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ORIGINAL PAPER

# A collocation method for the numerical solution of a two dimensional integral equation using a quadratic spline quasi-interpolant

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**Abstract** In this paper, we propose an interesting method for approximating the solution of a two dimensional second kind equation with a smooth kernel using a bivariate quadratic spline quasi-interpolant (abbr. QI) defined on a uniform criss-cross triangulation of a bounded rectangle. We study the approximation errors of this method together with its Sloan's iterated version and we illustrate the theoretical results by some numerical examples.

**Keywords** Collocation · Integral equations · Spline quasi-interpolant

## 1 Introdution

Let us consider the linear equation

$$u - \mathcal{K}u = (\mathcal{I} - \mathcal{K})u = f, \tag{1}$$

where $\mathcal{K}$ is a compact linear operator on the Banach space $\mathcal{X}$ and $f \in \mathcal{X}$. The operator $(\mathcal{I} - \mathcal{K})$ is assumed to be invertible, so that the equation has a

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unique solution $u \in \mathcal{X}$ for any given $f \in \mathcal{X}$. Let $\mathcal{K}$ be the integral compact linear operator defined by

$$\mathcal{K}u(s) := \int_{\Omega} k(s,t)u(t)dt, \quad s \in \Omega = [0,1] \times [0,1],$$

where, in this case, $\mathcal{X} := \mathcal{C}(\Omega)$ and the kernel $k \in \mathcal{C}(\Omega^2)$. A standard technique to solve (1) approximately is to replace $\mathcal{K}$ by a finite rank operator. The approximate solution of (1) is then obtained by solving a system of equations. The Galerkin, Nyström and degenerate kernel methods are the commonly used methods for this purpose. They have been extensively studied in the literature (see [3, 4]). Recently, Kulkarni introduced in [7] an efficient method for the approximate solution of integral equations defined on polygonal regions, that consists in approximating $\mathcal{K}$ by the finite rank operator

$$\mathcal{P}_n\mathcal{K} + \mathcal{K}\mathcal{P}_n - \mathcal{P}_n\mathcal{K}\mathcal{P}_n$$

where $\mathcal{P}_n$ is a sequence of projectors converging to the identity operator pointwise. Let $\mathcal{Q}_n$ be the bivariate quadratic spline QI introduced in [8] and defined on the uniform criss-cross triangulation of the domain $\Omega$ with meshlength $h = \frac{1}{n}$. In this paper, we propose to approximate $\mathcal{K}$ by one of the two following finite rank operators

$$\mathcal{K}_n := \mathcal{Q}_n\mathcal{K} + \mathcal{K}_{n,i} - \mathcal{Q}_n\mathcal{K}_{n,i}, \quad i = 1, 2, \tag{2}$$

where $\mathcal{K}_{n,1}$ is the degenerate kernel operator obtained by approximating the kernel $k(s,t)$ by $\mathcal{Q}_n$ with respect to the variable $t$, and $\mathcal{K}_{n,2}$ is the Nyström operator based on $\mathcal{Q}_n$. It was established that if the kernel is suitably smooth, then the order of convergence of the method is $\mathcal{O}(h^7)$ and that of its iterated version is $\mathcal{O}(h^8)$. The methods proposed here are similar to the Kulkarni's methods, but they are easier to implement and faster. They have been already introduced in [1] and [2] for eigenvalue problems and one dimensional integral equations respectively and they can be also extended to integral equations defined on a polygonal region in $\mathbb{R}^2$ using a piecewise polynomial interpolation as in [7]. This issue will be studied in a subsequent paper.

The paper has been arranged in the following way. In Section 2, we give the definition and the main properties of the spline QI $\mathcal{Q}_n$. In Section 3, we define collocation methods based on $\mathcal{Q}_n$ and we discuss the system of linear equations which needs to be solved to obtain the approximate solution. In Section 4 we analyze the convergence of these methods and their iterated versions. A discrete version of the proposed method is also defined. A numerical validation is given in Section 6.

## 2 Bivariate quadratic spline quasi-interpolant on a bounded domain

We recall the following notations from [8]. Let $\mathcal{T}_{mn}$ be a criss-cross triangulation of $\Omega$ based on the two partitions

$$\mathcal{X}_m := \{x_i,\ 0 \le i \le m\} \quad \text{and} \quad \mathcal{Y}_n := \{y_j,\ 0 \le j \le n\}$$

respectively of the segment $I = [0, 1] = [x_0, x_m] = [y_0, y_n]$, (see Fig. 1). For $1 \le i \le m$ and $1 \le j \le n$ we set $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, $I_i = [x_{i-1}, x_i]$, $J_j = [y_{j-1}, y_j]$, $s_i = \frac{1}{2}(x_{i-1} + x_i)$ and $t_j = \frac{1}{2}(y_{j-1} + y_j)$. Moreover, $s_0 = x_0$, $s_{m+1} = x_m$, $t_0 = y_0$, $t_{n+1} = y_n$.

We use the following notations

$$\sigma_i = \frac{h_i}{h_{i-1} + h_i},\ \sigma_i' = \frac{h_{i-1}}{h_{i-1} + h_i} = 1 - \sigma_i,$$

$$\tau_j = \frac{k_j}{k_{j-1} + k_j},\ \tau_j' = \frac{k_{j-1}}{k_{j-1} + k_j} = 1 - \tau_j,$$

for $1 \le i \le m$ and $1 \le j \le n$ with the convention $h_0 = h_{m+1} = k_0 = k_{n+1} = 0$,

$$a_i = -\frac{\sigma_i^2 \sigma_{i+1}'}{\sigma_i + \sigma_{i+1}'},\ b_i = 1 + \sigma_i \sigma_{i+1}',\ c_i = -\frac{\sigma_i (\sigma_{i+1}')^2}{\sigma_i + \sigma_{i+1}'},$$

$$\bar{a}_j = -\frac{\tau_j^2 \tau_{j+1}'}{\tau_j + \tau_{j+1}'},\ \bar{b}_j = 1 + \tau_j \tau_{j+1}',\ \bar{c}_j = -\frac{\tau_j (\tau_{j+1}')^2}{\tau_j + \tau_{j+1}'},$$

for $(i, j) \in \mathcal{A}_{mn}$, where $\mathcal{A}_{mn} = \{(i, j) : 0 \le i \le m + 1,\ 0 \le j \le n + 1\}$. The data sites are the $mn$ intersection points of diagonals in the subrectangles $\Omega_{ij} = I_i \times J_j$, the $2(m + n)$ midpoints of the subintervals on the four edges, and the four vertices of $\Omega$, i.e. the $(m + 2)(n + 2)$ points of the following set

$$\mathcal{D}_{mn} := \{M_{ij} = (s_i, t_j), (i, j) \in \mathcal{A}_{mn}\}.$$

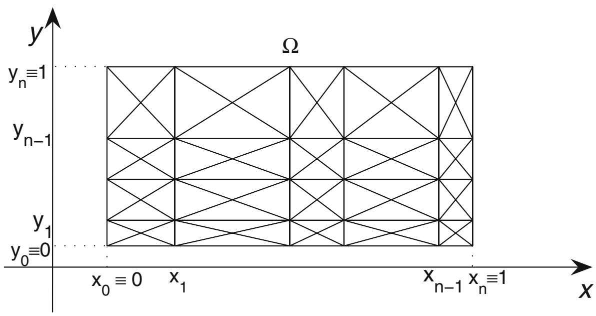


**Fig. 1** Triangulation $\mathcal{T}_{mn}$ of $\Omega = [0, 1] \times [0, 1]$

The simplest QI is the bivariate Schoenberg–Marsden operator given by

$$\mathcal{S} f := \sum_{(i,j)\in\mathcal{A}_{mn}} f(M_{ij}) B_{ij} \tag{3}$$

where

$$\mathcal{B}_{mn} := \{B_{ij},\ (i,j) \in \mathcal{A}_{mn}\}$$

is the collection of $(m+2)(n+2)$ B-splines with multiple knots (or generalized box-splines) generating the space $S_2(\mathcal{T}_{mn})$ of all $\mathcal{C}^1$ piecewise quadratic functions on the criss-cross triangulation $\mathcal{T}_{mn}$ associated with the partition $\mathcal{X}_m \times \mathcal{Y}_n$ of the domain $\Omega$ (see Fig. 2). The BB-coefficients of these B-splines can be found in the technical reports [9] and [10].

