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B-spline bases for unequally smooth quadratic spline spaces on non-uniform criss-cross triangulations

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Abstract In this paper, we investigate bivariate quadratic spline spaces on non-uniform criss-cross triangulations of a bounded domain with unequal smoothness across inner grid lines. We provide the dimension of the above spaces and we construct their local bases. Moreover, we propose a computational procedure to get such bases. Finally we introduce spline spaces with unequal smoothness also across oblique mesh segments.

Keywords Unequally smooth bivariate spline space · Non-uniform criss-cross triangulation · Bivariate B-spline basis

Mathematics Subject Classifications (2010) 65D07 · 41A15

1 Introduction

Let $\Omega = [a, b] \times [c, d]$ be a rectangular domain and m, n be positive integers. We consider the inner grid lines $u - \xi_i = 0,\ i = 1, \ldots, m$ and $v - \eta_j = 0,\ j = 1, \ldots, n$, where

$$a = \xi_0 < \xi_1 < \ldots < \xi_{m+1} = b \text{ and } c = \eta_0 < \eta_1 < \ldots < \eta_{n+1} = d \tag{1}$$

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partition Ω into $(m+1)(n+1)$ rectangular cells. By drawing both diagonals for each cell, we obtain a non-uniform criss-cross triangulation $\mathcal{T}_{mn}$, made of $4(m+1)(n+1)$ triangular cells.

The dimension and a B-spline basis for the space $\mathcal{S}_2^1(\mathcal{T}_{mn})$ of all quadratic splines on $\mathcal{T}_{mn}$, with maximum C^1 smoothness, were obtained in [12]. However some B-splines near the boundary of Ω have supports not completely contained in Ω. In [3, 8, 10] a local basis for $\mathcal{S}_2^1(\mathcal{T}_{mn})$ is given, with all supports included in Ω.

The interesting idea of getting unequally smooth quadratic spline spaces on $\mathcal{T}_{mn}$, i.e. with C^0 smoothness across some inner grid lines and with C^1 smoothness across the other ones, has been presented in [11, 12] and references therein. As remarked in [11], functions belonging to such spaces have total degree two and in some cases they are preferable to tensor product ones that may have some inflection points, due to their higher coordinate degree.

Since the above unequally spline spaces of total degree are useful in many applications [12] and, as far as we know, any theoretical analysis both on their dimension and on their local bases has not been provided in the literature, then in this paper we develop such an analysis, also considering possible jumps (C^{-1} smoothness) across inner grid lines.

Let $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$ be the space of bivariate quadratic piecewise polynomials on $\mathcal{T}_{mn}$, where

$$\bar{\mu}^\xi = (\mu_i^\xi)_{i=1}^m, \qquad \bar{\mu}^\eta = (\mu_j^\eta)_{j=1}^n \tag{2}$$

are vectors whose elements can be 1, 0, -1 and denote the C^1, C^0, C^{-1} smoothness, respectively, across the inner grid lines $u-\xi_i=0$, $i=1,\ldots,m$ and $v-\eta_j=0$, $j=1,\ldots,n$, while the smoothness across all oblique mesh segments[1] is C^1.

In case of jumps at $u=\xi_i$ and/or $v=\eta_j$, in order to uniquely define $s\in\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$, we set

$$s(\xi_i,v)=\begin{cases} s(\xi_i^+,v),\ i=0,\ldots,m,\\ s(\xi_i^-,v),\ i=m+1,\end{cases} \quad \text{and}\ s(u,\eta_j)=\begin{cases} s(u,\eta_j^+),\ j=0,\ldots,n,\\ s(u,\eta_j^-),\ j=n+1.\end{cases}$$

In Theorem 1 of Section 2 we get the dimension of $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$, that we express by a formula depending on m, n and the required smoothness. Then, we determine a finite set $\mathcal{B}$ of locally supported functions belonging to $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$, from which, in Theorem 2, we extract a basis for $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$. Beside the above theoretical analysis, in Section 3, we present a computational procedure for basis generation, also illustrated by some graphs of B-splines and an application. Finally, in Section 4 we consider unequal smoothness also across oblique mesh segments, we define a new spline space and, in Theorem 3,

[1] According to [12], we call *mesh segments* the line segments that form the boundary of each triangular cell of $\mathcal{T}_{mn}$.

we provide its dimension. Then, by using the "smoothing cofactor conformality method" [12], we construct some locally supported functions belonging to it.

2 On the construction of local bases for $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$

2.1 Dimension of $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$

Let $\mathcal{T}_{mn}$ be a non uniform criss-cross triangulation of a rectangular domain Ω and $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$ be the spline space defined as in Section 1.

Let L_u^0 (resp. L_u^{-1}) and L_v^0 (resp. L_v^{-1}) be the number of grid lines $u-\xi_i=0$, $i=1,\dots,m$ and $v-\eta_j=0$, $j=1,\dots,n$, respectively, across which we want $s\in\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$ has C^0 (resp. C^{-1}) smoothness, with $0\le L_u^0+L_u^{-1}\le m$ and $0\le L_v^0+L_v^{-1}\le n$.

We prove the following result concerning the dimension of $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$.

