Numer Algor (2014) 67:33–48
DOI 10.1007/s11075-013-9772-1

ORIGINAL PAPER

Complexity analysis of an interior-point algorithm for linear optimization based on a new proximity function

M. Reza Peyghami · S. Fathi Hafshejani

Received: 12 March 2013 / Accepted: 18 September 2013 / Published online: 14 January 2014

Abstract Kernel functions play an important role in the complexity analysis of the interior point methods for linear optimization. In this paper, we present a primal-dual interior point method for linear optimization based on a new kernel function consisting of a trigonometric function in its barrier term. By simple analysis, we show that the feasible primal-dual interior point methods based on the new proposed kernel function enjoys $O\left(\sqrt{n}\,(\log n)^2 \log \frac{n}{\epsilon}\right)$ worst case complexity result which improves the results obtained by El Ghami et al. (J Comput Appl Math 236:3613–3623, 2012) for the kernel functions with trigonometric barrier terms.

Keywords Kernel function · Linear optimization · Primal-dual interior-point methods · Large-update methods

1 Introduction

In this paper, we focus on the primal Linear Optimization (LO) problem as

$$(P) \qquad \min\left\{c^T x : Ax = b, x \geq 0\right\},$$

along with its dual problem as follows:

$$(D) \qquad \max\left\{b^T y : A^T y + s = c, s \geq 0\right\},$$

M. R. Peyghami (✉) · S. F. Hafshejani
Department of Mathematics, K.N. Toosi Univ. of Tech., P.O. Box 16315-1618, Tehran, Iran
e-mail: peyghami@kntu.ac.ir

M. R. Peyghami · S. F. Hafshejani
Scientific Computations in Optimization and Systems Engineering (SCOPE),
K.N. Toosi Univ. of Tech., P.O. Box 16315-1618, Tehran, Iran
e-mail: sfathi@mail.kntu.ac.ir

where $A \in \mathbb{R}^{m\times n}$ with $rank(A) = m$, $x, c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ and $s \in \mathbb{R}^n$.

Polynomial time Interior Point Methods (IPMs) for solving linear programming were first proposed by Karmarkar in [5]. The primal-dual IPMs have been developed by Kojima et al. in [6] and Megiddo in [7] for the first time. Nowadays, researches in this field spread out aiming at generating practically efficient methods with low complexity results in worst case.

Interior point methods based on barrier functions have been widely studied in the literature. The so called self-concordant barrier functions were first proposed by Nesterov and Nemirovskii in [9]. These functions allowed the IPMs for LO to be extended to the more general class of convex optimization problems, such as nonlinear complementarity problems, second order cone optimization (SOCO) and semidefinite optimization (SDO) problems. Note that the so far best known iteration complexity for LO in the large neighborhood of the central path, based on a self-concordant function, is $O\left(n \log \frac{n}{\epsilon}\right)$. This bound has been improved by Peng et al. in [10] by using the so called Self-Regular proximity functions. Indeed, they proposed a variant of primal-dual IPMs based on SR kernel functions for LO and its extension to SDO, SOCO and complementary problems and obtained the so far best known worst case iteration complexity as $O\left(\sqrt{n}\log n \log \frac{n}{\epsilon}\right)$. Since then, several attempts have been done for introducing non-SR kernel functions in order to at least meeting the SR based complexity results, see e.g. [2, 3, 11]. We refer the interested reader to the comparative study on the kernel functions in [1]. Recently, El Ghami et al. in [4] introduced a trigonometric kernel function for the first time and analyzed the feasible primal-dual IPMs based on this kernel function. They showed that the primal-dual interior point methods for solving LO enjoys $O\left(n^{\frac{3}{4}} \log \frac{n}{\epsilon}\right)$ as the worst case iteration complexity.

In this paper, we introduce the function

$$\psi(t) = \frac{t^2-1}{2} - \int_1^t e^{3(\tan(h(x))-1)}dx, \tag{1}$$

with

$$h(t) = \frac{\pi}{2+2t}, \tag{2}$$

as a new kernel function and provide a primal-dual interior point algorithm based on the proximity measure induced by this function. Our main focus is on analyzing the large-update version of the aforementioned algorithm in the large neighborhood of the central path. Using some easy to check conditions and simple analysis, we show that the worst case iteration complexity for primal-dual IPMs based on the proposed kernel function enjoys $O\left(\sqrt{n}\,(\log n)^2 \log \frac{n}{\epsilon}\right)$. This result improves significantly the iteration bound for linear optimization problems obtained by El Ghami et al. in [4].

The paper is organized as follows: In Section 2, we recall some basic concepts of interior point methods and the central path curve for LO. Some interesting and useful properties of the new kernel function are provided in Section 3. Section 4 is devoted to describe the proximity reduction during an inner iteration. An for the step size is discussed in Section 5. The worst case iteration bound for the primal-dual IPMs based on the new IPMs based on the new kernel function is given in Section 6. Finally, we end up the paper by some concluding remarks in Section 7.

