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Robust continuation methods for tracing solution curves of parameterized systems

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Abstract The continuation methods are efficient methods to trace solution curves of nonlinear systems with parameters, which are common in many fields of science and engineering. Existing continuation methods are unstable for some complicated cases in practice, such as the case that solution curves are close to each other or the case that the curve turns acutely at some points. In this paper, a more robust corrector strategy—sphere corrector is presented. Using this new strategy, combining various predictor strategies and various iterative methods with local quadratic or superlinear convergence rates, robust continuation procedures for tracing curves are given. When the predictor steplength is no more than the so-called granularity of solution curves, our procedure of tracing solution curve can avoid "curve-jumping" and trace the whole solution curve successfully. Numerical experiments illustrate our method is more robust and efficient than the existing continuation methods.

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1 Introduction

In many fields, we often need to solve parameterized systems of nonlinear equations:

$$F(x, t) = 0, x \in \mathbf{R}^n, t \in \mathbf{R}^m,$$

and under some basic assumptions, such a system implicitly defines some curves or manifolds of solution points, a basic and important problem in solving such parameterized systems is numerically tracing a curve among them from the given initial point on it to the target point.

Sometimes, although the original problem does not include any parameters, to construct a globally convergent method, one often introduces a parameter into the system such that it becomes a parameterized problem. For example, finding solutions of a nonlinear system without parameters is an important problem in applications, and Newton's method is a fundamental iterative method for finding successively better approximation to the solution of the nonlinear system without parameters, and it owns properties as follow: (1) Fast convergence rate. (2) Local convergence. Generally speaking, Newton's method possesses quadratic convergence rate, however, the convergence is guaranteed only when the initial point is close enough to the solution point, and every iterative point is in convergence region. Variants of Newton's method, such as Newton-like methods and Quasi-Newton methods, have fast and local quadratic or superlinear convergence rates also. For Newton's method and its variants, it is difficult to confirm whether the initial point satisfies the condition of local convergence for many nonlinear problems. Hence, it is significant to study globally convergent algorithms, such as continuation methods [1–4, 7, 11, 13, 20, 24], in which underdetermined systems with one parameter arose.

In this paper, the main theme is to design a robust and efficient continuation method to numerically trace solution curves of a parameterized system. For the convenience of our discussion, we make the following assumptions:

- The map $F : \mathbf{R}^{n+1} \to \mathbf{R}^n$ is smooth;
- there is a point $u_0 \in \mathbf{R}^{n+1}$ such that $F(u_0) = 0$;
- the Jacobian matrix $F'(u_0)$ has maximum rank.

Under these assumptions, the solution set of F contains a smooth curve $c(s)$, where s is the arclength, and there exists an open interval I such that for all $\alpha \in I$,

$$c(0) = u_0, \; c'(\alpha) \neq 0, \; \text{rank}\left(F'\left(c(\alpha)\right)\right) = n.$$

Although continuation methods have different forms, they have uniform basic framework: predictor and corrector. In the predictor step, we can adopt different strategies, mainly including two ways:

1. **Interpolation Predictors** [15]

   This class of predictors is generated by using the calculated points in the curve. Assume that the successive points $u_i, i = 0, 1, \ldots, n$ along the solution curve $c(s)$ have already been generated, the task is to construct a interpolating polynomial $p(s)$ satisfying $p(s_i) = u_i$, and the standard interpolating polynomial using Newton formula can be applied. In certain versions of the continuation method, also the corresponding tangents $F'(u_0), F'(u_1), \ldots, F'(u_n)$ are available, and the Hermite interpolating polynomial can be applied to predict.

2. **Talor Polynomial Predictors** [16, 23]

   This class of predictors is based on Taylor's formula and generated by exploiting the successive numerical difference at the point on the curves.

   Suppose u_k is the current point on the curve to be traced, v_{k+1} is the next predictor point, $h_{k+1} > 0$ is the steplength, then the well-known Euler predictor can be used:

$$v_{k+1} = u_k + h_{k+1}\zeta_k, \tag{1}$$

   where ζ_k is the tangent vector to the solution curve $c(s)$ at u_k, and is locally defined by the system

$$\begin{cases} F'(u_k)\zeta_k = 0, \\ \zeta_k^\top \zeta_k = 1. \end{cases}$$

   This system has exactly two solutions which correspond to the two possible directions of tracing the curve, to specify the direction of tracing, ζ_k is chosen to satisfy the sign of the determinant of the matrix

$$\begin{pmatrix} F'(u_k) \\ \zeta_k^\top \end{pmatrix}$$

   stays constant.

   In applying these two classes of strategy to find the predictor, we face a tradeoff. The former needs less computational work, and the latter is more accurate and stable. However, in this paper, we mainly study the different corrector strategies, so we adopt the same predictor strategy: Euler predictor.