Theorem 1 *The dimension of $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$ is*

$$\lambda=\dim\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})=d_1+d_2+d_3, \tag{3}$$

where

$$\begin{aligned}
d_1 &= mn+3m+3n+8,\\
d_2 &= (n+2)L_u^0+(m+2)L_v^0,\\
d_3 &= (2n+5+L_v^0+L_v^{-1})L_u^{-1}+(2m+5+L_u^0+L_u^{-1})L_v^{-1}+L_u^{-1}L_v^{-1}.
\end{aligned}$$

Proof For any two triangles $T=<v_1,v_2,v_3>$ and $\tilde{T}=<v_4,v_3,v_2>$ of $\mathcal{T}_{mn}$, sharing the edge $e=<v_2,v_3>$, let

$$p(v)=\sum_{i+j+k=2}c_{ijk}BE_{ijk}^2(v)\qquad\text{and}\qquad \tilde{p}(v)=\sum_{i+j+k=2}\tilde{c}_{ijk}\tilde{BE}_{ijk}^2(v)$$

where $\{BE_{ijk}^2\}$ and $\{\tilde{BE}_{ijk}^2\}$ are the Bernstein basis polynomials associated with T and $\tilde{T}$, respectively.

In Fig. 1 the two basic configurations (the other ones are obtained by symmetry) of the above situation are reported. Figure 1a shows two triangles belonging to two different rectangular cells of $\mathcal{T}_{mn}$, while Fig. 1b presents two triangles belonging to the same rectangular cell.

Then, from [7, Theorem 2.28, p. 39] the condition for C^0 continuity says that the B-coefficients of p and $\tilde{p}$ associated with domain points along the edge e must agree:

$$\tilde{c}_{0jk}=c_{0kj},\ \ j+k=2. \tag{4}$$

The condition for C^1 smoothness across the edge e is that (4) holds along with

$$\tilde{c}_{1jk}=b_1c_{1kj}+b_2c_{0,k+1,j}+b_3c_{0,k,j+1},\qquad j+k=1,$$

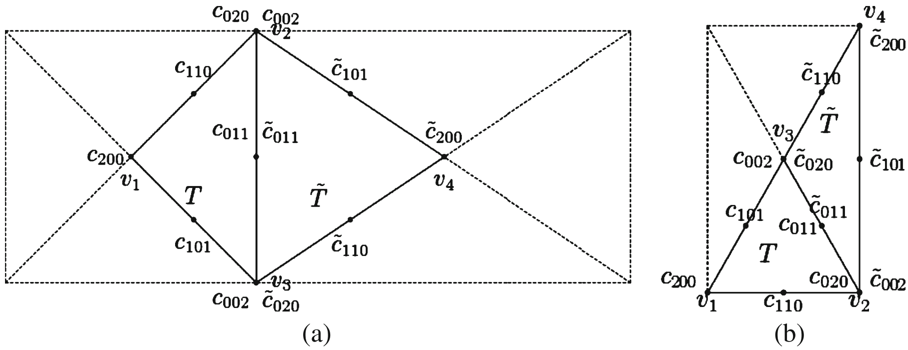


Fig. 1 B-coefficients of p and $\tilde{p}$ for the two basic configurations in $\mathcal{T}_{mn}$

where (b_1, b_2, b_3) are the barycentric coordinates of the vertex v_4 relative to the triangle T.

Finally C^{-1} smoothness across the edge e does not imply any condition between the B-coefficients of p and $\tilde{p}$.

Considering all pairs of adjacent triangles of $\mathcal{T}_{mn}$, together with the given C^ℓ smoothness, $\ell = -1, 0, 1$ across their common edge, we are able to count the constrained B-coefficients and therefore to detect the number of degrees of freedom, obtaining the space dimension (3). □

We remark that, if $s \in \mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$ is globally C^0 (i.e. $L_u^{-1} = L_v^{-1} = 0$), then $\dim\, \mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn}) = d_1 + d_2$, while, if it is globally C^1 (i.e. $L_u^{-1} = L_v^{-1} = L_u^0 = L_v^0 = 0$), then we obtain the well-known case $\dim\, \mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn}) = \dim\, \mathcal{S}_2^1(\mathcal{T}_{mn}) = d_1$ [12].

2.2 Spanning set and basis of $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$

We turn now to the problem of constructing a basis for $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$.

Setting

$$M = 3 + \sum_{i=1}^{m}(2 - \mu_i^\xi), \quad N = 3 + \sum_{j=1}^{n}(2 - \mu_j^\eta), \tag{5}$$

where μ_i^ξ and μ_j^η are defined as in (2), let $\bar{u} = (u_i)_{i=-2}^{M}$, $\bar{v} = (v_j)_{j=-2}^{N}$ be the nondecreasing sequences of knots, obtained from $\bar{\xi} = (\xi_i)_{i=0}^{m+1}$ and $\bar{\eta} = (\eta_j)_{j=0}^{n+1}$ by imposing the two following requirements:

(i) $u_{-2} = u_{-1} = u_0 = \xi_0 = a, \quad u_{M-2} = u_{M-1} = u_M = \xi_{m+1} = b,$
$v_{-2} = v_{-1} = v_0 = \eta_0 = c, \quad v_{N-2} = v_{N-1} = v_N = \eta_{n+1} = d;$

(ii) for $i = 1, \ldots, m$, the number ξ_i occurs exactly $2 - \mu_i^\xi$ times in $\bar{u}$ and for $j = 1, \ldots, n$, the number η_j occurs exactly $2 - \mu_j^\eta$ times in $\bar{v}$.