It is well known that $\mathcal{S}$ is exact on bilinear polynomials, i.e.

$$\mathcal{S} e_{rs} = e_{rs} \quad \text{for} \quad 0 \le r, s \le 1, \quad \text{where} \quad e_{rs}(x, y) = x^r y^s.$$

The QI used here is the following spline operator exact on the space $\Pi_2$ of polynomials of total degree 2 and defined in [8] by

$$\mathcal{Q} f := \sum_{(i,j)\in\mathcal{A}_{mn}} \mu_{ij}(f) B_{ij},$$

where the coefficient functionals $\mu_{ij}(f)$ are given by

$$\mu_{ij}(f) = (b_i + \bar{b}_j - 1) f(M_{ij}) + a_i f(M_{i-1,j}) + c_i f(M_{i+1,j})$$
$$+ \bar{a}_j f(M_{i,j-1}) + \bar{c}_j f(M_{i,j+1}).$$

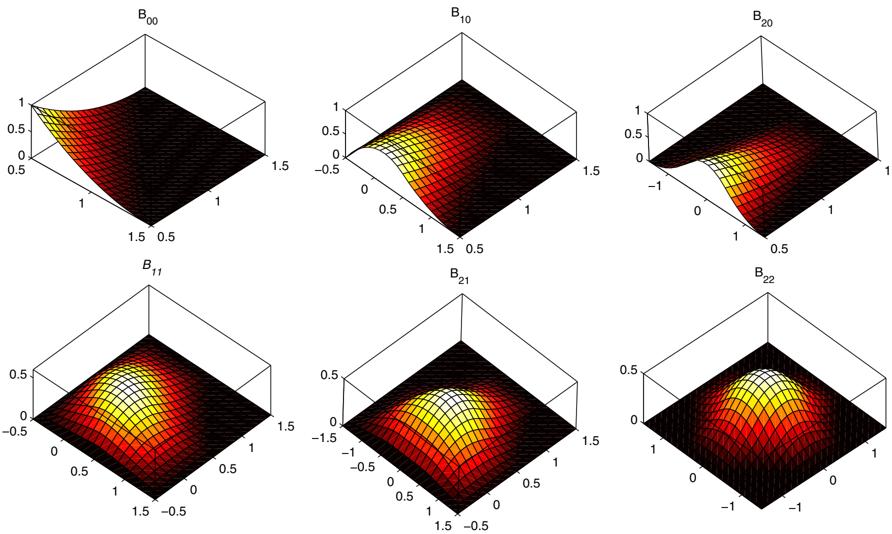


**Fig. 2** Some box-splines with multiple knots on the uniform triangulation

In terms of the quasi-Lagrange functions defined by

$$L_{ij} = (b_i + \bar{b}_j - 1)B_{ij} + a_{i+1}B_{i+1,j} + c_{i-1}B_{i-1,j} + \bar{a}_{j+1}B_{i,j+1} + \bar{c}_{j-1}B_{i,j-1},$$

$\mathcal{Q}$ can be written in the form

$$\mathcal{Q}f := \sum_{(i,j)\in\mathcal{A}_{mn}} f(M_{ij})L_{ij} \tag{4}$$

which is more convenient. For the norms of the derivatives, we set

$$\|D^k f\|_\Omega = \max\{|D^\alpha f|_{\infty,\Omega}\,;\,|\alpha| = k\}, \quad k \geq 1,$$

where

$$|D^\alpha f|_{\infty,\Omega} = \max\{|D^\alpha f(M)|_\infty\,;\,M \in \Omega\}$$

and

$$D^\alpha = D^{\alpha_1\alpha_2} = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1}x\partial^{\alpha_2}y}, \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2.$$

Note that throughout this paper $C_1, C_2, C_3$ and $C_4$ denote generic positive constants, which may take different values at their different occurrences, but will be independent on $n$.

**Theorem 1** *For $f \in \mathcal{C}(\Omega)$, we have the following error estimate*

$$\|f - \mathcal{Q}f\|_\Omega \leq C_1\omega(f,\Delta), \tag{5}$$

*and for $f \in \mathcal{C}^3(\Omega)$, we have*

$$\|f - \mathcal{Q}f\|_\Omega \leq \frac{1}{8}\Delta^3\|D^3 f\|_\Omega, \tag{6}$$

*where $\Delta = \max\{h_i, k_j\,;\,1 \leq i \leq m,\ 1 \leq j \leq n\}$ and $\omega(f,\Delta)$ is the modulus of continuity of $f$.*

*Proof* See [11]. □

For a uniform partition of $\Omega$ ($n = m,\ h_i = k_j = h$), we obtain

$$a_0 = \bar{a}_0 = c_0 = \bar{c}_0 = 0,\ b_0 = \bar{b}_0 = 1$$

$$a_{n+1} = \bar{a}_{n+1} = c_{n+1} = \bar{c}_{n+1} = 0,\ b_{n+1} = \bar{b}_{n+1} = 1$$

$$a_1 = \bar{a}_1 = c_n = \bar{c}_n = -\frac{1}{3}$$

$$b_1 = \bar{b}_1 = b_n = \bar{b}_n = \frac{3}{2}$$

$$c_1 = \bar{c}_1 = a_n = \bar{a}_n = -\frac{1}{6}$$

and for $2 \le i \le n-1$

$$a_i = \bar{a}_i = c_i = \bar{c}_i = -\frac{1}{8}, \quad b_i = \bar{b}_i = \frac{5}{4}.$$

In this case, $\mathcal{A}_{mn}$ and $\mathcal{Q}$ will be denoted by $\mathcal{A}_n$ and $\mathcal{Q}_n$ respectively. Let $M_{i,j}$ and $C_{i,j}$ be respectively the center and the midpoint of the horizontal edge $A_{i-1,j}A_{i,j}$ and let $D_{i,j}$ be the midpoint of the vertical edge $A_{i,j-1}A_{i,j}$, see Fig. 3. Using Taylor expansions of $f$ at the points

$$\{M_{ij},\ 4 \le i \le n-1,\ 4 \le j \le n-1\}, \quad \{A_{ij},\ 2 \le i \le n-2,\ 2 \le j \le n-2\},$$

and

$$\{C_{ij},\ 2 \le i \le n-2,\ 4 \le j \le n-1\}, \quad \{D_{ij},\ 4 \le i \le n-1,\ 2 \le j \le n-2\},$$

we obtain the superconvergence results

$$(f - \mathcal{Q}_n f)(M_{ij}) = \frac{h^4}{64}(D^{40} f + 2D^{22} + D^{04} f)(M_{ij}) + \mathcal{O}(h^5),$$

$$(f - \mathcal{Q}_n f)(A_{ij}) = \frac{h^4}{128}(3D^{40} f + 2D^{22} + 3D^{04} f)(A_{ij}) + \mathcal{O}(h^5),$$

$$(f - \mathcal{Q}_n f)(C_{ij}) = \frac{h^4}{128}(2D^{04} f + 2D^{22} + 3D^{40} f)(C_{ij}) + \mathcal{O}(h^5),$$

$$(f - \mathcal{Q}_n f)(D_{ij}) = \frac{h^4}{128}(3D^{04} f + 2D^{22} + 2D^{40} f)(D_{ij}) + \mathcal{O}(h^5). \tag{7}$$

**Theorem 2** *Let g be a differentiable function with bounded derivatives and* $f \in \mathcal{C}^4(\Omega)$, *then we have*

$$\mathcal{E}(f, g) = \int_\Omega (f - \mathcal{Q}_n f) g \le C_2 h^4 \left(\| D^3 f \|_\Omega + \| D^4 f \|_\Omega\right), \tag{8}$$

*where $C_2$ is a positive constant independent on n.*

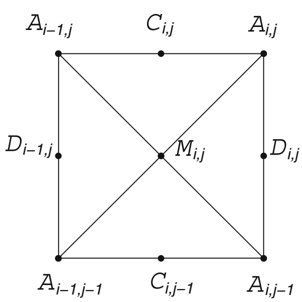


**Fig. 3** A square of the uniform criss-cross triangulation

*Proof* Let $\mathcal{L}f$ be the local Lagrange interpolant of $f$ and denote by $\{P_r, 1 \le r \le 8\}$ the eight interpolation points defining $\mathcal{L}f$, say the vertices and the midpoints of the edges of the subsquare $\Omega_{ij} = I_i \times I_j$, see Fig. 4. $\mathcal{L}f$ can be written as

$$\mathcal{L}f = \sum_{r=1}^{8} f(P_r)\ell_r,$$

where the basis functions $\ell_i$ satisfy $\ell_i(P_j) = \delta_{ij}$. We write

$$f - \mathcal{Q}_n f = f - \mathcal{L}f + \mathcal{L}f - \mathcal{Q}_n f,$$

then

$$\mathcal{E}(f, g) = \int_\Omega (f - \mathcal{L}f)g + \int_\Omega (\mathcal{L}f - \mathcal{Q}_n f)g. \tag{9}$$