We use the following notational conventions: Throughout the paper, $\|.\|$ denotes the Euclidian norm of a vector. The nonnegative and positive orthants are denoted by $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. For a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, x_* is the minimum component of x. For given vectors x and s, the vectors xs and $\frac{x}{s}$ denote the coordinate-wise operations on the vectors, i.e., whose components are $x_i s_i$ and $\frac{x_i}{s_i}$, respectively. We say that $f(t) = \Theta(g(t))$, if there exist positive constants ω_1 and ω_2 so that $\omega_1 g(t) \leq f(t) \leq \omega_2 g(t)$ satisfies for all $t \in \mathbb{R}_{++}$. We also say $f(t) = O(g(t))$, if there exists a positive constant ω so that $f(t) \leq \omega g(t)$, for all $t \in \mathbb{R}_{++}$.

2 Preliminaries

In this section, we briefly describe the idea behind the interior point methods based on kernel functions. We also provide the structure of the generic primal-dual IPMs when the kernel functions are used to induce the proximity measure.

One knows that for the feasible primal and dual linear optimization problems, i.e. problems (P) and (D), an optimal solution can be found by solving the following system:

$$\begin{aligned} Ax &= b, & x &\geq 0, \\ A^T y + s &= c, & s &\geq 0, \\ xs &= 0, \end{aligned} \tag{3}$$

where xs denotes the coordinate wise (Hadamard) product of the vectors x and s. In this system, the first and second equations are the primal and dual feasibility, respectively, while the third equation is the so called centering equation or complementary equation. Without loss of generality we assume that both problems (P) and (D) satisfy the Interior Point Condition (IPC) [12], i.e., there exist x^0 and $\left(y^0, s^0\right)$ such that

$$\begin{aligned} Ax^0 &= b, & x^0 &> 0, \\ A^T y^0 + s^0 &= c, & s^0 &> 0. \end{aligned}$$

In the primal-dual IPMs, the centering equation in (3) is replaced by a parametric equation $xs = \mu\mathbf{e}$, where μ is a positive parameter and $\mathbf{e}$ denotes the all-one vector, i.e., $\mathbf{e} = (1, 1, \ldots, 1)^T$. This leads us to the following parametric system:

$$\begin{aligned} Ax &= b, & x &> 0, \\ A^T y + s &= c, & s &> 0, \\ xs &= \mu\mathbf{e}. \end{aligned} \tag{4}$$

It is shown that, under IPC condition and $rank(A) = m$, system (4) has a unique solution, for each $\mu > 0$. This solution is denoted by $(x(\mu), y(\mu), s(\mu))$, where $x(\mu)$ and $(y(\mu), s(\mu))$ are called the μ-centers of (P) and (D), respectively. The set of all μ-centers, with $\mu > 0$, forms the so called *central path* for (P) and (D) [8, 13]. The central path for LO was first recognized by Sonnevend [13] and Megiddo [7]. It is proved that as $\mu \to 0$, the limit of the central path exists and converges to an analytic center of the optimal solutions set of (P) and (D).

Let $\mu > 0$ be fixed. A direct application of the Newton method on (4) provides the following system for Δx, Δy and Δs:

$$\begin{aligned} A\Delta x &= 0, \\ A^T\Delta y + \Delta s &= 0, \\ x\Delta s + s\Delta x &= \mu\mathbf{e} - xs. \end{aligned} \tag{5}$$

Thus, the new iterate is computed by

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \alpha\Delta y, \quad s_+ = s + \alpha\Delta s,$$

where $\alpha \in (0, 1]$ is chosen appropriately so that $(x_+, s_+) > 0$. For simplicity, let us change the vector space from (x, s) to the scaled vector space v which is defined by

$$v := \sqrt{\frac{xs}{\mu}}.$$

Therefore, the Newton system (5) in the v-space can be written as follows:

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T d_y + d_s &= 0, \\ d_x + d_s &= v^{-1} - v, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \bar{A} &:= \frac{1}{\mu} AV^{-1}X = AS^{-1}V \\ V &:= \mathrm{diag}(v),\, X := \mathrm{diag}(x),\, S := \mathrm{diag}(s) \\ d_x &= \frac{v\Delta x}{x}, \qquad d_s = \frac{v\Delta s}{s}. \end{aligned} \tag{7}$$

Note that $d_x = d_s = 0$ if and only if $v - v^{-1} = 0$ if and only if $x = e$ if and only if $x = x(\mu)$, $s = s(\mu)$. A crucial observation in (6) is that the right hand side of the third equation is the minus gradient of the proximity function $\Psi_c(v) = \sum_{i=1}^n \psi_c(v_i)$. This proximity function is induced by the kernel function $\psi_c(t) = \frac{t^2-1}{2} - \log t$, for $t > 0$, which is a strictly differentiable convex function on $\mathbb{R}^n_{++}$ with $\psi_c(1) = \psi_c'(1) = 0$.

