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Adapted Falkner-type methods solving oscillatory second-order differential equations

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Abstract The classical Falkner methods (Falkner, Phil Mag S 7:621, 1936) are well-known for solving second-order initial-value problems $u''(t) = f(t, u(t), u'(t))$. In this paper, we propose the adapted Falkner-type methods for the systems of oscillatory second-order differential equations $u''(t) + Mu(t) = g(t, u(t))$ and make a rigorous error analysis. The error bounds for the global errors on the solution and the derivative are presented. In particular, the error bound for the global error of the solution is shown to be independent of $\|M\|$. We also give a stability analysis and plot the regions of stability for our new methods. Numerical examples are included to show that our new methods are very competitive compared with the reformed Falkner methods in the scientific literature.

Keywords Adapted Falkner-type method · Error analysis · Oscillatory second-order differential equation

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1 Introduction

This paper is concerned with oscillatory second-order initial value problems of the form

$$\begin{cases} u''(t) + Mu(t) = g(t, u(t)), \quad t \in [t_0, T], \\ u(t_0) = u_0, \quad u'(t_0) = u'_0, \end{cases} \tag{1}$$

where $M \in R^{d\times d}$ is a symmetric positive semi-definite matrix that implicitly contains the frequencies of the problems. Such problems are frequently encountered in celestial mechanics, theoretical physics, chemistry, electronics, and so on. They are particularly attractive when these problems come from the spatial semi-discretizations of wave equations based on the method of lines. In practice, they can be integrated with general purpose methods or other codes adapted to the special structures of the problems under consideration. Generally the adapted methods are more efficient because they make full use of the information transpired from the special structures.

In the one-dimensional case $u''(t) + w^2u(t) = g(t, u(t))$ or the case where the single-frequency matrix $M = w^2 I$ is a diagonal matrix, many multi-step methods adapted to the problems (1) have been developed (see [1–3], for example). These methods share the fact that they integrate the unperturbed problems $u''(t) + w^2u(t) = 0$ exactly. Very recently, two-step extended Runge-Kutta-Nyström-type methods are proposed for the multidimensional systems $u''(t) + Mu(t) = g(t, u(t))$ (see [4]).

For a second-order initial value problem

$$u''(t) = f(t, u(t), u'(t)), \quad u(t_0) = u_0, \quad u'(t_0) = u'_0, \tag{2}$$

one of the effective multi-step integrators is due to Falkner [5] which can be written in the form

$$u_{n+1} = u_n + hu'_n + h^2 \sum_{j=0}^{k-1} \beta_j \nabla^j f_n, \quad u'_{n+1} = u'_n + h \sum_{j=0}^{k-1} \gamma_j \nabla^j f_n, \tag{3}$$

where $f_n = f(t_n, u_n, u'_n)$ and $\nabla^j f_n$ denotes the jth backward difference. The coefficients β_j and γ_j can be generated by the generating functions

$$G_\beta(t) = \sum_{j=0}^{\infty} \beta_j t^j = \frac{t + (1-t)\ln(1-t)}{(1-t)\ln^2(1-t)}, \quad G_\gamma(t) = \sum_{j=0}^{\infty} \gamma_j t^j = \frac{-t}{(1-t)\ln(1-t)}.$$

Analogously, there exist implicit formulas [6] that read

$$u_{n+1} = u_n + hu'_n + h^2 \sum_{j=0}^{k} \beta_j^* \nabla^j f_{n+1}, \quad u'_{n+1} = u'_n + h \sum_{j=0}^{k} \gamma_j^* \nabla^j f_{n+1}, \tag{4}$$

with generating functions for the coefficients given by

$$G_{\beta^*}(t) = \sum_{j=0}^{\infty} \beta_j^* t^j = \frac{t + (1-t)\ln(1-t)}{\ln^2(1-t)}, \quad G_{\gamma^*}(t) = \sum_{j=0}^{\infty} \gamma_j^* t^j = \frac{-t}{\ln(1-t)}.$$

For the variable step version, we refer the reader to [7].

