Numer Algor (2013) 62:337–354
DOI 10.1007/s11075-012-9587-5

ORIGINAL PAPER

Optimal homotopy analysis method for nonlinear differential equations in the boundary layer

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Received: 9 April 2012 / Accepted: 7 May 2012 /
Published online: 22 May 2012

Abstract Optimal homotopy analysis method is a powerful tool for nonlinear differential equations. In this method, the convergence of the series solutions is controlled by one or more parameters which can be determined by minimizing a certain function. There are several approaches to determine the optimal values of these parameters, which can be divided into two categories, i.e. global optimization approach and step-by-step optimization approach. In the global optimization approach, all the parameters are optimized simultaneously at the last order of approximation. However, this process leads to a system of coupled, nonlinear algebraic equations in multiple variables which are very difficult to solve. In the step-by-step approach, the optimal values of these parameters are determined sequentially, that is, they are determined one by one at different orders of approximation. In this way, the computational efficiency is significantly improved, especially when high order of approximation is needed. In this paper, we provide extensive examples arising in similarity and non-similarity boundary layer theory to investigate the performance of these approaches. The results reveal that with the step-by-step approach, convergent solutions of high order of approximation can be obtained within much less CPU time, compared with the global approach and the traditional HAM.

Keywords Optimal homotopy analysis method (OHAM) · Nonlinear differential equations · Similarity and non-similarity boundary layer equations

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1 Introduction

Homotopy Analysis Method (HAM, Liao [1]) is a tool for solving nonlinear differential equations. In this method, the solution of a nonlinear equation is expressed as the sum of an infinite series whose convergence is controlled by a non-zero parameter $\hbar$ which is now called convergence control parameter. Traditionally, this parameter is determined by means of the so-called $\hbar$-curve. The idea is to draw the curve of a certain quantity versus $\hbar$, from which an interval of $\hbar$ which guarantees the convergence of the solution can be identified. In the case of two coupled nonlinear equations, there may exist two convergence control parameters $\hbar_1, \hbar_2$. In this case, one can draw the iso-lines of a certain quantity on the plane of $(\hbar_1, \hbar_2)$, from which an area which ensures the convergence of the solutions may be identified. However, this approach works only when the number of the convergence control parameters is not more than two, since it is difficult to plot the iso-lines in a space with three or more dimensions.

Recently, some optimal approaches for the determination of the convergence control parameters are proposed by Yabushita [2] and Marinca et al. [3, 4], based on the minimization of the residual error of the nonlinear equation. In Yabushita [2], the minimization is accomplished by drawing the iso-lines of the residual error on the plane of $(\hbar_1, \hbar_2)$ from which the values of $\hbar_1, \hbar_2$ giving the minimum residual error can be determined optimally. However, due to the reason we have mentioned before, this approach works also when the number of the convergence control parameters is not more than two. Contrary to this graphical approach, in Marinca et al. [3, 4], the minimization is done by the stationary conditions of the residual error. In this approach, multiple parameters are introduced into the construction of the homotopy. These parameters can then be employed to control the convergence of the series solutions. In fact, this idea has its origin in a previously published paper by Liao [5] in 1999, where two embedding functions $A(p)$ and $B(p)$ are introduced into the construction of the homotopy. By expanding $A(p)$ and $B(p)$ into Maclaurin's series

$$A(p) = \sum_{k=0}^{+\infty} \alpha_k p^k, \quad B(p) = \sum_{k=0}^{+\infty} \beta_k p^k, \tag{1}$$

a number of parameters $\alpha_k, \beta_k, k = 0, 1, \cdots$, are introduced into the HAM, which can be used to control the convergence of the solutions. Marinca et al. [3, 4] proposed to determine these parameters optimally by minimizing the residual error of the nonlinear equation. In this approach, all the parameters are optimized simultaneously at the last order of approximation, which is a kind of global optimization approach. However, this approach leads to a system of coupled nonlinear algebraic equations with multiple variables which becomes more and more difficult to solve if the number of the convergence control parameters increases. Thus, the global optimization approach works quite well at low order of approximation with only a few convergence control

parameters. But certainly there exist cases where high order of approximation is needed. If this was the case, one would have to find some ways to get the optimal values of these parameters efficiently.

Recently, Niu and Wang [6] proposed a one-step approach to improve the computational efficiency of the optimal homotopy analysis method, which belongs to the category of step-by-step optimization approach. In this approach, the optimal values of the convergence control parameters are determined one by one at different order of approximation. By this way, we only need to solve a nonlinear algebraic equation in one variable at each order. Thus, the computational efficiency is greatly improved. In this paper, we further apply this method to nonlinear similarity and non-similarity boundary layer equations to investigate its efficiency when dealing with strongly nonlinear, coupled ordinary differential equations (ODEs) and partial differential equations (PDEs).

Optimal HAM has been applied to the boundary layer equation, i.e. the Blasius equation by Liao [7] most recently, where a new kind of average residual error is defined, which can be used to find the optimal convergence-control parameters much more efficiently. However, there are many other boundary layer equations (see [8–12] to name a few) which are far more complex than the Blasius equation. We still don't know the validity and efficiency of the optimal HAM in dealing with these equations. On the other hand, most boundary layer flows in nature are not similarity flows. Unlike similarity boundary layer flows, the non-similarity boundary layer flows are governed by nonlinear PDEs. Mathematically, it is much more difficult to solve PDEs than ODEs. Since there are very few attempts of optimal HAM on boundary layer equations, in this paper, we apply this method to some similarity and non-similarity boundary layer equations to investigate its performance in dealing with strongly nonlinear ODEs and PDEs. We believe the method presented in this paper will be useful for other boundary layer equations in fluid mechanics.

Section 2 describes the basic idea of the optimal HAM. In Section 3, two nonlinear PDEs and one set of coupled nonlinear ODEs are employed to investigate the convergence, accuracy and computational efficiency of different optimization approaches. Conclusions are given in the last section.