The new variant of primal-dual IPMs was developed by Peng et al. in [10] in which the proximity function $\Psi_c(v)$ is replaced by a proximity function $\Psi(v) = \sum_{i=1}^n \psi(v_i)$, where $\psi(t)$ is any strictly differentiable convex barrier function on $\mathbb{R}^n_{++}$ with $\psi(1) = \psi'(1) = 0$. In this case, system (6) is converted to the following system with new centering equation:

$$\begin{aligned} \bar{A}d_x &= 0, \\ \bar{A}^T d_y + d_s &= 0, \\ d_x + d_s &= -\nabla\Psi(v). \end{aligned} \tag{8}$$

Now, we describe one step of the IPMs based on the kernel function. Starting with the interior point (x_0, y_0, s_0), $\mu_0 > 0$, an accuracy $\epsilon > 0$ and the proximity function $\Psi(v)$, let a good approximation of the μ-center $(x(\mu), y(\mu), s(\mu))$ be known for $\mu > 0$. Then, the parameter μ is decreased by a factor $1 - \theta$, for $\theta \in (0, 1)$, and set $\mu := (1 - \theta)\mu$. An approximate solution of the μ-center is obtained by frequently using the Newton method. Indeed, we first solve system (8) for d_x and d_s and then find the Newton directions Δx, Δy and Δs by using (7). This procedure is repeated until we get to the point in which $x^T s < \epsilon$. In this case, we say that the current x and (y, s) are ϵ-approximate solutions of the primal and dual problems, respectively.

Now, we can outline the above procedure in the following primal-dual interior point scheme [10].

Algorithm 1 Generic Primal-dual IPM for LO

Input

a proximity function $\Psi(v)$

a threshold parameter $\tau > 0$

an accuracy parameter $\varepsilon > 0$

a barrier update parameter θ, $0 < \theta < 1$

begin

$x := \mathbf{e}$; $s := \mathbf{e}$; $\mu := 1$; $v := \mathbf{e}$;

while $n\mu > \varepsilon$ do

begin

$\mu := (1 - \theta)\mu$;

while $\Psi(v) > \tau$ do

begin

$x := x + \alpha \triangle x$

$s := s + \alpha \triangle s$

$y := y + \alpha \triangle y$

$v := \sqrt{\frac{xs}{\mu}}$

end

end

end

Algorithm 1 consists of inner and outer while loops (iterations). Each outer iteration includes an update of parameter μ and a sequence of (one or more) inner iteration. The total number of iterations is described as a function of the dimension n and ϵ. The choice of the barrier update parameter θ plays an important role in theory and practice of IPMs. For a constant θ, let say $\theta = \frac{1}{2}$, the algorithm is called the large-update method, while for the case when the θ is depended on n, let say $\theta = \frac{1}{\sqrt{n}}$, the algorithm is named the small-update method. Note that, small-update methods have the best iteration bound in theory while the large update methods are practically efficient [12].

3 The new kernel function and its properties

This section is devoted to provide some essential properties of the new kernel function, given by (1), which will be used in the complexity analysis of Algorithm 1. For this purpose, we need the first three derivatives of the function (1), which are:

$$\psi'(t) = t - e^{3(\tan(h(t))-1)} \tag{9}$$

$$\psi''(t) = 1 + \frac{6\pi}{(2+2t)^2}(1+\tan^2(h(t)))e^{3(\tan(h(t))-1)} \tag{10}$$

$$\psi'''(t) = \left(1+\tan^2(h(t))\right) e^{3(\tan(h(t))-1)} k(t), \tag{11}$$

where:

$$k(t) = -\frac{24\pi(2+2t) + 24\pi^2\tan(h(t)) + 36\pi^2(1+\tan^2(h(t)))}{(2+2t)^4}. \tag{12}$$

Lemma 3.1 *For the function $h(t)$, defined by (2), we have:*

$$\tan(h(t)) \geq \frac{1}{\pi t}, \qquad for\ all\ t \in (0, 1].$$

Proof The proof is similar to the proof of Lemma 2.1 in [4], however we restate it here. Let $g(t)$ be defined as

$$g(t) := \tan(h(t)) - \frac{1}{\pi t}.$$

For this function, we have:

$$g'(t) = \frac{h'(t)}{\cos^2(h(t))} + \frac{1}{\pi t^2} = \frac{1}{\pi t^2\cos^2(h(t))}(h'(t)\pi t^2 + \cos^2(h(t))),$$

where $h'(t) = \frac{-2\pi}{(2+2t)^2}$. Now, as $\frac{\pi}{4} \leq h(t) < \frac{\pi}{2}$, for all $t \in (0, 1]$, then we obtain

$$\sin\left(\frac{\pi}{2} - h(t)\right) = \cos(h(t)) \leq \frac{\pi}{2} - h(t).$$

This implies that

$$\begin{aligned} g'(t) &= \frac{1}{\pi t^2\cos^2(h(t))}\left(h'(t)\pi t^2 + \sin^2\left(\frac{\pi}{2} - h(t)\right)\right) \\ &\leq \frac{1}{\pi t^2\cos^2(h(t))}\left(h'(t)\pi t^2 + \left(\frac{\pi}{2} - h(t)\right)^2\right) \\ &\leq \frac{1}{\pi t^2\cos^2(h(t))}\left(\frac{-\pi^2 t^2}{(2+2t)^2}\right) < 0. \end{aligned}$$

Thus, $g(t)$ is a decreasing function on $(0, 1]$. Now, the proof is completed by considering the fact that $g(1) > 0$. □

Now, we provide some technical lemma about this function.