We have

$$\int_\Omega (\mathcal{L}f - \mathcal{Q}_n f)g = \sum_{i,j=0}^{n+1} \int_{\Omega_{ij}} (\mathcal{L}f - \mathcal{Q}_n f)g = \sum_{i,j=0}^{n+1} I_{ij}. \tag{10}$$

According to [5] and using the results of superconvergence given by (7), we obtain for $2 \le i \le n-1$ and $2 \le j \le n-1$

$$\|\mathcal{L}f - \mathcal{Q}_n f\|_{\Omega_{ij}} \le 3 \max_{1\le r\le 8} |(f - \mathcal{Q}_n f)(P_r)| \le \frac{3h^4}{16}\|D^4 f\|_\Omega + \mathcal{O}(h^5),$$

and therefore

$$I_{ij} \le \|g\|_\Omega \int_{\Omega_{ij}} |\mathcal{L}f - \mathcal{Q}_n f| \le \frac{3h^6}{16}\|D^4 f\|_\Omega \|g\|_\Omega. \tag{11}$$

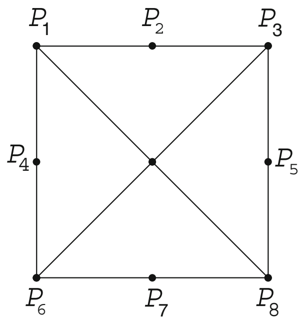


**Fig. 4** Lagrange interpolation points

For $i = 0, 1, n, n+1$ and $0 \leq j \leq n+1$, we have

$$I_{ij} \leq \|g\|_\Omega \int_{\Omega_{ij}} |\mathcal{L}f - \mathcal{Q}_n f| \leq \frac{3h^5}{8} \|D^3 f\|_\Omega \|g\|_\Omega \tag{12}$$

and similarly

$$I_{ji} \leq \frac{3h^5}{8} \|D^3 f\|_\Omega \|g\|_\Omega. \tag{13}$$

Now by combining (11)–(13) with (10), we obtain

$$\int_\Omega (\mathcal{L}f - \mathcal{Q}_n f) g \leq h^4 \left( 3 \left\| D^3 f \right\|_\Omega + \frac{3}{16} \left\| D^4 f \right\|_\Omega \right) \|g\|_\Omega. \tag{14}$$

Put $\lambda_r = \int_\Omega \ell_r$. It is easy to show that $\lambda_1 = \lambda_3 = \lambda_6 = \lambda_8 = -1$ and $\lambda_2 = \lambda_4 = \lambda_5 = \lambda_7 = 4$, then using the symmetries of the interpolation points $\{P_r, 1 \leq r \leq 8\}$ in the square $\Omega$ and the symmetries of the quadrature weights $\lambda_r$, we can show that

$$\int_\Omega (f - \mathcal{L}f) g \leq C_1 h^4 \left\| D^4 f \right\|_\Omega. \tag{15}$$

Then (8) follows by combining (14) and (15). □

## 3 Collocation methods

Let us define the following degenerate kernel

$$\mathcal{Q}_n k(s, .) = k_n(s, t) = \sum_{\alpha \in \mathcal{A}_n} k(s, M_\alpha) L_\alpha(t), \quad \text{with} \quad \alpha = (i, j).$$

Then, the associated degenerate kernel operator is given by

$$\mathcal{K}_{n,1}(u)(s) := \int_\Omega k_n(s, t) u(t) dt. \tag{16}$$

On the other hand, the Nyström operator based on $\mathcal{Q}_n$ is defined by

$$\mathcal{K}_{n,2}(u)(s) := \sum_{\alpha \in \mathcal{A}_n} w_\alpha k(s, M_\alpha) u(M_\alpha), \tag{17}$$

where the quadrature weights $w_\alpha := \int_\Omega L_\alpha,\ \alpha \in \mathcal{A}_n$, are given in the following table [6]

| j / i | 0 | 1 | 2 | 3 | ... | $n-2$ | $n-1$ | $n$ | $n+1$ |
|---|---|---|---|---|---|---|---|---|---|
| 0 | $-\frac{1}{12}$ | $\frac{7}{36}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | ... | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{7}{36}$ | $-\frac{1}{12}$ |
| 1 | $\frac{7}{36}$ | $\frac{2}{3}$ | $\frac{8}{9}$ | $\frac{7}{8}$ | ... | $\frac{7}{8}$ | $\frac{8}{9}$ | $\frac{2}{3}$ | $\frac{7}{36}$ |
| 2 | $\frac{1}{9}$ | $\frac{8}{9}$ | $\frac{37}{36}$ | $\frac{73}{72}$ | ... | $\frac{73}{72}$ | $\frac{37}{36}$ | $\frac{8}{9}$ | $\frac{1}{9}$ |
| 3 | $\frac{1}{9}$ | $\frac{7}{8}$ | $\frac{73}{72}$ | 1 | ... | 1 | $\frac{73}{72}$ | $\frac{7}{8}$ | $\frac{1}{9}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $n-2$ | $\frac{1}{9}$ | $\frac{7}{8}$ | $\frac{73}{72}$ | 1 | ... | 1 | $\frac{73}{72}$ | $\frac{7}{8}$ | $\frac{1}{9}$ |
| $n-1$ | $\frac{1}{9}$ | $\frac{8}{9}$ | $\frac{37}{36}$ | $\frac{73}{72}$ | ... | $\frac{73}{72}$ | $\frac{37}{36}$ | $\frac{8}{9}$ | $\frac{1}{9}$ |
| $n$ | $\frac{7}{36}$ | $\frac{2}{3}$ | $\frac{8}{9}$ | $\frac{7}{8}$ | ... | $\frac{7}{8}$ | $\frac{8}{9}$ | $\frac{2}{3}$ | $\frac{7}{36}$ |
| $n+1$ | $-\frac{1}{12}$ | $\frac{7}{36}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | ... | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{7}{36}$ | $-\frac{1}{12}$ |

We approximate

$$(\mathcal{I} - \mathcal{K})u = f$$

by

$$u_{n,i} - (\mathcal{Q}_n\mathcal{K} + \mathcal{K}_{n,i} - \mathcal{Q}_n\mathcal{K}_{n,i})u_{n,i} = f, \quad i = 1, 2, \tag{18}$$

that is,

$$(\mathcal{I} - \mathcal{K}_n)u_{n,i} = f,$$

and the iterated solution is defined by

$$\widetilde{u}_{n,i} = \mathcal{K}u_{n,i} + f. \tag{19}$$

In the next subsection, we consider the reduction of (18) to a system of linear equations and we give some details on the numerical implementation and the computational cost of the proposed method. Let first consider the following notations:

let $a$, $b$ and $\overline{b}$ be the vectors with components

$$a_\beta := \mathcal{K}f(M_\beta), \quad b_\beta := \langle f, L_\beta\rangle, \quad \text{and} \quad \overline{b}_\beta := f(M_\beta),$$

and $A, B, \overline{B}, C, \overline{C}, D, \overline{D}, E$ the matrices with respective entries

$$A_{\alpha,\beta} := \widetilde{L}_\beta(M_\alpha), \quad B_{\alpha,\beta} := \overline{k}_\beta(M_\alpha), \quad \overline{B}_{\alpha,\beta} := w_\beta \overline{k}_\beta(M_\alpha), \quad C_{\alpha,\beta} := \langle L_\alpha, L_\beta \rangle,$$

$$\overline{C}_{\alpha,\beta} := L_\beta(M_\alpha), \quad D_{\alpha,\beta} := k^*_\beta(M_\alpha), \quad \overline{D}_{\alpha,\beta} := w_\beta k^*_\beta(M_\alpha) \quad E_{\alpha,\beta} := \langle \overline{k}_\beta, L_\alpha \rangle$$

where $\overline{k}_\beta := k(., M_\beta)$, $k^*_\beta := \mathcal{K}\overline{k}_\beta$ and $\widetilde{L}_\beta := \mathcal{K} L_\beta$.

