u_0\|_{L^\infty(D)} + \gamma_1\|u_0 - \bar{u}_N\|_{L^\infty(D)}. \tag{76}$$

Now, by applying (74), we obtain

$$\begin{aligned}\|\hat{u}_N - u_0\|_{L^\infty(D)} &\leq \|u_0 - P_N u_0\|_{L^\infty(D)} \\ &+ \gamma_1(\gamma_2[\|Tu_0 - T_N u_0\|_{L^\infty(D)} + \gamma_3\|u_0 - P_N u_0\|_{L^\infty(D)}]) \\ &= (1+\gamma_1\gamma_3)\|u_0 - P_N u_0\|_{L^\infty(D)} + \gamma_1\gamma_2\|Tu_0 - T_N u_0\|_{L^\infty(D)}.\end{aligned} \tag{77}$$

Utilizing Lemma 4.1 and Remark 3, we conclude that

$$\|\hat{u}_N - u_0\|_{L^\infty(D)} \leq (1+\gamma_1\gamma_3)(1+\gamma_1)Ch_{X,D}^{q+1}|u_0|_{C^{q+1}(D^*)} + \gamma_1\gamma_2\frac{C_0}{M^{2m_N}}. \tag{78}$$

From Theorem 4.1 and the hypothesis H3, we find that $\|\hat{u}_N - u_0\|_{L^\infty(D)} \to 0$ as $N \to 0$. Therefore, there is a constant $M_1 > \bar{M} > 0$ such that for every $N > M_1$, $\|\hat{u}_N - u_0\|_{L^\infty(D)} < \hat{\varepsilon}$, where $\hat{\varepsilon} = \frac{\varepsilon}{1+\gamma_3}$. Also, from the hypothesis H3, there exists $M_2 > 0$ such that for every $N > M_2$, $\|T_N u_0 - Tu_0\|_{L^\infty(D)} < \hat{\varepsilon}$. Finally, by choosing $\hat{M} = \max\{M_1, M_2\}$, we deduce that $\hat{u}_N$, for $N > \hat{M}$, within $B(u_0, \hat{\varepsilon})$, is the unique solution of proposed method, because

$$\begin{aligned}\|\bar{u}_N - u_0\|_{L^\infty(D)} &\leq \|T_N\hat{u_N} - T_N u_0\|_{L^\infty(D)} + \|T_N u_0 - Tu_0\|_{L^\infty(D)} \\ &\leq \gamma_3\|u_0 - \hat{u}_N\|_{L^\infty(D)} + \|T_N u_0 - Tu_0\|_{L^\infty(D)} < \varepsilon, \quad N > \hat{M}.\end{aligned} \tag{79}$$

This completes the proof. □

Corollary 4.1 *The error of the presented method is mainly based upon:*

- *The MLS approximation error which is of* $O\left(h_{X,D}^{q+1}\right)$, *emerges from the first part of the right hand side of* (72).
- *The error caused by performing* m_N*-point numerical integration scheme over D, appears in the second part of the right hand side of* (72) *which is of* $O\left(\frac{1}{M^{2m_N}}\right)$.

It should be noted that for m_N *sufficiently large, the error of the MLS approximation is dominated over the integration error and so, increasing the number of nodes in the numerical integration method has no significant effect on the error. Therefore the proposed method will be of* $O\left(h_{X,D}^{q+1}\right)$. *Conversely, if the error of integration rule overcomes the MLS error, then the error of the method is of* $O\left(\frac{1}{M^{2m_N}}\right)$.

5 Numerical examples

In order to demonstrate the effectiveness of the proposed method, three nonlinear Fredholm integral equations on non-rectangular regions and one two-dimensional nonlinear Volterra integral equation are solved. Linear ($q = 1$ or $Q = 2$) and quadratic ($q = 2$ or $Q = 5$) basis functions and Gaussian and spline weight functions are utilized in the computations. Here, we employ the composite Gauss-Legendre quadrature formula with $M = 10$ and $m_N = 8$ when the kernel function is well-behaved (Examples 5.1, 5.2, 5.4) and the composite non-uniform Gauss-Legendre quadrature formula with $M = 10$ and $m_N = 8$ when the kernel function is weakly singular (Examples 5.3). Corollary 4.1 also confirms that increasing the number of integration points and subdivisions over the domain D does not improve results. Accuracy of the estimated solutions can be worked out by measuring $\|e\|_\infty$ and $\|e\|_2$ error norms which are defined by

$$\|e\|_\infty = \max_{(x,t)\in D}\{|u_{ex}(x,t) - \hat{u}(x,t)|\}, \tag{80}$$

$$\|e\|_2 = \left(\int_D |u_{ex}(x,t) - \hat{u}(x,t)|^2 \mathrm{d}x\mathrm{d}t\right)^{\frac{1}{2}}, \tag{81}$$

where $\hat{u}$ is the approximate solution of the exact solution u_{ex}. All routines are written in MAPLE software and run on a Pentium IV PC Laptop with a 2.10 GHz CPU and 2 GB RAM. The “Fsolve” command is used to solve the nonlinear system of algebraic equations that employs the floating-point arithmetic. In this command, the selection of initial guess is very important for convergence issue. Here, for $N \leq 20$, we choose the zero vector of length N as our initial guess, i.e., $\hat{u}^{(0)} = [0, 0, ..., 0]^t$. Also, to select the initial guess for $N > 20$, we apply the obtained solutions corresponding to the nodal points whose number is less than N. In other words, we assume that $\hat{u}_\tau$ is the approximate solution which is obtained by the presented method for $\tau < N$, then consider the following linear system of algebraic equations

$$\sum_{k=1}^{N} c_k^{(0)} \psi_k(x_i, t_i) = \hat{u}_\tau(x_i, t_i), \quad i = 1, ..., N. \tag{82}$$

The initial value may be chosen as the solution of system (82), i.e, $\hat{u}^{(0)} = \left[c_1^{(0)}, ..., c_N^{(0)}\right]^t$. Next, we increase the value of τ until a satisfactory convergence is achieved.