Recently, for the special second-order initial value problem

$$u''(t) = f(t, u(t)), \quad u(t_0) = u_0, \quad u'(t_0) = u_0', \tag{5}$$

Ramos et al. [8] proposed and studied a reformed Falkner scheme as follows

$$u_{n+1} = u_n + hu_n' + h^2 \sum_{j=0}^{k-1} \beta_j \nabla^j f_n, \quad u_{n+1}' = u_n' + h \sum_{j=0}^{k} \gamma_j^* \nabla^j f_{n+1}, \tag{6}$$

which evaluates u_{n+1} using the first formula in (3) and evaluates u_{n+1}' using the second formula in (4). Due to the absence of the first derivative on the function f, the value f_{n+1} can be obtained from u_{n+1} directly. Note that in this way the second formula of (6) is no longer implicit.

The authors in [8] show that the convergence order of the explicit scheme (6) is $k+1$ whereas it is well known that the classical explicit Falkner method (3) has only convergence of order k.

The method (6) is called "an unusual implementation of the explicit Falkner method" in [8]. For the sake of convenience, we call the method (6) "a k-step reformed Falkner method (RFMk)" in this paper. The purpose of this paper is to propose an extension of scheme (6) adapted to the oscillatory problems (1). The new methods (which will be called *adapted Falkner-type methods*) can integrate exactly the multidimensional unperturbed problems $u''(t) + Mu(t) = 0$, unfortunately, however, scheme (6) cannot. Moreover, we will give a rigorous error analysis for the new methods and propose uniform error bounds. Particularly, an interesting issue is that the error bounds for the global error of the solution are independent of $\|M\|$.

This paper is organized as follows: In Section 2, we formulate the adapted Falkner-type methods. In Section 3, we present a rigorous error analysis for the new methods and obtain the uniform error bounds. Section 4 is devoted to a stability analysis and the regions of stability for our new methods. Numerical examples are included in Section 5. We conclude the paper with some comments in the last section.

2 Formulation of the new methods

Firstly, we introduce the matrix-valued ϕ-functions appeared first in [9]

$$\phi_0(M) = \sum_{k=0}^{\infty} \frac{(-1)^k M^k}{(2k)!}, \quad \phi_1(M) = \sum_{k=0}^{\infty} \frac{(-1)^k M^k}{(2k+1)!}, \quad \forall M \in R^{d\times d}. \tag{7}$$

The interesting properties of these functions are established in the following proposition.

Proposition 2.1 *If M is symmetric and positive semi-definite, then*

$$\|\phi_0(M)\| \leq 1, \quad \|\phi_1(M)\| \leq 1,$$

where $\|\cdot\|$ denotes spectral norm.

Proof Since M is symmetric and positive semi-definite, we have $M = P^T\Omega^2 P$, where

$$\Omega^2 = \begin{pmatrix} w_1^2 & & & \\ & w_2^2 & & \\ & & \ddots & \\ & & & w_d^2 \end{pmatrix} \tag{8}$$

and P is an orthogonal matrix. So we have

$$\begin{aligned} \phi_0(M) &= \sum_{k=0}^{\infty} \frac{(-1)^k M^k}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k (P^T\Omega^2 P)^k}{(2k)!} \\ &= P^T \sum_{k=0}^{\infty} \frac{(-1)^k \Omega^{2k}}{(2k)!} P = P^T \phi_0(\Omega^2) P \end{aligned} \tag{9}$$

with

$$\phi_0(\Omega^2) = \begin{pmatrix} \cos w_1 & & & \\ & \cos w_2 & & \\ & & \ddots & \\ & & & \cos w_d \end{pmatrix}. \tag{10}$$

The definition of spectral norm results in $\left\|\phi_0(\Omega^2)\right\| \leq 1$ and

$$\|\phi_0(M)\| = \left\|P^T\phi_0(\Omega^2)P\right\| \leq \left\|\phi_0(\Omega^2)\right\| \leq 1.$$

$\|\phi_1(M)\| \leq 1$ can be obtained in a similar way. □

The true solutions to problems (1) and its derivatives satisfy the following equations [10]

$$\begin{aligned} u(t_n + h) &= \phi_0(V)u(t_n) + h\phi_1(V)u'(t_n) \\ &\quad + h^2 \int_0^1 (1-z)\phi_1((1-z)^2 V) g(t_n + hz, u(t_n + hz))dz, \\ u'(t_n + h) &= -hM\phi_1(V)u(t_n) + \phi_0(V)u'(t_n) \\ &\quad + h\int_0^1 \phi_0((1-z)^2 V) g(t_n + hz, u(t_n + hz))dz, \end{aligned} \tag{11}$$

where $V = h^2 M$. Here and in the sequel, the integral of a matrix-valued function or a vector-valued function is understood as componentwise.