### 3.1 Approximate solution for the operator $\mathcal{K}_{n,1}$

**Theorem 3** *The approximate solution of* (1) *is given by*

$$u_{n,1} = f + \sum_{\alpha \in \mathcal{A}_n} X_\alpha L_\alpha + \sum_{\beta \in \mathcal{A}_n} Y_\beta \overline{k}_\beta, \tag{20}$$

*where* $Z = [X\ Y]^T$ *is the solution of the following linear system of size* $N = 2(n+2)^2$

$$(I - F)Z = c \tag{21}$$

*with*

$$F := \begin{bmatrix} A & D - B \\ C & E \end{bmatrix} \quad \textit{and} \quad c := \begin{bmatrix} a \\ b \end{bmatrix}.$$

*Proof* Let

$$W_\alpha = \int_\Omega k(M_\alpha, t) u(t) dt \quad \text{and} \quad Y_\beta = \int_\Omega L_\beta(t) u(t) dt.$$

We obtain successively

$$\mathcal{Q}_n \mathcal{K} u = \sum_{\alpha \in \mathcal{A}_n} W_\alpha L_\alpha. \tag{22}$$

$$\mathcal{K}_{n,1} u = \sum_{\beta \in \mathcal{A}_n} Y_\beta k(., M_\beta).$$

Then

$$\mathcal{Q}_n \mathcal{K}_{n,1} u = \sum_{\alpha \in \mathcal{A}_n} \left( \sum_{\beta \in \mathcal{A}_n} Y_\beta k(M_\alpha, M_\beta) \right) L_\alpha.$$

By introducing the previous formulas in (18) with $i = 1$, the approximate solution can be written as

$$u_{n,1} = f + \sum_{\alpha \in \mathcal{A}_n} X_\alpha L_\alpha + \sum_{\beta \in \mathcal{A}_n} Y_\beta \overline{k}_\beta \tag{23}$$

with $X_\alpha = W_\alpha - \sum_{\beta \in \mathcal{A}_n} Y_\beta k(M_\alpha, M_\beta)$ and the iterated solution is given by

$$\widetilde{u}_{n,1} = f + \mathcal{K} f + \sum_{\alpha \in \mathcal{A}_n} X_\alpha \widetilde{L}_\alpha + \sum_{\beta \in \mathcal{A}_n} Y_\beta k^*_\beta.$$

The coefficients $X_\alpha$ and $Y_\beta$ are obtained by substituting $u_{n,1}$ from (23) in (18). Then, we have successively

$$
\begin{aligned}
\mathcal{Q}_n\mathcal{K}u_{n,1} &= \sum_{\alpha\in\mathcal{A}_n} \mathcal{K}u_{n,1}(M_\alpha)L_\alpha \\
&= \sum_{\alpha\in\mathcal{A}_n}\left(\mathcal{K}f(M_\alpha) + \sum_{\mu\in\mathcal{A}_n} X_\mu \widetilde{L}_\mu(M_\alpha) + \sum_{\nu\in\mathcal{A}_n} Y_\nu k_\nu^*(M_\alpha)\right)L_\alpha, \\
\mathcal{K}_{n,1}u_{n,1} &= \sum_{\beta\in\mathcal{A}_n} \overline{k}_\beta \int_\Omega L_\beta(t)u_{n,1}(t)dt = \sum_{\beta\in\mathcal{A}_n}\overline{k}_\beta\langle u_{n,1}, L_\beta\rangle \\
&= \sum_{\beta\in\mathcal{A}_n}\left(\langle f, L_\beta\rangle + \sum_{\mu\in\mathcal{A}_n} X_\mu\langle L_\mu, L_\beta\rangle + \sum_{\nu\in\mathcal{A}_n} Y_\nu\langle \overline{k}_\nu, L_\beta\rangle\right)\overline{k}_\beta, \\
\mathcal{Q}_n\mathcal{K}_{n,1}u_{n,1} &= \sum_{\alpha\in\mathcal{A}_n}\mathcal{K}_{n,1}u_{n,1}(M_\alpha)L_\alpha \\
&= \sum_{\alpha\in\mathcal{A}_n}\left(\sum_{\beta\in\mathcal{A}_n}\left(\langle f, L_\beta\rangle + \sum_{\mu\in\mathcal{A}_n} X_\mu\langle L_\mu, L_\beta\rangle\right.\right. \\
&\qquad \left.\left. + \sum_{\nu\in\mathcal{A}_n} Y_\nu\langle\overline{k}_\nu, L_\beta\rangle\right)\overline{k}_\beta(M_\alpha)\right)L_\alpha.
\end{aligned}
$$

By identifying the coefficients of $L_\alpha$ and $\overline{k}_\beta$ respectively in (18), we get

$$
\begin{aligned}
X_\alpha &= \mathcal{K}f(M_\alpha) + \sum_{\mu\in\mathcal{A}_n} X_\mu\widetilde{L}_\mu(M_\alpha) + \sum_{\nu\in\mathcal{A}_n} Y_\nu\mu_\nu^*(M_\alpha) \\
&\quad - \sum_{\beta\in\mathcal{A}_n}\left(\langle f, L_\beta\rangle + \sum_{\mu\in\mathcal{A}_n} X_\mu\langle L_\mu, L_\beta\rangle + \sum_{\nu\in\mathcal{A}_n} Y_\nu\langle\overline{k}_\nu, L_\beta\rangle\right)\overline{k}_\beta(M_\alpha), \\
Y_\beta &= \langle f, L_\beta\rangle + \sum_{\mu\in\mathcal{A}_n} X_\mu\langle L_\mu, L_\beta\rangle + \sum_{\nu\in\mathcal{A}_n} Y_\nu\langle\overline{k}_\nu, L_\beta\rangle.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
X &= a + AX + DY - B(b + CX + EY), \\
Y &= b + CX + EY. \qquad (24)
\end{aligned}
$$

Replacing $Y$ by its value in (24), we get

$$
\begin{aligned}
X &= a + AX + (D - B)Y, \\
Y &= b + CX + EY,
\end{aligned}
$$

which completes the proof. □

*Remark 1* In practice, the following integrals need to be evaluated numerically

$$a_\beta := \mathcal{K} f(M_\beta) = \int_\Omega k(M_\beta, s) f(s) ds,$$

$$b_\beta := \langle f, L_\alpha \rangle = \int_\Omega L_\alpha(s) f(s) ds,$$

$$A_{\alpha,\beta} := \widetilde{L}_\beta(t_\alpha) = \mathcal{K} L_\beta(M_\alpha) = \int_\Omega k(M_\alpha, s) L_\beta(s) ds,$$

$$D_{\alpha,\beta} := k^*_\beta(M_\alpha) = \mathcal{K}\overline{k}_\beta(M_\alpha) = \int_\Omega k(M_\alpha, s) k(s, M_\beta) ds,$$

$$E_{\alpha,\beta} := \langle \overline{k}_\beta, L_\alpha \rangle = \int_\Omega k(s, M_\beta) L_\alpha(s) dt.$$

For this purpose, we define in Section 5 a discrete version of the proposed method. Since $L_\alpha$ and $L_\beta$ for $\alpha, \beta \in \mathcal{A}_n$, are functions having small supports on $\Omega$ and are piecewise polynomials, the integrals

$$\int_\Omega L_\alpha(t) L_\beta(t) dt, \quad \alpha, \beta \in \mathcal{A}_n$$

appearing in the matrix $C$ can be evaluated exactly.

### 3.2 Approximate solution for the operator $\mathcal{K}_{n,2}$

**Theorem 4** *The approximate solution of* (1) *is given by*

$$u_{n,2} = f + \sum_{\alpha \in \mathcal{A}_n} X_\alpha L_\alpha + \sum_{\beta \in \mathcal{A}_n} w_\beta Y_\beta \overline{k}_\beta, \tag{25}$$

*where* $Z = [X\ Y]^T$ *is the solution of the following linear system of size* $N = 2(n+2)^2$

$$(I - \overline{F}) Z = \overline{c} \tag{26}$$

*with*

$$\overline{F} := \left[ \begin{array}{cc} A & \overline{D} - \overline{C} \\ \hline B & \overline{C} \end{array} \right] \quad \text{and} \quad \overline{c} := \left[ \begin{array}{c} a \\ \hline b \end{array} \right].$$

*Proof* From (22) and (17), we get

$$\mathcal{Q}_n \mathcal{K} u = \sum_{\alpha \in \mathcal{A}_n} W_\alpha L_\alpha$$

and

$$\mathcal{K}_{n,2} u = \sum_{\beta \in \mathcal{A}_n} w_\beta k(., M_\beta) u(M_\beta) = \sum_{\beta \in \mathcal{A}_n} w_\beta Y_\beta k(., M_\beta).$$

Then

$$\mathcal{Q}_n \mathcal{K}_{n,2} u = \sum_{\alpha \in \mathcal{A}_n} \left( \sum_{\beta \in \mathcal{A}_n} w_\beta Y_\beta k(M_\alpha, M_\beta) \right) L_\alpha .$$

