Numer Algor (2014) 65:339–354
DOI 10.1007/s11075-013-9708-9

ORIGINAL PAPER

Convergence analysis for a family of improved super-Halley methods under general convergence condition

Xiuhua Wang · Dongyang Shi · Jisheng Kou

Received: 27 January 2013 / Accepted: 17 March 2013 / Published online: 6 April 2013

Abstract In this paper, we focus on the semilocal convergence for a family of improved super-Halley methods for solving non-linear equations in Banach spaces. Different from the results in Wang et al. (J Optim Theory Appl 153:779–793, 2012), the condition of Hölder continuity of third-order Fréchet derivative is replaced by its general continuity condition, and the latter is weaker than former. Moreover, the R-order of the methods is also improved. By using the recurrence relations, we prove a convergence theorem to show the existence-uniqueness of the solution. The R-order of these methods is analyzed with the third-order Fréchet derivative of the operator satisfies general continuity condition and Hölder continuity condition.

Keywords Recurrence relations · Semilocal convergence · Nonlinear equations in Banach spaces · Super-Halley method · R-order of convergence

Mathematics Subject Classifications (2010) 65D10 · 65D99

1 Introduction

Solving the nonlinear equations in Banach spaces is an important problem in scientific and engineering computing areas, such equation can be given by

$$F(x) = 0, \tag{1.1}$$

X. Wang · D. Shi
Department of Mathematics, Zhengzhou University, Zhengzhou 450052, China

X. Wang · J. Kou (✉)
School of Mathematics and Statistics, Hubei Engineering University, Xiaogan 432100, Hubei, China
e-mail: kjsfine@yahoo.com.cn

where $F:\Omega\subseteq X\rightarrow Y$ is a nonlinear operator in a non-empty open convex subset Ω of a Banach space X with values in a Banach space Y.

The super-Halley method is a well-known third-order method for solving this equation. In [15], a class of modified super-Halley methods for solving the equation (1.1) is considered. This class of methods is given by

$$x_{n+1}=x_n-\left[I+\frac{1}{2}K_F(x_n)+\frac{1}{2}K_F(x_n)^2+K_F(x_n)^{\theta}\Phi(K_F(x_n))\right]\Gamma_nF(x_n), \quad (1.2)$$

where θ is a parameter, such that $\theta\geqslant 3$, $\Gamma_n=F'(x_n)^{-1}$, $K_F(x_n)$ is the following operator

$$K_F(x_n)=\Gamma_nF''\left(x_n-\frac{1}{3}\Gamma_nF(x_n)\right)\Gamma_nF(x_n). \quad (1.3)$$

In the methods given by (1.2), Φ is an operator which satisfies that there exists a real non-negative and non-decreasing function $\chi(t)$, such that $\|\Phi(K_F(x_n))\|\leqslant\chi(\|K_F(x_n)\|)$ and the function $\chi(t)$ is bounded for t in a suitable region. Under the conditions that

(A1) Γ_0 exist and $\|\Gamma_0\|\leqslant\beta$,
(A2) $\|\Gamma_0F(x_0)\|\leqslant\eta$,
(A3) $\|F''(x)\|\leqslant M,\quad x\in\Omega$,
(A4) $\|F'''(x)\|\leqslant N,\quad x\in\Omega$,
(A5) there exists a positive real number L such that

$$\|F'''(x)-F'''(y)\|\leqslant L\|x-y\|^q,\quad 0\leqslant q\leqslant 1,\quad \forall x,y\in\Omega, \quad (1.4)$$

the R-order of methods given by (1.2) is proved to be $3+q$.

But under the assumptions (A1)–(A5), we can not study the solution of some equations. Such as the nonlinear integral equation of mixed Hammerstein type [5, 8], which is given by

$$x(s)+\sum_{i=1}^{m}\int_a^b G_i(s,t)H_i(x(t))dt=u(s), s\in[a,b], \quad (1.5)$$

where $-\infty<a<b<+\infty$, u, G_i and H_i are known functions ($i=1,2,\ldots,m.$), x is the solution to be found. On the condition that $H_i'''(x(t))$ is (L_i,q_i)-Hölder continuous in Ω, $i=1,2,\ldots,m$, then the corresponding operator $F:\Omega\subseteq C[0,1]\rightarrow C[0,1]$,

$$[F(x)](s)=x(s)+\sum_{i=1}^{m}\int_a^b G_i(s,t)H_i(x(t))dt-u(s), s\in[a,b], \quad (1.6)$$

is such that its third Fréchet derivative is neither Lipschitz continuous nor Hölder continuous in Ω while, for an example, we consider the max-norm. In this case,

$$\left\| F'''(x) - F'''(y) \right\| \leqslant \sum_{i=1}^{m} L_i \|x - y\|^{q_i}, \, L_i \geqslant 0, q_i \in [0, 1], \ \ \forall x, y \in \Omega. \tag{1.7}$$

Obviously, the operator given by (1.7) does not satisfy the assumption (A5), so we can not study the solution of this equation under the assumptions(A1)–(A5). Since the importance of nonlinear integral equation of mixed Hammerstein type, it is considered in many papers, such as [1, 5, 6, 8], where in [5], Ezquerro and Hernández gave the following method

$$\begin{cases} y_n = x_n - \left[F'(x_n)\right]^{-1} F(x_n), \\ T(x_n, y_n) = \left[F'(x_n)\right]^{-1} \left(F'\left(x_n + \dfrac{2-3\theta}{3}(y_n - x_n)\right) - F'(x_n + \theta(y_n - x_n))\right), \\ \theta \in \left[0, \dfrac{1}{3}\right), \\ x_{n+1} = y_n + \left(\dfrac{-3}{4(1-3\theta)} T(x_n, y_n) + \dfrac{9}{8(1-3\theta)^2} T(x_n, y_n)^2\right)(y_n - x_n), n \geqslant 0. \end{cases} \tag{1.8}$$

They used the following assumption:

(B5) $\|F'''(x) - F'''(y)\| \leqslant \omega(\|x - y\|), \ \forall x, y \in \Omega$, where $\omega(z)$ is a non-decreasing continuous real function for $z > 0$ and $\omega(0) \geqslant 0$;

(B6) there exists a positive real function $\nu \in C[0, 1]$, with $\nu(t) \leqslant 1$, such that $\omega(tz) \leqslant \nu(t)\omega(z)$, for $t \in [0, 1], z \in (0, +\infty)$. Based on the assumptions (A1)–(A4) and (B5)–(B6), Ezquerro and Hernández [5] studied the semilocal convergence of the method given by (1.8). Choosing $\omega(t) = \sum_{i=1}^{m} L_i t^{q_i}$, they proved that the R-order of method (1.8) is $3 + q$ with the third Fréchet derivative of the operator F satisfies the condition (1.7), where $q = \min\{q_1, q_2..., q_m\}, q_i \in [0, 1], i = 1, 2...m$.