In the corrector step, we can adopt different strategies, combining suitable iterative methods with local quadratic or superlinear convergence rates, to bring the predictor point back to the solution curve. In practice, we often adopt corrector strategies as follows:

1. **Plane Corrector: Correct on the hyperplane that passes the predictor point and erects the predictor direction.**

   The intersection of the hyperplane P, going through the predictor point v_{k+1} and erecting the current tangent vector ζ_k, with the solution curve is the corrector point w need to be found, see Fig. 1a. The hyperplane P is defined by

$$P = \left\{ u \in \mathbf{R}^{n+1} \;\middle|\; \zeta_k^T (u - v_{k+1}) = 0 \right\}.$$

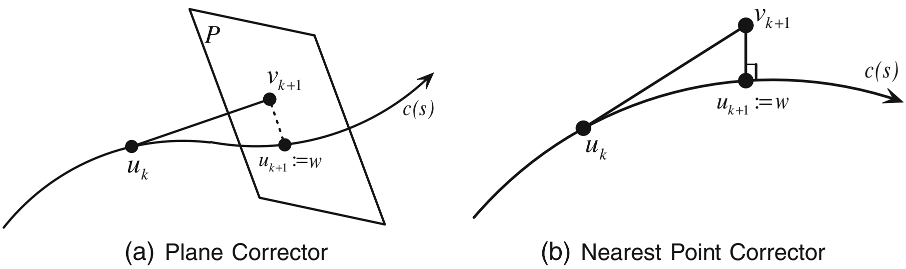

Fig. 1 Two corrector strategies

The corrector procedure is the procedure for solving the system:

$$\begin{cases} F(u) = 0, \\ \zeta_k^T (u - v_{k+1}) = 0, \end{cases}$$

e.g. Newton's method (or other suitable iterative methods with quadratic or superlinear convergence rates)

$$u^{(i+1)} = u^{(i)} - \begin{pmatrix} F'(u^{(i)}) \\ \zeta_k^T \end{pmatrix}^{-1} \begin{pmatrix} F(u^{(i)}) \\ \zeta_k^T (u^{(i)} - v_{k+1}) \end{pmatrix}, \ i = 0, 1, 2, \ldots$$

with the starting point $u^{(0)} = v_{k+1}$.

2. **Nearest Point Corrector: Take the point on the solution curve that is nearest to the predictor point as the corrector point** [3].

Let v_{k+1} be the predictor point and w be the solution of the minimization problem

$$\min_u \left\{ \|u - v_{k+1}\| \;\middle|\; F(u) = 0 \right\}.$$

The corrector procedure is solving w, see Fig. 1b. A straightforward way of solving the above minimization problem is Newton-type method

$$u^{(i+1)} = u^{(i)} - \left(F'\left(u^{(i)}\right)\right)^{\dagger} F\left(u^{(i)}\right), \ i = 0, 1, 2, \ldots$$

where $\left(F'\left(u^{(i)}\right)\right)^{\dagger}$ is the Moore-Penrose inverse of the matrix $F'(u^{(i)})$, with the starting point $u^{(0)} = v_{k+1}$. This Newton step is different from the classical Newton's method only in the form of the Moore-Penrose inverse replacing the classical inverse.

We have considerable continuation methods by combining various predictor strategies, various corrector strategies and various iterative methods with quadratic or superlinear convergence rates. All these methods can solve general problems in practice. However, if solution curves are close to each other or the curve turns acutely at some points, "curve-jumping", which means the continuation method jumps from a curve to another curve, may happen, these continuation methods are not robust for comparatively big predictor steplength. In such cases, they can succeed in tracing the curve only when the predictor steplength is very small, and thus the effectiveness of the curve-tracing will decrease. The algorithms in [10, 14, 17, 19] are designed for tracking homotopy paths arising in homotopy continuation methods for solving polynomial systems, this class of paths is monotone increasing with the parameter t and different from the complicated curves mentioned above. Recently, there are also some new strategies for continuation methods [6, 8, 9, 12, 15, 21, 22] are presented, and most of these strategies focus on the improvement of the predictor or the steplength control, however, the corrector step also plays an important role in the continuation method, in this paper, we present a simple but robust and efficient corrector strategy. Using this new strategy, combining various predictor strategies and various iterative methods, we present a more robust and efficient continuation method: Euler-sphere predictor-corrector method.

This paper is organized as follows. In Section 2, the new continuation method equipped with Euler Predictor and Sphere Corrector is constructed, and also the convergence theorem is presented. Numerical tests are demonstrated in Section 3 to show our new continuation method is more robust and efficient than the classical continuation methods.

2 A robust corrector strategy: sphere corrector

Let $c(s)$ be the parameterized presentation of the solution curve and u_k be a point on it. Applying the above mentioned predictors in Section 1, for instance, the well-known Euler predictor (1), we can obtain the predictor point v_{k+1}.

The sphere, whose diameter is the straight line segment joining the points u_k and v_{k+1}, intersects the solution curve $c(s)$ at only one point w (besides the start point u_k), see Fig. 2. The sphere is defined by

$$\left\{ u \in \mathbf{R}^{n+1} \;\middle|\; \left\| u - \frac{u_k + v_{k+1}}{2} \right\|^2 - \left(\frac{h_{k+1}}{2} \right)^2 = 0 \right\}.$$

The corrector procedure is the method for finding the solution of the system

$$M(u) = \begin{cases} F(u) = 0, \\ \left\| u - \dfrac{u_k + v_{k+1}}{2} \right\|^2 - \left(\dfrac{h_{k+1}}{2} \right)^2 = 0, \end{cases} \tag{2}$$

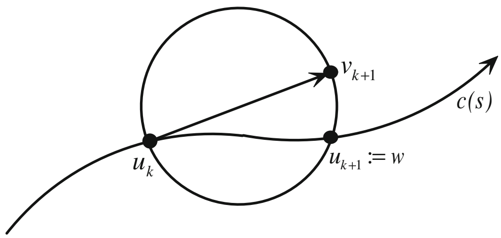

Fig. 2 Sphere Corrector

e.g. Newton's method, which takes the form:

$$u^{(i+1)} = u^{(i)} - \left(M'\left(u^{(i)}\right)\right)^{-1} M\left(u^{(i)}\right), i = 1, 2, \cdots .$$

or variants of Newton's method.