Lemma 3.2 *Let the function $\psi(t)$ be defined as in (1). Then, we have:*

i) $\psi''(t) > 1, \qquad for\ all\ t > 0.$
ii) $t\psi''(t) - \psi'(t) > 0, \qquad for\ all\ t > 1.$
iii) $t\psi''(t) + \psi'(t) > 0, \qquad for\ all\ t > 0.$
iv) $\psi'''(t) < 0, \qquad for\ all\ t > 0.$

Proof We first prove that (i) holds. To do so, we first note that, for all $t > 0$, we have $0 \le h(t) < \frac{\pi}{2}$, which implies that $\tan(h(t)) > 0$ and

$$\psi''(t) = 1 + \frac{6\pi}{(2+2t)^2}\left(1+\tan^2(h(t))\right)e^{3(\tan(h(t))-1)} \ge 1.$$

For proving (ii), we have:

$$t\psi''(t) - \psi'(t) = \left(\frac{6\pi t}{(2+2t)^2}\left(1+\tan^2(h(t))\right)+1\right)e^{3(\tan(h(t))-1)} > 0.$$

Now, we prove that (iii) holds. For this purpose, let $0 < t \le \frac{1}{2}$. Using Lemma 3.1 and the fact that $\tan(h(t)) \ge \sqrt{3}$ holds for all $t \in \left(0, \frac{1}{2}\right]$, we have

$$\begin{aligned} t\psi''(t) + \psi'(t) &= 2t + \left(\frac{6\pi t}{(2+2t)^2}\left(1+\tan^2(h(t))\right)-1\right)e^{3(\tan(h(t))-1)} \\ &\ge 2t + \left(\frac{6\pi t}{(2+2t)^2}+\frac{6\sqrt{3}}{(2+2t)^2}-1\right)e^{3(\tan(h(t))-1)} > 0. \end{aligned}$$

On the other hand, for $\frac{1}{2} < t \le 1$, we have $\left(1+\tan^2(h(t))\right) \ge 2$, which implies that

$$t\psi''(t) + \psi'(t) \ge 2t + \left(\frac{6\pi}{(2+2t)^2}-1\right)e^{3(\tan(h(t))-1)} > 0.$$

To complete the proof of (iii), it remains to show that the inequality holds for $t > 1$. Using the fact that $\psi'(1) = 0$ and $\psi'(t)$ is a strictly increasing function, for all $t > 0$, we obtain $\psi'(t) > 0$, for all $t > 1$. This implies that $t\psi''(t) + \psi'(t) > 0$, for all $t > 1$. Therefore the proof is completed for the item (iii).

The proof of (iv) is trivial from (12).

□

The third part of Lemma 3.2 is known as exponential convexity (or simply e-convexity) property of the kernel function $\psi(t)$. The following lemma provides equivalent forms of the e-convexity property [1].

Lemma 3.3 *(Lemma 2.1.2 in [10]) Let $\psi(t)$ be a twice differentiable kernel function for $t > 0$. Then, the following three properties are equivalent:*

i) $\psi(\sqrt{t_1 t_2}) \le \frac{1}{2}(\psi(t_1) + \psi(t_2)), \qquad for \quad t_1, t_2 > 0.$
ii) $\psi\prime(t) + t\psi''(t) \ge 0, \qquad for \quad t > 0.$
iii) $\psi(e^{\xi})$ *is a convex function.*

The e-convexity property plays an important role in the analysis of the primal-dual interior point methods based on kernel functions.

In what follows, we explore some results related to the new kernel function which are important in the complexity analysis of Algorithm 1. To do so, we first define the norm-based proximity measure $\delta(v)$ as follows:

$$\delta(v) := \frac{1}{2}\|\nabla\Psi(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n}(\psi'(v_i))^2}, \qquad v \in \mathbb{R}^n_{++}. \tag{13}$$

Based on this measure and the fact that from $\psi(1) = \psi'(1) = 0$, the function $\psi(t)$ can be described by its second derivative according to:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi, \tag{14}$$

we have:

Lemma 3.4 *Let the kernel function $\psi(t)$ be defined as in (1). Then, we have:*

i) $\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi\prime(t)^2$, *for all* $t > 0$.
ii) $\Psi(v) \leq 2\delta(v)^2$.
iii) $\|v\| \leq \sqrt{n} + \sqrt{2\Psi(v)} \leq \sqrt{n} + 2\delta(v)$.

Proof The proof is similar to the proof of Lemma 3.4 in [11] using (14) and Lemma 3.2. Therefore, we omit it here. □

In the next section the effect of updating the barrier parameter μ on the value of the proximity measure is investigated.