Example 5.1 As the first example let

$$u(x,t) - \int_D \frac{e^{xy} + \sin(x+s)}{e^{xt}(1+u^3(y,s))} \mathrm{d}y\mathrm{d}s = f(x,t), \quad (x,t) \in D, \tag{83}$$

where

$$f(x,t) = \cos(x+t) \; - \; e^{-xt}(0.1233708861e^{0.7288455458x} - 0.0565348496xe^{0.1643128022x} \\ - \; 0.1368525534\cos(x) - 0.2400535639\sin(x) - 0.1584353894),$$

and D is the asteroid domain which is drawn in Fig. 1. The exact solution for this equation is $u_{ex}(x,t) = \cos(x+t)$. We can separate the domain D as $D = D_1 \cup D_2 \cup D_3 \cup D_4$, where

$$\begin{aligned}
D_1 &= \left\{(x,t) \in \mathbb{R}^2 : \; 0 < x < \frac{1}{3}, \; \frac{1}{2} + \frac{1}{2}x < t < \frac{1}{2} - \frac{1}{2}x\right\}, \\
D_2 &= \left\{(x,t) \in \mathbb{R}^2 : \; \frac{1}{3} < x < \frac{1}{2}, \; 2x < t < 1 - 2x\right\}, \\
D_3 &= \left\{(x,t) \in \mathbb{R}^2 : \; \frac{1}{2} < x < \frac{2}{3}, \; 2 - 2x < t < -1 + 2x\right\}, \\
D_4 &= \left\{(x,t) \in \mathbb{R}^2 : \; \frac{2}{3} < x < 1, \; 1 - \frac{1}{2}x < t < \frac{1}{2}x\right\}.
\end{aligned}$$

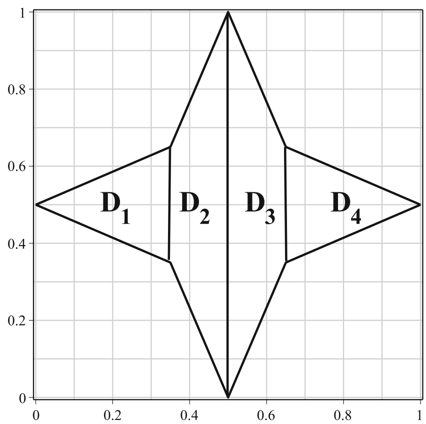


Fig. 1 The consideration domain D for Example 5.1

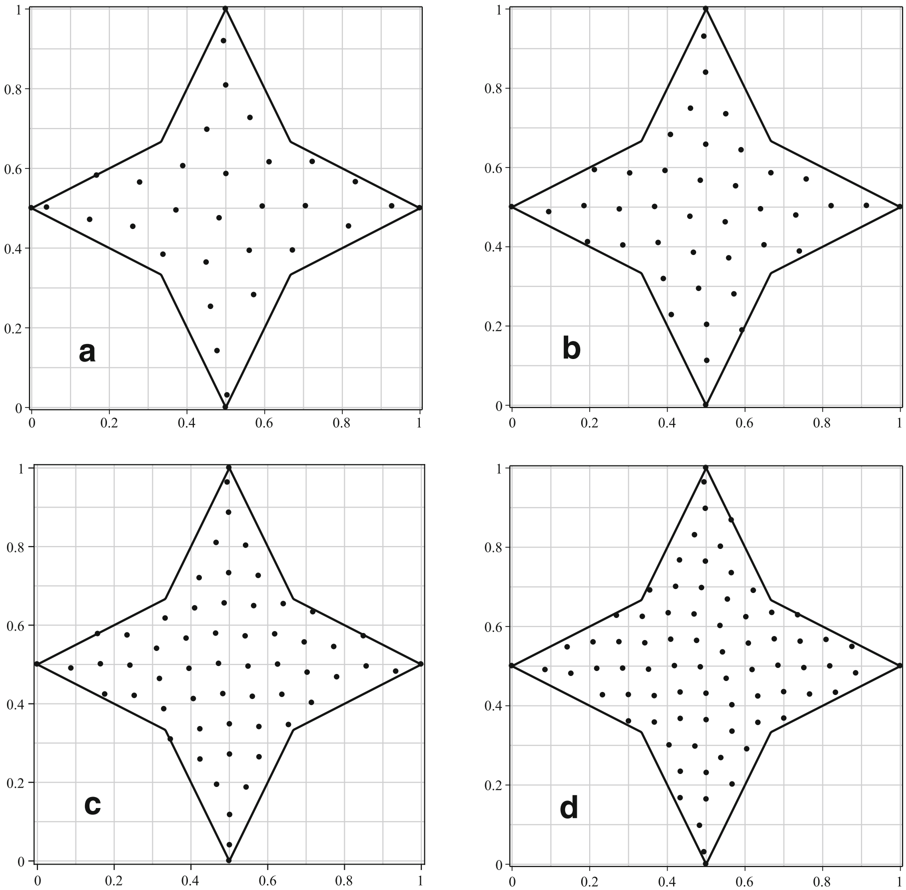


Fig. 2 Node distribution (**a**: 32, **b**: 42, **c**: 59 and **d**: 78 nodes) for Example 5.1