We can now briefly describe how to generate the points along the curve $c(s)$ under the assumption that a point $u_k \in \mathbf{R}^{n+1}$ has been accepted such that $\|F(u_k)\| \leq \varepsilon$, where ε is the given tolerance. To obtain a new point u_{k+1} along the curve, we first make a predictor step, for instance, the well-known Euler predictor step (1), and then choose a suitable iterative method with local quadratic or superlinear convergence rates to get the solution, which is the next corrector point u_{k+1}, of the system (2).

The following algorithm describes a particular version of the continuation method equipped with the Euler predictor and sphere corrector.

Algorithm 1: Euler-Sphere Predictor-Corrector Method:

input

- *The initial point* $u_0 \in \mathbf{R}^{n+1}$ *such that* $F(u_0) = 0$;
- *The tolerance* $\varepsilon > 0$;
- *The initial steplength* $h > 0$, *the maximal steplength* $h_{\max} > 0$;
- *The positive integers* T *and* L *for steplength adaptation.*

begin

$u = u_0$; *print* u.

repeat

- *Predictor step. Calculate the tangent vector* ζ *to the solution curve at* u, *and apply Euler predictor to obtain the predictor point* $v = u + h\zeta$;
- *Corrector step. Choose a suitable iterative method, such as Newton's method or Newton's method with either "line search" or "trust region" step control, to solve the nonlinear system* (2),
  - *if the generated sequence by the iterative method converges to* w, *that is* $\|F(w)\| \leq \varepsilon$, *then set* $u = w$ *and print* u;
- *Steplength adaptation.*
  - *In the corrector step, if the iterative method converges in no more than* T *steps, then* $h := \min(Lh, h_{\max})$
  - *If the iterative method in corrector step does not converge, then* $h := h/L$;

until traversing is stopped.

To trace a curve successfully, we should adjust the steplength to make the sphere intersect solution curves at only two points, including the start point.

Lemma 1 *Suppose $F : \mathbf{R}^{n+1} \to \mathbf{R}^n$ is a smooth map, zero is its regular value, then there exists a regular point $u_0 \in \mathbf{R}^{n+1}$ satisfying $F(u_0) = 0$ and a scalar $h_{\max} > 0$, such that for any $h \in (0, h_{\max}]$, the system*

$$M(u) = \begin{cases} F(u) = 0, \\ \left\| u - u_0 - \frac{h}{2}\zeta_0 \right\|^2 - \left(\frac{h}{2}\right)^2 = 0, \end{cases}$$

where ζ_0 is the tangent vector to the solution curve of F *at u_0, has only one nonsingular solution* w *(besides u_0,).*

Proof Since 0 is a regular value of F, then there exists a point $u_0 \in \mathbf{R}^{n+1}$ such that $F(u_0) = 0$ and $F'(u_0)$ is full row-rank. From the Implicit Function Theorem, the solution set of F contains a smooth curve $c(s)$, which is locally parameterized about u_0 with respect to the parameter s.

Firstly, we prove the existence of the solution w. u_0 is a regular point of F, then there is an open neighborhood $U(u_0-\delta, u_0+\delta) \subset R^{n+1}$ of u_0 such that $F'(u)$ is full row-rank for any $u \in U \cap c(s)$. Since the predictor direction ζ_0 points to the interior of the sphere, there must exist a regular point of F in $\partial U(u_0, u_0 + \delta) \cap c(s)$. From the Implicit Function Theorem, there exists a scalar $h_{\max} > 0$, and for any $h < h_{\max}$, the intersection point w of B_h, where

$$B_h = \left\{ u \,\middle|\, \left\| u - u_0 - \frac{h}{2}\zeta_0 \right\|^2 - \left(\frac{h}{2}\right)^2 = 0 \right\}, \tag{3}$$

with the solution curve $c(s)$ is a regular point, that is, w is a nonsingular solution of the system $M(u) = 0$.

Secondly, we prove the uniqueness of the solution w. Suppose for any predictor steplength h, the sphere intersects the solution curve at no less than three points. Without loss of generality, we assume there are three points $u_0 = c(s_0)$, $w_1 = c(s_1)$, $w_2 = c(s_2)$. Taking u_0 as a fixed point and w_1, w_2 as moving points, then there is a constant ε, if $h < \varepsilon$, w_1, w_2 are both sufficiently close to u_0. Joining any two points, we can obtain three segments $[u_0, w_1]$, $[u_0, w_2]$ and $[w_1, w_2]$. The slopes of the segments $[u_0, w_1]$ and $[u_0, w_2]$ are respectively as follows:

$$\lim_{w_1 \to u_0} \frac{w_1 - u_0}{s_1 - s_0} = c'(s_0), \ \lim_{w_2 \to u_0} \frac{w_2 - u_0}{s_2 - s_0} = c'(s_0).$$