Recently, some sixth-order variants are developed in [10, 11]. These variants focus on finding a simple root of a non-linear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ for an open interval D is a scalar function. In order to relax the condition (A5) considered in [15], to improve the R-order of convergence, in this paper, we consider the semilocal convergence for a family of improved super-Halley methods in Banach space given by

$$\begin{cases} z_n = x_n - \left[I + \dfrac{1}{2}K_F(x_n) + \dfrac{1}{2}K_F(x_n)^2 + K_F(x_n)^\theta \Phi(K_F(x_n))\right] \Gamma_n F(x_n), \\ x_{n+1} = z_n - \left[I - \Gamma_n F''(v_n)(z_n - x_n) + \delta \left[\Gamma_n F''(v_n)(z_n - x_n)\right]^2\right] \Gamma_n F(z_n), \end{cases} \tag{1.9}$$

where I is the identity operator, θ, δ are two parameters, such that $\theta \geqslant 3$ and $\delta \in [-1, 1]$, $K_F(x_n)$ is defined by (1.3), $\Gamma_n = F'(x_n)^{-1}$, $v_n = x_n - \frac{1}{3}\Gamma_n F(x_n)$, Φ is also an operator which satisfies that there exists a real non-negative and non-decreasing function $\chi(t)$, such that $\|\Phi(K_F(x_n))\| \leqslant \chi(\|K_F(x_n)\|)$ and the function $\chi(t)$ is bounded for $t \in (0, s)$, where s will be defined in the latter development. This family of methods given by (1.9) can be derived by adding an evaluation of the function at another point in the procedure iterated by methods (1.2), but the R-order of convergence can be improved from $3 + q$ for methods (1.2) to $5 + q$ for methods (1.9) under the above conditions (A1)–(A5).

Here, to solve the problem that the third-order derivative of an operator is neither Lipschitz nor Hölder continuous, we assume that F''' satisfies the general continuity condition (B5) employed in [5], instead of the Hölder continuity condition (A5) used in [15]. By using the recurrence relations, we establish the semilocal convergence of the methods given by (1.9). This approach has been successfully used in establishing the convergence of some methods, such as the references [2–5, 7, 9, 12–18]. We prove a convergence theorem to show the existence-uniqueness of the solution. The R-order of methods (1.9) is also analyzed with F''' satisfies general continuity condition and Hölder continuity condition. Finally, we give some numerical results to show our approach.

2 Some preliminary results

Let X and Y be Banach spaces, the nonlinear operator $F : \Omega \subset X \to Y$ be three times Fréchet differentiable in a non-empty open convex subset Ω. Taking $x_0 \in \Omega$ and furthermore, we make the following assumptions:

(C1) There exists $\Gamma_0 = F'(x_0)^{-1}$ and $\|\Gamma_0\| \leqslant \beta$,
(C2) $\|\Gamma_0 F(x_0)\| \leqslant \eta$,
(C3) $\|F''(x)\| \leqslant M, \quad x \in \Omega$,
(C4) $\|F'''(x)\| \leqslant N, \quad x \in \Omega$,
(C5) $\|F'''(x) - F'''(y)\| \leqslant \omega(\|x - y\|), \quad \forall x, y \in \Omega$, where $\omega(\mu)$ is a non-decreasing continuous real function for $\mu > 0$ and satisfy $\omega(0) \geqslant 0$,
(C6) there exists a non-negative real function $\phi \in C[0, 1]$, with $\phi(t) \leqslant 1$, such that $\omega(t\mu) \leqslant \phi(t)\omega(\mu)$, for $t \in [0, 1]$, $\mu \in (0, +\infty)$.

Remark The conditions (C5) and (C6), which have been used in [5], are the generalization for the Hölder continuity of F''' by choosing $\omega(\mu) = L\mu^q$ and $\phi(t) = t^q$.

We define $B(x, r) = \{y \in X : \|y-x\| < r\}$ and $\overline{B(x, r)} = \{y \in X : \|y-x\| \leqslant r\}$ in this paper. The following lemma gives an approximation of the operator F which will be used in the latter development.

Lemma 1 *Assume that the nonlinear operator $F : \Omega \subset X \to Y$ is three times Fréchet differentiable in a non-empty open convex subset Ω, where X and Y are Banach spaces. Then we have*

$$\begin{aligned} F(x_{n+1}) = & \int_0^1 F'''(x_n + t(v_n - x_n))(v_n - x_n)dt(z_n - x_n)\Gamma_n F(z_n) \\ & - \delta F''(v_n)(z_n - x_n)\Gamma_n F''(v_n)(z_n - x_n)\Gamma_n F(z_n) \\ & - \int_0^1 \left[F''(x_n + t(z_n - x_n)) - F''(x_n)\right](z_n - x_n)dt\Gamma_n F(z_n) \\ & + \int_0^1 F''(x_n + t(z_n - x_n))(z_n - x_n)dt\Gamma_n F''(v_n)(z_n - x_n)\Gamma_n F(z_n) \\ & - \delta \int_0^1 F''(x_n+t(z_n-x_n))(z_n-x_n)dt \left[\Gamma_n F''(v_n)(z_n-x_n)\right]^2 \Gamma_n F(z_n) \\ & + \int_0^1 F''(z_n + t(x_{n+1} - z_n))(x_{n+1} - z_n)^2(1 - t)dt, \qquad (2.1) \end{aligned}$$

where z_n, x_n are given by (1.9), $y_n = x_n - \Gamma_n F(x_n)$, the definitions of v_n and δ are as same as the ones in (1.9).