4 The effect of barrier update on proximity function

In this section, we study the growth behavior of the proximity function after a μ-update. For given τ, at the start of any outer iteration of Algorithm 1 and just before updating of μ, we have $\Psi(v) \leq \tau$. For $0 < \theta < 1$, and due to the updating strategy of μ, i.e. $\mu := (1-\theta)\mu$, the vector v is divided by a factor $\sqrt{1-\theta}$. This leads to an increase in the value of $\Psi(v)$, in general. The subsequent inner iterations are performed in order to bring the values of $\Psi(v)$ back to the situation where we have $\Psi(v) \leq \tau$. Therefore, just after updating of μ, the largest values of $\Psi(v)$ occur. Therefore, we have the following results which are important in deriving the iteration complexity of the Algorithm 1.

Lemma 4.1 *For given $\beta \geq 1$, we have*

$$\psi(\beta t) \leq \psi(t) + \frac{1}{2}\left(\beta^2 - 1\right)t^2.$$

Proof Let us define $\psi(t)$ as

$$\psi(t) = \frac{1}{2}\left(t^2 - 1\right) + p(t),$$

where the function $p(t)$ is defined as follows:

$$p(t) = -\int_1^t e^{3(\tan(h(x))-1)}dx.$$

Then, we have

$$\psi(\beta t) - \psi(t) = \frac{1}{2}\left(\beta^2 - 1\right)t^2 + p(\beta t) - p(t).$$

Since $\beta \geq 1$, to complete the proof, it is sufficient to show that the function $p(t)$ is a decreasing function. For this purpose, we have

$$p'(t) = -e^{3(\tan(h(t))-1)},$$

which is negative for all $t > 0$. □

Lemma 4.2 *Let* $0 < \theta < 1$ *and* $v_+ = \frac{v}{\sqrt{1-\theta}}$. *Then, one has*

$$\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}\left(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n\right).$$

Proof Applying Lemma 4.1 with $\beta = \frac{1}{\sqrt{1-\theta}}$, we have

$$\Psi(\beta v) \leq \Psi(v) + \frac{1}{2}\sum_{i=1}^{n}\left(\beta^2 - 1\right)v_i^2 = \Psi(v) + \frac{\theta\|v\|^2}{2(1-\theta)}.$$

Now, the lemma is easily followed by using the third item of Lemma 3.4 on this inequality. □

5 An estimation for the step size

This section is devoted to obtain a default value for the step size during an inner iteration. After an inner iteration, the new point is computed by

$$x_+ = x + \alpha\Delta x, \quad y_+ = y + \alpha\Delta y, \quad s_+ = s + \alpha\Delta s,$$

where α is the step size. From the centering equation in (7), we have:

$$x_+ = \frac{x}{v}(v + \alpha d_x), \quad y_+ = y + \alpha\Delta y, \quad s_+ = \frac{s}{v}(v + \alpha d_s).$$

Thus,

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s).$$

The e-convexity property of the function $\Psi(v)$ implies that

$$\Psi(v_+) = \Psi\left(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}\right) \leq \frac{1}{2}\left[\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)\right].$$

Let

$$f(\alpha) = \Psi(v_+) - \Psi(v), \tag{15}$$

$$f_1(\alpha) = \frac{1}{2}\left[\Psi(v+\alpha d_x) + \Psi(v+\alpha d_s)\right] - \Psi(v). \tag{16}$$

The function $f_1(\alpha)$ is a convex function with respect to α since Ψ is a convex function and its arguments in the first two terms are linear with respect to α. The first and second derivatives of the function $f_1(\alpha)$ with respect to α are:

$$f_1'(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi'(v_i+\alpha d_{x_i})d_{x_i} + \psi'(v_i+\alpha d_{s_i})d_{s_i}\right),$$

$$f_1''(\alpha) = \frac{1}{2}\sum_{i=1}^{n}\left(\psi''(v_i+\alpha d_{x_i})d_{x_i}^2 + \psi''(v_i+\alpha d_{s_i})d_{s_i}^2\right).$$

Note that

$$f_1'(0) = \frac{1}{2}\nabla\Psi(v)^T(d_x+d_s) = -\frac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2,$$

where the last equality is obtained from (13). For simplicity, in the sequel, we use the following notations:

$$v_* = \min(v), \qquad \delta := \delta(v).$$

The following results introduce the conditions on α in which we have $f_1'(\alpha) < 0$ which implies that the function $f(\alpha)$ decreases during an inner iteration using the fact that $f(0) = f_1(0) = 0$ and $f(\alpha) \le f_1(\alpha)$.