Taylor's formula yields

$$c(s_2) = c(s_1) + c'(s_1)(s_2 - s_1) + o(s_2 - s_1),$$

and hence the slope of segment $[w_1, w_2]$ is

$$\lim_{w_1, w_2 \to u_0} \frac{w_1 - w_2}{s_1 - s_2} = c'(s_0).$$

Thus u_0, w_1, w_2 are collinear. However, when h is sufficiently small, $[u_0, w_1]$ (or $[u_0, w_2]$) is the diameter, and hence $[u_0, w_2] \perp [w_1, w_2]$ (or $[u_0, w_1] \perp [w_1, w_2]$) which is inconsistent to the collinearity of u_0, w_1, w_2. This completes the proof. □

We call $h_{\max}$ in Lemma 1 as the granularity of the solution curve, which shows the proximity of the solution curves of $F(x) = 0$.

Definition 1 Let $c(s)$ be a solution curve of $F(u) = 0$, $u_0 \in c(s)$,

$$h_{\max}(u_0) = \max\left\{h \left| \textit{The sphere } B_h \textit{ intersects } F^{-1}(0) \textit{ at only two points} \right.\right\},$$

where B_h is defined as (3), then

$$h_{\max} := \inf_{u \in c(s)} h_{\max}(u)$$

is defined as the granularity of $c(s)$.

On the convergence of the new continuation method, we have the following results, which show that our method indeed approximates a solution curve if the steplength is small enough.

Theorem 1 *Suppose* $F : \mathbf{R}^{n+1} \to \mathbf{R}^n$ *is a smooth map, and zero is a regular value of* F*, and* $F(u_0) = 0$*,* $c_p(s)$ *is the polygonal path, which starts at* u_0 *and goes through all points* u_i *generated by Algorithm 1,* $c(s)$ *is the solution curve starting at* u_0*, and both curves are parameterized with respect to the arclength* s*. Then, for a given maximal arclength* $s_{\max}$*, and for the given constant* $\varepsilon > 0$ *as in the Algorithm 1, there exists the constant* $h_{\max} > 0$ *such that*

(1) $\|F(u_i)\| \le O(h^2)$ *for* $0 < h \le h_{\max}$*;*
(2) $\|F(c_p(s))\| \le O(h^2)$ *for* $0 < h \le h_{\max}$*;*
(3) $\|c_p(s) - c(s)\| \le O(h^2)$ *for* $0 < h \le h_{\max}$

holds for all $s \in [0, s_{\max}]$.

Proof Let U be a compact neighborhood of $c([0, s_{\max}])$ consisting of regular points of F, then for any $v \in U$, the following Frobenius norms

$$\left\|F'(v)\right\|, \left\|(M'(v))^{-1}\right\|, \left\|F''(v)\right\|, \left\|M''(v)\right\|$$

are bounded. From Lemma 1, there exists $h_{\max} > 0$, for any predictor steplength $h \in (0, h_{\max}]$, the corrector sphere, going through u_i and the next predictor point v_{i+1}, with the diameter h intersects the solution curves at only one point, besides u_i.

The proof of assertion (1) proceeds by induction. Suppose the estimate in assertion (1) is true for the current corrector point u_i, that is

$$F(u_i) \le O\left(h^2\right).$$

The next predictor point is $v_{i+1} = u_i + h\zeta_i$, where ζ_i is the chosen tangent vector to the solution curve at u_i, Taylor's formula yields

$$F(v_{i+1}) = F(u_i) + hF'(u_i)\zeta_i + \frac{h^2}{2}A_1\langle\zeta_i, \zeta_i\rangle,$$

where

$$A_1 = \int_0^1 F''(u_i + \xi h \zeta_i) 2(1-\xi) d\xi$$

is the mean value of F'' on the segment $[u_i, v_{i+1}]$. From the definition of ζ_i, we can get $F'(u_i)\zeta_i = 0$, thus from the induction hypothesis, we obtain the estimate

$$F(v_{i+1}) \le O\left(h^2\right).$$

Since predictor point v is on the sphere, then

$$\|M(v_{i+1})\| = \|F(v_{i+1})\| \le O\left(h^2\right). \tag{4}$$

Suppose the generated finite sequence by the iterative method in the corrector step is $u_i^{(0)}, \ldots, u_i^{(l)}$ satisfying $u_i^{(0)} = v_{i+1}, u_i^{(l)} = u_{i+1}$, then combining (4), we get the following estimate

$$\begin{aligned} & \|u_{i+1} - v_{i+1}\| = \|u_i^{(l)} - u_i^{(0)}\| \\ \le\ & \|u_i^{(l)} - u_i^{(l-1)}\| + \ldots + \|u_i^{(1)} - u_i^{(0)}\| \\ \le\ & \left\|(M'(u_i^{(l-1)}))^{-1}\right\| \left\|M(u_i^{(l-1)})\right\| + \ldots + \left\|(M'(u_i^{(0)}))^{-1}\right\| \left\|M(u_i^{(0)})\right\| \\ \le\ & O(h^2). \end{aligned} \tag{5}$$

Taylor's formula yields

$$\begin{aligned} M(u_{i+1}) = M(v_{i+1}) &+ M'(v_{i+1})(u_{i+1} - v_{i+1}) \\ &+ \frac{1}{2} A_2 \langle u_{i+1} - v_{i+1}, u_{i+1} - v_{i+1} \rangle, \end{aligned} \tag{6}$$

where

$$A_2 = \int_0^1 M''(v_{i+1} + \xi(u_{i+1} - v_{i+1})) 2(1-\xi) d\xi$$

is the mean value of F'' on the segment $[v_{i+1}, u_{i+1}]$. Now from (4), (5) and (6), when h is sufficiently small, we obtain

$$\|M(u_{i+1})\| \le O\left(h^2\right).$$

Obviously, $\|F(u_{i+1})\| \le \|M(u_{i+1})\| \le O\left(h^2\right)$. Consequently, we complete the inductive step for proving assertion (1).