Proof By Taylor expansion, we have

$$\begin{aligned} F(x_{n+1}) = & F(z_n) + F'(z_n)(x_{n+1} - z_n) \\ & + \int_0^1 F''(z_n + t(x_{n+1} - z_n))(x_{n+1} - z_n)^2(1 - t)dt, \qquad (2.2) \end{aligned}$$

$$\begin{aligned} F'(z_n) = & F'(x_n) + F''(x_n)(z_n - x_n) \\ & + \int_0^1 \left[F''(x_n + t(z_n - x_n)) - F''(x_n)\right](z_n - x_n)dt. \qquad (2.3) \end{aligned}$$

□

Then we obtain

$$
\begin{aligned}
&F(z_n) + F'(z_n)(x_{n+1} - z_n)\\
&\quad = F(z_n) - F'(z_n)\left[I - \Gamma_n F''(v_n)(z_n - x_n) + \delta\left[\Gamma_n F''(v_n)(z_n - x_n)\right]^2\right]\Gamma_n F(z_n)\\
&\quad = F(z_n) - \left\{F'(x_n) + F''(x_n)(z_n - x_n) + \int_0^1 [F''(x_n + t(z_n - x_n)) - F''(x_n)](z_n - x_n)dt\right\}\\
&\qquad \times \left[I - \Gamma_n F''(v_n)(z_n - x_n) + \delta\left[\Gamma_n F''(v_n)(z_n - x_n)\right]^2\right]\Gamma_n F(z_n)\\
&\quad = F(z_n) - F(z_n) + F''(v_n)(z_n - x_n)\Gamma_n F(z_n) - \delta F''(v_n)(z_n - x_n)\Gamma_n F''(v_n)(z_n - x_n)\Gamma_n F(z_n)\\
&\qquad - F''(x_n)(z_n - x_n)\Gamma_n F(z_n) + F''(x_n)(z_n - x_n)\Gamma_n F''(v_n)(z_n - x_n)\Gamma_n F(z_n)\\
&\qquad - \delta F''(x_n)(z_n - x_n)[\Gamma_n F''(v_n)(z_n - x_n)]^2\Gamma_n F(z_n)\\
&\qquad - \int_0^1 \left[F''(x_n + t(z_n - x_n)) - F''(x_n)\right](z_n - x_n)dt\\
&\qquad \times \left[I - \Gamma_n F''(v_n)(z_n - x_n) + \delta\left[\Gamma_n F''(v_n)(z_n - x_n)\right]^2\right]\Gamma_n F(z_n). \qquad (2.4)
\end{aligned}
$$

Since

$$
F''(v_n) = F''(x_n) + \int_0^1 F'''(x_n + t(v_n - x_n))(v_n - x_n)dt, \qquad (2.5)
$$

we have

$$
\begin{aligned}
&F(z_n) + F'(z_n)(x_{n+1} - z_n)\\
&\quad = \int_0^1 F'''(x_n + t(v_n - x_n))(v_n - x_n)dt(z_n - x_n)\Gamma_n F(z_n)\\
&\qquad -\delta F''(v_n)(z_n - x_n)\Gamma_n F''(v_n)(z_n - x_n)\Gamma_n F(z_n)\\
&\qquad -\delta F''(x_n)(z_n - x_n)[\Gamma_n F''(v_n)(z_n - x_n)]^2\Gamma_n F(z_n)\\
&\qquad - \int_0^1 [F''(x_n + t(z_n - x_n)) - F''(x_n)](z_n - x_n)dt\Gamma_n F(z_n)\\
&\qquad + \int_0^1 F''(x_n + t(z_n - x_n))(z_n - x_n)dt\Gamma_n F''(v_n)(z_n - x_n)\Gamma_n F(z_n)\\
&\qquad -\delta\int_0^1 [F''(x_n + t(z_n - x_n)) - F''(x_n)](z_n - x_n)dt[\Gamma_n F''(v_n)(z_n - x_n)]^2\Gamma_n F(z_n).
\end{aligned}
\qquad (2.6)
$$

Substituting (2.6) into (2.2), we can obtain (2.1).

Now we give the definitions of the following functions.

$$
p(t) = g(t) + \frac{1}{2}t\left[1 + tg(t) + |\delta|(tg(t))^2\right]\left[1 + t + 2t^{\theta-1}\chi(t) + g(t)^2\right], \qquad (2.7)
$$

$$h(t) = \frac{1}{1 - tp(t)}, \tag{2.8}$$

$$\varphi(t,u,v) = \left[\frac{1}{3}g(t)u + (1+|\delta|)(g(t)t)^2 + \frac{1}{2}g(t)^2u + |\delta|(g(t)t)^3\right]\psi(t,u,v) + \frac{1}{2}t\left[1 + tg(t) + |\delta|(tg(t))^2\right]^2\psi(t,u,v)^2, \tag{2.9}$$

where

$$g(t) = 1 + \frac{1}{2}t + \frac{1}{2}t^2 + t^\theta\chi(t),$$

$$\psi(t,u,v) = t^\theta\chi(t) + t^3\left[\frac{1}{2} + t^{\theta-2}\chi(t)\right] + \frac{1}{2}t(g(t)-1)^2 + \frac{1}{6}tu + \frac{1}{2}u(g(t)-1) + \left(\frac{1}{6}J_1 + \frac{1}{2}J_2\right)v,$$

$$J_1 = \int_0^1 \phi\left(\frac{t}{3}\right)dt, \qquad J_2 = \int_0^1 \phi(t)(1-t)^2dt.$$

The functions defined above will be used frequently in the later developments, so next we study some of their properties. Let $f(t) = p(t)t - 1$. Since $f(0) = -1 < 0$ and $f(\frac{1}{2}) \geqslant \frac{3299}{8192} > 0$, then we can conclude that $f(t) = 0$ has at least a root in $(0, \frac{1}{2})$. Let s be the smallest positive root of the equation $p(t)t - 1 = 0$, then $s < \frac{1}{2}$.

Lemma 2 *Let the functions p, h and φ be given in (2.7)–(2.9). Then*

(a) *$p(t)$ and $h(t)$ are increasing and $p(t) > 1$, $h(t) > 1$ for $t \in (0, s)$;*
(b) *For $t \in (0, s)$, a fixed $u > 0$, and a fixed $v > 0$, $\varphi(t,u,v)$ is increasing as the function of t; similarly, for $u > 0$, a fixed $t \in (0, s)$ and a fixed $v > 0$, $\varphi(t,u,v)$ is increasing as the function of u; for $v > 0$, a fixed $u > 0$ and a fixed $t \in (0, s)$, $\varphi(t,u,v)$ is increasing as the function of v.*

Define $\eta_0 = \eta$, $\beta_0 = \beta$, $a_0 = M\beta\eta$, $b_0 = N\beta\eta^2$, $c_0 = \beta\eta^2\omega(\eta)$ and $d_0 = h(a_0)\varphi(a_0, b_0, c_0)$, and moreover, we define the following sequences as

$$\eta_{n+1} = d_n\eta_n, \tag{2.10}$$

$$\beta_{n+1} = h(a_n)\beta_n, \tag{2.11}$$

$$a_{n+1} = M\beta_{n+1}\eta_{n+1}, \tag{2.12}$$

$$b_{n+1} = N\beta_{n+1}\eta_{n+1}^2, \tag{2.13}$$

$$c_{n+1} = \beta_{n+1}\eta_{n+1}^2 w(\eta_{n+1}), \tag{2.14}$$

$$d_{n+1} = h(a_{n+1})\varphi(a_{n+1}, b_{n+1}, c_{n+1}), \tag{2.15}$$

where $n \geqslant 0$. From the definitions of $a_{n+1}, b_{n+1}, c_{n+1}$ and (2.10)–(2.11), we can get

$$a_{n+1} = h(a_n)d_n a_n, \tag{2.16}$$

$$b_{n+1} = h(a_n)d_n^2 b_n, \tag{2.17}$$

$$c_{n+1} \leqslant h(a_n)d_n^2 \phi(d_n)c_n. \tag{2.18}$$

Now we give the following lemma to show some important properties of the previous sequences.

