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A novel approach to construct numerical methods for stochastic differential equations

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Abstract In this paper we propose a new numerical method for solving stochastic differential equations (SDEs). As an application of this method we propose an explicit numerical scheme for a super linear SDE for which the usual Euler scheme diverges.

Keywords Explicit numerical scheme · Super linear stochastic differential equations.

Mathematical Subject Classifications (2010) 60H10 · 60H35

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ be a complete probability space with a filtration and let a Wiener process $(W_t)_{t\geq 0}$ defined on this space. Consider the following stochastic differential equation,

$$x_t = x_0 + \int_0^t a(x_s)ds + \int_0^t b(x_s)dW_s, \tag{1}$$

where $a, b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are measurable functions and x_0 such that is $\mathcal{F}_0$-measurable and square integrable. Suppose that this problem has a unique strong solution x_t.

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Assumption A Assume that there exist $f(x, y), g(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $f(x, x) = a(x)$, $g(x, x) = b(x)$. Furthermore, assume that for any $R > 0$ and any $|x_1|, |x_2|, |y_1|, |y_2| \leq R$ we have

$$|f(x_1, y_1) - f(x_2, y_2)| + |g(x_1, y_1) - g(x_2, y_2)| \leq C_R(|x_1 - x_2| + |y_1 - y_2|),$$

for some C_R depending on R and on a, b, f, g.

Let $0 = t_0 < t_1 < \ldots < t_n = T$ and set $\Delta = \frac{T}{n}$. For any $t \in [t_k, t_{k+1}]$ consider the following stochastic differential equation,

$$y_t = y_{t_k} + \int_{t_k}^{t} f(y_s, y_{t_k})ds + \int_{t_k}^{t} g(y_s, y_{t_k})dW_s, \quad t \in [t_k, t_{k+1}] \tag{2}$$

where we assume that SDE (2), in each step, has a unique strong solution which we denote by y_t and $y_0 = x_0$. Note, that the coefficients in (2) are random. Let us call (2) a "semi-discrete" numerical scheme because, in general, we discretize only a part of our original SDE. In practice we will choose f, g such that this numerical scheme will have a known explicit solution, for example, the "semi-discrete" SDE (2) may be linear. We can write the above numerical scheme more compactly,

$$y_t = x_0 + \int_0^t f(y_s, y_{\hat{s}})ds + \int_0^t g(y_s, y_{\hat{s}})dW_s, \tag{3}$$

where

$$\hat{s} = t_k, \quad \text{when} \quad s \in [t_k, t_{k+1}].$$

The solution y_t of (3) depends on Δ and we should use a notation like this, y_t^{Δ}, but we don't do here for simplicity. Note that we use the second variable in f, g to denote the discretized part of the original SDE.

The SDEs (3), i.e. the "semi-discrete" SDEs, are not algebraic equations, because in order to solve for y_t we have to solve a stochastic differential equation. In this setting, we can not reproduce the known implicit numerical schemes but we can reproduce for example the Euler scheme. So the usual Euler scheme belongs to our setting choosing $f(x, y) = a(y)$ and $g(x, y) = b(y)$. Another interesting way to choose f, g is $f(x, y) = -\frac{1}{2}b'(y)b(y) + a(y) + \frac{1}{2}b'(x)b(x)$ and $g(x, y) = b(x)$ (see [1]). This comes from the fact that the following SDE,

$$x_t = x_0 + \int_0^t \frac{1}{2}b'(x_s)b(x_s)ds + \int_0^t b(x_s)dW_s,$$

has a known explicit solution (see [7], p. 117). We can use also other, more sophisticated, SDEs with known explicit solutions as these described in [7]. Thus, our method here is more general than [1] because we can arrive to other SDEs, like linear SDES, choosing suitable f, g. The main advantage of our method is that we produce always explicit numerical schemes in contrast to other interesting but implicit methods (see for example [4–6]).

Another suitable choice, that we are going to use in our example in this paper, is $f(x, y) = \frac{a(y)}{y}x$ and $g(x, y) = \frac{b(y)}{y}x$. Then the resulting "semi-discrete" SDE will be a linear stochastic differential equation with known explicit solution.

Let us point out that our main result and setting seems to be true for the multidimensional case and we shall discuss this in our future work.