From the definition of the corrector sphere, we can obtain the estimate

$$\|u_i - u_{i+1}\| = O(h) \tag{7}$$

when the predictor steplength is sufficiently small.

To prove assertion (2), we consider using Taylor's formulas

$$\begin{aligned} F(u_i) &= F(u_\sigma) + F'(u_\sigma)(u_i - u_\sigma) + \frac{1}{2} D_1 \langle u_i - u_\sigma, u_i - u_\sigma \rangle, \\ F(u_{i+1}) &= F(u_\sigma) + F'(u_\sigma)(u_{i+1} - u_\sigma) + \frac{1}{2} D_2 \langle u_{i+1} - u_\sigma, u_{i+1} - u_\sigma \rangle, \end{aligned} \tag{8}$$

where $u_\sigma = \sigma u_i + (1-\sigma)u_{i+1}$ for $0 \le \sigma \le 1$, and

$$D_1 = \int_0^1 F''(u_\sigma + \xi(u_i - u_\sigma))2(1-\xi)d\xi,$$
$$D_2 = \int_0^1 F''(u_\sigma + \xi(u_{i+1} - u_\sigma))2(1-\xi)d\xi$$

are the mean values of F'' on the segments $[u_i, u_\sigma]$ and $[u_\sigma, u_{i+1}]$ respectively. Multiplying the first equation in (8) by σ and the second by $1-\sigma$, summing them yields

$$\begin{aligned} F(u_\sigma) &= \sigma F(u_i) + (1-\sigma)F(u_{i+1}) \\ &\quad - \frac{1}{2}\sigma D_1 \|u_i - u_\sigma\|^2 - \frac{1}{2}(1-\sigma)D_2 \|u_{i+1} - u_\sigma\|^2 \\ &= \sigma F(u_i) + (1-\sigma)F(u_{i+1}) - \frac{1}{2}\sigma(1-\sigma)\|u_i - u_{i+1}\|^2 \\ &\quad \times \big((1-\sigma)D_1 + \sigma D_2\big). \end{aligned} \tag{9}$$

Assertion (2) now follows from the estimates (7) and (9) and the assertion (1) for sufficiently small h.

The following is the proof of assertion (3). Obviously, when the predictor steplength is sufficiently small, for each corrector point u_i, there is a unique s_i such that

$$\|u_i - c(s_i)\| = \min_{s\in\mathbf{R}} \|u_i - c(s)\|,$$

therefore, the following orthogonality

$$\big(u_i - c(s_i)\big) \perp \dot{c}(s_i)$$

holds.

Taylor's formula yields

$$F(u_i) = F(c(s_i)) + F'(c(s_i))\big(u_i - c(s_i)\big) + O\left(\|u_i - c(s_i)\|^2\right). \tag{10}$$

If we multiply through (10) with the Moore-Penrose inverse $(F'(c(s_i)))^\dagger$, and from $F(c(s_i)) = 0$, it follows that

$$(F'(c(s_i)))^\dagger F(u_i) = \big(u_i - c(s_i)\big) + O\left(\|u_i - c(s_i)\|^2\right).$$

Thus the assertion

$$\|c(s_i) - u_i\| \le O\big(h^2\big) \tag{11}$$

follows from $\|F(u_i)\| \le O(h^2)$.

Setting $\triangle s_i = s_{i+1} - s_i$, and using the fact that $\dot{c}(s)^\top \dot{c}(s) = 1$ implies $\dot{c}(s) \perp \ddot{c}(s)$, Taylor's formula yields

$$\begin{aligned} \|c(s_{i+1}) - c(s_i)\|^2 &= \|\dot{c}(s_i)\triangle s_i + \frac{1}{2}\ddot{c}(s_i)(\triangle s_i)^2 + O((\triangle s_i)^3)\|^2 \\ &= (\triangle s_i)^2 + O\left((\triangle s_i)^4\right) \end{aligned}$$

and consequently

$$\|c(s_{i+1}) - c(s_i)\| = \triangle s_i(1 + O((\triangle s_i)^2)). \tag{12}$$

From estimates (7), (11) and (12), it is straightforward to get the terms $O(h)$ and $O(\triangle s_i)$ can be used interchangeably, thus it is justified to replace $O(\triangle s_i)$ by $O(h)$ in the estimates below.