Lemma 3 *If*

$$a_0 < s \quad and \quad h(a_0)d_0 < 1, \tag{2.19}$$

then we have

(a) $h(a_n) > 1$ *and* $d_n < 1$ *for* $n \geqslant 0$,
(b) *the sequences* $\{\eta_n\}, \{a_n\}, \{b_n\}, \{c_n\}$ *and* $\{d_n\}$ *are decreasing,*
(c) $p(a_n)a_n < 1$ *and* $h(a_n)d_n < 1$ *for* $n \geqslant 0$.

The proof of Lemma 3 can be obtained by induction.

Lemma 4 *Let the functions* p, h *and* φ *be given in* (2.7)–(2.9). *Let* $\alpha \in (0, 1)$, *then* $p(\alpha t) < p(t)$, $h(\alpha t) < h(t)$, $\varphi(\alpha t, \alpha^2 u, \alpha^2 v) < \alpha^4 \varphi(t, u, v)$ *and* $\varphi(\alpha t, \alpha^2 u, \alpha^{(2+q)} v) < \alpha^{(4+q)} \varphi(t, u, v)$ *for* $t \in (0, s)$, *where* s *is the smallest positive root of the equation* $p(t)t - 1 = 0$.

3 Recurrence relations for the methods

For $n = 0$, the existence of Γ_0 implies the existence of v_0, y_0, and furthermore, we have

$$\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leqslant \eta_0, \tag{3.1}$$

$$\|v_0 - x_0\| = \| - \frac{1}{3}\Gamma_0 F(x_0)\| \leqslant \frac{1}{3}\eta_0. \tag{3.2}$$

This implies that $y_0, v_0 \in B(x_0, R\eta)$, where $R = \frac{p(a_0)}{1-d_0}$.

Moreover, we have

$$\|K_F(x_0)\| \leqslant \|\Gamma_0\| \left\| F''\left(x_0 + \frac{1}{3}(y_0 - x_0)\right)\right\| \|\Gamma_0 F(x_0)\| \leqslant M\beta_0\eta_0 = a_0. \tag{3.3}$$

Then we obtain

$$\begin{aligned}\|z_0 - x_0\| &= \left\| -\left[I + \frac{1}{2}K_F(x_0) + \frac{1}{2}K_F(x_0)^2 + K_F(x_0)^{\theta}\Phi(K_F(x_0)) \right] \Gamma_0 F(x_0) \right\| \\ &\leqslant \left[1 + \frac{1}{2}\|K_F(x_0)\| + \frac{1}{2}\|K_F(x_0)\|^2 + \|K_F(x_0)\|^{\theta}\chi(\|K_F(x_0)\|) \right] \\ &\quad \times \|\Gamma_0 F(x_0)\| \leqslant \left[1 + \frac{1}{2}a_0 + \frac{1}{2}a_0^2 + a_0^{\theta}\chi(a_0) \right] \|\Gamma_0 F(x_0)\| \\ &= g(a_0)\|\Gamma_0 F(x_0)\| \leqslant g(a_0)\eta_0. \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\|z_0 - y_0\| &= \left\| -\left[\frac{1}{2}K_F(x_0) + \frac{1}{2}K_F(x_0)^2 + K_F(x_0)^{\theta}\Phi(K_F(x_0)) \right] \Gamma_0 F(x_0) \right\| \\ &\leqslant \left[\frac{1}{2}\|K_F(x_0)\| + \frac{1}{2}\|K_F(x_0)\|^2 + \|K_F(x_0)\|^{\theta}\chi(\|K_F(x_0)\|) \right] \|\Gamma_0 F(x_0)\| \\ &\leqslant \left[\frac{1}{2}a_0 + \frac{1}{2}a_0^2 + a_0^{\theta}\chi(a_0) \right] \|\Gamma_0 F(x_0)\| \\ &= [g(a_0) - 1]\|\Gamma_0 F(x_0)\| \leqslant [g(a_0) - 1]\eta_0. \end{aligned} \tag{3.5}$$

Furthermore, we have

$$\begin{aligned}\|x_1 - z_0\| &= \left\| -[I - \Gamma_0 F''(v_0)(z_0 - x_0) + \delta[\Gamma_0 F''(v_0)(z_0 - x_0)]^2]\Gamma_0 F(z_0) \right\| \\ &\leqslant [1 + a_0 g(a_0) + |\delta|(a_0 g(a_0))^2]\beta_0 \|F(z_0)\|. \end{aligned} \tag{3.6}$$

By Taylor expansion, we have

$$\begin{aligned} F(z_0) &= F(x_0) + F'(x_0)(z_0 - x_0) \\ &\quad + \int_0^1 F''(x_0 + t(z_0 - x_0))(z_0 - x_0)^2(1-t)dt \\ &= \left[-\frac{1}{2}F''(v_0)\Gamma_0 F(x_0) - \frac{1}{2}F''(v_0)\Gamma_0 F(x_0)K_F(x_0) \right] \Gamma_0 F(x_0) \\ &\quad - F''(v_0)\Gamma_0 F(x_0)K_F(x_0)^{\theta-1}\Phi(K_F(x_0))\Gamma_0 F(x_0) \\ &\quad + \int_0^1 F''(x_0 + t(z_0 - x_0))(z_0 - x_0)^2(1-t)dt. \end{aligned} \tag{3.7}$$

Then we can obtain

$$\|F(z_0)\| \leqslant \frac{1}{2}M\eta_0^2\left[1 + a_0 + 2a_0^{\theta-1}\chi(a_0) \right] + \frac{1}{2}M\|z_0 - x_0\|^2. \tag{3.8}$$

Consequently, we have

$$\|x_1 - x_0\| \leqslant \|x_1 - z_0\| + \|z_0 - x_0\| \leqslant p(a_0)\|\Gamma_0 F(x_0)\| \leqslant p(a_0)\eta_0, \tag{3.9}$$

which shows that $x_1 \in B(x_0, R\eta)$ because of the assumption $d_0 < 1/h(a_0) < 1$.