By introducing the previous formulas in (18) with $i = 2$, the approximate solution can be written as

$$u_{n,2} = f + \sum_{\alpha \in \mathcal{A}_n} X_\alpha L_\alpha + \sum_{\beta \in \mathcal{A}_n} w_\beta Y_\beta \overline{k}_\beta , \tag{27}$$

with $X_\alpha = W_\alpha - \sum_{\beta \in \mathcal{A}_n} w_\beta Y_\beta k(M_\alpha, M_\beta)$. Then, the iterated solution is given by

$$\widetilde{u}_{n,2} = f + \mathcal{K} f + \sum_{\alpha \in \mathcal{A}_n} X_\alpha \widetilde{L}_\alpha + \sum_{\beta \in \mathcal{A}_n} w_\beta Y_\beta k^*_\beta .$$

The coefficients $X_\alpha$ and $Y_\beta$ are obtained by substituting $u_{n,2}$ from (27) in (18). Then, we have successively

$$\begin{aligned}
\mathcal{Q}_n \mathcal{K} u_{n,2} &= \sum_{\alpha \in \mathcal{A}_n} \mathcal{K} u_{n,2}(M_\alpha) L_\alpha \\
&= \sum_{\alpha \in \mathcal{A}_n} \left( \mathcal{K} f(M_\alpha) + \sum_{\mu \in \mathcal{A}_n} X_\mu \widetilde{L}_\mu(M_\alpha) + \sum_{\nu \in \mathcal{A}_n} w_\nu Y_\nu k^*_\nu(M_\alpha) \right) L_\alpha , \\
\mathcal{K}_{n,2} u_{n,2} &= \sum_{\beta \in \mathcal{A}_n} w_\beta \overline{k}_\beta u_{n,2}(M_\beta) \\
&= \sum_{\beta \in \mathcal{A}_n} w_\beta \left( f(M_\beta) + \sum_{\mu \in \mathcal{A}_n} X_\mu L_\mu(M_\beta) + \sum_{\nu \in \mathcal{A}_n} w_\nu Y_\nu \overline{k}_\nu(M_\beta) \right) \overline{k}_\beta , \\
\mathcal{Q}_n \mathcal{K}_{n,2} u_{n,2} &= \sum_{\alpha \in \mathcal{A}_n} \mathcal{K}_{n,2} u_{n,2}(M_\alpha) L_\alpha \\
&= \sum_{\alpha \in \mathcal{A}_n} \left( \sum_{\beta \in \mathcal{A}_n} w_\beta \left( f(M_\beta) + \sum_{\mu \in \mathcal{A}_n} X_\mu L_\mu(M_\beta) \right.\right. \\
&\qquad \left.\left. + \sum_{\nu \in \mathcal{A}_n} w_\nu Y_\nu \overline{k}_\nu(M_\beta) \right) \overline{k}_\beta(M_\alpha) \right) L_\alpha .
\end{aligned}$$

By identifying the coefficients of $L_\alpha$ and $\overline{k}_\beta$ respectively in (18), we obtain

$$X_\alpha = \mathcal{K} f(M_\alpha) + \sum_{\mu\in\mathcal{A}_n} X_\mu \widetilde{L}_\mu(M_\alpha) + \sum_{\nu\in\mathcal{A}_n} w_\nu Y_\nu k^*_\nu(M_\alpha)$$

$$- \sum_{\beta\in\mathcal{A}_n} w_\beta \left( f(M_\beta) + \sum_{\mu\in\mathcal{A}_n} X_\mu L_\mu(M_\beta) + \sum_{\nu\in\mathcal{A}_n} w_\nu Y_\nu \overline{k}_\nu(M_\beta)\right) \overline{k}_\beta(M_\alpha),$$

$$Y_\beta = f(M_\beta) + \sum_{\mu\in\mathcal{A}_n} X_\mu L_\mu(M_\beta) + \sum_{\nu\in\mathcal{A}_n} w_\nu Y_\nu \overline{k}_\nu(M_\beta).$$

Then, we have

$$X = a + AX + \overline{D}Y - C(b + \overline{B}X + \overline{C}Y), \qquad (28)$$
$$Y = \overline{b} + \overline{B}X + \overline{C}Y.$$

Replacing $Y$ by its value in (28), we get

$$X = a + AX + (\overline{D} - \overline{C})Y,$$
$$Y = \overline{b} + \overline{B}X + \overline{C}Y,$$

which completes the proof. □

### 3.3 Comparison with Kulkarni's method

In the Kulkarni's method, the operator $\mathcal{K}$ is replaced by the finite rank operator

$$\mathcal{Q}_n\mathcal{K} + \mathcal{K}\mathcal{Q}_n - \mathcal{Q}_n\mathcal{K}\mathcal{Q}_n.$$

Then, by proceeding as before, we can show that the matrix of the linear system that will be solved to obtain the approximate solution is given by

$$H := \begin{bmatrix} A & S - A \\ \overline{C} & A \end{bmatrix}, \qquad (29)$$

where $S$ is the matrix with entries

$$S_{\alpha,\beta} := \mathcal{K}^2 L_\beta(M_\alpha) = \mathcal{K}\widetilde{L}_\beta(M_\alpha) = \int_\Omega \int_\Omega k(M_\alpha, s)k(s,t) L_\beta(t) ds dt.$$

A comparison of (29) with (21) and (26) shows that the matrices in the present methods are simpler since there are only double integrals to evaluate. The approximate solution corresponding to the Kulkarni's method have the following expression

$$u_n = f + \sum_{\alpha\in\mathcal{A}_n} X_\alpha L_\alpha + \sum_{\beta\in\mathcal{A}_n} Y_\beta \widetilde{L}_\beta. \qquad (30)$$

It can be seen that the solutions given by (20) and (25) are simpler to obtain since one has just to evaluate the functions $\bar{k}_\beta$ instead of the double integrals

$\widetilde{L}_\beta = \mathcal{K} L_\beta$. For completeness, we give in Section 5 the computational costs of the the discrete versions of Kulkarni's and the proposed methods. In the next section, we precise the convergence orders of our method for both operators $\mathcal{K}_{n,1}$ and $\mathcal{K}_{n,2}$.

## 4 Orders of convergence

In this section, we prove that, under certain conditions, $\widetilde{u}_{n,i}$ converges to $u$ faster than $u_{n,i}$. The error estimates for $u_{n,i}$ and $\widetilde{u}_{n,i},\ i = 1, 2$ can be summarized as follows.

**Theorem 5** *For all integer n large enough and* $i = 1, 2$*,*

$$\|u - u_{n,i}\|_\Omega \leq C_1 \|(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_{n,i})u\|_\Omega \tag{31}$$

*and*

$$\|u - \widetilde{u}_{n,i}\|_\Omega \leq \|(\mathcal{I} - \mathcal{K})^{-1}\|_\Omega \Big( \|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_{n,i})u\|_\Omega + \|\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_{n,i})\|_\Omega \|u - u_{n,i}\|_\Omega \Big) \tag{32}$$

*where* $C_1$ *is a constant independent on n.*

*Proof* Since

$$\|\mathcal{K} - \mathcal{K}_n\|_\Omega = \|(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_{n,i})\|_\Omega \to 0, \quad \text{when} \quad n \to \infty.$$

for all $n$ large, $(\mathcal{I} - \mathcal{K}_n)$ is invertible and $\|(\mathcal{I} - \mathcal{K}_n)^{-1}\|_\infty \leq C_1$, with $C_1$ a constant independent on $n$.

For $i = 1, 2$, we have

$$\begin{aligned} u - u_{n,i} &= [(\mathcal{I} - \mathcal{K})^{-1} - (\mathcal{I} - \mathcal{K}_n)^{-1}] f \\ &= (\mathcal{I} - \mathcal{K}_n)^{-1}(\mathcal{K} - \mathcal{K}_n)u. \end{aligned}$$

Thus

$$\begin{aligned} \|u - u_{n,i}\|_\Omega &\leq \|(\mathcal{I} - \mathcal{K}_n)^{-1}\|_\Omega \|(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_{n,i})u\|_\Omega \\ &\leq C_1 \|(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_{n,i})u\|_\Omega, \end{aligned}$$

which completes the proof of (31). On the other hand we have

$$\begin{aligned} u - \widetilde{u}_{n,i} &= \mathcal{K}(u - u_{n,i}) \\ &= \mathcal{K}(I - \mathcal{K})^{-1}(\mathcal{K} - \mathcal{K}_n)(\mathcal{I} - \mathcal{K}_n)^{-1} f \\ &= (\mathcal{I} - \mathcal{K})^{-1}\mathcal{K}(\mathcal{I} - \mathcal{Q}_n)(\mathcal{K} - \mathcal{K}_{n,i})(u + u_{n,i} - u), \end{aligned}$$

and the estimate (32) follows. □

**Proposition 1** *Assume that $u$ is differentiable with bounded derivatives. For $k(.,.) \in \mathcal{C}^3(\Omega^2)$, we have*

$$\|(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_{n,1})u\|_\Omega \leq C_3 h^7 \tag{33}$$