Given approximations $u_j \approx u(t_j)$, the interpolation polynomial p_n through the points $(t_{n-k+1}, g(t_{n-k+1}, u_{n-k+1})), \ldots, (t_n, g(t_n, u_n))$ is given by

$$p_n(t_n + \theta h) = \sum_{j=0}^{k-1} (-1)^j \binom{-\theta}{j} \nabla^j g_n, \quad g_j = g(t_j, u_j).$$

Here, $\nabla^j g_n$ denotes the jth backward difference, defined recursively by

$$\nabla^0 g_n = g_n, \quad \nabla^j g_n = \nabla^{j-1} g_n - \nabla^{j-1} g_{n-1}, \quad j = 1, 2, \ldots.$$

Similarly, we consider the interpolation polynomial p_n^* through the points

$$(t_{n-k+1}, g(t_{n-k+1}, u_{n-k+1})), \ldots, (t_{n+1}, g(t_{n+1}, u_{n+1}))$$

which reads

$$p_n^*(t_n + \theta h) = \sum_{j=0}^{k} (-1)^j \binom{-\theta + 1}{j} \nabla^j g_{n+1}.$$

Replacing $g(t_n + hz, u(t_n + hz))$ in the first equation of (11) by the interpolation polynomial $p_n(t_n + zh)$ and replacing $g(t_n + hz, u(t_n + hz))$ in the second equation of (11) by the interpolation polynomial $p_n^*(t_n + zh)$, respectively, define a new numerical method

$$\begin{aligned} u_{n+1} &= \phi_0(V) u_n + h \phi_1(V) u_n' + h^2 \int_0^1 (1 - z) \phi_1((1 - z)^2 V) p_n(t_n + zh) dz, \\ u_{n+1}' &= -h M \phi_1(V) u_n + \phi_0(V) u_n' + h \int_0^1 \phi_0((1 - z)^2 V) p_n^*(t_n + zh) dz. \end{aligned} \tag{12}$$

By inserting the interpolation polynomials p_n and p_n^* into (12), we get a new scheme

$$\begin{aligned} u_{n+1} &= \phi_0(V) u_n + h \phi_1(V) u_n' + h^2 \sum_{j=0}^{k-1} \beta_j(V) \nabla^j g_n, \\ u_{n+1}' &= -h M \phi_1(V) u_n + \phi_0(V) u_n' + h \sum_{j=0}^{k} \gamma_j^*(V) \nabla^j g_{n+1}, \end{aligned} \tag{13}$$

where the coefficients $\beta_j(V)$ and $\gamma_j^*(V)$ are defined by

$$\begin{aligned} \beta_j(V) &= (-1)^j \int_0^1 (1 - z) \phi_1((1 - z)^2 V) \binom{-z}{j} dz, \\ \gamma_j^*(V) &= (-1)^j \int_0^1 \phi_0((1 - z)^2 V) \binom{-z + 1}{j} dz. \end{aligned}$$

Obviously, the new scheme (13) is an explicit scheme. The scheme (13) is called a *k-step adapted Falkner-type method* (*AFMk*).

It can be observed that when $M \to \mathbf{0}$ ($V \to \mathbf{0}$), (13) reduces to the k-step reformed Falkner method (6) proposed in [8].

The coefficients $\beta_j(V)$ are found from the generating function

$$\begin{aligned} G_\beta(t, V) &= \sum_{j=0}^{\infty} \beta_j(V) t^j \\ &= \left[\phi_1(V)(1-t)\ln(1-t) + I - (1-t)\phi_0(V)\right] \\ &\quad \times \left[(1-t)(I \ln^2(1-t) + V)\right]^{-1}, \end{aligned} \tag{14}$$

and can be determined recursively as follows:

$$\begin{aligned} &\beta_0(V) = V^{-1}[I - \phi_0(V)], \quad \beta_1(V) = V^{-1}[I - \phi_1(V)], \\ &\beta_n(V) = V^{-1}\left[I - \frac{1}{n}\phi_1(V) - \beta_0(V)S_n - \beta_1(V)S_{n-1} - \ldots - \beta_{n-2}(V)S_2\right], \\ &\qquad n \geq 2, \end{aligned}$$