Since $a_0 < s$ and $p(a_0) < p(s)$, we obtain

$$\|I - \Gamma_0 F'(x_1)\| \leqslant \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \leqslant M\beta_0\|x_1 - x_0\| \leqslant a_0 p(a_0) < 1.$$

By the Banach lemma, we have that $\Gamma_1 = [F'(x_1)]^{-1}$ exists and

$$\begin{aligned} \|\Gamma_1\| &\leqslant \frac{\|\Gamma_0\|}{1 - \|\Gamma_0\| \|F'(x_0) - F'(x_1)\|} \\ &\leqslant \frac{\|\Gamma_0\|}{1 - a_0 p(a_0)}\| = h(a_0)\|\Gamma_0\| \leqslant h(a_0)\beta_0 = \beta_1. \end{aligned} \tag{3.10}$$

So y_1, v_1 are well defined. Using the results in [15]

$$\begin{aligned} F(z_n) =& \frac{1}{2}(F''(v_n) - F''(x_n))\Gamma_n F(x_n)K_F(x_n)(y_n - x_n) \\ &+ F''(v_n)\Gamma_n F(x_n)K_F(x_n)^{\theta-1}\Phi(K_F(x_n))(y_n - x_n) \\ &- F''(x_n)\Gamma_n F(x_n)\left[\frac{1}{2}K_F(x_n)^2 + K_F(x_n)^\theta \Phi(K_F(x_n))\right](y_n - x_n) \\ &- \frac{1}{6}\int_0^1 \left[F'''\left(x_n + \frac{1}{3}t(y_n - x_n)\right) - F'''(x_n)\right](y_n - x_n)dt(y_n - x_n)^2 \\ &+ \frac{1}{2}\int_0^1 \left[F'''(x_n + t(y_n - x_n)) - F'''(x_n)\right](y_n - x_n)^3(1-t)^2 dt \\ &+ \int_0^1 F''(y_n + t(z_n - y_n))(z_n - y_n)^2(1-t)dt \\ &+ \int_0^1 F'''(x_n + t(y_n - x_n))(y_n - x_n)^2(1-t)dt(z_n - y_n), \end{aligned} \tag{3.11}$$

we have

$$\begin{aligned} \|F(z_0)\| \leqslant& \left[Ma_0^{\theta-1}\chi(a_0)\eta_0 + Ma_0^2\eta_0\left[\frac{1}{2} + a_0^{\theta-2}\chi(a_0)\right] + \frac{M}{2}[g(a_0) - 1]^2\eta_0\right]\|\Gamma_0 F(x_0)\| \\ &+ \left[\frac{N}{6}a_0\eta_0^2 + \frac{N}{2}[g(a_0) - 1]\eta_0^2 + \left(\frac{1}{6}J_1 + \frac{1}{2}J_2\right)\omega(\eta_0)\eta_0^2\right]\|\Gamma_0 F(x_0)\|, \end{aligned} \tag{3.12}$$

where $J_1 = \int_0^1 \phi(\frac{t}{3})dt$, $J_2 = \int_0^1 \phi(t)(1-t)^2 dt$. Therefore

$$\begin{aligned} \|x_1 - z_0\| &\leqslant \left[1 + a_0 g(a_0) + |\delta|(a_0 g(a_0))^2\right]\beta_0 \|F(z_0)\| \\ &\leqslant \left[1 + a_0 g(a_0) + |\delta|(a_0 g(a_0))^2\right]\psi(a_0, b_0, c_0)\|\Gamma_0 F(x_0)\| \\ &\leqslant \left[1 + a_0 g(a_0) + |\delta|(a_0 g(a_0))^2\right]\psi(a_0, b_0, c_0)\eta_0. \end{aligned} \tag{3.13}$$

From Lemma 1, we can obtain

$$\begin{aligned} \|F(x_1)\| \leqslant& \frac{1}{3}g(a_0)N\eta_0^2\beta_0\|F(z_0)\| + |\delta|g(a_0)^2 a_0 M\eta_0\beta_0\|F(z_0)\| \\ &+ \frac{1}{2}g(a_0)^2 N\eta_0^2\beta_0\|F(z_0)\| + g(a_0)^2 a_0 M\eta_0\beta_0\|F(z_0)\| \\ &+ |\delta|g(a_0)^3 a_0^2 M\eta_0\beta_0\|F(z_0)\| + \frac{1}{2}M\|x_1 - z_0\|^2. \end{aligned} \tag{3.14}$$

From (3.10) and (3.14), we have

$$\begin{aligned} \|y_1 - x_1\| &= \|\Gamma_1 F(x_1)\| \leqslant \|\Gamma_1\|\|F(x_1)\| \\ &\leqslant h(a_0)\varphi(a_0, b_0, c_0)\|\Gamma_0 F(x_0)\| \\ &\leqslant h(a_0)\varphi(a_0, b_0, c_0)\eta_0 = d_0\eta_0 = \eta_1. \end{aligned} \tag{3.15}$$

Since $p(a_0) > 1$, we obtain

$$\begin{aligned} \|y_1 - x_0\| &\leqslant \|y_1 - x_1\| + \|x_1 - x_0\| \\ &\leqslant (p(a_0) + d_0)\eta_0 \\ &< p(a_0)(1 + d_0)\eta < R\eta, \end{aligned} \tag{3.16}$$

$$\begin{aligned} \|v_1 - x_0\| &\leqslant \|x_1 - x_0\| + \|\frac{1}{3}(y_1 - x_1)\| \\ &< (p(a_0) + d_0)\eta_0 < R\eta, \end{aligned} \tag{3.17}$$

which implies $y_1, v_1 \in B(x_0, R\eta)$.