Let $\bar{B}_{ij}(u, v)$ be the quadratic C^1 B-spline belonging to the space $\mathcal{S}_2^1(\mathcal{T}_{mn})$, for which the B-form is given in [4, 9] (see Fig. 2, where we report the $\bar{B}_{ij}$'s support and its B-coefficients). For the above sequences $\bar{u}$ and $\bar{v}$, we consider the following set of functions

$$\mathcal{B} = \{B_{ij}(u, v)\}_{(i,j)\in\mathcal{K}_{MN}}, \tag{6}$$

where $\mathcal{K}_{MN} = \{(i, j) : 0 \le i \le M-1, 0 \le j \le N-1\}$ and any B_{ij} is obtained in B-form by the $\bar{B}_{ij}$'s one, conveniently setting $h_p = u_p - u_{p-1}$, $p = i-1, i, i+1$, and/or $k_q = v_q - v_{q-1}$, $q = j-1, j, j+1$, equal to zero (see Fig. 2), if there are double (or triple) knots in $\bar{u}$, $\bar{v}$ [4, 9]. When $\frac{0}{0}$ occurs, we set the corresponding value equal to zero.

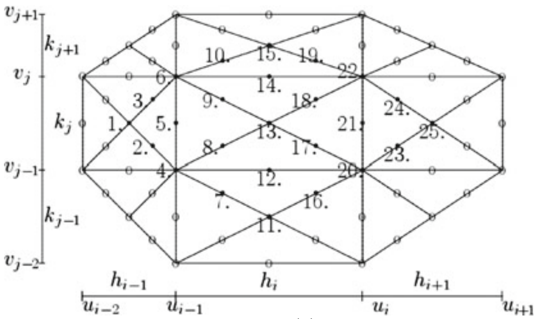
If both/either $\bar{u}$ and/or $\bar{v}$ have/has double (or triple) knots, then the B_{ij} smoothness will change and the support will change as well.

In [4] supports and B-coefficients of such B-splines are reported.

The functions B_{ij}'s belong to $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$, have a local support, are non negative and, by an argument similar to the one used in [8, 10], we can show that they form a partition of unity.

In $\mathcal{B}$ we find different types of spline functions. There are

$$\rho = 2M(1 + L_v^{-1}) + 2N(1 + L_u^{-1}) - 4(1 + L_u^{-1})(1 + L_v^{-1})$$



(a)

1. $\frac{\sigma_i'}{4}$, 2. $\frac{\sigma_i'}{2}$, 3. $\frac{\sigma_i'}{2}$, 4. $\sigma_i'\tau_j'$, 5. σ_i',
6. $\sigma_i'\tau_{j+1}$, 7. $\frac{\tau_j'}{2}$, 8. $\frac{\sigma_i' + \tau_j'}{2}$, 9. $\frac{\sigma_i' + \tau_{j+1}}{2}$, 10. $\frac{\tau_{j+1}}{2}$,
11. $\frac{\tau_j'}{4}$, 12. τ_j', 13. $\frac{\sigma_i' + \sigma_{i+1} + \tau_{j+1} + \tau_j'}{4}$, 14. τ_{j+1}, 15. $\frac{\tau_{j+1}}{4}$,
16. $\frac{\tau_j'}{2}$, 17. $\frac{\sigma_{i+1} + \tau_j'}{2}$, 18. $\frac{\sigma_{i+1} + \tau_{j+1}}{2}$, 19. $\frac{\tau_{j+1}}{2}$, 20. $\sigma_{i+1}\tau_j'$,
21. σ_{i+1}, 22. $\sigma_{i+1}\tau_{j+1}$, 23. $\frac{\sigma_{i+1}}{2}$, 24. $\frac{\sigma_{i+1}}{2}$, 25. $\frac{\sigma_{i+1}}{4}$,

(b)

Fig. 2 **a** Support and **b** B-coefficients of $\bar{B}_{ij}(u, v)$, where "O" denotes a zero B-coefficient and $\sigma_{i+1} = \frac{h_{i+1}}{h_i+h_{i+1}}$, $\sigma_i' = \frac{h_{i-1}}{h_{i-1}+h_i}$, $\tau_{j+1} = \frac{k_{j+1}}{k_j+k_{j+1}}$, $\tau_j' = \frac{k_{j-1}}{k_{j-1}+k_j}$, with $h_i = u_i - u_{i-1}$, $k_j = v_j - v_{j-1}$

unequally smooth functions, that we call *boundary B-splines*, whose restrictions to the boundary $\partial\Omega$ of Ω and to the grid lines with associated C^{-1} smoothness are univariate quadratic B-splines. The remaining $MN-\rho$ functions, called *inner B-splines*, are such that their restrictions to $\partial\Omega$ and to the C^{-1} smoothness grid lines are equal to zero.

Then the following theorem holds.

Theorem 2 *Let:*

(i) $\{\Omega^r\}_{r=1}^{\gamma}$ *be a partition of* Ω *into rectangular subdomains, generated by the grid lines with associated* C^0 *and* C^{-1} *smoothness, with*

$$\gamma = (L_u^0 + L_u^{-1} + 1)(L_v^0 + L_v^{-1} + 1); \tag{7}$$

(ii) $\mathcal{B}$ *be defined as in* (6)*;*
(iii) $\mathcal{B}_1 \subset \mathcal{B}$ *be the set of inner B-splines with* C^1 *smoothness everywhere or with* C^0 *smoothness only on the boundary of their support;*
(iv) $\{\mathcal{B}^{(r)}\}_{r=1}^{\gamma}$ *be a partition of* $\mathcal{B}_1$*, where each* $\mathcal{B}^{(r)}$ *contains B-splines with support in* Ω^r.