where

$$S_m = \sum_{j=1}^{m-1} \frac{1}{j(m-j)}, \quad m \geq 2.$$

Similarly, the generating function corresponding to $\gamma_j^*(V)$ is given by

$$\begin{aligned} G_{\gamma^*}(t, V) &= \sum_{j=0}^{\infty} \gamma_j^*(V) t^j \\ &= -\left[I \ln(1-t) + (-1+t)\phi_0(V)\ln(1-t) + (-1+t)V\phi_1(V)\right] \\ &\quad \times \left[I \ln^2(1-t) + V\right]^{-1} \end{aligned} \tag{15}$$

which gives the following recursive expressions:

$$\begin{aligned} &\gamma_0^*(V) = \phi_1(V), \quad \gamma_1^*(V) = V^{-1}[I - \phi_0(V)] - \phi_1(V), \\ &\gamma_n^*(V) = V^{-1}\left[\frac{1}{n}I + \frac{1}{n(n-1)}\phi_0(V) - \gamma_0^*(V)S_n - \gamma_1^*(V)S_{n-1} - \ldots - \gamma_{n-2}^*(V)S_2\right], \\ &\qquad n \geq 2. \end{aligned}$$

The derivation of the generating functions in (14) and (15) has an explanation in Appendix A.

Remark 2.1 The scheme (13) was discovered for $y'' + w^2 y = g(t, y)$ in [11]. In this paper, we consider the new scheme (13) for the multidimensional oscillatory systems and give the generating functions.

Example 2.1 The case of $k = 1$ gives

$$\begin{aligned} u_{n+1} &= \phi_0(V)u_n + h\phi_1(V)u'_n + h^2V^{-1}\left[I - \phi_0(V)\right]g_n, \\ u'_{n+1} &= -hM\phi_1(V)u_n + \phi_0(V)u'_n \\ &\quad + h\left\{V^{-1}\left[I - \phi_0(V)\right]g_{n+1} + \left(\phi_1(V) - V^{-1}\left[I - \phi_0(V)\right]\right)g_n\right\}, \quad (16) \end{aligned}$$

which will be shown to be convergence of order two. It can be observed that when $M \to \mathbf{0}$ $(V \to \mathbf{0})$, (16) reduces to the well-known Velocity Verlet formula [12]. So (16) is an extension of the Velocity Verlet formula for oscillatory problems (1).

3 Error analysis

The purpose of this section is to give an error analysis of scheme (13) when applied to oscillatory problems (1). We will derive uniform error bounds on bounded time intervals. In what follows, we use Euclidean norm and its induced matrix norm (spectral norm) and denote them by $\|\cdot\|$.

Before going on with our work, we restate the discrete Gronwall's lemma (Lemma 2.4 in [13]) which is useful to our error analysis.

Lemma 3.1 *Let α, ϕ, ψ and χ be nonnegative functions defined for $t = n\Delta t$, $n = 0, 1, \ldots, M$, and assume χ is nondecreasing. If*

$$\phi_k + \psi_k \le \chi_k + \Delta t \sum_{n=1}^{k-1} \alpha_n \phi_n, \qquad k = 0, 1, \ldots, M,$$

and if there is a positive constant $\hat{c}$ such that $\Delta t \sum_{n=1}^{k-1} \alpha_n \le \hat{c}$, then

$$\phi_k + \psi_k \le \chi_k e^{\hat{c}k\Delta t}, \qquad k = 0, 1, \ldots, M,$$

where the subscript indices k and n denote the values of functions at $t_k = k\Delta t$ and $t_n = n\Delta t$, respectively.

Now we are ready to state our main convergence result for the adapted Falkner-type method (13).