From the orthogonality relations $(u_i - c(s_i)) \perp \dot{c}(s_i)$ and $(u_{i+1} - c(s_{i+1})) \perp \dot{c}(s_{i+1})$ and Taylor's formula we obtain

$$\begin{aligned} &\big(u_i - c(s_i)\big)^\top \big(c(s_{i+1}) - c(s_i)\big) \\ &= \big(u_i - c(s_i)\big)^\top \big(\dot{c}(s_i)\triangle s_i + O\big((\triangle s_i)^2\big)\big) \le O\big(h^2\big) O\big((\triangle s_i)^2\big) = O\big(h^4\big). \end{aligned}$$

Similarly, $\big(u_{i+1} - c(s_{i+1})\big)^\top \big(c(s_{i+1}) - c(s_i)\big) \le O\big(h^4\big)$. Thus we obtain

$$\begin{aligned} \|u_{i+1} - u_i\|^2 &= \|(u_{i+1} - c(s_{i+1})) + (c(s_{i+1}) - c(s_i)) + (c(s_i) - u_i)\|^2 \\ &= \|c(s_{i+1}) - c(s_i)\|^2 + O\big(h^4\big). \end{aligned}$$

Taking square roots and using (12), we get

$$\begin{aligned} \|u_{i+1} - u_i\| &= \|c(s_{i+1}) - c(s_i)\| + \frac{1}{2} \frac{O\big(h^4\big)}{\|c(s_{i+1}) - c(s_i)\|} \\ &= \|c(s_{i+1}) - c(s_i)\| + O\big(h^3\big). \end{aligned} \tag{13}$$

Summing up all terms $\|u_{i+1} - u_i\|$, $\|c(s_{i+1}) - c(s_i)\|$, and arclengths between the nodes of c_p, using (12) and (13), we can get

$$\begin{aligned} \sum \big\|u_{i+1} - u_i\big\| &= \sum \big\|c(s_{i+1}) - c(s_i)\big\| + O\big(h^2\big), \\ \sum \big\|c(s_{i+1}) - c(s_i)\big\| &= \sum \triangle s_i + O\big(h^2\big). \end{aligned}$$

This implies

$$\|c_p(s_i) - u_i\| \le O\big(h^2\big). \tag{14}$$

Let

$$s = \tau s_i + (1 - \tau)s_{i+1}, \ \tau \in [0, 1],$$

then from Taylor's formula and the estimates (11) and (14), it follows that

$$\begin{aligned} &\|c(s) - c_p(s)\| \\ &= \|c(\tau s_i + (1 - \tau)s_{i+1}) - c_p(\tau s_i + (1 - \tau)s_{i+1})\| \\ &\le \|[\tau c(s_i) + (1 - \tau)c(s_{i+1})] - c_p(\tau s_i + (1 - \tau)s_{i+1})\| + O\big(h^2\big) \\ &\le \|[\tau c(s_i) + (1 - \tau)c(s_{i+1})] - [\tau u_i + (1 - \tau)u_{i+1}]\| + O\big(h^2\big) \\ &\le \tau\|c(s_i) - u_i\| + (1 - \tau)\|c(s_{i+1}) - u_{i+1}\| + O\big(h^2\big) \\ &\le O\big(h^2\big). \end{aligned}$$

This completes the proof of assertion (3). □

When the step length h is no more than the granularity $h_{\max}$ of $c(s)$, that is $h \in (0, h_{\max}]$, theoretically, the nonlinear system $M(u) = 0$ in (2) has only one solution, besides u_k. Numerically, we can find such a point by certain suitable iterative method, such as Newton's method or Newton's method with either "line search" or "trust region" step control in [5, 18]. Therefore, "curve-jumping" does not happen in the process of curve-tracing. In contrast with other continuation methods, the new continuation method is more robust and efficient for tracing some complex curves in practice.

3 Numerical examples

In this section, numerical results are presented to prove our new corrector strategy better than other corrector strategies. Our main contribution in this paper is on the corrector, so for the different continuation methods, we adopt the same predictor strategy, iterative method and strategy for the steplength selection.

Our numerical tests are carried out using Matlab 2008a running on a PC with Windows XP operation system, Intel(R) Core(TM)2 Duo P8600 2.40GHz processor and 2GB of memory.

Throughout the remainder of this section, it will be convenient to use the following notations.

- A: Continuation method with Plane Corrector;
- B: Continuation method with Nearest Point Corrector;
- C: New continuation method with Sphere Corrector;
- Y: "Curve-jumping" happens in the process of curve-tracing;
- N: "Curve-jumping" does not happen in the process of curve-tracing;
- Number1: Number of the corrector points generated by continuation methods;
- Number2: Total number of Newton iteration in the corrector step;
- T: Elapsed time of tracing the whole curve.

Example 1 Consider the two homocentric circles defined by $F(x, y) = 0$, where

$$F(x, y) = \prod_{i=1}^{2} \left(x^2 + y^2 - r_i^2\right), \quad r_1 = 1, \quad r_2 > 1. \tag{15}$$

A smaller r_2 implies two closer homocentric circles. Our goal is numerically tracing the circle defined by $x^2 + y^2 - 1 = 0$, denoted by S. Two numerical tests are designed to answer the following two questions:

- For the given step length, which method of algorithms A, B, C can successfully trace one of two homocentric circles closer to each other?
- For the given radius r_2, which method of algorithms A, B, C can successfully trace one of two homocentric circles by a larger step length?