Lemma 5.1 *(Lemma 4.1 in* [1]*) Assume that f_1 is defined by (16). Then, the second derivative of the function f_1 with respect to α satisfies the following inequality:*

$$f_1''(\alpha) \le 2\delta^2\psi''(v_* - 2\alpha\delta).$$

Lemma 5.2 *(Lemma 4.2 in* [1]*) The inequality $f_1'(\alpha) \le 0$ holds if α satisfies the following inequality:*

$$-\psi'(v_* - 2\alpha\delta) + \psi'(v_*) \le 2\delta. \tag{17}$$

Lemma 5.3 *(Lemma 4.3 in* [1]*) Let $\rho : [0,\infty) \to (0,1]$ be the inverse of the function $-\frac{1}{2}\psi'(t)$ in the interval $(0,1]$. Then, the largest possible value for α in order to satisfy* (17) *is given by*

$$\overline{\alpha} = \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)). \tag{18}$$

Lemma 5.4 *(Lemma 4.4 in* [1]*) Let $\overline{\alpha}$ be defined as in* (18)*. Then, it yields the following inequality:*

$$\overline{\alpha} \ge \frac{1}{\psi''(\rho(2\delta))}. \tag{19}$$

Due to Lemmas 5.3 and 5.4, in the sequel, we set the following value for the step size in Algorithm 1 as the default value:

$$\tilde{\alpha} = \frac{1}{\psi''(\rho(2\delta))}. \tag{20}$$

It is easily seen that $\tilde{\alpha} \leq \overline{\alpha}$. The following lemma helps us to provide an upper bound for the decrease of the proximity function during an inner iteration.

Lemma 5.5 *(Lemma 3.5 in [4]) Suppose that the step size α is such that $\alpha \leq \bar{\alpha}$, thus we have:*

$$f(\alpha) \leq -\alpha\delta^2.$$

Now, we are ready to provide the amount of decrease in the proximity function during an inner iteration by considering the default value for the step size, i.e. $\alpha = \tilde{\alpha}$.

Lemma 5.6 *Let $\Psi(v) \geq 1$, and ρ and $\tilde{\alpha}$ be defined as in Lemma 5.3 and (20), respectively. Then, we have*

$$f(\tilde{\alpha}) \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq -\Theta\left(\frac{\delta}{\left(1+\frac{1}{3}\log(4\delta+1)\right)^2}\right). \tag{21}$$

Proof From Lemma 5.5 and the fact that $\tilde{\alpha} \leq \overline{\alpha}$ imply that $f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2$, which gives the first inequality.

To prove the last inequality, we need to compute the inverse function of $-\frac{1}{2}\psi'(t)$ in the interval $(0, 1]$. For this purpose, we solve the equation $-\frac{1}{2}\psi'(t) = s$ for t. We have

$$-\left(t - e^{3(\tan(h(t))-1)}\right) = 2s.$$

This implies that,

$$e^{3(\tan(h(t))-1)} \leq 2s + 1, \tag{22}$$

where the last inequality is obtained from the fact that $t \leq 1$. Now, letting $t = \rho(2\delta)$, we get $4\delta = -\psi'(t)$. Thus, from (22), we have

$$e^{3(\tan(h(t))-1)} \leq 4\delta + 1 \tag{23}$$

$$\Rightarrow 3(\tan(h(t)) - 1) \leq \log(4\delta + 1)$$

$$\Rightarrow \tan(h(t)) \leq \left(1 + \frac{1}{3}\log(4\delta + 1)\right). \tag{24}$$

Now, we obtain a lower bound for $\tilde{\alpha}$. Using (23) and (24), for all $0 < t \leq 1$, we have:

$$
\begin{aligned}
\tilde{\alpha} = \frac{1}{\psi''(t)} &= \frac{1}{1 + \frac{6\pi}{(2+2t)^2}(1+\tan^2(h(t)))e^{3(\tan(h(t))-1)}} \\
&\geq \frac{1}{1 + \frac{3\pi}{2}(4\delta+1) + \frac{3\pi}{2}(4\delta+1)\left(1+\frac{1}{3}\log(4\delta+1)\right)^2} \\
&\geq \frac{1}{2\delta + 9\pi\delta + 9\pi\delta\left(1+\frac{1}{3}\log(4\delta+1)\right)^2} \\
&\geq \frac{1}{2(1+9\pi)\delta\left(1+\frac{1}{3}\log(4\delta+1)\right)^2} \\
&\geq \Theta\left(\frac{1}{\delta\left(1+\frac{1}{3}\log(4\delta+1)\right)^2}\right),
\end{aligned}
$$

where the second inequality is obtained from the second part of Lemma 3.4 with $\Psi(v) \geq 1$. This implies that

$$
f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 \leq -\frac{\delta^2}{\psi''(\rho(2\delta))} \leq -\Theta\left(\frac{\delta}{\left(1+\frac{1}{3}\log(4\delta+1)\right)^2}\right).
$$

This completes the proof of lemma. □

Note that, using the second part of Lemma 3.4, we have $\delta \geq \frac{1}{\sqrt{2}}\Psi^{\frac{1}{2}}$. Therefore, an immediate consequence of Lemma 5.6 is:

$$
f(\tilde{\alpha}) \leq -\Theta\left(\frac{\delta}{\left(1+\frac{1}{3}\log(4\delta+1)\right)^2}\right) \leq -\Theta\left(\frac{\Psi^{\frac{1}{2}}}{\left(1+\frac{1}{3}\log\Psi\right)^2}\right). \tag{25}
$$

6 Iteration complexity

In this section, the worst case total iteration complexity for the Algorithm 1 based on the proximity measure Ψ induced from the kernel function ψ defined by (1) is derived. As before, the value $\tilde{\alpha}$, defined by (20), is utilized as a default value for the step size during an inner iteration.