Theorem 3.1 *Suppose that M be symmetric and positive semi-definite in the initial value problem (1) and $\frac{\partial g}{\partial u}$ be uniformly bounded in a strip along the exact solution u, and consider the k-step adapted Falkner-type method (13) with step length h satisfying $0 < h < H$ with H sufficiently small. Let $f(t) = g(t, u(t))$ and assume $f(t) \in C^{k+1}_{[t_0,T]}$. Then, for*

$$\|u_j - u(t_j)\| \le c_0 h^{k+1}, \quad j = 1, \ldots, k-1, \quad \|u'_{k-1} - u'(t_{k-1})\| \le c_0 h^{k+1},$$

the error bounds

$$\|u_n - u(t_n)\| \le Ch^{k+1}, \quad \|u_n' - u'(t_n)\| \le C'h^{k+1}$$

hold uniformly for $k \le n \le \frac{T-t_0}{h}$. *The constant* C *depends on* T, k, $\sup\limits_{0\le t\le t_{n+1}} \| f^{(k)}(t)\|$ *and* $\sup\limits_{0\le t\le t_{n+1}} \| f^{(k+1)}(t)\|$, *but is independent of* $\|M\|$, *n and h. The constant* C' *depends on* T ,*k*, $\|M\|$, $\sup\limits_{0\le t\le t_{n+1}} \| f^{(k)}(t)\|$ *and* $\sup\limits_{0\le t\le t_{n+1}} \| f^{(k+1)}(t)\|$, *but is independent of n and h.*

Proof Let $\tilde{p}_n$ denote the interpolation polynomial through the exact data

$$(t_{n-k+1},\ f(t_{n-k+1})), \ldots, (t_n,\ f(t_n)),$$

where $f(t) = g(t, u(t))$. This polynomial has the form

$$\tilde{p}_n(t_n + \theta h) = \sum_{j=0}^{k-1} (-1)^j \binom{-\theta}{j} \nabla^j f(t_n), \tag{17}$$

where the backward differences are defined recursively by

$$\nabla^0 f(t_n) = f(t_n), \quad \nabla^j f(t_n) = \nabla^{j-1} f(t_n) - \nabla^{j-1} f(t_{n-1}), \quad j = 1, 2, \ldots .$$

Its interpolation error is given by

$$f(t_n + \theta h) - \tilde{p}_n(t_n + \theta h) = h^k (-1)^k \binom{-\theta}{k} f^{(k)}(\zeta(\theta)) \tag{18}$$

with $\zeta(\theta) \in [t_{n-k+1}, t_{n+1}]$. Similarly, let $\tilde{p}_n^*$ denote the interpolation polynomial through the exact data

$$(t_{n-k+1},\ f(t_{n-k+1})), \ldots, (t_{n+1},\ f(t_{n+1})),$$

and we have

$$\tilde{p}_n^*(t_n + \theta h) = \sum_{j=0}^{k} (-1)^j \binom{-\theta + 1}{j} \nabla^j f(t_{n+1}). \tag{19}$$

Its interpolation error is given by

$$f(t_n + \theta h) - \tilde{p}_n^*(t_n + \theta h) = h^{k+1} (-1)^{k+1} \binom{-\theta + 1}{k+1} f^{(k+1)}(\xi(\theta)) \tag{20}$$

with $\xi(\theta) \in [t_{n-k+1}, t_{n+1}]$.

Replacing $p_n(t_n+zh)$ by $\tilde{p}_n(t_n+zh)$ and replacing $p_n^*(t_n+zh)$ by $\tilde{p}_n^*(t_n+zh)$ in (12), respectively, we have

$$\begin{aligned} u(t_n+h) &= \phi_0(V)u(t_n) + h\phi_1(V)u'(t_n) \\ &\qquad + h^2\int_0^1 (1-z)\phi_1((1-z)^2V)\tilde{p}_n(t_n+zh)dz + \delta_{n+1}, \\ u'(t_n+h) &= -hM\phi_1(V)u(t_n) + \phi_0(V)u'(t_n) \\ &\qquad + h\int_0^1 \phi_0((1-z)^2V)\tilde{p}_n^*(t_n+zh)dz + \delta'_{n+1} \end{aligned} \tag{21}$$

with defects

$$\begin{aligned} \delta_{n+1} &= h^2\int_0^1 (1-z)\phi_1((1-z)^2V)[f(t_n+zh) - \tilde{p}_n(t_n+zh)]dz, \\ \delta'_{n+1} &= h\int_0^1 \phi_0((1-z)^2V)[f(t_n+zh) - \tilde{p}_n^*(t_n+zh)]dz. \end{aligned} \tag{22}$$

Due to (18) and (20) and Proposition 2.1, these defects are bounded by

$$\|\delta_{n+1}\| \le C_1h^{k+2}\sup_{0\le t\le t_{n+1}}\left\|f^{(k)}(t)\right\|, \quad \|\delta'_{n+1}\| \le C'_1h^{k+2}\sup_{0\le t\le t_{n+1}}\left\|f^{(k+1)}(t)\right\|,$$

where C_1 and C'_1 only depend on k.