Table 1 Some numerical results of Example 5.1

N	h	$\|e\|_2$ with Gaussian weight function		$\|e\|_2$ with spline weight function	
		$q=1$	$q=2$	$q=1$	$q=2$
13	0.250	8.52×10^{-3}	1.64×10^{-3}	5.90×10^{-3}	1.44×10^{-3}
20	0.166	4.08×10^{-3}	5.46×10^{-4}	2.46×10^{-3}	4.77×10^{-4}
32	0.125	2.18×10^{-3}	2.71×10^{-4}	1.28×10^{-3}	1.71×10^{-4}
42	0.100	1.39×10^{-3}	1.25×10^{-4}	9.01×10^{-4}	9.34×10^{-5}
59	0.083	9.43×10^{-4}	6.11×10^{-5}	4.25×10^{-4}	5.15×10^{-5}
78	0.071	6.93×10^{-4}	4.08×10^{-5}	2.82×10^{-4}	3.36×10^{-5}

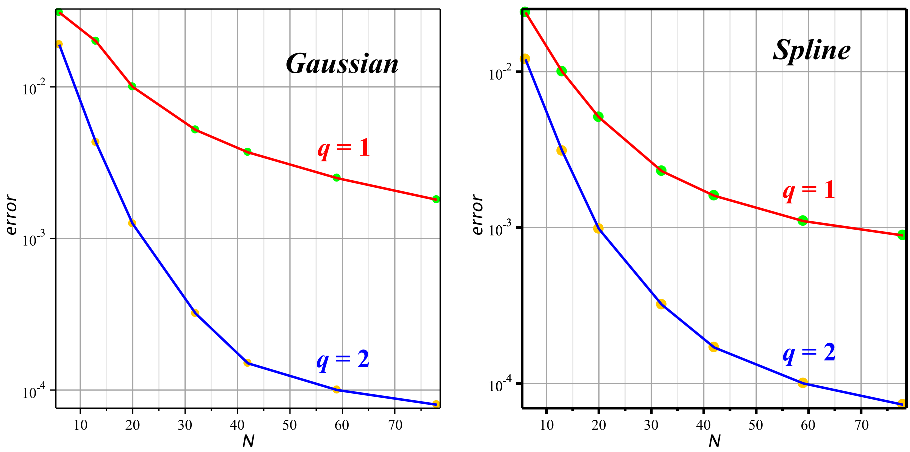


Fig. 3 Maximum error distributions of Example 5.1

The distribution of nodes is depicted in Fig. 2. Numerical results are presented in Table 1 in terms of $\|e\|_2$ at different numbers of N for the linear and quadratic basis functions using Gaussian and spline weight functions. In computations, we put $\delta = 2 \times h$ for the linear case and $3 \times h$ for the quadratic case, where h is the distance between two consecutive nodes [39, 53], to ensure the regularity of the moment matrix A in the MLS approximation [55]. In Gaussian weight function, we chose $\alpha = 0.5 \times h$. From Corollary 4.1, we know that the results by the linear basis gradually converge to the exact values along with the increase of the nodes i.e, $\|u_{ex} - \hat{u}\| \approx O(h^2)$. Also, for the quadratic basis, we have $\|u_{ex} - \hat{u}\| \approx O(h^3)$, but for large N, the error near the boundary increases which effects on the global error [39, 54]. The maximum errors for $q = 1$ and $q = 2$ with Gaussian and spline weight functions are graphically shown in Fig. 3.

Example 5.2 In this example, we solve integral equation

$$u(x,t) - \int_D \frac{x(1-y^2)}{(1+t)(1+s^2)} \left(1 - e^{-u(y,s)}\right) \mathrm{d}y\mathrm{d}s = -\ln\left(1 + \frac{xt}{1+t^2}\right) + \frac{0.052x}{1+t}, \quad (x,t) \in D, \tag{84}$$

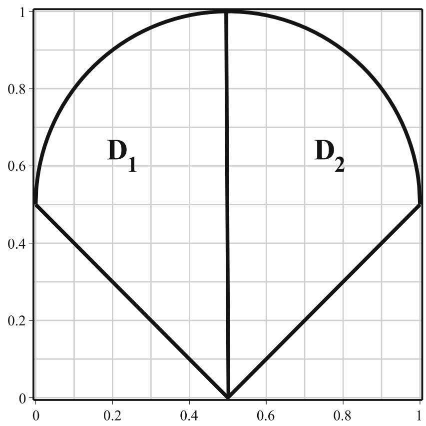
where $D = D_1 \cup D_2$ is the wedge-like domain which is shown in Fig. 4 and given as follows:

$$D_1 = \left\{(x,t) \in \mathbb{R}^2 : 0 < x < \frac{1}{2},\ \frac{1}{2} - x < t < \sqrt{\left(\frac{1}{4} - \left(x - \frac{1}{2}\right)^2\right)} + \frac{1}{2}\right\},$$

$$D_2 = \left\{(x,t) \in \mathbb{R}^2 : \frac{1}{2} < x < 1,\ x - \frac{1}{2} < t < \sqrt{\left(\frac{1}{4} - \left(x - \frac{1}{2}\right)^2\right)} + \frac{1}{2}\right\}.$$

The exact solution for this equation is $u_{ex}(x,t) = -\ln\left(1 + \frac{xt}{1+t^2}\right)$. The integral equation (84) is also solved on the square domain $D = [0,1] \times [0,1]$ in [27]. The traditional methods take difficulty for the numerical solution of this problem due to