2 Optimal homotopy analysis method

Consider the following nonlinear equation

$$\mathcal{N}[u(\mathbf{r})] = 0, \tag{2}$$

where $\mathcal{N}$ is a nonlinear differential operator and $u(\mathbf{r})$ is a function of the independent variables $\mathbf{r} = \{r_1, r_2, r_3, \cdots\}$. Here, boundary and initial conditions are omitted for simplicity. From (2), we construct an equation

$$(1-p)\mathcal{L}[U(\mathbf{r}, p) - u_0(\mathbf{r})] = p\,\hbar(p)\mathcal{N}[U(\mathbf{r}, p)], \tag{3}$$

where $\mathcal{L}$ is an auxiliary linear operator, $u_0(\mathbf{r})$ is an initial approximation of $u(\mathbf{r})$, p is an embedding parameter and $\hbar(p)$ is the convergence control function satisfying $\hbar(1) \neq 0$. Equation (3) is only a special case of (2.5) in Liao [5] when $B(p) = p$ and $\hbar A(p) = p\,\hbar(p)$.

From (3), we readily have

$$U(\mathbf{r}, 0) = u_0(\mathbf{r}), \quad U(\mathbf{r}, 1) = u(\mathbf{r}). \tag{4}$$

Thus, as the embedding parameter p varies from 0 to 1, $U(\mathbf{r}, p)$ varies (or deforms) from the initial approximation $u_0(\mathbf{r})$ to $u(\mathbf{r})$, the solution of (2). We expand $U(\mathbf{r}, p)$ and $\hbar(p)$ into Maclaurin's series as

$$U(\mathbf{r}, p) = \sum_{m=0}^{+\infty} u_m(\mathbf{r})\, p^m, \qquad \hbar(p) = \sum_{k=0}^{+\infty} \hbar_k p^k, \tag{5}$$

where

$$u_m(\mathbf{r}) = \frac{1}{m!} \frac{\partial^m U(\mathbf{r}, p)}{\partial p^m}\bigg|_{p=0}, \qquad \hbar_k = \frac{1}{m!} \frac{\partial^m \hbar(p)}{\partial p^m}\bigg|_{p=0}. \tag{6}$$

Assuming that the above series converge at $p = 1$, we then have, due to (4), that

$$u(\mathbf{r}) = u_0(\mathbf{r}) + \sum_{m=1}^{+\infty} u_m(\mathbf{r}). \tag{7}$$

Now we explain how to get $u_m(\mathbf{r})$. Differentiating (3) m times with respect to p, then dividing it by $m!$ and finally setting $p = 0$, we obtain the governing equation for $u_m(\mathbf{r})$ which reads

$$\mathcal{L}\left[u_m(\mathbf{r}) - \chi_m u_{m-1}(\mathbf{r})\right] = \sum_{k=0}^{m-1} \hbar_k R_{m-1-k}(\mathbf{r}), \tag{8}$$

where $m = 1, 2, \cdots$,

$$\chi_m = \begin{cases} 0, & \text{when } m = 1, \\ 1, & \text{when } m > 1, \end{cases} \tag{9}$$

and

$$R_i(\mathbf{r}) = \frac{1}{i!} \frac{\partial^i \mathcal{N}[U(\mathbf{r}, p)]}{\partial p^k}\bigg|_{p=0}, \tag{10}$$

in which $i = 0, 1, 2, \cdots$.

Equation (8) is a linear equation and can be solved easily. Solving (8) one by one for $m = 1, 2, \cdots, M$, we obtain the M-th order approximation of $u(\mathbf{r})$

$$\tilde{u}_M(\mathbf{r}, \hbar_M) = u_0(\mathbf{r}) + \sum_{m=1}^{M} u_m(\mathbf{r}, \hbar_m), \tag{11}$$

where $\boldsymbol{\hbar}_m = \{\hbar_0, \hbar_1, \cdots, \hbar_{m-1}\}$. It is seen that the convergence of $\tilde{u}_M$ depends on the vector $\boldsymbol{\hbar}_M$ which is now called the convergence control vector and can be determined optimally by minimizing a certain function $J(\tilde{u}_M(\mathbf{r}, \boldsymbol{\hbar}_M))$. A possible, also most obvious choice of such a function is the integration of the square residual of the original equation over the whole domain, as seen in the next section. The stationary condition of $J(\tilde{u}_M(\mathbf{r}, \boldsymbol{\hbar}_M))$ gives

$$\frac{\partial J(\tilde{u}_M(\mathbf{r}, \boldsymbol{\hbar}_M))}{\partial \hbar_m} = 0, \quad m = 0, 1, 2, \cdots, M-1. \tag{12}$$

In this way, the values of the components in $\boldsymbol{\hbar}_M$ can be determined optimally in sense of minimum residual. Since all the parameters are optimized simultaneously at the Mth order of approximation, it is a kind of global optimization approach. Note that (12) leads to a system of highly nonlinear algebraic equations in multiple variables $\hbar_0, \hbar_1, \cdots, \hbar_{M-1}$ which becomes more and more difficult to solve if M increases. Thus, the global optimization approach works quite well when the number of convergence control parameters is usually not more than three, as seen in Marinca et al. [3, 4] and also in Liao [15]. But certainly, there exist cases where more control parameters are needed, for example, in the cases of strongly nonlinear problems where high order of approximation is necessary to improve the accuracy. If this was the case, one has to face the problem on how to get the optimal values of $\hbar_0, \hbar_1, \cdots, \hbar_{M-1}$ efficiently.

To improve the computational efficiency, Niu and Wang [6] proposed a one-step approach to obtain the optimal values of the components in $\boldsymbol{\hbar}_M$. In this step-by-step approach, the values of $\hbar_0, \hbar_1, \cdots, \hbar_{M-1}$ are determined one by one at different orders of approximation. The idea is to determine the value of $\hbar_0$ when $M = 1$, $\hbar_1$ is then determined when $M = 2$, etc. Although $\hbar_m, m = 0, 1, 2, \cdots$, obtained in this way may not be globally optimal, the computation can be expected to be more efficient and the accuracy can also be guaranteed if high order of approximation is employed. In Section 3, the accuracy, convergency and the efficiency of different optimization approaches are investigated with examples occurring in similarity and non-similarity boundary layer flows.

3 Examples and results analysis

In this section, two nonlinear PDEs and one system of coupled nonlinear ODEs occurring in similarity and non-similarity boundary layer theory, are employed to demonstrate the performance of the optimal homotopy analysis method (OHAM) described in Section 2. The convergency, accuracy and efficiency of different optimization approaches are investigated. The calculations are carried out on a personal computer with 2GB RAM and 3GHz CPU. The code is developed using symbolic computing software MATHEMATICA [13].