Besides, we have

$$M\|\Gamma_1\|\|\Gamma_1 F(x_1)\| \leqslant h(a_0)d_0a_0 = a_1, \tag{3.18}$$

$$N\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^2 \leqslant h(a_0)d_0^2 b_0 = b_1, \tag{3.19}$$

$$\|\Gamma_1\|\|\Gamma_1 F(x_1)\|^2 w(\|\Gamma_1 F(x_1)\|) \leqslant \beta_1\eta_1^2 w(\eta_1) = c_1. \tag{3.20}$$

Using the induction, we can obtain the following items:

(I) There exists $\Gamma_n = [F'(x_n)]^{-1}$ and $\|\Gamma_n\| \leqslant h(a_{n-1})\|\Gamma_{n-1}\| \leqslant h(a_{n-1})\beta_{n-1} = \beta_n$,
(II) $\|\Gamma_n F(x_n)\| \leqslant h(a_{n-1})\varphi(a_{n-1}, b_{n-1}, c_{n-1})\|\Gamma_{n-1} F(x_{n-1})\| \leqslant d_{n-1}\eta_{n-1} = \eta_n$,
(III) $M\|\Gamma_n\|\|\Gamma_n F(x_n)\| \leqslant a_n$,
(IV) $N\|\Gamma_n\|\|\Gamma_n F(x_n)\|^2 \leqslant b_n$,
(V) $\|\Gamma_n\|\|\Gamma_n F(x_n)\|^2 w(\|\Gamma_n F(x_n)\|) \leqslant c_n$,
(VI) $\|z_n - x_n\| \leqslant g(a_n)\|\Gamma_n F(x_n)\| \leqslant g(a_n)\eta_n$,
(VII) $\|x_{n+1} - x_n\| \leqslant p(a_n)\|\Gamma_n F(x_n)\| \leqslant p(a_n)\eta_n$.

Lemma 5 *Let the assumptions of Lemma* 3 *and the conditions* (C1)–(C6) *hold, then* $\|v_n - x_0\| \leqslant R\eta$, $\|z_n - x_0\| \leqslant R\eta$, $\|x_{n+1} - x_0\| \leqslant R\eta$, *where* $R = \frac{p(a_0)}{1-d_0}$.

Proof By (II), (VI) and (VII), we obtain

$$\begin{aligned} \|v_n - x_0\| &\leqslant \|v_n - x_n\| + \|x_n - x_0\| \leqslant \left\|\frac{1}{3}\Gamma_n F(x_n)\right\| + \sum_{i=0}^{n-1}\|x_{i+1} - x_i\| \\ &\leqslant \eta_n + \sum_{i=0}^{n-1} p(a_i)\eta_i \leqslant p(a_0)\sum_{i=0}^{n}\eta_i \end{aligned} \tag{3.21}$$

$$\begin{aligned} \|z_n - x_0\| &\leqslant \|z_n - x_n\| + \|x_n - x_0\| \leqslant g(a_n)\eta_n + \sum_{i=0}^{n-1}\|x_{i+1} - x_i\| \\ &\leqslant p(a_n)\eta_n + \sum_{i=0}^{n-1} p(a_i)\eta_i \leqslant p(a_0)\sum_{i=0}^{n}\eta_i \end{aligned} \tag{3.22}$$

and

$$\begin{aligned}\|x_{n+1}-x_0\| &\leqslant \sum_{i=0}^{n}\|x_{i+1}-x_i\| \\ &\leqslant \sum_{i=0}^{n} p(a_i)\eta_i \leqslant p(a_0)\sum_{i=0}^{n}\eta_i \\ &= p(a_0)\sum_{i=0}^{n}\eta\left(\prod_{j=0}^{i-1} d_j\right). \end{aligned} \tag{3.23}$$

□

Let $\gamma = h(a_0)d_0$ and $\lambda = 1/h(a_0)$; then we have $a_1 = \gamma a_0$, $b_1 = h(a_0)d_0^2 b_0 < \gamma^2 b_0$, $c_1 \leqslant h(a_0)d_0^2\phi(d_0)c_0 < \gamma^2 c_0$, by Lemma 4, we have that

$$d_1 < h(\gamma a_0)\varphi(\gamma a_0, \gamma^2 b_0, \gamma^2 c_0) < \gamma^4 d_0 = \gamma^{5^1-1} d_0 = \lambda\gamma^{5^1}.$$

Suppose $d_k \leqslant \lambda\gamma^{5^k}$, $k \geqslant 1$. Then by Lemma 3, we have $a_{k+1} < a_k$ and $h(a_k)d_k < 1$. Consequently, we obtain

$$\begin{aligned} d_{k+1} &< h(a_k)\varphi(h(a_k)d_k a_k, h(a_k)d_k^2 b_k, h(a_k)d_k^2\phi(d_k)c_k) \\ &< h(a_k)\varphi(h(a_k)d_k a_k, h(a_k)^2 d_k^2 b_k, h(a_k)^2 d_k^2 c_k) \\ &< h(a_k)^4 d_k^5 < \lambda\gamma^{5^{k+1}}. \end{aligned} \tag{3.24}$$

This yields that $d_j \leqslant \lambda\gamma^{5^j}$, $j \geqslant 0$. Thus from (3.23) and (3.24), we have

$$\begin{aligned}\|x_{n+1}-x_0\| &\leqslant p(a_0)\sum_{i=0}^{n}\eta\left(\prod_{j=0}^{i-1}\lambda\gamma^{5^j}\right) \\ &= p(a_0)\eta\sum_{i=0}^{n}\lambda^i\gamma^{\frac{5^i-1}{4}} \\ &\leqslant p(a_0)\eta\frac{1-\lambda^{n+1}\gamma^{\frac{5^n+3}{4}}}{1-d_0} < R\eta. \end{aligned} \tag{3.25}$$

Similarly, $\|v_n - x_0\| \leqslant R\eta$ and $\|z_n - x_0\| \leqslant R\eta$.

Lemma 6 *Let* $R = \frac{p(a_0)}{1-d_0}$. *If* $a_0 < s$ *and* $h(a_0)d_0 < 1$, *then* $R < 1/a_0$.

4 Semilocal convergence

4.1 Convergence theorem

Next we prove the following theorem to show the existence and uniqueness of the solution.

Theorem 1 *Let X and Y be two Banach spaces, the nonlinear operator* $F : \Omega \subseteq X \to Y$ *be three times Fréchet differentiable in a non-empty open convex subset* Ω. *Assume that* $x_0 \in \Omega$ *and all conditions* (C1)–(C6) *hold. Let* $a_0 = M\beta\eta$, $b_0 = N\beta\eta^2$, $c_0 = \beta\eta^2\omega(\eta)$ *and* $d_0 =$

$h(a_0)\varphi(a_0, b_0, c_0)$ satisfy $a_0 < s$ and $h(a_0)d_0 < 1$, where s is the smallest positive root of the equation $p(t)t - 1 = 0$ and p, h, φ are defined by (2.7)–(2.9). Let $\overline{B(x_0, R\eta)} \subseteq \Omega$ where $R = \frac{p(a_0)}{1-d_0}$, then starting from x_0, the sequence $\{x_n\}$ generated by the methods (1.9) converges to a solution x^* of $F(x) = 0$ with x_n, x^* belong to $\overline{B(x_0, R\eta)}$ and x^* is the unique solution of $F(x) = 0$ in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$.