Fig. 4 The consideration domain D for Example 5.2



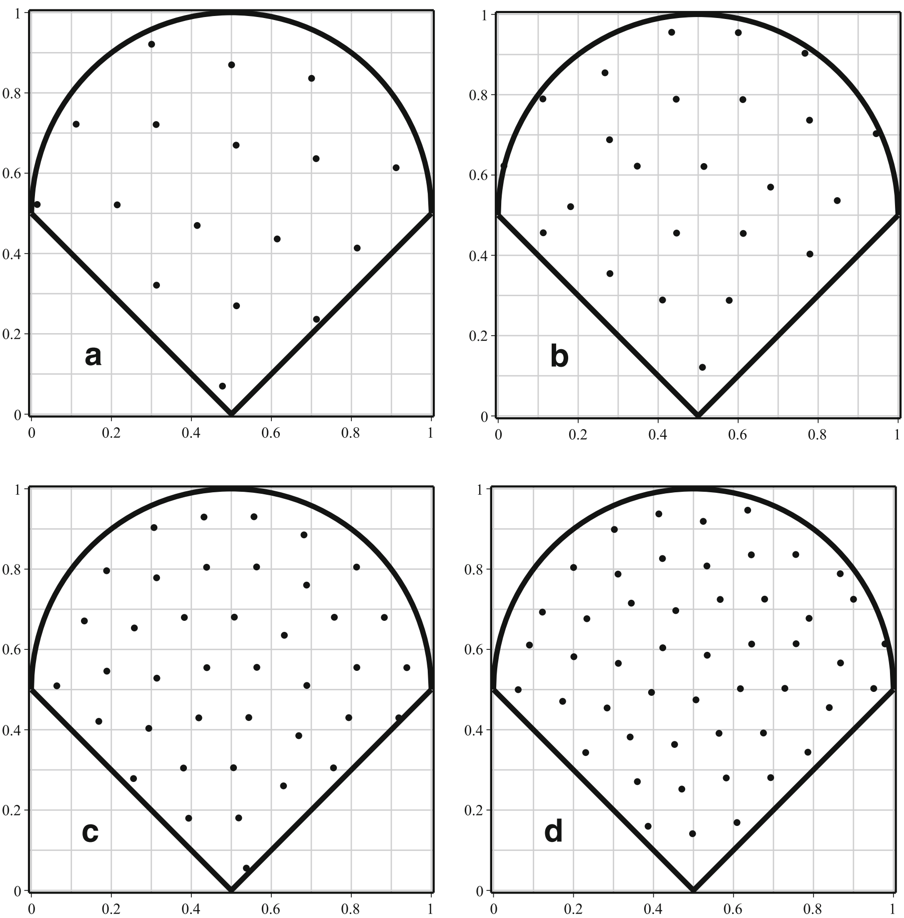


Fig. 5 Node distribution (**a**: 17, **b**: 24, **c**: 40 and **d**: 50 nodes) for Example 5.2

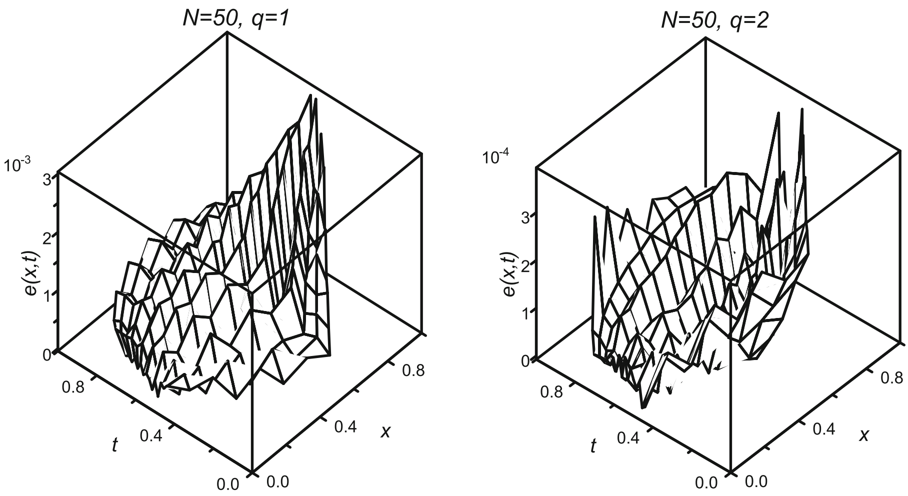


Fig. 6 Absolute error distributions of Example 5.2 with $N = 50$ and $q = 1, 2$ via Gaussian weight functions

the irregular domain. But the problem can be solved using the proposed method in this parer based on use of some nodes scattered over the domain D. The distribution of nodes is depicted in Fig. 5.