3.1 Example 1

First, we consider the following nonlinear PDE [14]

$$\frac{\partial^3 f}{\partial \eta^3} + \frac{1}{2} f \frac{\partial^2 f}{\partial \eta^2} = (1-\xi)\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}\right) - \xi(1-\xi), \tag{13}$$

subject to the boundary conditions

$$f(\xi, 0) = 0, \quad f_\eta(\xi, 0) = 0, \quad f_\eta(\xi, +\infty) = \xi, \tag{14}$$

where ξ, η are non-similarity variables. This equation describes the non-similarity boundary-layer flow of a Newtonian fluid near an asymmetric plane stagnation point where the dimensionless external flow velocity is $u_e(\xi) = \xi = x/(1+x)$.

From physical point of view, it is well known that most of the boundary-layer flows decay exponentially at infinity. Thus, it is reasonable to assume that $f(\xi, \eta)$ can be expressed by the following series

$$f(\xi, \eta) = \sum_{k=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} a_k^{m,n} \xi^k \eta^m \exp(-n\gamma\eta), \tag{15}$$

where $a_k^{m,n}$ are coefficients to be determined. Due to (15) and the boundary conditions (14), we choose the initial approximation of $f(\xi, \eta)$ as

$$f_0(\xi, \eta) = \xi\left(\eta - \frac{1 - e^{-\gamma\eta}}{\gamma}\right), \tag{16}$$

and the linear operator as

$$\mathcal{L}[F(\xi, \eta; p)] = \frac{\partial^3 F}{\partial \eta^3} + \gamma \frac{\partial^2 F}{\partial \eta^2}. \tag{17}$$

According to (13), we define the nonlinear operator

$$\mathcal{N}[F(\xi, \eta; p)] = \frac{\partial^3 F}{\partial \eta^3} + \frac{1}{2} F \frac{\partial^2 F}{\partial \eta^2} - (1-\xi)\left(\frac{\partial F}{\partial \eta}\frac{\partial^2 F}{\partial \xi \partial \eta} - \frac{\partial F}{\partial \xi}\frac{\partial^2 F}{\partial \eta^2}\right) + \xi(1-\xi). \tag{18}$$

Then we construct the so-called zeroth-order deformation equation

$$(1-p)\mathcal{L}[F(\xi, \eta; p) - f_0(\xi, \eta)] = p\, \hbar(p) \mathcal{N}[F(\xi, \eta; p)], \tag{19}$$

subject to the boundary conditions

$$F(\xi, 0; p) = 0, \quad \left.\frac{\partial F(\xi, \eta; p)}{\partial \eta}\right|_{\eta=0} = 0, \quad \left.\frac{\partial F(\xi, \eta; p)}{\partial \eta}\right|_{\eta\to+\infty} = \xi. \tag{20}$$

From (19) and (20), we readily have

$$F(\xi,\eta;0)=f_0(\xi,\eta),\quad F(\xi,\eta;1)=f(\xi,\eta), \tag{21}$$

thus $F(\xi,\eta;p)$ deforms from $f_0(\xi,\eta)$ to $f(\xi,\eta)$ as p goes from 0 to 1.

The corresponding governing equations for the high-order derivatives in the Maclaurin's series are

$$\mathcal{L}[f_m(\xi,\eta)-\chi_m\ f_{m-1}(\xi,\eta)]=\sum_{k=0}^{m-1}\hbar_k R_{m-1-k}(\xi,\eta), \tag{22}$$

subject to the boundary conditions

$$f_m(\xi,0)=0,\quad \left.\frac{\partial f_m(\xi,\eta)}{\partial\eta}\right|_{\eta=0}=0,\quad \left.\frac{\partial f_m(\xi,\eta)}{\partial\eta}\right|_{\eta\to+\infty}=0, \tag{23}$$

where $m=1,2,\cdots$, and

$$\begin{aligned}R_i(t)=&\frac{1}{i!}\left.\frac{\partial^i\mathcal{N}[F(\xi,\eta;p)]}{\partial p^i}\right|_{p=0}=\frac{\partial^3 f_i}{\partial\eta^3}+\frac{1}{2}\sum_{j=1}^{i}f_{i-j}\frac{\partial^2 f_j}{\partial\eta^2}\\&-(1-\xi)\sum_{j=1}^{i}\left(\frac{\partial f_j}{\partial\eta}\frac{\partial^2 f_{i-j}}{\partial\xi\,\partial\eta}-\frac{\partial f_j}{\partial\xi}\frac{\partial^2 f_{i-j}}{\partial\eta^2}\right)\\&+(1-\chi_{i+1})\xi(1-\xi),\end{aligned} \tag{24}$$

in which $i=0,1,2,\cdots$. Solving (22) and (23) one by one for $m=1,2,\cdots,M$, and at the Mth order approximation, we get

$$f(\xi,\eta)\approx\tilde{f}_M(\xi,\eta)=\sum_{m=0}^{M}f_m(\xi,\eta). \tag{25}$$

Note that the approximate solution $\tilde{f}_M(\xi,\eta)$ contains $\boldsymbol{\hbar}_M=\{\hbar_1,\hbar_2,\cdots,\hbar_{M-1}\}$ which is called the convergence control vector and is employed to control the convergence of the solution. To determine $\boldsymbol{\hbar}_M$, we define the following function

$$\begin{aligned}J(\boldsymbol{\hbar}_M)=\int_0^1\Bigg\{\int_0^{+\infty}\Bigg[&\frac{\partial^3\tilde{f}_M}{\partial\eta^3}+\frac{1}{2}\tilde{f}_M\frac{\partial^2\tilde{f}_M}{\partial\eta^2}\\&-(1-\xi)\left(\frac{\partial\tilde{f}_M}{\partial\eta}\frac{\partial^2\tilde{f}_M}{\partial\xi\,\partial\eta}-\frac{\partial\tilde{f}_M}{\partial\xi}\frac{\partial^2\tilde{f}_M}{\partial\eta^2}\right)+\xi(1-\xi)\Bigg]^2\mathrm{d}\eta\Bigg\}\mathrm{d}\xi.\end{aligned} \tag{26}$$