Furthermore, a priori error estimate is given by

$$\|x_n - x^*\| \leqslant p(a_0)\eta\lambda^n \gamma^{\frac{5^n-1}{4}} \frac{1}{1 - \lambda\gamma^{5^n}}, \tag{4.1}$$

where $\gamma = h(a_0)d_0$ and $\lambda = 1/h(a_0)$.

Proof Lemma 5 shows that the sequence $\{x_n\}$ is well-defined in $\overline{B(x_0, R\eta)}$. Now we prove that $\{x_n\}$ is a Cauchy sequence. Note that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leqslant \sum_{i=n}^{n+m-1} \|x_{i+1} - x_i\| \\ &\leqslant p(a_0) \sum_{i=n}^{n+m-1} \eta_i \\ &\leqslant p(a_0)\eta\lambda^n\gamma^{\frac{5^n-1}{4}} \frac{1 - \lambda^m \gamma^{\frac{5^n(5^{m-1}+3)}{4}}}{1 - \lambda\gamma^{5^n}}. \end{aligned} \tag{4.2}$$

This shows that there exists a x^* such that $\lim_{n\to\infty} x_n = x^*$. ☐

Letting $n = 0, m \to \infty$ in (4.2), we have

$$\| x^* - x_0 \| \leqslant R\eta, \tag{4.3}$$

which shows that $x^* \in \overline{B(x_0, R\eta)}$.

From Lemma 1, we can obtain

$$\begin{aligned} \|F(x_{n+1})\| \leqslant & \left[\frac{1}{3}g(a_0)N\eta_n^2 + |\delta|g(a_0)^2 a_0 M\eta_n + \frac{1}{2}g(a_0)^2 N\eta_n^2\right]\psi(a_0, b_0, c_0)\eta_n \\ & + [g(a_0)^2 a_0 M\eta_n + |\delta|g(a_0)^3 a_0^2 M\eta_n]\psi(a_0, b_0, c_0)\eta_n \\ & + \frac{1}{2}M[1 + a_0 g(a_0) + |\delta|(a_0 g(a_0))^2]^2 \psi(a_0, b_0, c_0)^2 \eta_n^2. \end{aligned} \tag{4.4}$$

Let $n \to \infty$ in (4.4), then we obtain $\|F(x_n)\| \to 0$ since $\eta_n \to 0$. Notice that $F(x)$ is continuous in Ω, then we have $F(x^*) = 0$.

Now we prove the uniqueness of x^* in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$. From Lemma 6, we can obtain

$$\frac{2}{M\beta} - R\eta = \left(\frac{2}{a_0} - R\right)\eta > \frac{1}{a_0}\eta > R\eta,$$

and then $\overline{B(x_0, R\eta)} \subseteq B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$, so $x^* \in B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$.

Let x^{**} be another root of $F(x) = 0$ in $B(x_0, \frac{2}{M\beta} - R\eta) \bigcap \Omega$. By Taylor theorem, we have

$$0 = F(x^{**}) - F(x^*) = \int_0^1 F'((1-t)x^* + tx^{**})dt(x^{**} - x^*). \tag{4.5}$$

Since

$$\begin{aligned}
&\|\Gamma_0\| \left\| \int_0^1 [F'((1-t)x^* + tx^{**}) - F'(x_0)]dt \right\| \\
&\leqslant M\beta \int_0^1 [(1-t)\|x^* - x_0\| + t\|x^{**} - x_0\|]dt \\
&< \frac{M\beta}{2}\left[R\eta + \frac{2}{M\beta} - R\eta\right] = 1.
\end{aligned} \tag{4.6}$$

by the Banach lemma, we have that $\int_0^1 F'((1-t)x^* + tx^{**})dt$ is invertible and hence $x^{**} = x^*$.

Finally, letting $m \to \infty$ in (4.2), we can get (4.1).

4.2 *R*-order of convergence

Now we analyze the R-order of convergence of methods (1.9) under the condition that $F'''(x)$ is Hölder continuous; that is, $\omega(s) = Ls^q$ and $\phi(t) = t^q$, $q \in [0, 1]$. Let $\gamma = h(a_0)d_0$ and $\lambda = 1/h(a_0)$, then we can get the following lemma.

Lemma 7 *Under the assumptions of Lemma 3, let $\gamma = h(a_0)d_0$ and $\lambda = 1/h(a_0)$, then we have*

$$d_n \leqslant \lambda\gamma^{(5+q)^n}, \quad n \geqslant 0, \tag{4.7}$$

$$\prod_{i=0}^{n} d_i \leqslant \lambda^{n+1}\gamma^{\frac{(5+q)^{n+1}-1}{4+q}}, \tag{4.8}$$

$$\eta_n \leqslant \eta\lambda^n\gamma^{\frac{(5+q)^n-1}{4+q}}, \quad n \geqslant 0, \tag{4.9}$$

$$\sum_{i=n}^{n+m} \eta_i \leqslant \eta\lambda^n\gamma^{\frac{(5+q)^n-1}{(4+q)}} \frac{1 - \lambda^{m+1}\gamma^{\frac{(5+q)^n((5+q)^m+4)}{4+q}}}{1 - \lambda\gamma^{(5+q)^n}}, \quad n \geqslant 0, \; m \geqslant 1. \tag{4.10}$$

Furthermore, we can derive a priori error estimate

$$\|x_n - x^*\| \leqslant \frac{p(a_0)\eta}{\gamma^{1/(4+q)}(1-d_0)}(\gamma^{1/(4+q)})^{(5+q)^n}. \tag{4.11}$$

This shows that for the case of Hölder continuity of F''', the R-order of convergence of methods (1.9) is at least $5 + q$, and especially when F''' is Lipschitz continuous, the R-order of convergence becomes six.

Remark For the mixed condition (1.7)

$$\|F'''(x) - F'''(y)\| \leqslant \sum_{i=1}^{m} L_i\|x - y\|^{q_i}, L_i \geqslant 0, q_i \in [0, 1], \quad \forall x, y \in \Omega,$$

taking $\omega(s) = \sum_{i=1}^{m}(L_i s^{q_i})$, then that $\omega(ts) = \sum_{i=1}^{m}(L_i t^{q_i} s^{q_i})$. Since $t \in [0, 1]$, $q_i \in [0, 1]$, we have that $\phi(t) = t^q$, where $q = \min\{q_1, q_2, \dots, q_m\}$.