Here, $\delta = 2 \times h$ for linear basis and $\delta = 3 \times h$ for quadratic ones, where h is the distance between two consecutive nodes. Also in Gaussian weight functions $\alpha = 0.5 \times h$. The absolute error for $q = 1, 2$ and $N = 50$ with Gaussian and spline weight functions are graphically shown in Figs. 6 and 7, correspondingly. Tables 2 and 3 show $\|e\|_\infty$ and $\|e\|_2$ at different numbers of N for Gaussian and spline weight functions, respectively. Moreover, the rates of convergence are presented in these

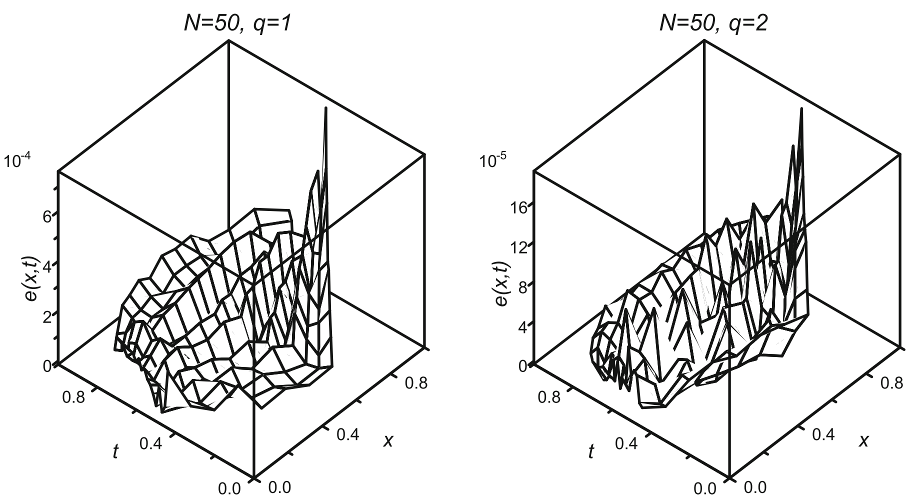


Fig. 7 Absolute error distributions of Example 5.2 with $N = 50$ and $q = 1, 2$ via spline weight functions

Table 2 Some numerical results for Example 5.2 with Gaussian weight function

N	h	$\|e\|_2$		$\|e\|_\infty$			
		$q=1$	$q=2$	$q=1$	Ratio	$q=2$	Ratio
9	0.50	9.36×10^{-3}	3.01×10^{-4}	3.94×10^{-2}	–	1.34×10^{-2}	–
17	0.25	2.21×10^{-4}	1.78×10^{-5}	9.34×10^{-3}	4.21	1.82×10^{-3}	7.36
24	0.20	1.37×10^{-4}	1.94×10^{-5}	5.98×10^{-3}	1.56	8.78×10^{-4}	2.07
40	0.15	9.59×10^{-5}	1.35×10^{-5}	4.01×10^{-3}	1.49	5.21×10^{-4}	1.68
50	0.14	6.93×10^{-5}	8.68×10^{-6}	3.02×10^{-3}	1.32	3.79×10^{-4}	1.37

Table 3 Some numerical results for Example 5.2 with spline weight function

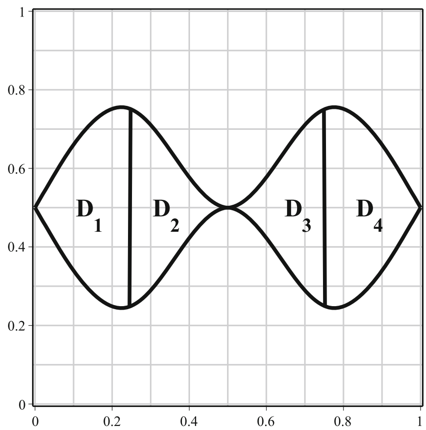
N	h	$\|e\|_2$		$\|e\|_\infty$			
		$q=1$	$q=2$	$q=1$	Ratio	$q=2$	Ratio
9	0.50	3.12×10^{-4}	3.12×10^{-4}	1.34×10^{-2}	–	1.02×10^{-2}	–
17	0.25	3.64×10^{-5}	1.71×10^{-5}	3.42×10^{-3}	3.92	1.27×10^{-3}	8.04
24	0.20	2.18×10^{-5}	1.23×10^{-5}	1.31×10^{-3}	2.61	6.49×10^{-4}	1.95
40	0.15	1.42×10^{-5}	4.68×10^{-6}	9.79×10^{-4}	1.33	3.39×10^{-4}	1.91
50	0.14	9.89×10^{-6}	3.41×10^{-6}	7.46×10^{-4}	1.31	1.91×10^{-4}	1.77

tables for each two consecutive rows. As can be seen, the error ratio is approximately $O(h^2)$ for the linear case and $O(h^3)$ for the quadratic case.

Example 5.3 Consider the following weakly singular two-dimensional integral equation

$$u(x,t) - \int_D \ln\sqrt{(x-y)^2+(t-s)^2}\,\sin(y^2+u(y,s))\mathrm{d}y\mathrm{d}s = f(x,t), \quad (x,t)\in D, \tag{85}$$