Then, a B-spline basis for $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$ *can be extracted from* $\mathcal{B}$*, by removing* γ *B-splines, one in each* $\mathcal{B}^{(r)}$, $r = 1, \ldots, \gamma$.

Proof Taking into account the knot multiplicities, we remark that (5) can be written in the equivalent form

$$M = 3 + m + L_u^0 + 2L_u^{-1}, \quad N = 3 + n + L_v^0 + 2L_v^{-1}. \tag{8}$$

Then, from (8), it is easy to prove that $\sharp\mathcal{B} = M \cdot N > \lambda$, with λ defined by (3), i.e. the elements of $\mathcal{B}$ are linearly dependent.

Now, we can show that in $\mathcal{B}$ we find a set of λ linearly independent (l.i.) B-splines.

We consider the set of boundary B-splines belonging to $\mathcal{B}$ and we denote it by $\mathcal{B}_2$. Since their restrictions to $\partial\Omega$ and to the grid lines with associated C^{-1} smoothness are univariate quadratic B-splines, then they will be l.i. as the univariate ones.

Let $\mathcal{B}_3$ be the set of inner B-splines, belonging to $\mathcal{B}$ and having C^0 smoothness across either horizontal or vertical edges inside their support, that are l.i. as well, because their restrictions on such lines are quadratic piecewise polynomials having the same B-coefficients as the univariate B-splines [4].

The set $\mathcal{B}\backslash(\mathcal{B}_2 \cup \mathcal{B}_3)$ is the set $\mathcal{B}_1$, that we partition into the subsets $\mathcal{B}^{(r)}$, $r = 1, \ldots, \gamma$. If in each $\mathcal{B}^{(r)}$, $r = 1, \ldots, \gamma$, we delete any one element, from [1], we get a set of l.i. B-splines, denoted by $\tilde{\mathcal{B}}^{(r)}$. Moreover, thanks to the local support property of the B-splines, we can deduce that the elements of the set $\bigcup_{r=1}^{\gamma} \tilde{\mathcal{B}}^{(r)}$ are l.i..

From the same property of the B_{ij}'s, we can also get that the $\lambda^* = M \cdot N - \gamma$ functions belonging to the set $\bigcup_{r=1}^{\gamma} \tilde{\mathcal{B}}^{(r)} \cup \mathcal{B}_2 \cup \mathcal{B}_3$ are l.i..

Therefore, in $\mathcal{B}$ we have detected λ^* l.i. B-splines. From (7) and (8), after some algebra, it is easy to show that $\lambda^* = \lambda$.

Then, we can conclude that the B-splines belonging to the set $\bigcup_{r=1}^{\gamma} \tilde{\mathcal{B}}^{(r)} \cup \mathcal{B}_2 \cup \mathcal{B}_3$ are a basis for the space $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$. □

We remark that the set $\mathcal{B}$, defined in (6), is a spanning set of $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$ endowed with the partition unity property.

However, in several problems, for example in the application of Galerkin method to PDEs in isogeometric analysis [6], it is essential the management of a basis for the space where the approximating solution is looked for. This subject is very interesting, we are working on it and we have already obtained some results [2].

Instead, in CAGD applications is more convenient to use $\mathcal{B}$, in order to get a surface having the convex hull property.

Example 1 Given $\bar{\xi} = (0, 2, 4, 6, 8, 10)$, $\bar{\eta} = (0, 1, 2, 3, 4)$, $\bar{\mu}^\xi = (0, 1, 1, -1)$, $\bar{\mu}^\eta = (0, 1, 1)$, we want to construct a basis for the corresponding spline space $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{43})$, whose dimension, from (3), is $\lambda = 64$.

Such a space is made of functions that are C^1 inside $\Omega = [0, 10] \times [0, 4]$, except across the lines $u = 2$ and $v = 1$, where they are only continuous and across $u = 8$, where they have a jump.

From (5), we compute $M = 10$, $N = 7$ and consequently

$$\bar{u} = (0, 0, 0, 2, 2, 4, 6, 8, 8, 8, 10, 10, 10), \quad \bar{v} = (0, 0, 0, 1, 1, 2, 3, 4, 4, 4),$$

on which the spanning set $\mathcal{B} = \{B_{ij}\}_{i=0,\, j=0}^{9,6}$ is defined.

In order to detect a B-spline basis, we notice that the domain Ω is subdivided into six subdomains Ω^r, $r = 1, \ldots, 6$, as shown in Fig. 3. Therefore, according to Theorem 2, we have:

– $\mathcal{B}_1 = \bigcup_{r=1}^{6} \mathcal{B}^{(r)}$, with

$$\mathcal{B}^{(1)} = \{B_{11}\}, \mathcal{B}^{(2)} = \{B_{1j},\ j = 3, 4, 5\}, \mathcal{B}^{(3)} = \{B_{i1},\ i = 3, 4, 5\},$$
$$\mathcal{B}^{(4)} = \{B_{ij},\ i, j = 3, 4, 5\}, \mathcal{B}^{(5)} = \{B_{81}\}, \mathcal{B}^{(6)} = \{B_{8j},\ j = 3, 4, 5\};$$