In the second section we will state and prove a convergence result and in the third section we will give an application. For a general study of the numerical analysis of stochastic differential equations one can see [7].

2 Convergence of the semi-discrete numerical scheme

In this section we shall prove that our numerical scheme converges to the true solution.

Theorem 1 *Assume that Assumption A holds and that* (2) *has a unique strong solution in each interval. Suppose that*

$$\mathbb{E}\left(\sup_{0\le t\le T}|x_t|^p\right) < A, \quad \mathbb{E}\left(\sup_{0\le t\le T}|y_t|^p\right) < A$$

for some $p > 2$, $A > 0$ *independent of* Δ. *Then*

$$\lim_{\Delta\to 0}\mathbb{E}\sup_{0\le t\le T}|y(t) - x(t)|^2 = 0.$$

Proof Set $\rho_R = \inf\{t \in [0, T] : |x_t| \ge R\}$ and $\tau_R = \inf\{t \in [0, T] : |y_t| \ge R\}$. Let $\theta_R = \min\{\tau_R, \rho_R\}$.

At first, let us estimate the following probability

$$\mathbb{P}(\tau_R \le T) = \mathbb{E}\left[\mathbb{I}_{\{\tau_R\le T\}}\frac{|y^p_{\tau_R}|}{R^p}\right] \le \frac{A}{R^p}.$$

Therefore we can prove that $\mathbb{P}(\tau_r \le T \text{ or } \rho_R \le T) \le \frac{2A}{R^p}$. Using the Young inequality we obtain, for any $\delta > 0$,

$$\mathbb{E}\left(\sup_{0\le t\le T}|y_t - x_t|^2\right) \le \mathbb{E}\left(\sup_{0\le t\le T}\left|y_{t\wedge\theta_R} - x_{t\wedge\theta_R}\right|^2\right) + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)2A}{p\delta^{\frac{2}{p-2}}R^p}.$$

We shall estimate now the term $|x_{t\wedge\theta_R} - y_{t\wedge\theta_R}|^2$ as follows, using Assumption A for $f(\cdot,\cdot)$,

$$
\begin{aligned}
&|x_{t\wedge\theta_R} - y_{t\wedge\theta_R}|^2 = \\
&\left|\int_0^{t\wedge\theta_R}(f(x_s,x_s)-f(y_s,y_{\hat{s}}))ds + \int_0^{t\wedge\theta_R}(g(x_s,x_s)-g(y_s,y_{\hat{s}}))dW_s\right|^2 \\
&\le 2\int_0^{t\wedge\theta_R}\left|f(x_s,x_s)-f(y_s,y_{\hat{s}})\right|^2 ds + 2\left|\int_0^{t\wedge\theta_R}(g(x_s,x_s)-g(y_s,y_{\hat{s}}))dW_s\right|^2 \\
&\le C_R\int_0^{t\wedge\theta_R}\left(|x_s-y_s|^2+|x_s-y_{\hat{s}}|^2\right)ds + 2\left|\int_0^{t\wedge\theta_R}(g(x_s,x_s)-g(y_s,y_{\hat{s}}))dW_s\right|^2
\end{aligned}
$$

Note that C_R will be different from line to line. We can write now,

$$
\begin{aligned}
\sup_{0\le t\le s}|x_{t\wedge\theta_R}-y_{t\wedge\theta_R}|^2 \le C_R\int_0^s\left(|x_{r\wedge\theta_R}-y_{r\wedge\theta_R}|^2+|x_{r\wedge\theta_R}-y_{\widehat{r\wedge\theta_R}}|^2\right)dr + \\
2\sup_{0\le t\le s}\left|\int_0^{t\wedge\theta_R}\big(g(x_s,x_s)-g(y_s,y_{\hat{s}})\big)dW_s\right|^2.
\end{aligned}
$$

Taking expectations on both sides, using Doob's martingale inequality for the second term at the right hand side and Assumption A for $g(\cdot,\cdot)$ we arrive at,