Fig. 8 The consideration domain D for Example 5.3



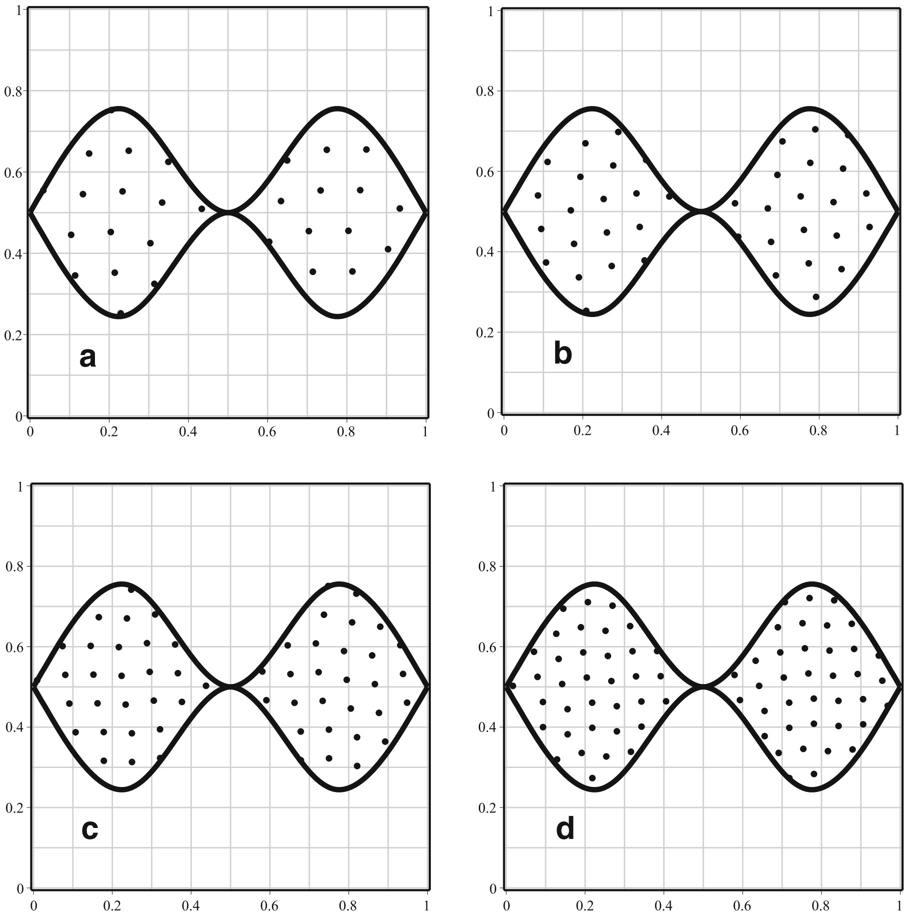


Fig. 9 Node distribution (**a**: 29, **b**: 40, **c**: 57 and **d**: 74 nodes) for Example 5.3

Table 4 Some numerical results of Example 5.3

N	h	$\Vert e\Vert_2$ with Gaussian weight function		$\Vert e\Vert_2$ with spline weight function	
		$q=1$	$q=2$	$q=1$	$q=2$
9	0.200	2.28×10^{-3}	3.38×10^{-4}	8.92×10^{-4}	2.32×10^{-4}
18	0.142	1.23×10^{-3}	1.38×10^{-4}	4.61×10^{-4}	8.77×10^{-4}
29	0.111	9.11×10^{-4}	6.73×10^{-5}	2.72×10^{-4}	3.65×10^{-5}
40	0.090	5.91×10^{-4}	4.04×10^{-5}	1.91×10^{-4}	2.37×10^{-5}
57	0.076	4.26×10^{-4}	1.92×10^{-5}	1.31×10^{-4}	1.35×10^{-5}
74	0.066	3.09×10^{-4}	1.88×10^{-5}	9.93×10^{-5}	9.56×10^{-6}

where the function $f(x,t)$ has been so chosen that the exact solution of the integral equation (85) is $u_{ex}(x,t) = \frac{1}{x+t+1}$. Here the bow tie shaped domain D is shown in Fig. 8 and the decomposition of $D = D_1 \cup D_2 \cup D_3 \cup D_4$ is determined as follows:

$$
\begin{aligned}
D_1 &= \left\{(x,t) \in \mathbb{R}^2 : \ 0 < x < \frac{1}{4}, \ \alpha(x) < t < 1 - \alpha(x)\right\}, \\
D_2 &= \left\{(x,t) \in \mathbb{R}^2 : \ \frac{1}{4} < x < \frac{1}{2}, \ \beta(x) < t < 1 - \beta(x)\right\}, \\
D_3 &= \left\{(x,t) \in \mathbb{R}^2 : \ \frac{1}{2} < x < \frac{3}{4}, \ \beta(-x+1) < t < 1 - \beta(-x+1)\right\}, \\
D_4 &= \left\{(x,t) \in \mathbb{R}^2 : \ \frac{3}{4} < x < 1, \ \alpha(-x+1) < t < 1 - \alpha(-x+1)\right\},
\end{aligned}
$$

where

$$
\alpha(x) = \frac{1}{2} + \frac{12x}{7} - \frac{80x^3}{7} \quad \text{and} \quad \beta(x) = \frac{-1}{14} + \frac{60x}{7} - \frac{92x^2}{7} + \frac{176x^3}{7}.
$$

The distribution of nodes is depicted in Fig. 9. Table 4 is presented the numerical results in terms of $\|e\|_2$ at different numbers of N for the linear and quadratic basis functions using Gaussian and spline weight functions. In computations, we put $\delta = 2 \times h$ for the linear case and $3 \times h$ for the quadratic case, where h is the distance between two consecutive nodes [39, 53]. In Gaussian weight function, we chose $\alpha = 0.5 \times h$. As we expected, from Theorem 4.3, the results gradually converge to the exact values as the number of nodes increases. The ratio of error remains approximately constant for the linear case (≈ 4) and for the quadratic case (≈ 8) so, the numerical results confirm the theoretical error estimates. The maximum error for $q = 1$ and $q = 2$ with Gaussian and spline weight functions are graphically shown in Fig. 10.