5 Numerical results

Now we consider a non-linear integral equation of mixed Hammerstein type, which has been used in reference [5]. This equation is given by

$$x(s) = 1 + \frac{1}{3}\int_0^1 G(s,t)\left(x(t)^{10/3} + x(t)^4\right)dt, s \in [0, 1], \tag{5.1}$$

where $x \in C[0, 1]$, $t \in [0, 1]$ and G is the Green function

$$G(s,t) = \begin{cases} (1-s)t, & t \leqslant s, \\ s(1-t), & s \leqslant t. \end{cases}$$

In order to find the solution of (5.1), we need to solve the equation $F(x) = 0$, where $F : \Omega \subseteq C[0, 1] \to C[0, 1]$,

$$[F(x)](s) = x(s) - 1 - \frac{1}{3}\int_0^1 G(s,t)\left(x(t)^{10/3} + x(t)^4\right)dt, s \in [0, 1]. \tag{5.2}$$

Here, we take $\Omega = B(0, 3/2)$. The Fréchet derivatives of F are given by

$$F'(x)y(s) = y(s) - \frac{1}{3}\int_0^1 G(s,t)\left(\frac{10}{3}x(t)^{7/3} + 4x(t)^3\right)y(t)dt, \quad y \in \Omega,$$

$$F''(x)yz(s) = -\frac{1}{3}\int_0^1 G(s,t)\left(\frac{70}{9}x(t)^{4/3} + 12x(t)^2\right)y(t)z(t)dt, \quad y, z \in \Omega,$$

$$F'''(x)yzu(s) = -\frac{1}{3}\int_0^1 G(s,t)\left(\frac{280}{27}x(t)^{1/3} + 24x(t)\right)y(t)z(t)u(t)dt, \quad y, z, u \in \Omega.$$

Notice that F''' is neither Lipschitz continuous nor Hölder continuous, but the operator F can satisfy the conditions of Theorem 1. Actually, we can define

$$\omega(\mu) = \frac{35}{81}\mu^{\frac{1}{3}} + \mu,$$

and

$$\phi(t) = t^{\frac{1}{3}}.$$

Apparently, the functions $\omega(\mu)$ and $\phi(t)$ given above satisfy the assumptions (C5) and (C6). Taking the constant function $x_0(t) = 1$ as the initial approximate solution, and furthermore, we choose $\Phi(K_F(x_n)) = 0, \delta = 1$, then we have that

$$\|F(x_0)\| = \frac{1}{12},$$

$$\|\Gamma_0\| \leqslant 1.44 \equiv \beta,$$

$$\|\Gamma_0 F(x_0)\| \leqslant 0.12 \equiv \eta,$$

$$\|F''(x)\| \leqslant 1.68145831\cdots \equiv M,$$

$$\|F'''(x)\| \leqslant 1.99462961\cdots \equiv N.$$

Here, we use the max norm. Moreover, we compute

$$a_0 \simeq 0.2906,$$

Since $p(a_0)a_0 \simeq 0.5121 < 1$, we have that $a_0 < s$, where s is the smallest positive root of the equation $p(t)t - 1 = 0$. Furthermore, we compute

$$h(a_0)d_0 \simeq 0.0342 < 1.$$

So the conditions of Theorem 1 are satisfied. Moreover, we obtain that the solution x^* belongs to $\overline{B(x_0, R\eta)} = \overline{B(1, 0.215\cdots)} \subset \Omega$ and it is unique in $B(1, 0.611\cdots) \bigcap \Omega$.

Acknowledgements This work is supported by NSF of China (No.11271340), by the Key Project of Chinese Ministry of Education (No.212109).

References

1. Bruns, D.D., Bailey, J.E.: Nonlinear feedback control for operating a nonisothermal CSTR near an unstable steady state. Chem. Eng. Sci. **32**, 257–264 (1977)
2. Argyros, I.K., Hilout, S.: A convergence analysis for directional two-step Newton methods. Numer. Algor. **55**(4), 503–528 (2010)
3. Candela, V., Marquina, A.: Recurrence relations for rational cubic methods I: the Halley method. Computing **44**(2), 169–184 (1990)
4. Candela, V., Marquina, A.: Recurrence relations for rational cubic methods II: the Chebyshev method. Computing **45**(4), 355–367 (1990)
5. Ezquerro, J.A., Hernández, M.A.: New iterations of R-order four with reduced computational cost. BIT Numer. Math. **49**, 325–342 (2009)
6. Ezquerro, J.A., Hernández, M.A.: On the R-order of the Halley method. J. Math. Anal. Appl **303**, 591–601 (2005)
7. Gutiérrez, J.M., Hernández, M.A.: Recurrence relations for the super-Halley method. Comput. Math. Appl. **36**(7), 1–8 (1998)
8. Ganesh, M., Joshi, M.C.: Numerical solvability of Hammerstein integral equations of mixed type. IMA J. Numer. Anal. **11**, 21–31 (1991)
9. Hernández, M.A.: Chebyshev's approximation algorithms and applications. Comput. Math. Appl. **41**(3–4), 433–445 (2001)
10. Kou, J.: Some new sixth-order methods for solving non-linear equations. Appl. Math. Comput. **189**, 647–651 (2007)
11. Kou, J.: On Chebyshev–Halley methods with sixth-order convergence for solving non-linear equations. Appl. Math. Comput. **190**, 126–131 (2007)
12. Parhi, S.K., Gupta, D.K.: Semilocal convergence of a stirling-like method in Banach spaces. Int. J. Comput. Methods **7**(02), 215–228 (2010)
13. Parida, P.K., Gupta, D.K.: Recurrence relations for a Newton-like method in Banach spaces. J. Comput. Appl. Math. **206**(2), 873–887 (2007)
14. Rall, L.B.: Computational Solution of Nonlinear Operator Equations. Robert E. Krieger, New York (1979)
15. Wang, X., Kou, J., Gu, C.: Semilocal convergence of a class of modified super-Halley methods in Banach spaces. J. Optim. Theory Appl. **153**, 779–793 (2012)
16. Ye, X., Li, C.: Convergence of the family of the deformed Euler–Halley iterations under the Hölder condition of the second derivative. J. Comput. Appl. Math. **194**(2), 294–308 (2006)
17. Ye, X., Li, C., Shen, W.: Convergence of the variants of the Chebyshev-Halley iteration family under the Hölder condition of the first derivative. J. Comput. Appl. Math. **203**(1), 279–288 (2007)
18. Zhao, Y., Wu, Q.: Newton-Kantorovich theorem for a family of modified Halley's method under Hölder continuity conditions in Banach space. Appl. Math. Comput. **202**(1), 243–251 (2008)