The first two orders of $J(\hbar_M)$ for $M = 1, 2$ are

$$\begin{aligned} J(\hbar_1) &= 1.06875 + 3.15344\hbar_0 + 2.5068\hbar_0^2 + 0.00997743\hbar_0^3 \\ &\quad + 0.000803807\hbar_0^4, \end{aligned} \tag{27}$$

$$\begin{aligned} J(\hbar_2) &= 1.06875 + 6.30688\hbar_0 + 13.9692\hbar_0^2 + 13.2073\hbar_0^3 + 4.55707\hbar_0^4 \\ &\quad + 0.050348\hbar_0^5 + 0.0134552\hbar_0^6 + 0.00211213\hbar_0^7 + 0.000212093\hbar_0^8 \\ &\quad + 3.16651\hbar_1 + 10.0429\hbar_0\hbar_1 + 6.70136\hbar_0^2\hbar_1 + 0.122038\hbar_0^3\hbar_1 \\ &\quad + 0.0467669\hbar_0^4\hbar_1 + 0.0100448\hbar_0^5\hbar_1 + 0.00125619\hbar_0^6\hbar_1 + 2.51095\hbar_1^2 \\ &\quad + 0.0836952\hbar_0\hbar_1^2 + 0.0498557\hbar_0^2\hbar_1^2 + 0.0142378\hbar_0^3\hbar_1^2 + 0.0025031\hbar_0^4\hbar_1^2 \\ &\quad + 0.0172266\hbar_1^3 + 0.00568101\hbar_0\hbar_1^3 + 0.00203715\hbar_0^2\hbar_1^3 \\ &\quad + 0.000250338\hbar_1^4, \end{aligned} \tag{28}$$

respectively. The convergence control vector $\hbar_M$ can be optimally identified from the conditions

$$\frac{\partial J(\hbar_M)}{\partial \hbar_1} = 0, \quad \frac{\partial J(\hbar_M)}{\partial \hbar_2} = 0, \quad \cdots, \quad \frac{\partial J(\hbar_M)}{\partial \hbar_{M-1}} = 0, \tag{29}$$

which consists of a system of nonlinear coupled algebraic equations in multiple variables $\hbar_k, k = 0, 1, \cdots, M-1$. For example, when $M = 2$, (28) and (29) will result in two algebraic equations in two variables $\hbar_0, \hbar_1$, in which the highest power is seven since the highest power in $J(\hbar_2)$ is eight on $\hbar_0$. One can imagine that, as the order M increases, the nonlinearity of this set of equations will become more and more stronger, which might eventually leave them unsolvable. As a consequence, this approach usually works under low order of approximation. Table 1 lists the values of $\hbar_M$, the function $J(\hbar_M)$ and the CPU time T(in seconds) when $\gamma = 2$ given by this global optimization approach. It is seen that the computing becomes more and more time consuming when the order of approximation M increases.

To improve the accuracy and the computational efficiency at high order of approximation, we can obtain the optimal values of $\hbar_k, k = 0, 1, 2, \cdots, M-1$, sequentially, which in fact corresponds to a kind of step-by-step optimization approach. Specifically, we solve $\hbar_0$ from the stationary condition of $J(\hbar_1)$ to get $\hbar_0 = -0.631195$ and then substitute it into $J(\hbar_2)$ to obtain

$$\begin{aligned} J(\hbar_1) &= 0.0510936 - 0.526875\hbar_1 + 2.4748\hbar_1^2 \\ &\quad + 0.0144524\hbar_1^3 + 0.000250338\hbar_1^4. \end{aligned} \tag{30}$$

Table 1 $\hbar_M$, $J(\hbar_M)$ and CPU time given by the global optimization approach for Example 1

Order M	$\hbar_0$	$\hbar_1$	$\hbar_2$	$J(\hbar_M)$	CPU time $T(s)$
1	−0.631195	–	–	7.46594E-2	2.609
2	−0.794511	0.126987	–	1.50282E-2	34.297
3	−0.930669	0.12139	0.0584152	3.72218E-3	1,699.08

Unlike (28), $J(\hbar_1)$ in (30) is only function of variable $\hbar_1$, the stationary condition of which gives $\hbar_1 = 0.106348$. Similarly, we substitute $\hbar_0$ and $\hbar_1$ into $J(\boldsymbol{\hbar}_3)$ to obtain

$$J(\hbar_2) = 0.016908 - 0.238386\hbar_2 + 2.47566\hbar_2^2 + 0.0137907\hbar_2^3 + 0.000250338\hbar_2^4, \tag{31}$$

from which we get $\hbar_2 = 0.0481264$. In this way, components in $\boldsymbol{\hbar}_M$ are obtained one by one. Since each time we need only to solve one nonlinear algebraic equation in one variable, the computation can be expected to be more efficient. Although the values of $\hbar_k, k = 0, 1, 2, \cdots, M-1$, thus obtained may not be globally optimal, accurate enough results can still be reached if high order of approximation is computed.

It should be pointed out that, in the above optimal HAM, the convergence of the series solution is controlled by multiple parameters, i.e. $\hbar_k, k = 0, 1, 2, \cdots, M-1$, which is a remarkable improvement of HAM. In the traditional HAM, the convergence of the solution is globally controlled by a unique parameter $\hbar$ whose value is usually determined by the so-called $\hbar$-curve, as seen in Liao [15]. Here we point out that this parameter can also be determined by the stationary condition of the function. With definition in (26), the function $J_M(\hbar)$ for $M = 1, 2$ are given as

$$J_1(\hbar) = 0.075 + 0.128705\hbar + 0.0588279\hbar^2 + 0.000397482\hbar^3 + 0.0000424453\hbar^4, \tag{32}$$

$$J_2(\hbar) = 0.075 + 0.257411\hbar + 0.344456\hbar^2 + 0.211722\hbar^3 + 0.053664\hbar^4 + 0.00110055\hbar^5 + 0.000458398\hbar^6 + 0.000180168\hbar^7 + 0.0000297256\hbar^8, \tag{33}$$

respectively, where the subscript on J means the order of approximation. We can see that this approach also results in highly nonlinear algebraic equations in $\hbar$, which also becomes more and more time consuming in computing if the order of approximation M increases.

Table 2 lists the values of $\hbar_n$, the function $J(\hbar_n)$ and the CPU time (s) given by the step-by-step optimization approach, compared with that given by the traditional HAM. It is seen that both approaches can give convergent, accurate results at high order approximation. However, due to the strong nonlinearity of the function used (see (33) for instance), computing by means of the traditional HAM costs much longer CPU time if the same accuracy is pursued. This can also be seen clearly in Fig. 1.