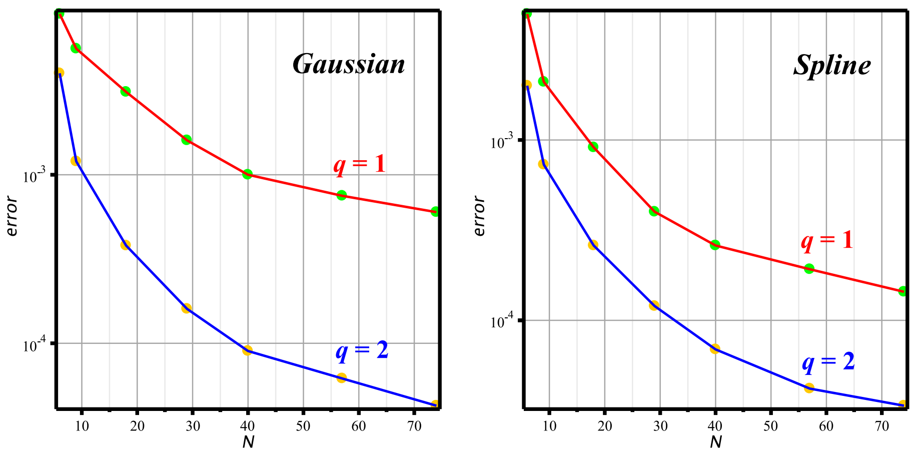


Fig. 10 Maximum error distributions of Example 5.3

Example 5.4 **Application to Volterra integral equations.**

A particular kind of the two-dimensional Volterra integral equation was motivated by the Darboux problem for hyperbolic partial differential equations of the form

$$u_{xt} = K(x, t, u(x, t)), \quad (x, t) \in D, \tag{86}$$

where $D = \{(x, t) \in \mathbb{R}^2 : \ 0 \le x \le 1, \ 0 \le t \le 1\}$. The values of the solution $u = u(x, t)$ are prescribed on part of the boundary as

$$\begin{aligned} u(x, 0) &= g_1(x) \ , \qquad 0 \le x \le 1, \\ u(0, t) &= g_2(t) \ , \qquad 0 \le t \le 1. \end{aligned}$$

The given functions g_1 and g_2 are assumed to satisfy $g_1(0) = g_2(0)$. This problem is equivalent to the Volterra integral equation [21]

$$u(x, t) - \int_0^x \int_0^t K(y, s, u(y, s))\mathrm{d}y\mathrm{d}s = g(x, t), \quad (x, t) \in D, \tag{87}$$

where the non-homogeneous term $g(x, y)$ contains the given initial values

$$g(x, t) = g_1(x) + g_2(t) - g_1(0). \tag{88}$$

Now the intervals $[0, x]$ and $[0, y]$ can be transferred to a fixed interval $[0, 1]$ by using simple linear transformations

$$y(x, \theta) = \theta x, \quad s(t, \eta) = \eta t,$$

Table 5 Some numerical results of Example 5.4 with Gaussian weight function

N	h	$\|e\|_2$		$\|e\|_\infty$			
		$q = 1$	$q = 2$	$q = 1$	Ratio	$q = 2$	Ratio
9	0.50	5.76×10^{-3}	4.29×10^{-4}	1.67×10^{-1}	–	1.88×10^{-2}	–
25	0.25	1.36×10^{-3}	5.49×10^{-5}	4.80×10^{-2}	3.47	2.81×10^{-3}	6.69
36	0.20	8.60×10^{-4}	2.75×10^{-5}	3.24×10^{-2}	1.48	1.62×10^{-3}	1.73
49	0.16	5.96×10^{-4}	4.26×10^{-5}	2.15×10^{-2}	1.50	4.89×10^{-3}	0.33
64	0.14	4.35×10^{-4}	4.39×10^{-5}	1.53×10^{-2}	1.40	5.76×10^{-3}	0.89

Table 6 Some numerical results of Example 5.4 with spline weight function

N	h	$\|e\|_2$		$\|e\|_\infty$			
		$q = 1$	$q = 2$	$q = 1$	Ratio	$q = 2$	Ratio
9	0.50	4.13×10^{-4}	2.61×10^{-4}	1.16×10^{-2}	–	9.75×10^{-3}	–
25	0.25	3.53×10^{-5}	3.19×10^{-5}	2.95×10^{-3}	3.92	1.69×10^{-3}	5.76
36	0.20	5.16×10^{-5}	1.63×10^{-5}	1.66×10^{-3}	1.78	6.02×10^{-4}	2.80
49	0.16	2.73×10^{-5}	8.62×10^{-6}	1.10×10^{-3}	1.49	3.79×10^{-4}	1.58
64	0.14	2.01×10^{-5}	5.00×10^{-6}	8.25×10^{-4}	1.34	2.16×10^{-4}	1.75

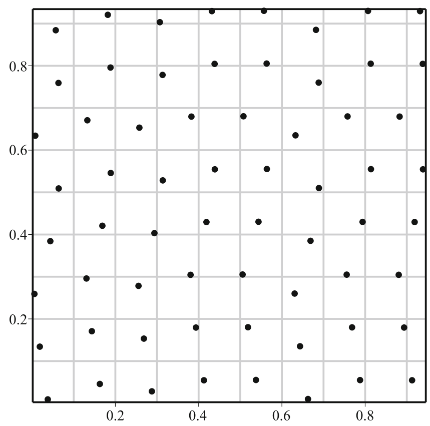


Fig. 11 Node distribution for Example 5.4 with $N = 64$

and so, (87) takes the following form

$$u(x,t) - \int_0^1 \int_0^1 xtK(y(x,\theta), s(t,\eta), u(y(x,\theta), s(t,\eta)))\mathrm{d}\theta\mathrm{d}\eta = g(x,t), \quad 0 \le x, t \le 1. \quad (89)$$

Now, the nonlinear Fredholm integral equation (89) can be solved by the proposed method.