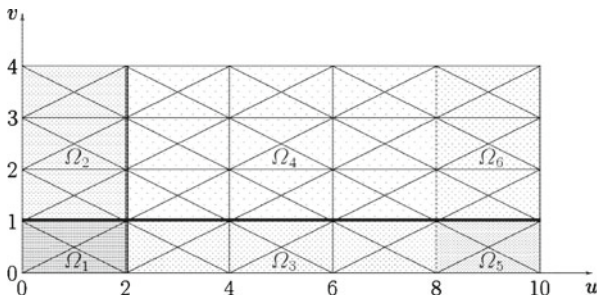


Fig. 3 The domain $\Omega = [0, 10] \times [0, 4]$ subdivided into the six subdomains Ω^r, $r = 1, \ldots, 6$, where a *thick line* corresponds to a double knot and a *dotted line* corresponds to a triple knot

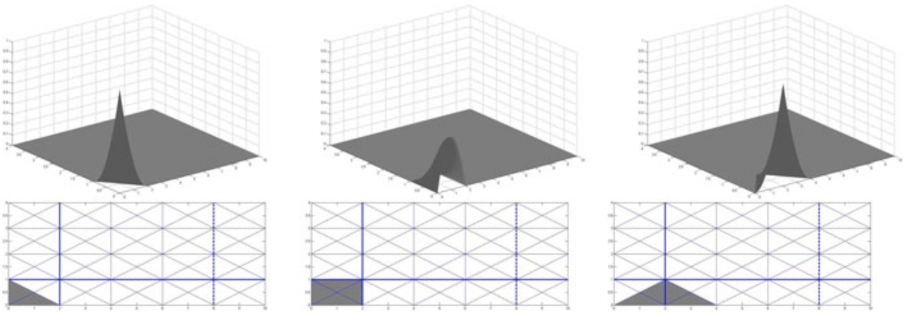


Fig. 4 Graphs and supports of B_{00}, B_{10} and B_{20}

– $\mathcal{B}_2 = \{B_{i0}, B_{i6}\ i = 0, \ldots, 9,\ B_{0j}, B_{6j}, B_{7j}, B_{9j},\ \ j = 1, \ldots, 5\}$;
– $\mathcal{B}_3 = \{B_{i2},\ i = 1, \ldots, 5, 8,\ B_{2j}\ j = 1, 3, 4, 5\}$.

Now, since we have to delete any one B-spline from each $\mathcal{B}^{(r)}$, $r = 1, \ldots, 6$, we choose to remove

B_{11} from $\mathcal{B}^{(1)}$, B_{13} from $\mathcal{B}^{(2)}$, B_{31} from $\mathcal{B}^{(3)}$,
B_{33} from $\mathcal{B}^{(4)}$, B_{81} from $\mathcal{B}^{(5)}$ and B_{83} from $\mathcal{B}^{(6)}$,

obtaining the following sets $\tilde{\mathcal{B}}^{(r)}$, $r = 1, \ldots, 6$:

$$\tilde{\mathcal{B}}^{(1)} = \emptyset, \tilde{\mathcal{B}}^{(2)} = \{B_{1j},\ \ j = 4, 5\}, \tilde{\mathcal{B}}^{(3)} = \{B_{i1},\ i = 4, 5\},$$
$$\tilde{\mathcal{B}}^{(4)} = \{B_{ij},\ i, j = 3, 4, 5,\ \ (i, j) \neq (3, 3)\}, \tilde{\mathcal{B}}^{(5)} = \emptyset, \tilde{\mathcal{B}}^{(6)} = \{B_{8j},\ \ j = 4, 5\}.$$

Therefore, we get the basis, given by the sixty-four B-splines belonging to the set $(\bigcup_{r=1}^{6} \tilde{\mathcal{B}}^{(r)}) \cup \mathcal{B}_2 \cup \mathcal{B}_3$.

In Figs. 4, 5, 6, 7 and 8 some of the above basis functions are reported.

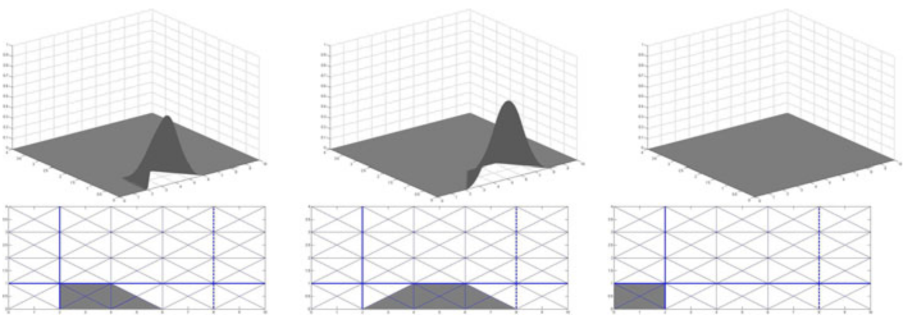
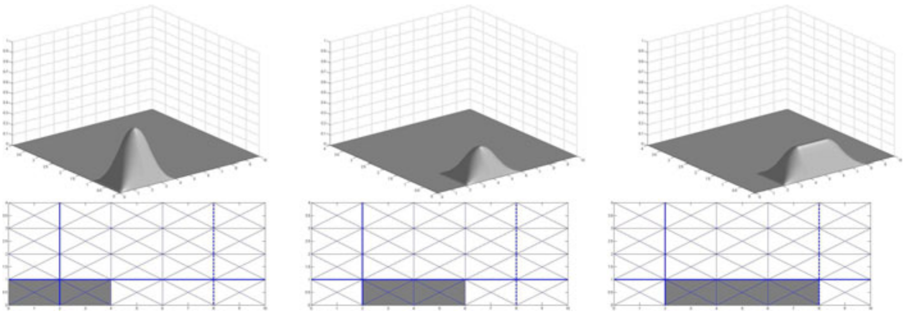