Using Lemma 4.2, we have

$$
\Psi(v_+) \leq \Psi(v) + \frac{\theta}{2(1-\theta)}(2\Psi(v) + 2\sqrt{2n\Psi(v)} + n), \tag{26}
$$

right after updating the parameter μ to $(1-\theta)\mu$, for $\theta \in (0, 1)$. Since we are interested to work in large neighborhood of the central path, we assume that $\tau = O(n) \geq 1$. Moreover, in the large update methods we have $\theta = \Theta(1)$.

At the start of an outer iteration and just before updating of the parameter μ, we have $\Psi(v) \leq \tau$. Based on (26), the $\Psi(v)$ exceeds the threshold τ after updating of μ. Therefore, we need to compute the number of inner iterations that are required to return the iterations back to the situation where $\Psi(v) \leq \tau$. Let us denote the value of $\Psi(v)$ after μ-update by Ψ_0, and the subsequent values by Ψ_j, for $j = 1, \ldots, L-1$, where L is the total number of inner iterations in an outer iteration. Relation (26) together with $\Psi(v) \leq \tau = O(n)$ imply that

$$\Psi_0 \leq \tau + \frac{\theta}{2(1-\theta)}\left(2\tau + 2\sqrt{2n\tau} + n\right) = O(n). \tag{27}$$

Now, from (15) and (25) and using the fact that in the inner iteration we have $\Psi_j > \tau \geq 1$, the decrease of Ψ in each inner iteration is given by

$$\Psi_{j+1} \leq \Psi_j - \kappa \Delta\Psi_j \qquad j = 0, 1, \ldots, L-1, \tag{28}$$

where κ is some positive constant and $\Delta\Psi_j$ is defined by

$$\Delta\Psi_j = \frac{\Psi_j^{\frac{1}{2}}}{\left(1 + \frac{1}{3}\log \Psi_j\right)^2}. \tag{29}$$

Now, we are in situation to state the inner iteration complexity result in an outer iteration. The following technical lemma helps us in this purpose. One can find its proof in [10].

Lemma 6.1 *Given $\alpha \in [0, 1]$ and $t \geq -1$, one has*

$$(1+t)^{\alpha} \leq 1 + \alpha t.$$

Now, the worst case upper bound for the total number of inner iterations in an outer iteration is provided in the following theorem.

Theorem 6.1 *Let μ be updated by $\mu := (1-\theta)\mu$ and $\tau \geq 1$. Then, the number of inner iterations that are required to return the iterations back to the situation where $\Psi(v) \leq \tau$ is bounded above by*

$$L \leq 1 + \frac{2\left(1 + \frac{1}{3}\log \Psi_0\right)^2}{\kappa}\Psi_0^{\frac{1}{2}}. \tag{30}$$

Proof Using (28), for all $j = 0, 1, \ldots, L-1$, we have

$$\begin{aligned}
0 \le \Psi_{j+1}^{\frac{1}{2}} &\le \left(\Psi_j - \kappa \frac{\Psi_j^{\frac{1}{2}}}{\left(1+\frac{1}{3}\log\Psi_j\right)^2}\right)^{\frac{1}{2}} \\
&= \Psi_j^{\frac{1}{2}} \left(1 - \kappa \frac{\Psi_j^{-\frac{1}{2}}}{\left(1+\frac{1}{3}\log\Psi_j\right)^2}\right)^{\frac{1}{2}} \\
&\le \Psi_j^{\frac{1}{2}} \left(1 - \kappa \frac{\Psi_j^{-\frac{1}{2}}}{2\left(1+\frac{1}{3}\log\Psi_j\right)^2}\right) \\
&= \Psi_j^{\frac{1}{2}} - \frac{\kappa}{2\left(1+\frac{1}{3}\log\Psi_j\right)^2},
\end{aligned} \tag{31}$$

where the last inequality is obtained from Lemma 6.1. By subsequently using (31), we obtain

$$\Psi_{j+1}^{\frac{1}{2}} \le \Psi_0^{\frac{1}{2}} - \frac{j\kappa}{2\left(1+\frac{1}{3}\log\Psi_0\right)^2}.$$

Letting $j = L-1$, we obtain

$$0 \le \Psi_L^{\frac{1}{2}} \le \Psi_0^{\frac{1}{2}} - \frac{(L-1)\kappa}{2\left(1+\frac{1}{3}\log\Psi_0\right)^2},$$

which implies that

$$L \le 1 + \frac{2\left(1+\frac{1}{3}\log\Psi_0\right)^2}{\kappa}\Psi_0^{\frac{1}{2}}.$$