*and for $k(.,.) \in \mathcal{C}^4(\Omega^2)$, we have*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_{n,1})u\|_\Omega \leq C_4 h^8. \tag{34}$$

*Proof* For a fixed $\alpha = (\alpha_1, \alpha_2)$ such that $0 \leq |\alpha| = \alpha_1 + \alpha_2 \leq 4$, we denote

$$\ell(x, y, s, t) = \frac{\partial^{|\alpha|} k(x, y, s, t)}{\partial x^{\alpha_1} \partial y^{\alpha_2}}, \quad (x, y, s, t) \in \Omega^2.$$

Let $(x, y) \in \Omega$, we denote $\ell_{(x,y)}(s, t) = \ell(x, y, s, t)$, $(s, t) \in \Omega$. Then, for each $(x, y) \in \Omega$ we have

$$D^\alpha[(\mathcal{K}-\mathcal{K}_{n,1})u](x, y) = \int_\Omega u(s, t)(\mathcal{I}-\mathcal{Q}_n)\ell_{(x,y)}(s, t)dsdt$$

$$\begin{aligned}\left|D^\alpha[(\mathcal{K}-\mathcal{K}_{n,1})u]\right|_\Omega &= \max_{(x,y)\in\Omega} |\mathcal{E}(\ell_{(x,y)}, u)| \\ &\leq C_2 h^4 \left(\|D^3 k\|_\Omega + \|D^4 k\|_\Omega\right),\end{aligned}$$

where

$$D^\alpha k = D^{\alpha_1\alpha_2\alpha_3\alpha_4} k = \frac{\partial^{|\alpha|} k}{\partial^{\alpha_1} x \partial^{\alpha_2} y \partial^{\alpha_3} s \partial^{\alpha_4} t}, \quad \text{with} \quad |\alpha| = \sum_{i=1}^{4} \alpha_i.$$

Hence taking supremum, we obtain

$$\|D^\alpha[(\mathcal{K}-\mathcal{K}_{n,1})u]\|_\Omega \leq C_2 h^4 \left(\|D^3 k\|_\Omega + \|D^4 k\|_\Omega\right). \tag{35}$$

By the estimate (6) we get

$$\begin{aligned}\|(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_{n,1})u\|_\Omega &\leq C_1 h^3 \|D^3[(\mathcal{K}-\mathcal{K}_{n,1})u]\|_\Omega \\ &\leq C_1 C_2 h^7 (\|D^3 k\|_\Omega + \|D^4 k\|_\Omega)\end{aligned}$$

which completes the proof of (33) with $C_3 = C_1 C_2(\|D^3 k\|_\Omega + \|D^4 k\|_\Omega)$.

On the other hand we have

$$\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)u(x, y) = \int_\Omega k(x, y, s, t)(\mathcal{I}-\mathcal{Q}_n)u(s, t)dsdt.$$

Then, by (8) we get

$$\begin{aligned}\|[\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)u]\|_\Omega &= \max_{(x,y)\in\Omega} |\mathcal{E}(u, k(x, y, ., .)| \\ &\leq C_2 h^4 \left(\|D^3 u\|_\Omega + \|D^4 u\|_\Omega\right).\end{aligned} \tag{36}$$

Now, using (35) and (36) we obtain

$$\begin{aligned}\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_{n,1})u\|_\Omega &\le C_2h^4(\|D^3[(\mathcal{K}-\mathcal{K}_{n,1})u]\|_\Omega+\|D^4[(\mathcal{K}-\mathcal{K}_{n,1})u]\|_\Omega)\\ &\le 2(C_2)^2h^8(\|D^3k\|_\Omega+\|D^4k\|_\Omega)\end{aligned}$$

which completes the proof of (34) with $C_4 = 2(C_2)^2(\|D^3k\|_\Omega + \|D^4k\|_\Omega)$. □

**Proposition 2** *For $u \in \mathcal{C}^3(\Omega)$ and $k(.,.) \in \mathcal{C}^4(\Omega^2)$ we have*

$$\|(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_{n,2})u\|_\Omega \le C_3h^7. \tag{37}$$

*For $u \in \mathcal{C}^4(\Omega)$ and $k(.,.) \in \mathcal{C}^4(\Omega^2)$ we have*

$$\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_{n,2})u\|_\Omega \le C_4h^8. \tag{38}$$

*Proof* For a fixed $\alpha = (\alpha_1, \alpha_2)$ such that $0 \le |\alpha| = \alpha_1 + \alpha_2 \le 4$, and for a fixed $(x, y) \in \Omega$ we have

$$D^\alpha[(\mathcal{K}-\mathcal{K}_{n,2})u](x,y) = \int_\Omega (\mathcal{I}-\mathcal{Q}_n)(\ell_{(x,y)}u)(s,t)dsdt.$$

Then,

$$\begin{aligned}\|D^\alpha[(\mathcal{K}-\mathcal{K}_{n,2})u]\|_\Omega &= \max_{(x,y)\in\Omega}|\mathcal{E}(\ell_{(x,y)}u,1)|\\ &\le C_2h^4\left(\|D^3[\ell_{(x,y)}u]\|_\Omega+\|D^4[\ell_{(x,y)}u]\|_\Omega\right)\\ &\le C_2h^4\left(\|D^3u\|_\Omega\|D^3k\|_\Omega+\|D^4u\|_\Omega\|D^4k\|_\Omega\right)\end{aligned}$$

and consequently

$$\|D^\alpha[(\mathcal{K}-\mathcal{K}_{n,2})u]\|_\Omega \le C_2h^4(\|D^3u\|_\Omega\|D^3k\|_\Omega+\|D^4u\|_\Omega\|D^4k\|_\Omega). \tag{39}$$

By the estimate (6) we get

$$\begin{aligned}\|(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_{n,2})u\|_\Omega &\le C_1h^3\|D^3[(\mathcal{K}-\mathcal{K}_{n,2})u]\|_\Omega\\ &\le C_1C_2h^7(\|D^3u\|_\Omega\|D^3k\|_\Omega+\|D^4u\|_\Omega\|D^4k\|_\Omega)\end{aligned}$$

which completes the proof of (37) with $C_3 = C_1C_2(\|D^3u\|_\Omega\|D^3k\|_\Omega + \|D^4u\|_\Omega\|D^4k\|_\Omega)$.

Now, using (34) and (37) we obtain

$$\begin{aligned}\|\mathcal{K}(\mathcal{I}-\mathcal{Q}_n)(\mathcal{K}-\mathcal{K}_{n,2})u\|_\Omega &\le C_2h^4\left(\|D^3[(\mathcal{K}-\mathcal{K}_{n,2})u]\|_\Omega+\|D^4[(\mathcal{K}-\mathcal{K}_{n,2})u]\|_\Omega\right)\\ &\le 2(C_2)^2h^8\left(\|D^3u\|_\Omega\|D^3k\|_\Omega+\|D^4u\|_\Omega\|D^4k\|_\Omega\right),\end{aligned}$$

which completes the proof of (38) with $C_4 = 2(C_2)^2(\|D^3u\|_\Omega\|D^3k\|_\Omega + \|D^4u\|_\Omega\|D^4k\|_\Omega)$. □

**Theorem 6** *Let $u_{n,i}$ and $\widetilde{u}_{n,i}, i = 1, 2,$ be the approximate solutions of (1) defined by (18) and (19), respectively. In the case of the degenerate kernel operator we assume that $k(.,.) \in \mathcal{C}^4(\Omega^2)$ and $u$ is differentiable with bounded* *derivatives, while in the case of the Nyström operator we assume that* $k(.,.) \in \mathcal{C}^4(\Omega^2)$ *and* $u \in \mathcal{C}^4(\Omega)$*. Then we have*

$$\|u - u_{n,i}\|_\Omega = \mathcal{O}(h^7) \tag{40}$$

*and*

$$\|u - \widetilde{u}_{n,i}\|_\Omega = \mathcal{O}(h^8). \tag{41}$$

*Proof* In the case of the degenerate kernel operator, (40) follows from the estimate (31) of Theorem 1 and the estimate (33) of Proposition 2. In the case of the Nyström operator, we use the estimates (31) and (39) to deduce (40). Since from (6) we have

$$\begin{aligned}\|[(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}u]\|_\Omega &\leq C_1 \|D^3(\mathcal{K}u)\|_\Omega h^3 \\ &\leq C_1 \|D^3 k\|_\Omega \|u\|_\Omega h^3\end{aligned}$$

it follows that

$$\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}\|_\Omega \leq C_1 \|D^3 k\|_\Omega h^3. \tag{42}$$