Fig. 5 Graphs and supports of B_{30}, B_{40} and B_{11}

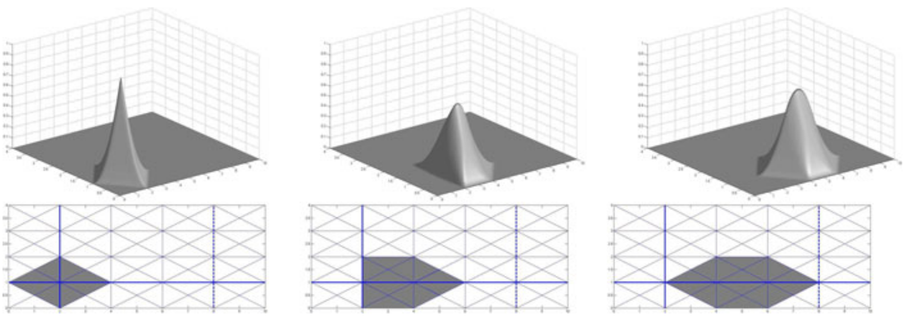


$i = 2, \quad j = 1$ $h_i = k_{j-1} = k_{j+1} = 0$

$i = 3, \quad j = 1$ $h_{i-1} = k_{j-1} = k_{j+1} = 0$

$i = 4, \quad j = 1$ $k_{j-1} = k_{j+1} = 0$

Fig. 6 Graphs and supports of B_{21}, B_{31} and B_{41}

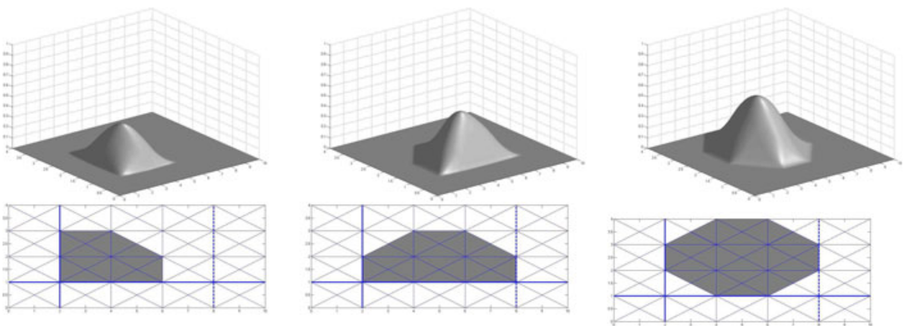


$i = 2, \quad j = 2$ $h_i = k_j = 0$

$i = 3, \quad j = 2$ $h_{i-1} = k_j = 0$

$i = 4, \quad j = 2$ $k_j = 0$

Fig. 7 Graphs and supports of B_{22}, B_{32} and B_{42}



$i = 3, \quad j = 3$ $h_{i-1} = k_{j-1} = 0$

$i = 4, \quad j = 3$ $k_{j-1} = 0$

$i = 4, \quad j = 4$

Fig. 8 Graphs and supports of B_{33}, B_{43} and B_{44}

Remark 1 We remark that we could also consider tensor product spaces with unequal smoothness across horizontal and vertical grid lines, but the splines belonging to them would have higher coordinate degree four, with possible inflection points, instead of the total degree two, typical of splines on criss-cross triangulations, as remarked also in [11].

3 A computational procedure for $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$ basis generation

In this section we present a computational procedure to generate the B-spline basis of $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta)}(\mathcal{T}_{mn})$, for any given criss-cross triangulation $\mathcal{T}_{mn}$ of a rectangular domain Ω.

The code, developed in the convenient interactive environment that MATLAB provides, is available in [5]. All computations have been carried out on a personal computer with a 16-digit arithmetic.

Our main user-callable function M-file is `bijdec`. Given in input:

- the vectors of points $\bar{\xi} = (\xi_i)_{i=0}^{m+1}$ and $\bar{\eta} = (\eta_j)_{j=0}^{n+1}$;
- the vectors of smoothnesses $\bar{\mu}^\xi = (\mu_i^\xi)_{i=1}^m$, $\bar{\mu}^\eta = (\mu_j^\eta)_{j=1}^n$;
- a point (u, v) in Ω;
- two integers i and j, $(i, j) \in \mathcal{K}_{MN}$;

the procedure returns the value of $B_{ij}(u, v)$, computed by means of its B-coefficients [4] and the de Casteljau algorithm for triangular surfaces [7]. Then, the B-spline basis is obtained for convenient values $(i, j) \in \mathcal{K}_{MN}$, as described in Theorem 2.

Other minor M-files, like `bijplt` (calling `bijdec`) or `oij`, are utilities that perform the visualization of all B_{ij}'s and the construction and the visualization of their supports, respectively.

The B-coefficients of any B_{ij} are computed by the ones of the B-spline $\bar{B}_{ij}$, putting to zero the length of the support intervals that are "degenerate" because of multiple knots, induced by the required smoothnesses.