It is interesting that $|\hbar_k|$ decreases with the increasing of k, as seen in Tables 1 and 2. In fact, the idea behind optimal HAM is to approach the minimum of a certain function by progressing in the direction opposite to its gradient. Here, $\hbar_k$ plays the role of step-size in the direction of the gradient. Therefore, when the function approaches to its minimum, the step-size should be smaller and smaller.

Table 2 Comparison of results given by the step-by-step optimization approach and the traditional HAM for Example 1

Step-by-step optimization approach				Traditional HAM		
Order M	$\hbar_{M-1}$	$J(\hbar_{M-1})$	$T(s)$	$\hbar$	$J_M(\hbar)$	$T(s)$
1	−0.631195	7.46594E-2	3.469	−0.631195	7.46594E-2	2.323
2	0.106348	2.30687E-2	48.375	−0.489156	4.00114E-2	14.819
3	4.81264E-2	1.11709E-2	253.688	−0.416675	2.85722E-2	76.969
4	3.11532E-2	5.36579E-3	1,125.89	−1.059	2.8917E-3	532.149
5	2.70381E-2	2.64302E-5	7,988.72	−0.98227	1.1319E-3	2,873.91
6	–	–	–	−0.857074	7.51812E-4	1,1640.7
7	–	–	–	−0.753433	6.04018E-4	4,0437.8

3.2 Example 2

Second, we consider the following nonlinear PDE [16]

$$\frac{\partial^3 f}{\partial \eta^3}+\frac{1}{2} f\frac{\partial^2 f}{\partial \eta^2}=(1-\xi)\left(\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta}-\frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2}\right)$$
$$-\lambda(1-\xi)^2\left(\frac{\partial^2 f}{\partial \xi \partial \eta}\frac{\partial^3 f}{\partial \eta^3}+\frac{\partial f}{\partial \eta}\frac{\partial^4 f}{\partial \xi \partial \eta^3}-\frac{\partial^2 f}{\partial \eta^2}\frac{\partial^3 f}{\partial \xi \partial \eta^2}-\frac{\partial f}{\partial \xi}\frac{\partial^4 f}{\partial \eta^4}\right)$$
$$+\lambda(1-\xi)\left[\frac{\partial f}{\partial \eta}\frac{\partial^3 f}{\partial \eta^3}+\frac{1}{2} f\frac{\partial^4 f}{\partial \eta^4}-\frac{1}{2}\left(\frac{\partial^2 f}{\partial \eta^2}\right)^2\right], \tag{34}$$

subject to the boundary conditions

$$f(\xi,0)=0,\quad f_\eta(\xi,0)=\xi,\quad f_\eta(\xi,+\infty)=0, \tag{35}$$

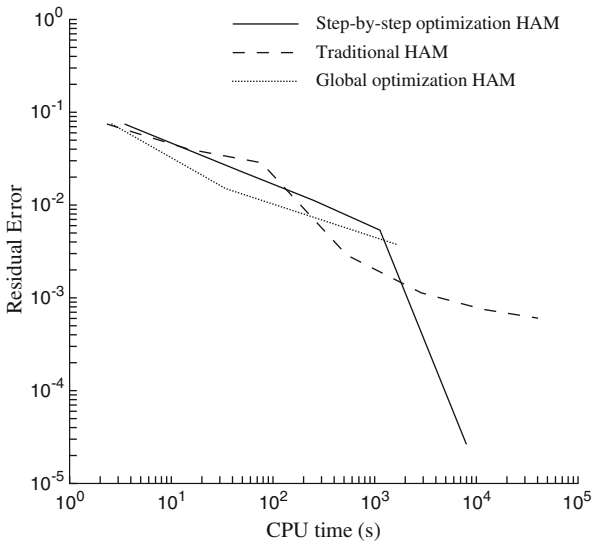

Fig. 1 Comparison of CPU time (seconds) consumed by different optimization approaches for Example 1

where λ is a constant and ξ, η are non-similarity variables. This equation describes the non-similarity boundary-layer flow of second-order fluid over a flat sheet with arbitrary stretching velocity.

Solution $f(\xi, \eta)$ can be expressed by

$$f(\xi, \eta) = \sum_{k=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} a_k^{m,n} \xi^k \eta^m \exp(-n\eta), \tag{36}$$

where $a_k^{m,n}$ are coefficients. The initial approximation of $f(\xi, \eta)$ is chosen as

$$f_0(\xi, \eta) = \xi \left[1 - \exp(-\eta)\right], \tag{37}$$

which satisfies the boundary conditions (35) and the solution expression (36). The linear operator in (17) is adopted while fixing $\gamma = 1$. The governing equations for $f_m(\xi, \eta)$ are

$$\mathcal{L}[f_m(\xi, \eta) - \chi_m f_{m-1}(\xi, \eta)] = \sum_{k=0}^{m-1} \hbar_k R_{m-1-k}(\xi, \eta), \tag{38}$$

subject to boundary conditions

$$f_m(\xi, 0) = 0, \quad \left.\frac{\partial f_m(\xi, \eta)}{\partial \eta}\right|_{\eta=0} = 0, \quad \left.\frac{\partial f_m(\xi, \eta)}{\partial \eta}\right|_{\eta=+\infty} = 0, \tag{39}$$

where $m = 1, 2, \cdots$, and

$$\begin{aligned} R_i(\xi, \eta) = {} & \frac{\partial^3 f_i}{\partial \eta^3} + \frac{1}{2} \sum_{j=0}^{i} f_{i-j} \frac{\partial^2 f_j}{\partial \eta^2} \\ & - (1 - \xi) \sum_{j=0}^{i} \left(\frac{\partial f_{i-j}}{\partial \eta} \frac{\partial^2 f_j}{\partial \xi \partial \eta} - \frac{\partial f_{i-j}}{\partial \xi} \frac{\partial^2 f_j}{\partial \eta^2} \right) \\ & + \lambda (1 - \xi)^2 \sum_{j=0}^{i} \left(\frac{\partial^2 f_{i-j}}{\partial \xi \partial \eta} \frac{\partial^3 f_j}{\partial \eta^3} + \frac{\partial f_{i-j}}{\partial \eta} \frac{\partial^4 f_j}{\partial \xi \partial \eta^3} \right) \\ & - \lambda (1 - \xi)^2 \sum_{j=0}^{i} \left(\frac{\partial^2 f_{i-j}}{\partial \eta^2} \frac{\partial^3 f_j}{\partial \xi \partial \eta^2} + \frac{\partial f_{i-j}}{\partial \xi} \frac{\partial^4 f_j}{\partial \eta^4} \right) \\ & - \lambda (1 - \xi) \sum_{j=0}^{i} \left(\frac{\partial f_{i-j}}{\partial \eta} \frac{\partial^3 f_j}{\partial \eta^3} + \frac{1}{2} f_{i-j} \frac{\partial^4 f_j}{\partial \eta^4} - \frac{1}{2} \frac{\partial^2 f_{i-j}}{\partial \eta^2} \frac{\partial^2 f_j}{\partial \eta^2} \right), \end{aligned} \tag{40}$$

in which $i = 0, 1, 2, \cdots$.