On the other hand, we can easily show that

$$\|(\mathcal{I} - \mathcal{Q}_n)\mathcal{K}_{n,i}\|_\Omega \leq C_1 \|D^3 k\|_\Omega h^3, \ i = 1, 2. \tag{43}$$

We now deduce (41) from (32), (34), (38), (42), (40) and (43). □

**Theorem 7** *Let* $u_{n,i}$ *and* $\widetilde{u}_{n,i}, i = 1, 2,$ *be the approximate solutions of* (1) *defined by* (18) *and* (19)*, respectively and obtained by using the Schoenberg operator* $\mathcal{S}_n$ *given by* (3)*. In the case of the degenerate kernel operator we assume that* $k(.,.) \in \mathcal{C}^2(\Omega^2)$ *and* $u \in \mathcal{C}(\Omega)$*, while in the case of the Nyström operator we assume that* $k(.,.) \in \mathcal{C}^2(\Omega^2)$ *and* $u \in \mathcal{C}^2(\Omega)$*. Then we have*

$$\|u - u_{n,i}\|_\Omega = \mathcal{O}(h^4) \tag{44}$$

*and*

$$\|u - \widetilde{u}_{n,i}\|_\Omega = \mathcal{O}(h^4). \tag{45}$$

*Proof* Since $\mathcal{S}_n$ is exact on bilinear polynomials we have $f - \mathcal{S}_n f = \mathcal{O}(h^2)$ for $f \in \mathcal{C}^2(\Omega)$ which implies that $\int_\Omega (f - \mathcal{S}_n f) g = \mathcal{O}(h^2)$ with $g \in \mathcal{C}(\Omega)$. By proceeding exactly as for $\mathcal{Q}_n$, in the above theorem, to obtain (44) and (45). □

## 5 Discrete methods

In the discretized version of the proposed method, the operator $\mathcal{K}_n$ defined by (2) is replaced by

$$\mathcal{K}_n^D = \mathcal{Q}_n \mathcal{K}_{m,2} + (\mathcal{I} - \mathcal{Q}_n)\mathcal{K}_{n,i}, \quad i = 1, 2$$

where $\mathcal{K}_{m,2}$ is the Nyström operator based on $\mathcal{Q}_m$ given by (17)

$$\mathcal{K}_{m,2}(u)(s) := \sum_{\alpha \in \mathcal{A}_m} w_\alpha k(s, M_\alpha) u(M_\alpha), \quad \text{for some} \quad m \geq n. \tag{46}$$

Let

$$(\mathcal{I} - \mathcal{K}_n^D) u_{n,i}^D = f \tag{47}$$

and

$$\tilde{u}_{n,i}^D = \mathcal{K}_{m,2} u_{n,i}^D + f. \tag{48}$$

Let $u_m$ be the solution of the Nyström equation $(\mathcal{I} - \mathcal{K}_{m,2}) u_m = f$. It is easy to show that

$$\|u - u_m\|_\Omega = \mathcal{O}(\tilde{h}^4), \text{ with } \tilde{h} = \frac{1}{m}.$$

On the other hand, the estimates (36), (37) and (40) are valid when $\mathcal{K}$ is replaced by $\mathcal{K}_{m,2}$. Hence,

$$\|u_m - u_{n,i}^D\|_\Omega = \mathcal{O}(h^7)$$

and

$$\|u_m - \tilde{u}_{n,i}^D\|_\Omega = \mathcal{O}(h^8), \text{ with } h = \frac{1}{n}.$$

**Theorem 8** *Let* $u_{n,i}^D$ *and* $\tilde{u}_{n,i}^D, i = 1, 2,$ *be the approximate solutions of* (1) *defined by* (47) *and* (48)*, respectively. Assume that the conditions of Theorem* 6 *hold, then we have*

$$\|u - u_{n,i}^D\|_\Omega = \mathcal{O}(\max\{\tilde{h}^4, h^7\})$$

*and*

$$\|u - \tilde{u}_{n,i}^D\|_\Omega = \mathcal{O}(\max\{\tilde{h}^4, h^8\}).$$

Thus, if $m \geq \left[n^{\frac{7}{4}}\right] + 1$ (respectively $m \geq n^2$), then the order of convergence in (40) (respectively in (41)) is retained. Similarly, for the Schoenberg operator $\mathcal{S}_n$ the associated Nyström operator is given by

$$\overline{\mathcal{K}}_{m,2}(u)(s) := \sum_{\alpha \in \mathcal{A}_m} \xi_\alpha k(s, M_\alpha) u(M_\alpha), \tag{49}$$

where the quadrature weights $\xi_\alpha := \int_\Omega B_\alpha,\ \alpha \in \mathcal{A}_m$ are given in the following table

| j / i | 0 | 1 | 2 | ... | $m-1$ | $m$ | $m+1$ |
|---|---|---|---|---|---|---|---|
| 0 | $\frac{1}{12}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | ... | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{12}$ |
| 1 | $\frac{1}{4}$ | $\frac{5}{12}$ | $\frac{2}{3}$ | ... | $\frac{2}{3}$ | $\frac{5}{12}$ | $\frac{1}{4}$ |
| 2 | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | ... | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $m-1$ | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | ... | 1 | $\frac{2}{3}$ | $\frac{1}{3}$ |
| $m$ | $\frac{1}{4}$ | $\frac{5}{12}$ | $\frac{2}{3}$ | ... | $\frac{2}{3}$ | $\frac{5}{12}$ | $\frac{1}{4}$ |
| $m+1$ | $\frac{1}{12}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | ... | $\frac{1}{3}$ | $\frac{1}{4}$ | $\frac{1}{12}$ |

In this case, the operator $\mathcal{K}_n$ is replaced by

$$\overline{\mathcal{K}}_n^D = \mathcal{S}_n \overline{\mathcal{K}}_{m,2} + (\mathcal{I} - \mathcal{S}_n)\mathcal{K}_{n,i}, \quad i = 1, 2.$$

**Theorem 9** *Let $u_{n,i}^D$ and $\tilde{u}_{n,i}^D, i = 1, 2,$ be the approximate solutions of* (1) *defined by* (47) *and* (48)*, respectively and obtained by using the Schoenberg operator $\mathcal{S}_n$ given by* (3)*. Assume that the conditions of Theorem 7 hold, then we have*

$$\|u - u_{n,i}^D\|_\Omega = \mathcal{O}(\max\{\tilde{h}^2, h^4\})$$

*and*

$$\|u - \tilde{u}_{n,i}^D\|_\Omega = \mathcal{O}(\max\{\tilde{h}^2, h^4\}).$$

Thus, in order to retain the orders of convergence of $u_{n,i}$ and $\tilde{u}_{n,i}$, we need to choose $m \geq n^2$.

Now, we look at the number of arithmetic operations used in computing the approximate solutions $u_{n,i}^D, i = 1, 2,\ u_n^D$ obtained respectively by discretized collocation and kulkarni's methods on a point $t \in \Omega$. Let $\mathbf{n} = (n+2)^2$ and $\mathbf{m} = (m+2)^2$.

- The calculation of each one of the vectors $a$ and $b$ requires approximately $3\mathbf{nm}$ flops.
- The calculation of each one of the matrices $A, D, E$ requires approximately $3\mathbf{n}^2\mathbf{m}$ flops, while the calculation of the matrices $\overline{B}, \overline{D}, S$ requires respectively $\mathbf{n}^2$, $4\mathbf{n}^2\mathbf{m}$, $5\mathbf{n}^2\mathbf{m}^2$ flops.

- The evaluation of each one of the matrices $F, \overline{F}, \overline{H}$ in the linear systems (21), (26) and (29) with their LU-factorization requires approximately $\mathbf{n}^2 + \frac{2}{3}\mathbf{n}^3$ flops.
- The computation of the solution of each one of the linear systems (21), (26) and (29) requires approximately $2(2\mathbf{n})^3$ flops.
- The final step is the evaluation of $u_{n,1}^D(t)$, $u_{n,2}^D(t)$ and $u_n^D(t)$ which requires respectively $2\mathbf{n}$, $3\mathbf{n}$, $\mathbf{n}(\mathbf{m}+1)$ flops.