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le s}|x_{t\wedge\theta_R}-y_{t\wedge\theta_R}|^2\right) &\le C_R\mathbb{E}\int_0^s\left(|x_{r\wedge\theta_R}-y_{r\wedge\theta_R}|^2+|x_{r\wedge\theta_R}-y_{\widehat{r\wedge\theta_R}}|^2\right)dr \\
&\le C_R\int_0^s\mathbb{E}\sup_{0\le l\le r}|x_{l\wedge\theta_R}-y_{l\wedge\theta_R}|^2dr + C_R\int_0^s\mathbb{E}|y_{r\wedge\theta_R}-y_{\widehat{r\wedge\theta_R}}|^2dr.
\end{aligned}
$$

We shall estimate the term $\mathbb{E}|y_{t\wedge\theta_R}-y_{\widehat{t\wedge\theta_R}}|^2$. We begin with,

$$
\begin{aligned}
|y_{t\wedge\theta}-y_{\widehat{t\wedge\theta_R}}|^2 &= \left|\int_{\widehat{t\wedge\theta_R}}^{t\wedge\theta_R}f(y_s,y_{\hat{s}})\,ds + \int_{\widehat{t\wedge\theta_R}}^{t\wedge\theta_R}g(y_s,y_{\hat{s}})\,dW_s\right|^2 \\
&\le 2\left(\int_{\widehat{t\wedge\theta_R}}^{t\wedge\theta_R}(y_s,y_{\hat{s}})\,ds\right)^2 + 2\left|\int_{\widehat{t\wedge\theta_R}}^{t\wedge\theta_R}g(y_s,y_{\hat{s}})dW_s\right|^2.
\end{aligned}
$$

Taking expectations, using Ito's isometry and the fact that $|f(y_s,y_{\hat{s}})|, |g(y_s,y_{\hat{s}})| \le C_R$ we have that,

$$
\mathbb{E}|y_{t\wedge\theta}-y_{\widehat{t\wedge\theta}}|^2 \le C_R\Delta.
$$

Collecting all together and using the Gronwall inequality we arrive at,

$$
\mathbb{E}\left(\sup_{0\le t\le T}|y_t-x_t|^2\right) \le C_R\Delta + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)2A}{p\delta^{\frac{2}{p-2}}R^p}.
$$

Now, given $\varepsilon > 0$ choose $\delta > 0$ such that $\frac{2^{p+1}\delta A}{p} < \varepsilon/3$ and then choose R such that $\frac{(p-2)2A}{p\delta^{\frac{2}{p-2}}R^p} < \varepsilon/3$. Finally, choose Δ small enough to get the desired result. □

3 Example

Below we will propose an explicit numerical scheme for a super linear SDE as an application of Theorem 1. Consider the following SDE,

$$x_t = x_0 - \int_0^t x_s^3 ds + \int_0^t b x_s dW_s,$$

with $x_0 \in \mathbb{R}_+$ and $b \in \mathbb{R}$. This SDE has a unique strong solution and we know that the usual Euler scheme diverges (see [3]).

We propose the following "semi-discrete" numerical scheme which is explicit,

$$y_t = x_0 + \int_0^t y_s(-y_{\hat{s}}^2) ds + \int_0^t b y_s dW_s.$$

This semi discrete scheme is a linear SDE with unique strong solution,

$$y_t = x_0 e^{-\int_0^t (y_{\hat{s}}^2 + \frac{b^2}{2}) ds + bW_t}.$$

We need the moment bounds which we will prove in the next lemma.

Lemma 1 *Suppose that* $x_0 > 0$ *and* $x_0 \in \mathbb{R}$. *Then there exists some* $A > 0$ *such that*

$$\mathbb{E}\left(\sup_{0 \le t \le T} |y_t|^p\right) < A, \quad \mathbb{E}\left(\sup_{0 \le t \le T} |x_t|^p\right) < A, \quad \mathbb{E}\left(\sup_{0 \le t \le T} \frac{1}{|x_t|^2}\right) < A,$$

for any $p \geq 2$.

Proof Set $r = \min\{\rho, \tau\}$ where $\rho = \inf\{t \in [0, T] : |x_t| > R\}$ and $\tau = \inf\{t \in [0, T] : |y_t| > R\}$ for some $R > 0$.