The function to be minimized is

$$J(\hbar_M) = \int_0^1 \left(\int_0^{+\infty} \mathrm{Res}_M^2 \mathrm{d}\eta \right) \mathrm{d}\xi, \tag{41}$$

where Res_M is the residual of the equation at Mth order of approximation defined by

$$\begin{aligned}\mathrm{Res}_M = &\left(\frac{\partial^3 \tilde{f}_M}{\partial \eta^3} + \frac{1}{2}\tilde{f}_M \frac{\partial^2 \tilde{f}_M}{\partial \eta^2}\right) - (1-\xi)\left(\frac{\partial \tilde{f}_M}{\partial \eta}\frac{\partial^2 \tilde{f}_M}{\partial \xi \partial \eta} - \frac{\partial \tilde{f}_M}{\partial \xi}\frac{\partial^2 \tilde{f}_M}{\partial \eta^2}\right)\\ &+ \lambda(1-\xi)^2\left(\frac{\partial^2 \tilde{f}_M}{\partial \xi \partial \eta}\frac{\partial^3 \tilde{f}_M}{\partial \eta^3} + \frac{\partial \tilde{f}_M}{\partial \eta}\frac{\partial^4 \tilde{f}_M}{\partial \xi \partial \eta^3} - \frac{\partial^2 \tilde{f}_M}{\partial \eta^2}\frac{\partial^3 \tilde{f}_M}{\partial \xi \partial \eta^2} - \frac{\partial \tilde{f}_M}{\partial \xi}\frac{\partial^4 \tilde{f}_M}{\partial \eta^4}\right)\\ &- \lambda(1-\xi)\left[\frac{\partial \tilde{f}_M}{\partial \eta}\frac{\partial^3 \tilde{f}_M}{\partial \eta^3} + \frac{1}{2}\tilde{f}_M \frac{\partial^4 \tilde{f}_M}{\partial \eta^4} - \frac{1}{2}\left(\frac{\partial^2 \tilde{f}_M}{\partial \eta^2}\right)^2\right]. \end{aligned} \tag{42}$$

The first two orders of $J(\hbar_M)$ for $M = 1, 2$ are given as

$$\begin{aligned} J(\hbar_1) = &\, 0.0759921 + 0.246881\hbar_0 + 0.20039\hbar_0^2 - 0.00757623\hbar_0^3\\ &+ 0.000582034\hbar_0^4, \end{aligned} \tag{43}$$

$$\begin{aligned} J(\hbar_2) = &\, 0.0759921 + 0.493762\hbar_0 + 1.17136\hbar_0^2 + 1.17521\hbar_0^3 + 0.373116\hbar_0^4\\ &- 0.0656679\hbar_0^5 + 0.000812542\hbar_0^6 + 0.0105274\hbar_0^7 + 0.00193622\hbar_0^8\\ &+ 0.246881\hbar_1 + 0.80156\hbar_0\hbar_1 + 0.526993\hbar_0^2\hbar_1 - 0.105828\hbar_0^3\hbar_1\\ &- 0.00974547\hbar_0^4\hbar_1 + 0.0227231\hbar_0^5\hbar_1 + 0.00526368\hbar_0^6\hbar_1 + 0.20039\hbar_1^2\\ &- 0.0454574\hbar_0\hbar_1^2 - 0.0171444\hbar_0^2\hbar_1^2 + 0.0173164\hbar_0^3\hbar_1^2 + 0.00568077\hbar_0^4\hbar_1^2\\ &- 0.00757623\hbar_1^3 + 0.00465627\hbar_0\hbar_1^3 + 0.00288606\hbar_0^2\hbar_1^3\\ &+ 0.000582034\hbar_1^4, \end{aligned} \tag{44}$$

respectively.

Table 3 gives the values of $\hbar_M$, the function $J(\hbar_M)$ and the CPU time $T(s)$ obtained by the global optimization approach for example 2. It is seen that accurate enough results can be obtained at 3rd order of approximation. However, due to the strong nonlinearity of the coupled algebraic equations in multiple variables, the CPU time is increased dramatically. Thus only low orders of approximation are available.

Results given by the step-by-step optimization approach are shown in Table 4 in the case of $\lambda = 1/2$, compared with that given by the traditional HAM. From this table, one can observe the same scenario, that is, both approaches give accurate results, while traditional HAM is much more time consuming

Table 3 $\hbar_M$, $J(\hbar_M)$ and CPU time T (in seconds) given by global optimization approach for Example 2

Order M	$\hbar_0$	$\hbar_1$	$\hbar_2$	$J(\hbar_M)$	CPU time $T(s)$
1	−0.594722	–	–	1.70962E-3	3.031
2	−0.58047	0.0517985	–	1.36521E-4	31.312
3	−0.426366	−0.0339004	0.037401	7.39286E-5	1,441.84

Table 4 Comparison between results given by the step-by-step optimization approach and the traditional HAM for Example 2

Step-by-step optimization approach				Traditional HAM		
Order M	$\hbar_{M-1}$	$J(\hbar_{M-1})$	$T(s)$	$\hbar$	$J_M(\hbar)$	$T(s)$
1	−0.594722	1.70962E-3	3.762	−0.594722	1.70962E-3	2.875
2	5.47753E-2	1.4045E-4	32.203	−0.478105	3.33207E-4	19.234
3	2.47251E-3	6.19396E-5	142.875	−0.40835	1.18622E-4	96.312
4	–	–	–	−0.36229	5.84801E-5	628.406

than the step-by-step approach, especially at high order of approximation, as also shown graphically in Fig. 2. This is expected, since in the traditional HAM, the convergence control parameter $\hbar$ is obtained by solving a nonlinear algebraic equation in which the power of the unknown is extremely high.