Example 2 We want to use such procedures to represent some B-splines belonging to the set $\mathcal{B} = \{B_{ij}\}_{i=0,\,j=0}^{9,6}$ of Example 1.

In Figs. 4–8 we report graphs (obtained with `bijplt` on 55×55 evaluation points $(u, v) \in \Omega$) and supports (by using `oij`) of fifteen B-splines belonging to $\mathcal{B}$ for given i and j, $(i, j) \in \mathcal{K}_{10,7}$, while the other fifty-five ones can be obtained via affine transformations by the first ones.

We remark that a thick line corresponds to a double knot and a dotted line corresponds to a triple knot.

Application 1 We consider the test function

$$f(u, v) = \begin{cases} (-\mid u - 0.2 \mid +0.4)F(u, v) - 0.4 & \text{if } 0 \le u \le 1,\ 0 \le v < 0.5 \\ (-\mid u - 0.2 \mid +0.4)F(u, v) & \text{if } 0 \le u \le 1,\ 0.5 \le v \le 1 \end{cases}$$

where

$$F(u,v) = \frac{3}{4}e^{\left(-\frac{1}{4}\left((9u-2)^2+(9v-2)^2\right)\right)} + \frac{3}{4}e^{\left(-\left(\frac{(9u+1)^2}{49}+\frac{(9v+1)}{10}\right)\right)}$$
$$+\frac{1}{2}e^{\left(-\frac{1}{4}\left((9u-7)^2+(9v-3)^2\right)\right)} - \frac{1}{5}e^{\left(-\left((9u-4)^2+(9v-7)^2\right)\right)}$$

is the well-known Franke's function.

Here we propose an example of approximation of the function f with singularities, by considering the bivariate Schoenberg–Marsden operator [3, 8, 10]

$$S_1 f(u,v) = \sum_{(i,j)\in\mathcal{K}_{MN}} f(s_i,t_j)\, B_{ij}(u,v), \quad (u,v)\in\Omega=[0,1]\times[0,1], \tag{9}$$

where

$$s_i = \frac{u_{i-1}+u_i}{2}, \qquad t_j = \frac{v_{j-1}+v_j}{2}, \qquad (i,j)\in\mathcal{K}_{MN}.$$

In order to simulate C^0 smoothness across the line $u = 0.2$ and the jump across $v = 0.5$, we consider

$$\bar{\xi} = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1), \quad \bar{\eta} = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.75, 1)$$

and

$$\bar{\mu}^{\xi} = (1, 0, 1, 1, 1, 1, 1), \qquad \bar{\mu}^{\eta} = (1, 1, 1, 1, -1, 1).$$

Then, from (5), we obtain $M = N = 11$ and

$$\bar{u} = (0, 0, 0, 0.1, 0.2, 0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 1, 1, 1),$$
$$\bar{v} = (0, 0, 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.5, 0.5, 0.75, 1, 1, 1).$$

We set

$$f(s_i,t_6) = (-\mid s_i - 0.2 \mid +0.4)F(s_i,t_6) - 0.4, \qquad i = 0,\ldots,10$$

and the graph of the corresponding surface (9), evaluated on a 100×100 uniform rectangular grid of points in Ω, is given in Fig. 9.

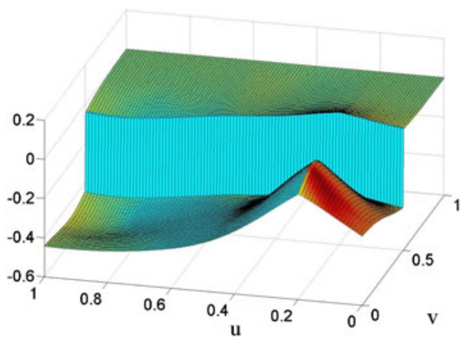


Fig. 9 The graph of $S_1 f$ given in Application 1

The proposed example only wants to show how the use of multiple knots in $\bar{u}$ and $\bar{v}$ allows to simulate singularities of f. Obviously, in order to get better function approximations, we should increase the number of knots in $\bar{\xi}$ and $\bar{\eta}$.

4 On the spline space $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn})$

The study of the unequal smoothness case not only across grid lines of $\mathcal{T}_{mn}$, but also across oblique mesh segments, could be an interesting extension of the above results.

In this section we investigate this problem, defining a new spline space, providing its dimension and constructing some locally supported functions belonging to it.

First of all, we restrict our attention to the case of uniform partitions (1), i.e. $\xi_i - \xi_{i-1} = h$, $i = 1, \ldots, m+1$ and $\eta_j - \eta_{j-1} = k$, $j = 1, \ldots, n+1$.

Let $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn})$ be the space of bivariate quadratic piecewise polynomials on $\mathcal{T}_{mn}$, where $\bar{\mu}^\xi$, $\bar{\mu}^\eta$ are defined as in (2) and

$$\bar{\mu}^{ob} = \left\{\mu_l^{ob}\right\}_{l=1}^{2(m+n+1)}$$

is the vector whose elements can be $1, 0, -1$ and denote the C^1, C^0, C^{-1} smoothness, respectively, across the oblique cross-cuts [12].

If we consider the cross-cuts with associated C^{-1} smoothness, we can partition Ω into a set of subdomains $\{\Omega_{-1}^r\}$, where only C^0 and C^1 smoothnesses occur, and we denote with $\mathcal{T}_{mn}^r$ the restriction of $\mathcal{T}_{mn}$ to Ω_{-1}^r.