Using Ito's formula on $|y_{t\wedge r}|^p$ we obtain,

$$\begin{aligned} |y_{t\wedge r}|^p &= |x_0|^p + \int_0^t \left(p|y_s|^p(-y_{\hat{s}}^2) + \frac{b^2 p(p-1)}{2}|y_s|^p\right) \mathbb{I}_{\{(0,r)\}}(s) ds \\ &\quad + \int_0^t bp|y_s|^p \mathbb{I}_{\{(0,r)\}}(s) dW_s \\ &\le |x_0|^p + \frac{b^2 p(p-1)}{2} \int_0^t |y_s|^p \mathbb{I}_{\{(0,r)\}}(s) ds \\ &\quad + \int_0^t bp|y_s|^p \mathbb{I}_{\{(0,r)\}}(s) dW_s \end{aligned}$$

Taking expectations we obtain,

$$\mathbb{E}|y_{t\wedge r}|^p \le |x_0|^p + \frac{b^2 p(p-1)}{2} \int_0^t \mathbb{E}|y_{s\wedge r}|^p ds.$$

Therefore, using the Gronwall inequality, we arrive at,

$$\mathbb{E}|y_{t\wedge r}|^p \le C(p)|x_0|^p, \tag{4}$$

with $C(p)$ independent of $R > 0$. But $\mathbb{E}|y_{t\wedge r}|^p = \mathbb{E}\left(|y_{t\wedge r}|^p \mathbb{I}_{\{r\geq t\}}\right) + R^p P(r < t)$. That means that $P(t \wedge r < t) = P(r < t) \to 0$ as $R \to \infty$ so $t \wedge r \to t$ in probability and noting that r increases as R increases we have that $t \wedge r \to t$ almost surely too, as $R \to \infty$. Going back to (4) and using Fatou's lemma we obtain,

$$\mathbb{E}|y_t|^p \leq C(p)|x_0|^p,$$

for any $p \geq 2$ and this is crucial to ensure that $\mathbb{E}\int_0^t bp|y_s|^p dW_s = 0$ in the next step. Using Ito's formula again on $|y_t|^p$, taking supremum and then expectations and finally Doob's martingale inequality we arrive at,

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |y_t|^p\right) < A,$$

for some $A > 0$. To prove now that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |x_t|^p\right) < A$$

we use Theorem 2.4.1 of [9] and obtain first the bound,

$$\mathbb{E}|x_t|^p \leq A,$$

for any $p \geq 2$ and then using Ito's formula on $|x_t|^p$ taking supremum and then expectations and finally Doob's martingale inequality we obtain the desired bound.

Finally, we shall prove that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \frac{1}{x_t^2}\right) < A.$$

Set $l = \inf\{t \geq 0 : \frac{1}{x_t} \geq m\}$. Using Ito's formula on $(\frac{1}{x_{t\wedge l}})^2$ we have,

$$\mathbb{E}\left(\frac{1}{x_{t\wedge l}}\right)^2 = \mathbb{E}\left(\frac{1}{x_0}\right)^2 + \mathbb{E}\int_0^t \left(2 + 3\left(\frac{b^2}{x_{s\wedge l}}\right)^2\right) ds.$$

Using Gronwall's inequality we can prove that $\mathbb{E}\left(\frac{1}{x_{t\wedge l}}\right)^2 < A$ with A independent of m. As before we deduce that $\mathbb{E}\left(\frac{1}{x_t}\right)^2 < A$ and then that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} \frac{1}{x_t^2}\right) < A.$$

□

Now we can apply Theorem 1 to prove that our explicit numerical scheme converges in the mean square sense. Let us note that from the above moment bounds for the true solution we deduce that $x_t \geq 0$ a.s. Indeed, we have proved that

$$\mathbb{E}\left(\frac{1}{x_{t\wedge l}^2}\right) = \mathbb{E}\left(\frac{1}{x_{t\wedge l}^2}\mathbb{I}_{\{l>t\}}\right) + m^2 P(l \leq t) < A.$$

Therefore, letting $m \to \infty$ we obtain that $P(l \leq t) \to 0$, noting that

$$P(x_t \leq 0) = P\left(\bigcap_{m=1}^{\infty}\left\{x_t \leq \frac{1}{m}\right\}\right) = \lim_{n\to\infty} P\left(\left\{x_t \leq \frac{1}{m}\right\}\right) \leq \lim_{m\to\infty} P(l \leq t) = 0.$$

Thus, we need our numerical scheme to be positivity preserving. As we can see easily our scheme has this advantage. For this kind of problems there is also the tamed Euler scheme as the authors in [2] proposes. This kind of method behaves very well and in comparison with our "semi-discrete" method seems that the tamed Euler method is less expensive because we have in each step to compute an exponential. However, the tamed Euler scheme does not seem to preserve positivity, at least, for any $\Delta > 0$.