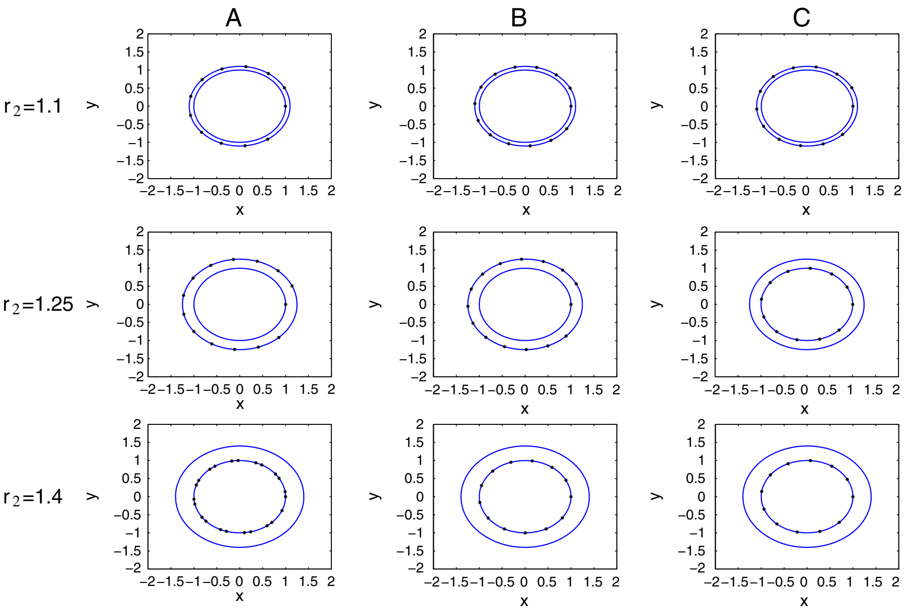

Fig. 3 For the different radius r_2, figures of curve-tracing with the fixed steplength

In the following figures, the two homocentric circles are the curves defined by $F(x, y) = 0$ in (15) with different r_2, the circle with smaller radius is the curve need to be traced and the points denoted by '$*$' are the generated points by algorithms A, B, C.

Test 1. In this test, for different values of the radius r_2, we apply algorithms A, B, C with the same steplength to trace the circle S from the start point $(1, 0)$. The figures in Fig. 3 show whether "curve-jumping" happens in the process of curve-tracing by algorithms A, B, C respectively.

Table 1 shows whether "curve-jumping" happens for different radii, and if "curve-jumping" does not happen, it presents the number of corrector points and the elapsed time of tracing the whole curve by algorithms A, B, C respectively.

Table 1 For the different radius r_2, results of curve-tracing with the fixed steplength h

r_2	Jumping			Number1			T (seconds)		
	A	B	C	A	B	C	A	B	C
1.10	Y	Y	Y	–	–	–	–	–	–
1.25	Y	Y	N	–	–	11	–	–	0.031
1.40	N	N	N	22	12	11	0.0037	0.0064	0.0027

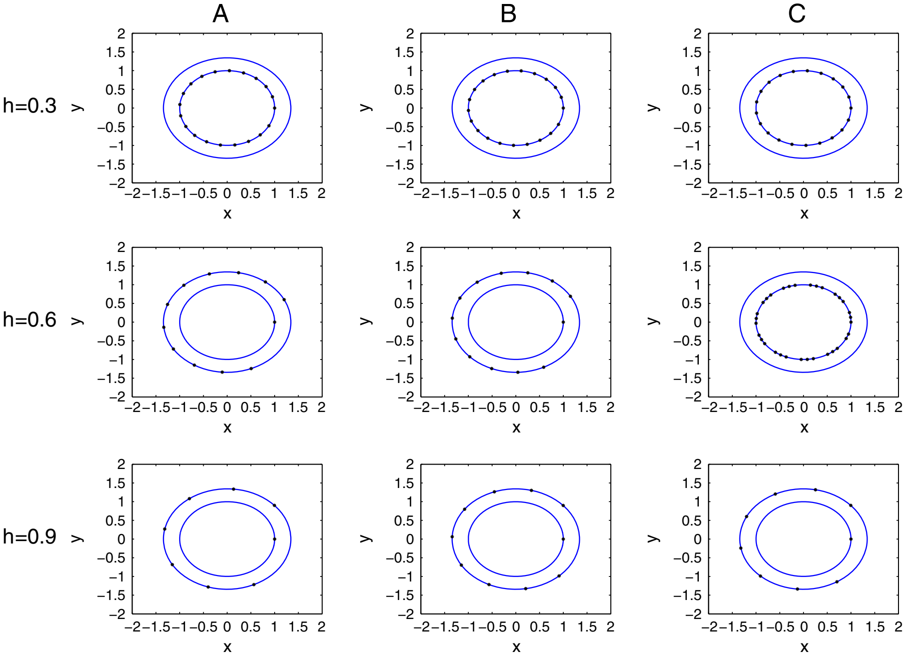

Fig. 4 For the fixed radius r_2, figures of curve-tracing with the different steplengths

Test 2. In this test, for the same radius r_2, we apply algorithms A, B, C with different steplengths to trace the circle S from the start point $(1, 0)$. The figures in Fig. 4 show whether the "curve-jumping" happens in the process of curve-tracing by algorithms A, B, C respectively.