Now, let $\{V^r\}$ be the set of the intersection points of the cross-cuts inside Ω_{-1}^r, called “inner grid points”. They are the intersection of exactly either two or four cross-cuts of $\mathcal{T}_{mn}^r$.

Then, we define eight sets $V_{ij}^r \subset V^r$, $i, j \geq 0$ and $i + j = 2, 4$, where the elements of each V_{ij}^r are the inner grid points, intersection of i cross-cuts with associated C^0 smoothness and j with C^1 smoothness.

We set

$$\bar{\mu} = \bar{\mu}^\xi \cup \bar{\mu}^\eta \cup \bar{\mu}^{ob} = \{\mu_i\}_{i=1}^L,$$

with $L = m + n + 2(m + n + 1)$ the number of all cross-cuts in Ω.

For any inner grid point $\nu \in V^r$, let $\bar{\mu}_\nu \subset \bar{\mu}$ be the smoothness set associated with the cross-cuts around ν and let $\mu_a = \min_i\{\mu_i \in \bar{\mu}_\nu\}$.

From [12, Chap. 2] we can write that the dimension of the vector space of solutions of local conformality equation at ν is

$$d_2^{\bar{\mu}_\nu} = \sum_{\ell=1}^{2-\mu_a} \left[-1 - \mu_a + \sum_{\mu_i \in \bar{\mu}_\nu,\ i \neq a} [2 - \mu_i - \ell + 1]_+ \right]_+ ,$$

where $[\,\cdot\,]_+$ is the usual truncation function and in our case we have

$$\begin{array}{ll} d_2^{\bar{\mu}_\nu} = 1,\ \nu \in V_{20}^r, & d_2^{\bar{\mu}_\nu} = 0,\ \nu \in V_{11}^r, \\ d_2^{\bar{\mu}_\nu} = 0,\ \nu \in V_{02}^r, & d_2^{\bar{\mu}_\nu} = 7,\ \nu \in V_{40}^r, \\ d_2^{\bar{\mu}_\nu} = 5,\ \nu \in V_{31}^r, & d_2^{\bar{\mu}_\nu} = 3,\ \nu \in V_{22}^r, \\ d_2^{\bar{\mu}_\nu} = 2,\ \nu \in V_{13}^r, & d_2^{\bar{\mu}_\nu} = 1,\ \nu \in V_{04}^r. \end{array}$$

Then, from [12, Chap. 2], we immediately obtain the following theorem.

Theorem 3 *The dimension of $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn})$ is*

$$\dim \mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn}) = \sum_r \dim \mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn}^r),$$

where

$$\dim \mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn}^r) = 6 + \sum_{i=1}^{L^r} \binom{2-\mu_i+1}{2} + \sum_{j=1}^{\sharp V^r} d_2^{\bar{\mu}_\nu},$$

with L^r the number of cross-cuts in Ω_{-1}^r.

If the partitions (1) are not uniform, the oblique cross-cuts might become piecewise straight lines. We still denote by μ_l^{ob} the common smoothness across all mesh segments of the l-th piecewise straight line and by $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn})$ the spline space.

By using the same logical scheme of Theorem 1 proof, i.e. by counting the constrained B-coefficients, and by [7, p. 238], we can deduce that the dimension of $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn})$ does not change with respect to the above uniform case.

Finally, we construct two locally supported functions in $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn})$ in case of uniform partitions (1).

By the "smoothing cofactor conformality method" [12, Chap. 1] and by imposing the C^0 smoothness across the oblique mesh segments belonging to

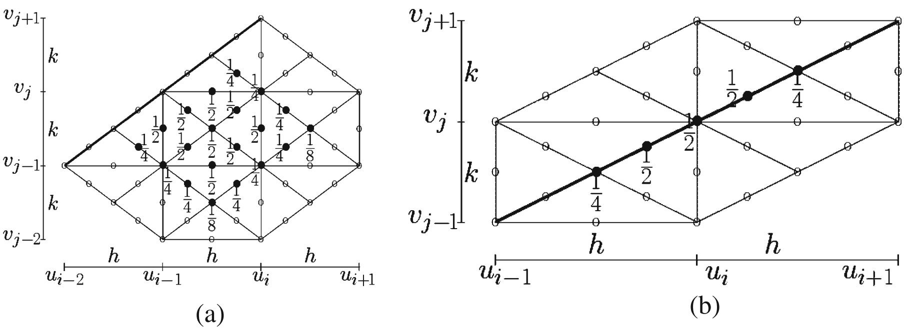


Fig. 10 Supports and B-coefficients of two locally supported functions belonging to $\mathcal{S}_2^{(\bar{\mu}^\xi,\bar{\mu}^\eta,\bar{\mu}^{ob})}(\mathcal{T}_{mn})$. A *thick line* denotes C^0 smoothness

the same straight line of the $\bar{B}_{ij}$ support (Fig. 2), firstly we obtain the local function, whose support and B-coefficients are shown in Fig. 10a.

Then, we get another locally supported function, with C^0 smoothness across the oblique mesh segments belonging to the same straight line inside its support (see Fig. 10b).

By using the same technique, it is possible to generate other locally supported functions with different supports and smoothnesses, also in the non-uniform case.

A more general treatment related to the basis generation for $\mathcal{S}_2^{(\bar{\mu}^{\xi},\bar{\mu}^{\eta},\bar{\mu}^{ob})}(\mathcal{T}_{mn})$ will be developed in a further paper.

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