We shall give a simulation to show that the two numerical schemes are close. So, let the following SDE,

$$x_t = 1 + \int_0^t -x_s^3 ds + \int_0^t x_s dW_s, \quad t \in [0, 1].$$

We apply the tamed Euler scheme and the semi discrete scheme to this problem for $\Delta = 10^{-4}$ and we plot the difference between these methods, i.e. if $z_t = y_t^{tamed} - y_t^{semi}$ we plot z_t on [0, 1] (Figs. 1 and 2).

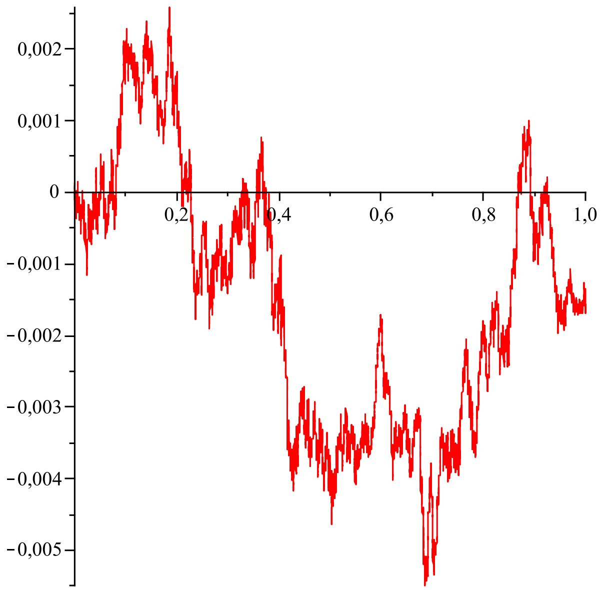

Fig. 1 Difference between the semi discrete scheme and tamed Euler scheme for $x_0 = 1$, $\Delta = 10^{-4}$, $b = 1$, $T = 1$

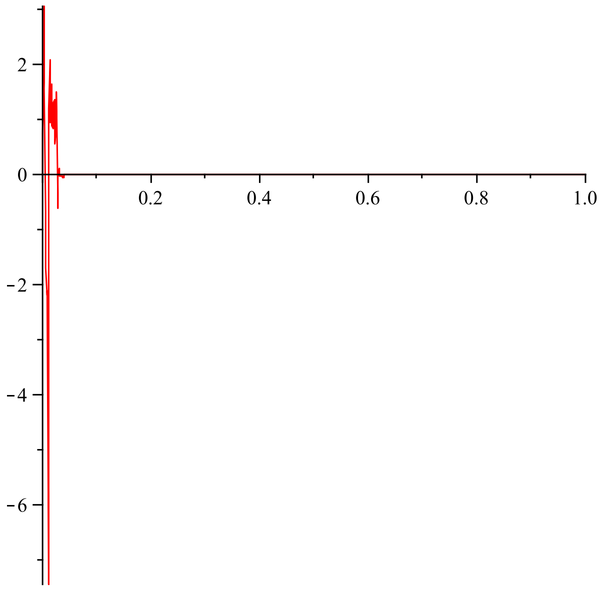

Fig. 2 Tamed Euler method does not preserve positivity, $x_0 = 1$, $\Delta = 10^{-3}$, $b = 20$, $T = 1$

4 Conclusion

In this paper we propose a new numerical method for solving stochastic differential equations. We apply our method to a super-linear SDE and compare with the tamed Euler method. We see that our method preserves positivity of the true solution. Our method seems to behave very well when applied to super-linear problems. In order to manage other interesting problems a suitable transformation and then the application of this "semi-discrete" method maybe the answer. Another possibility is to introduce and study tamed semi-discrete methods. Our goal in the future is to apply our method to other stochastic differential equations arising in financial mathematics (see for example [8]) and to give a more detailed study of this method studying the rate of convergence, the case where the choice f, g are not locally Lipschitz etc.

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