This completes the proof of the theorem. □

Now, using (27), for the large update interior point methods, we have $\Psi_0 = O(n)$. Therefore, from Theorem 6.1, we obtain the following upper bound for the total number of inner iterations in an outer iteration:

$$L \le \left\lceil 1 + \frac{2\left(1+\frac{1}{3}\log\Psi_0\right)^2}{\kappa}\Psi_0^{\frac{1}{2}} \right\rceil = \left\lceil O\left(\sqrt{n}(\log n)^2\right)\right\rceil. \tag{32}$$

On the other hand, for $\theta = \Theta(1)$ and given accuracy parameter $\epsilon > 0$, the total number of outer iterations for getting $n\mu \le \epsilon$ are bounded above by $O\left(\frac{1}{\theta}\log\frac{n}{\epsilon}\right)$, see Lemma I.36 in [12]. Now, the total number of iterations for the Algorithm 1 is obtained by multiplying the total number of inner and outer iterations. Therefore, by omitting the integer bracket in (32) which does not change the order of complexity, we then derive the following total number of iterations to get an ϵ solution, i.e. a solution that satisfies $x^T s = n\mu \leq \epsilon$, as follows:

$$O\left(\sqrt{n}(\log n)^2 \log \frac{n}{\epsilon}\right).$$

The iteration complexity for the small-update methods is straight and we rest it for the interested readers.

7 Concluding remarks

In this paper, we propose a new kernel function consisting of a trigonometric function in its barrier term for the large-update primal-dual interior point methods for linear optimization in large neighborhood of the central path. Using some mild and easy to check conditions, a simple analysis for the primal dual IPMs based on the proximity function induced by the new kernel function is provided. As usual, the e-convexity property of the kernel function plays an important role in deriving a default value for the step size. We finally derive the worst case iteration complexity of Algorithm 1 which is $O\left(\sqrt{n}(\log n)^2 \log \frac{n}{\epsilon}\right)$. This bound improves significantly the results obtained by El Ghami et al. in [4] for the kernel functions with trigonometric function in their barrier term.

Acknowledgments The authors would like to thank the Research Council of K.N. Toosi University of Technology and the SCOPE research center for supporting the work.

References

1. Bai, Y.Q., El Ghami, M., Roos, C.: A comparative study of kernel functions for primal-dual interior-point algorithms in linear optimization. SIAM J. Optim. **15**(1), 101–128 (2004)
2. Bai, Y.Q., El Ghami, M., Roos, C.: A new efficient large-update primal-dual interior-point methods based on a finite barrier. SIAM J. Optim. **13**(3), 766–782 (2003)
3. El Ghami, M., Roos, C.: Generic primal-dual interior point methods based on a new kernel function. Int. J. RAIRO Oper. Res. **42**(2), 199–213 (2008)
4. El Ghami, M., Guennoun, Z.A., Boula, S., Steihaug, T.: Interior-point methods for linear optimization based on a kernel function with a trigonometric barrier term. J. Comput. Appl. Math. **236**, 3613–3623 (2012)
5. Karmarkar, N.K.: A new polynomial-time algorithm for linear programming. Combinatorica **4**, 375–395 (1984)
6. Kojima, M., Mizuno, S., Yoshise, A.: A primal-dual interior point algorithm for linear programming. In: Megiddo, N. (ed.) Progress in Mathematical Programming: Interior Point and Related Methods, pp. 29–47. Springer, New York (1989)
7. Megiddo, N.: Pathways to the optimal set in linear programming. In: Megiddo, N. (ed.) Progress in Mathematical Programming: Interior Point and Related Methods, pp. 131–158. Springer, New York (1989)
8. Monteiro, R.D.C., Adler, I.: Interior-point path following primal-dual algorithms: Part I: linear programming. Math. Program. **44**, 27–41 (1989)
9. Nesterov, Y.E., Nemirovskii, A.S.: Interior Point Polynomial Algorithms in Convex Programming, SIAM Studies in Applied Mathematics, vol. 13. SIAM, Philadelphia (1994)

10. Peng, J., Roos, C., Terlaky, T.: Self-Regularity: A New Paradigm for Primal-Dual Interior-Point Algorithms. Princeton University Press, Princeton, NJ (2002)
11. Peyghami, M., Amini, K.: A kernel function based interior-point methods for solving $P_*(\kappa)$-linear complementarity problem. Acta Math. Sin. (Engl. Ser.) **26**(9), 1761–1778 (2010)
12. Roos, C., Terlaky, T., Vial, J.-Ph.: Theory and Algorithms for Linear Optimization: An Interior Point Approach. Springer, New York (2005)
13. Sonnevend, G.: An analytic center for polyhedrons and new classes of global algorithms for linear (smooth, convex) programming. In: Prakopa, A., Szelezsan, J., Strazicky, B. (eds.) Lecture Notes in Control and Information Sciences, vol. 84, pp. 866–876. Springer, Berlin (1986)