3.3 Example 3

At last, we consider the following coupled nonlinear ODEs [17]

$$f''' + \frac{3}{4} f f'' - \frac{1}{2} f'^2 + \theta = 0, \tag{45}$$

$$\mathrm{Pr}^{-1}\theta'' + \frac{3}{4} f\theta' = 0, \tag{46}$$

subject to the boundary conditions

$$f(0) = f'(0) = 0, \quad g(0) = 1, \tag{47}$$

$$f'(+\infty) = 0, \quad g(+\infty) = 0, \tag{48}$$

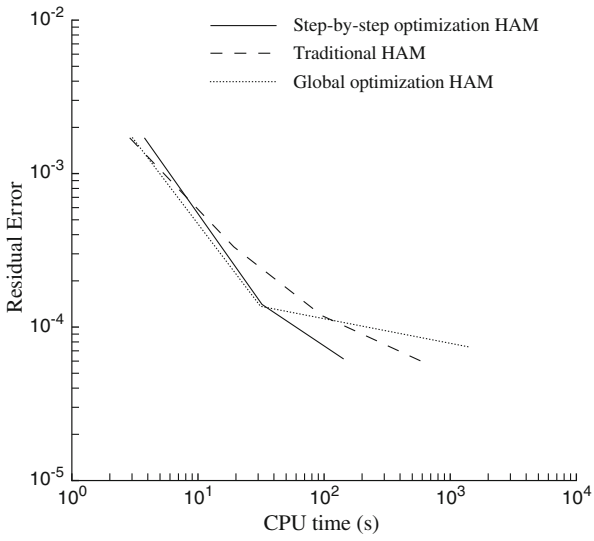

Fig. 2 Comparison of CPU time (seconds) consumed by different optimization approaches for Example 2

These equations describe the laminar free convection similarity boundary layer flow on a vertical plate.

The solutions $f(\eta), \theta(\eta)$ are expressed in the form of

$$f(\eta) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} a_{m,n} \eta^m \exp(-n\eta), \tag{49}$$

$$\theta(\eta) = \sum_{m=0}^{+\infty} \sum_{n=0}^{+\infty} b_{m,n} \eta^m \exp(-n\eta), \tag{50}$$

respectively, where $a_{m,n}, b_{m,n}$ are coefficients. The linear operators are chosen as

$$\mathcal{L}_f = \frac{\partial^3}{\partial \eta^3} - \frac{\partial}{\partial \eta}, \quad \mathcal{L}_\theta = \frac{\partial^2}{\partial \eta^2} - 1, \tag{51}$$

and the initial approximations are chosen as

$$f_0(\eta) = 1 - e^{-\eta} - \eta e^{-\eta}, \tag{52}$$

$$\theta_0(\eta) = e^{-\eta} + \eta e^{-\eta}, \tag{53}$$

which satisfy the boundary conditions (47), (48) and the solution expressions (49), (50). The governing equations for $f_m(\eta), \theta_m(\eta)$ are

$$\mathcal{L}_f[f_m(\eta) - \chi_m f_{m-1}(\eta)] = \sum_{k=0}^{m-1} \hbar_k^f R_{m-1-k}^f(\eta), \tag{54}$$

$$\mathcal{L}_\theta[\theta_m(\eta) - \chi_m \theta_{m-1}(\eta)] = \sum_{k=0}^{m-1} \hbar_k^\theta R_{m-1-k}^\theta(\eta), \tag{55}$$

subject to boundary conditions

$$f_m(0) = f_m'(0) = 0, \quad g_m(0) = 0, \tag{56}$$

$$f_m'(+\infty) = 0, \quad g_m(+\infty) = 0, \tag{57}$$

Table 5 $\hbar_M$, $J(\hbar_M)$ and CPU time T (in seconds) given by the global optimization approach for Example 3

Order M	$\hbar_0^f$	$\hbar_0^\theta$	$\hbar_1^f$	$\hbar_1^\theta$	$\hbar_2^f$	$\hbar_2^\theta$	$J(\hbar_M)$	CPU time $T(s)$
1	−1.16	−1.07	–	–	–	–	1.66E-2	3.235
2	−0.45	−1.47	−0.37	−0.17	–	–	6.85E-3	45.61
3	−1.46	−0.95	0.24	−0.24	0.28	0.032	8.90E-4	4,085.3

Table 6 Comparison between results given by the step-by-step optimization approach and the traditional HAM for Example 3

Step-by-step optimization approach					Traditional HAM			
M	$\hbar^f_{M-1}$	$\hbar^\theta_{M-1}$	$J(\hbar^f_{M-1}, \hbar^\theta_{M-1})$	$T(s)$	$\hbar^f$	$\hbar^\theta$	$J(\hbar^f, \hbar^\theta)$	$T(s)$
1	−1.16	−1.07	6.97E-2	3.16	−0.90	−1.07	6.53E-2	2.454
2	1.79E-1	−1.97E-2	2.05E-2	15.03	−1.42	−1.25	1.59E-2	23.923
3	1.49E-1	−1.14E-2	6.22E-3	36.75	−1.36	−1.09	3.94E-3	96.751
4	3.10E-2	−6.02E-3	1.10E-3	73.70	−1.24	−0.80	3.06E-3	666.47
5	−3.08E-2	−2.31E-3	7.20E-4	134.34	−1.06	−0.68	2.85E-3	3,845.53
6	−3.37E-2	−1.19E-4	3.62E-4	184.89	−0.84	−0.66	2.76E-3	16,927.6
7	−1.27E-2	9.13E-4	1.51E-4	256.30	−0.73	−0.60	2.62E-3	60,042.2
8	5.20E-3	1.15E-3	1.37E-4	294.02	−0.65	−0.55	2.49E-3	193,249
9	1.14E-2	8.86E-4	1.17E-4	337.56	–	–	–	–
10	8.18E-3	3.80E-4	7.05E-5	390.14	–	–	–	–

where $m = 1, 2, \cdots$, and

$$R_i^f(\eta) = f_i''' + \frac{3}{4}\sum_{j=0}^{i} f_{i-j} f_j'' - \frac{1}{2}\sum_{j=0}^{i} f_{i-j}' f_j' + \theta_i, \tag{58}$$

$$R_i^\theta(\eta) = \mathrm{Pr}^{-1}\theta_i'' + \frac{3}{4}\sum_{j=0}^{i} f_{i-j}\theta_j', \tag{59}$$

in which $i = 0, 1, 2, \cdots$.