Table 2 shows whether "curve-jumping" happens in the process of curve-tracing by algorithms A, B, C with different steplengths, and if "curve-jumping" does not happen, it presents the number of corrector points and the elapsed time of tracing the whole curve by algorithms A, B, C.

Conclusion: From the figures in Fig. 3 and results in Table 1, we can see that for a given steplength, say $r_2 = 1.2$, Algorithm A and Algorithm B both jump from the circle S to another circle. From the figures in Fig. 4

Table 2 For the fixed radius r_2, results of curve-tracing with the different steplengths

Steplength h	Jumping			Number1			T (seconds)		
	A	B	C	A	B	C	A	B	C
0.3	N	N	N	19	20	20	0.0030	0.0083	0.0039
0.6	Y	Y	N	–	–	31	–	–	0.0072
0.9	Y	Y	Y	–	–	–	–	–	–

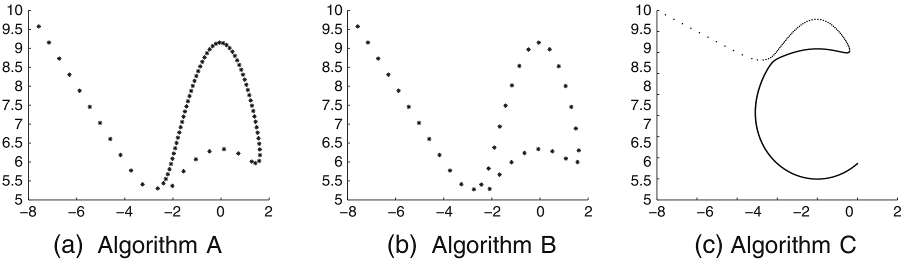

Fig. 5 Corrector points generated by algorithms A, B, C

and results in Table 2, we can see that for the fixed two homocentric circles, only Algorithm C can successfully trace the curve S with a comparatively large steplength, say $h = 0.6$. Therefore, in some sense, we can say that Algorithm C is more robust and suitable for tracing curves which are close to each other.

Example 2 Consider the curve defined by

$$F(x, y) = \left(e^{\frac{-x^2-y+9}{\alpha}} + e^{\frac{-y-x+2}{\alpha}}\right)^{\beta} + e^{\frac{-4x^2-y^2+36}{5}} - 1,$$

where α, β are real numbers.

Test 1. $\alpha = 2.5, \beta = -\frac{1}{2}$.

We trace the solution curve of $F(x, y) = 0$ by algorithms A, B, C with the same steplength $h = 0.6$ from the start point (-8,10) respectively, see Fig. 5a–c.

When we trace the curve by Algorithm A and Algorithm B, both processes happen catastrophe, that is, the methods trace the curve repeatedly, accordingly they fail to trace the whole curve. However, Algorithm C can trace the curve successfully. It is shown that Algorithm C is more robust for tracing these complex curves (with turning points).

Test 2. $\alpha = 3, \beta = -\frac{3}{5}$.

We trace the solution curve by algorithms A, B, C with the same steplength $h = 0.8$ from the start point (-8,10) respectively, see Fig. 6a–c.

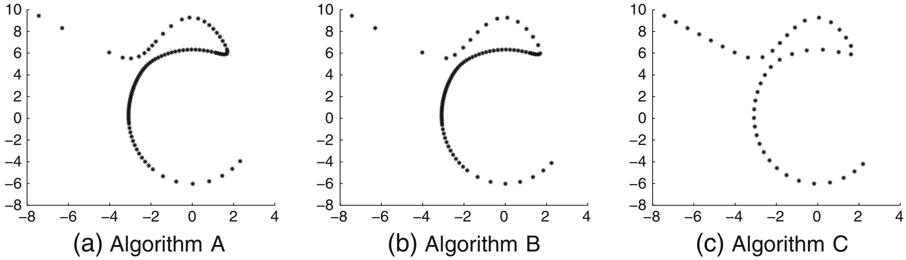

Fig. 6 Corrector points generated by algorithms A, B, C

Table 3 Tracing the curve by algorithms A, B, C

Algorithms	Jumping	Number1	Number2	T (seconds)
A	N	96	381	0.025
B	N	88	354	0.021
C	N	45	224	0.014

From the figures, we can see that algorithms A, B, C all trace the solution curve successfully, and the numbers of corrector points generated by Algorithm A and Algorithm B are both far bigger than that by Algorithm C. The reason is that the predictor steplength is very small at the turning points when we apply algorithms A, B to trace the curve. However, by Algorithm C, the steplength can keep larger in the process of curve-tracing by algorithms A, B, C. It is shown that Algorithm C is more efficient for tracing this class of complex curves, comparing with Algorithm A and Algorithm B. The detail of the results of curve-tracing by algorithms A, B, C is shown in Table 3.

4 Conclusion

The continuation methods have long served as powerful tools to trace solution curves of parameterized systems. However, for some complicated cases in practice, such as the case that solution curves are close to each other or the case that the curve turns acutely at some points, the classical continuation methods are unstable.

A simple but robust corrector–sphere corrector is presented in this paper, and combining the well-known Euler predictor, we describe a particular version of continuation method: Euler-Sphere Predictor-Corrector Method. As our discussion in this paper, our new continuation method is more robust and can avoid "curve-jumping" in the process of tracing curves close to each other, and more efficient than the classical continuation methods for tracing curves containing many inflection points.

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