Consider the following second kind nonlinear two-dimensional Volterra integral equation

$$u(x,t) - 4\int_0^x \int_0^t y^2 + e^{-2s}u^2(y,s)\mathrm{d}y\mathrm{d}s = x^2e^t + \frac{1}{14}x^7 - \frac{1}{14}e^{2t}x^7 - \frac{1}{5}x^5t, \quad 0 \le x, t \le 1, \quad (90)$$

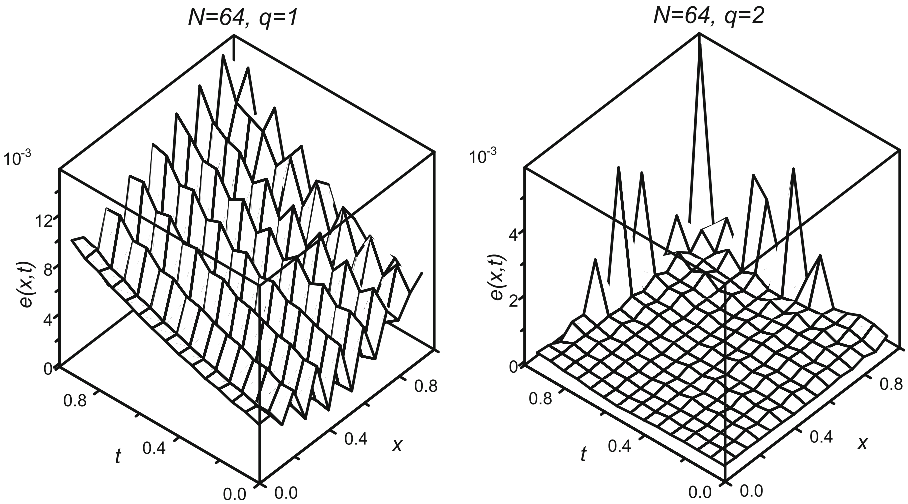


Fig. 12 Absolute error distributions of Example 5.4 with $N = 64$ and $q = 1, 2$ via Gaussian weight functions

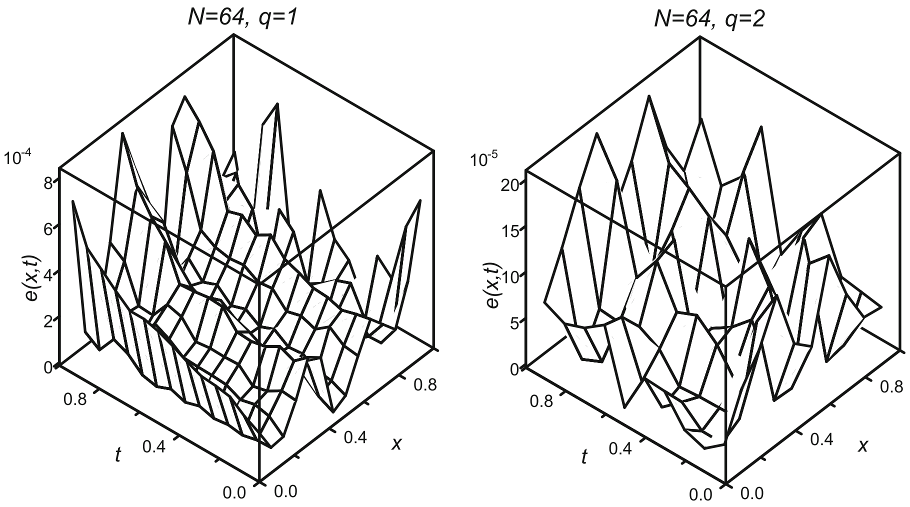


Fig. 13 Absolute error distributions of Example 5.4 with $N = 64$ and $q = 1, 2$ via spline weight functions

with the exact solution $u_{ex}(x,t) = x^2 e^t$ [49]. Tables 5 and 6 show $\|e\|_\infty$ and $\|e\|_2$ at the different number of the nodes that are regularly employed in unit square, for linear and quadratic basis functions and Gaussian and spline weight functions, respectively. The ratio of error, as $N \to \infty$, remains approximately constant for the linear case $O(h^2)$ and degrades for the quadratic case expectedly [54]. Here, $\delta = 2 \times h$ for linear basis and $\delta = 3 \times h$ for quadratic ones, where $h = \frac{1}{\sqrt{N}-1}$. In Gaussian weight functions $\alpha = 0.5 \times h$. The distribution of nodes is depicted in Fig. 11 for $N = 64$. Also, the absolute error for $q = 1, 2$ and $N = 64$ with Gaussian and spline weight functions are graphically shown in Figs. 12 and 13, respectively. Note that "INTSOLVE" command (the only command for solving integral equations in MAPLE) just solves one-dimensional linear Volterra integral equations whereas is unable to find the solution of two-dimensional nonlinear Volterra integral equations.

6 Conclusion

In this study, we developed an efficient and computationally attractive method to solve two-dimensional nonlinear integral equation of the second kind on non-rectangular domains. The method is based on the use of the shape functions of the moving least squares (MLS) approximation. This approach does not need any background interpolation or approximation cells and so it is a meshless method. The proposed scheme can be used in various kinds of regions, because it dose not depend to the geometry of the domain. The error analysis is provided for the method. We can also expand this scheme to 3D problems and other classes of integral equations such as integro-differential and first kind integral equations with little additional work.

The convergence accuracy of the new method was examined in three nonlinear Fredholm integral equations on non-rectangular domains and one 2D nonlinear Volterra integral equation.

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