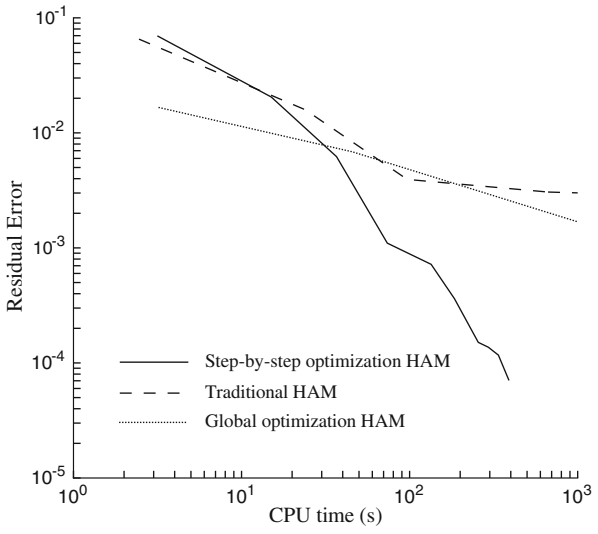

Fig. 3 Comparison of CPU time (seconds) consumed by different optimization approaches for Example 3

The convergence control vector $\boldsymbol{\hbar}_M = (\hbar_1^f, \hbar_2^f, \cdots, \hbar_{M-1}^f, \hbar_1^\theta, \hbar_2^\theta, \cdots, \hbar_{M-1}^\theta)$ is determined by the stationary condition of the following function

$$J(\boldsymbol{\hbar}_M) = \int_0^1 \left(\mathrm{Res}_{f,M}^2 + \mathrm{Res}_{\theta,M}^2\right) \mathrm{d}\eta, \tag{60}$$

where

$$\mathrm{Res}_{f,M} = \tilde{f}_M''' + \frac{3}{4}\tilde{f}_M \tilde{f}_M'' - \frac{1}{2}\tilde{f}_M'^2 + \tilde{\theta}_M, \tag{61}$$

$$\mathrm{Res}_{\theta,M} = \mathrm{Pr}^{-1}\tilde{\theta}_M'' + \frac{3}{4}\tilde{f}_M \tilde{\theta}_M', \tag{62}$$

in which $\tilde{f}_M(\eta)$, $\tilde{\theta}_M(\eta)$ are the Mth order approximations of $f(\eta)$ and $\theta(\eta)$, respectively.

Results obtained by the global optimization approach at low order of approximation are given in Table 5. Results obtained by the step-by-step approach and the traditional HAM are listed in Table 6. Compared with the traditional HAM, the CPU time costed by the step-by-step approach is remarkably reduced. For example, to obtain the 8th order of approximation, the CPU time consumed by the traditional HAM is more than 193249 seconds, while by the step-by-step approach it is only about 294 seconds. It is worth mentioning that computing $\hbar$ consumed most of the CPU time in the traditional HAM. The efficiency of the step-by-step approach is also depicted in Fig. 3.

4 Conclusions

In this paper, we investigate the performance of different optimization approaches of the optimal homotopy analysis method (OHAM) by means of three examples of similarity and non-similarity boundary layer equations. In the global optimization approach, the solution convergence control parameters $\hbar_0, \hbar_1, \hbar_2, \cdots$, are determined simultaneously by minimizing a certain function at the last order of approximation, while in the step-by-step approach, they are determined sequentially one by one by minimizing the function at different orders of approximation. Both the global optimization approach and the step-by-step approach employ multiple parameters to perform the optimization, while in the traditional HAM, the optimization is performed with respect to a unique parameter.

With the global optimization approach, one has to solve a set of highly nonlinear, coupled algebraic equations in multiple variables which is very difficult to solve if the number of the parameters increases. Thus, the global optimization approach usually works well at low order of approximation. Similarly, in the traditional HAM, due to the highly nonlinearity of the resulted algebraic equation, the computing of $\hbar$ is also very time consuming, especially at high order of approximation. On the contrary, the step-by-step optimization approach results in a series of decoupled algebraic equations in one variable

which can be solved much more easily. Thus, with the step-by-step approach, the CPU time is remarkably reduced, especially at high order approximation. Although the values of the parameters obtained with this approach may not be globally optimal, accurate enough results may still be reached if high order of approximation is computed. All our three examples support these conclusions.

We make a few remarks on the optimization approaches. Firstly, it is found that in general the absolute values of parameters $\hbar_0, \hbar_1, \hbar_2, \cdots, \hbar_k$ decreases with the increasing of k, which might be explained by an analogy with the optimal gradient method or residual method widely used in linear algebra. The idea behind optimal HAM is to approach the minimum of a certain function by progressing in the direction of descent. Here, $\hbar_k, k = 0, 1, 2, \cdots$, play the role of step-size in the direction of descent. Therefore, when the function is approaching to its minimum, the step-size should be of course smaller and smaller.

Secondly, the weakness of the optimal descent algorithm which minimizes a certain function is that the minimization following the direction of descent is forgotten at the next step. This is also the case in the step-by-step approach. Since all the parameters $\hbar_i, i = 0, 1, \cdots, m-2$, are already known when solving u_m, they can not be adjusted again to minimize the function at the mth order of approximation. Thus, the convergence of the solution is not globally controlled. Whereas in the global approach, as well as the traditional HAM, the convergence of the solution can be globally controlled by the vector $\hbar_M$ or the scalar $\hbar$.

Thirdly, we mention that due to the nonlinearity of the algebraic equations, there may exist more than one real solutions for $\hbar_m, m = 0, 1, 2, \cdots$, in the optimal HAM. In this case, we only take those real and select the one which gives the minimum residual.

Finally, it is worth to mention that although three examples of boundary layer flows are taken to illustrate the performance of the optimal HAM, to fully evaluate this method, we still need more applications, especially to those strongly nonlinear equations. We believe that the method presented in this paper will be useful for other boundary layer equations widely occurring in fluid mechanics.

Acknowledgements Thanks for the financial supports of the National Natural Science Foundation of China (Approval No.10972136) and Science Foundation of China University of Petroleum, Beijing (No.YJRC-2011-6).

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