Thus the total cost in operations in the three methods are given in the following table

| | |
|---|---|
| Collocation 1 | $3\mathbf{m}(3\mathbf{n}^2 + 2\mathbf{n}) + \frac{14}{3}\mathbf{n}^3 + \mathbf{n}^2 + 2\mathbf{n}$ |
| Collocation 2 | $\mathbf{m}(7\mathbf{n}^2 + 3\mathbf{n}) + \frac{14}{3}\mathbf{n}^3 + 2\mathbf{n}^2 + 3\mathbf{n}$ |
| Kulkarni′s method | $5\mathbf{m}^2\mathbf{n}^2 + \mathbf{m}(3\mathbf{n}^2 + 5\mathbf{n}) + \frac{14}{3}\mathbf{n}^3 + \mathbf{n}^2 + 2\mathbf{n}$ |

where for $i = 1, 2$, *Collocation* **i** is our method based on the operator $\mathcal{K}_{n,i}$.

*Remark 2* For $\mathbf{m} >> \mathbf{n}$, the collocation methods 1 and 2 have respectively costs of approximately $3\mathbf{m}(3\mathbf{n}^2 + 2\mathbf{n})$ and $\mathbf{m}(7\mathbf{n}^2 + 3\mathbf{n})$ arithmetic operations, while the Kulkarni's method has a cost of approximately $5\mathbf{m}^2\mathbf{n}^2 + \mathbf{m}(3\mathbf{n}^2 + 5\mathbf{n})$ which is more expensive.

## 6 Numerical results

In this section we give the results obtained by the above collocation methods and their iterated versions using the QIs $\mathcal{Q}_n$ and $\mathcal{S}_n$ in the case of the Nyström operator $\mathcal{K}_{n,2}$. We first consider the following integral equation quoted from [3]

$$u(x, y) - \int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi}} \cos(xt)\cos(ys)u(t, s)dtds = f(x, y), \quad 0 \le x, y \le \sqrt{\pi}.$$

For illustrative purpose we choose as exact solution $u(x, y) = 1$ and we define $f$ accordingly. Numerical results are given in Tables 1, 2, 3 and the computed

**Table 1** Collocation and iterated collocation methods using $\mathcal{S}_n$

| $n$ | $m$ | $N$ | $\|u - u_{n,2}^D\|_\Omega$ | | $\|u - \widetilde{u}_{n,2}^D\|_\Omega$ | |
|---|---|---|---|---|---|---|
| 4 | 16 | 72 | 1.07(−02) | – | 7.87(−03) | – |
| 8 | 64 | 200 | 7.97(−04) | 3.75 | 4.55(−04) | 4.11 |
| 16 | 256 | 648 | 5.31(−05) | 3.91 | 2.65(−05) | 4.10 |
| 32 | 1024 | 2312 | 3.52(−06) | 3.91 | 1.74(−06) | 3.92 |

**Table 2** Collocation method using $\mathcal{Q}_n$

| $n$ | $m$ | $N$ | $\|u - u^D_{n,2}\|_\Omega$ | |
|---|---|---|---|---|
| 4 | 12 | 72 | 4.75(−05) | – |
| 8 | 39 | 200 | 3.43(−07) | 7.11 |
| 16 | 129 | 648 | 2.59(−09) | 7.05 |
| 32 | 431 | 2312 | 1.93(−11) | 7.03 |

**Table 3** Iterated collocation method using $\mathcal{Q}_n$

| $n$ | $m$ | $N$ | $\|u - \tilde{u}^D_{n,2}\|_\Omega$ | |
|---|---|---|---|---|
| 4 | 16 | 72 | 2.87(−05) | – |
| 8 | 64 | 200 | 1.18(−07) | 7.92 |
| 16 | 256 | 648 | 4.81(−10) | 7.94 |
| 32 | 1024 | 2312 | 1.92(−12) | 7.97 |

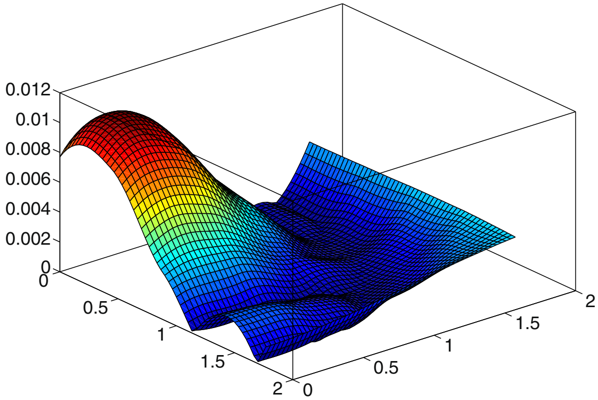


**Fig. 5** Schoenberg operator $\mathcal{S}_n$

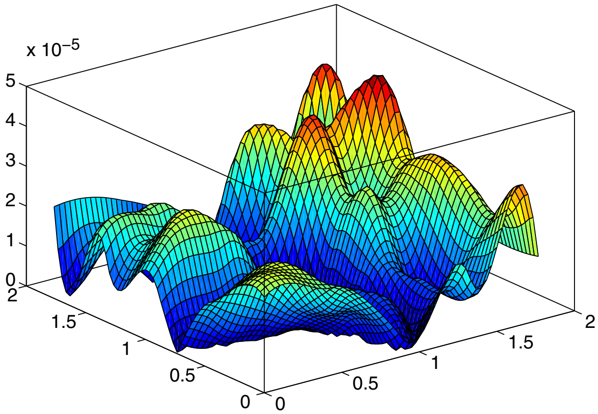


**Fig. 6** Quadratic quasi-interpolant $\mathcal{Q}_n$

**Table 4** Collocation and iterated collocation methods using $\mathcal{S}_n$

| $n$ | $m$ | $N$ | $\|u - u^D_{n,2}\|_\Omega$ | | $\|u - \widetilde{u}^D_{n,2}\|_\Omega$ | |
|---|---|---|---|---|---|---|
| 4 | 16 | 72 | 5.21(−01) | – | 8.15(−02) | – |
| 8 | 64 | 200 | 3.94(−02) | 3.72 | 5.02(−03) | 4.02 |
| 16 | 256 | 648 | 2.78(−03) | 3.82 | 2.98(−04) | 4.07 |
| 32 | 1024 | 2312 | 1.83(−04) | 3.93 | 1.94(−05) | 3.94 |

**Table 5** Collocation method using $\mathcal{Q}_n$

| $n$ | $m$ | $N$ | $\|u - u_{n,2}^D\|_\Omega$ | |
|---|---|---|---|---|
| 4 | 12 | 72 | 9.70(−05) | – |
| 8 | 39 | 200 | 9.03(−07) | 6.75 |
| 16 | 129 | 648 | 7.88(−09) | 6.84 |
| 32 | 431 | 2312 | 6.30(−11) | 6.97 |

convergence orders (last column) agree with the theoretical results. The numerical algorithm was run on a PC with Intel Pentium 2, 26 × 2 GHz CPU, 4GB RAM, and the programs were compiled by using MATLAB.

Figures 5 and 6 show the graphs of the errors obtained by the collocation methods, based respectively on $\mathcal{S}_n$ and $\mathcal{Q}_n$, and using the Nyström operator with $n = 4$.

As a second example, we consider the following integral equation quoted from [12]

$$u(x, y) - \int_{-1}^{1}\int_{-1}^{1} k(x, y, s, t)u(t, s)dtds = f(x, y), \quad -1 \leq x, y \leq 1,$$

where

$$k(x, y, s, t) = \frac{1}{4}\exp\left(-\frac{(1+x)}{2}\frac{(1+s)}{2} - \frac{(1+y)}{2}\frac{(1+t)}{2}\right)$$

and

$$g(x, y) = 1 - \frac{e^{-2-x-y}(-1+e^{1+x})(-1+e^{1+y})}{(1+x)(1+y)}.$$

The true solution of this equation is $u(x, y) = 1$. The numerical results are given in Tables 4, 5 and 6 and agree with the theoretical results.

*Remark 3* In Tables 1–6, $m$ is the integer that defines the Nyström operators given by (46) and (49). In Tables 1, 3, 4 and 6 we have chosen $m = n^2$, while in Tables 2 and 5 we have taken $m = \left[n^{\frac{7}{4}}\right] + 1$. On the other hand, $N = 2(n+2)^2$ is the size of the linear system associated with the collocation method. This also illustrates a difficulty with problems in more than one variable: the size of the linear system increases quite rapidly, and for $N$ very large the system must be solved iteratively (see [3, Chapter 6]).

**Table 6** Iterated collocation method using $\mathcal{Q}_n$

| $n$ | $m$ | $N$ | $\|u - \tilde{u}_{n,2}^D\|_\Omega$ | |
|---|---|---|---|---|
| 4 | 16 | 72 | 6.18(−05) | – |
| 8 | 64 | 200 | 2.74(−07) | 7.82 |
| 16 | 256 | 648 | 1.11(−09) | 7.95 |
| 32 | 1024 | 2312 | 4.32(−12) | 8.01 |

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