Let $e_n = u_n - u(t_n)$ and $e'_n = u'_n - u'(t_n)$. Subtracting (21) from (12) yields the error recursions

$$\begin{aligned} e_{n+1} &= \phi_0(V)e_n + h\phi_1(V)e'_n \\ &\qquad + h^2\int_0^1 (1-z)\phi_1((1-z)^2V)[p_n(t_n+zh) - \tilde{p}_n(t_n+zh)]dz - \delta_{n+1}, \\ e'_{n+1} &= -hM\phi_1(V)e_n + \phi_0(V)e'_n \\ &\qquad + h\int_0^1 \phi_0((1-z)^2V)[p_n^*(t_n+zh) - \tilde{p}_n^*(t_n+zh)]dz - \delta'_{n+1}, \end{aligned}$$

which can also be expressed in the form

$$\begin{pmatrix} e_{n+1} \\ e'_{n+1} \end{pmatrix} = Q\begin{pmatrix} e_n \\ e'_n \end{pmatrix} + \begin{pmatrix} h^2A_n - \delta_{n+1} \\ hB_n - \delta'_{n+1} \end{pmatrix} \tag{23}$$

with

$$\begin{aligned} Q &= \begin{pmatrix} \phi_0(V) & h\phi_1(V) \\ -hM\phi_1(V) & \phi_0(V) \end{pmatrix}, \\ A_n &= \int_0^1 (1-z)\phi_1((1-z)^2V)[p_n(t_n+zh) - \tilde{p}_n(t_n+zh)]dz, \\ B_n &= \int_0^1 \phi_0((1-z)^2V)[p_n^*(t_n+zh) - \tilde{p}_n^*(t_n+zh)]dz. \end{aligned} \tag{24}$$

Solving the recursion (23) yields

$$\begin{pmatrix} e_n \\ e_n' \end{pmatrix} = Q^{n-k+1} \begin{pmatrix} e_{k-1} \\ e_{k-1}' \end{pmatrix} + \sum_{j=k-1}^{n-1} Q^{n-j-1} \begin{pmatrix} h^2 A_j - \delta_{j+1} \\ hB_j - \delta_{j+1}' \end{pmatrix}.$$

By Appendix B, we have

$$Q^m = \begin{pmatrix} \phi_0(m^2V) & mh\phi_1(m^2V) \\ -mhM\phi_1(m^2V) & \phi_0(m^2V) \end{pmatrix}. \tag{25}$$

From (25) and

$$\left\| mh\phi_1(m^2V) \right\| \leq (T - t_0) \left\| \phi_1(m^2V) \right\| \leq T - t_0,$$

for $n \geq k$, it follows that

$$\begin{aligned}
\|e_n\| \leq{}& \|e_{k-1}\| + (T - t_0) \left\| e_{k-1}' \right\| \\
&+ \sum_{j=k-1}^{n-1} \Big\{ \left(h^2 \left\| p_j\left(t_j + zh\right) - \tilde{p}_j\left(t_j + zh\right) \right\| + \left\| \delta_{j+1} \right\| \right) \\
&\qquad + (T - t_0) \left(h \left\| p_j^*\left(t_j + zh\right) - \tilde{p}_j^*\left(t_j + zh\right) \right\| + \left\| \delta_{j+1}' \right\| \right) \Big\} \\
\leq{}& \|e_{k-1}\| + (T - t_0) \left\| e_{k-1}' \right\| \\
&+ \sum_{j=k-1}^{n-1} \left\{ \left(h^2 C_2 \sum_{l=j-k+1}^{j} \|g(t_l, u_l) - g(t_l, u(t_l))\| + \|\delta_{j+1}\| \right) \right. \\
&\qquad \left. + (T - t_0) \left(hC_2' \sum_{l=j-k+1}^{j+1} \|g\left(t_l, u_l\right) - g\left(t_l, u\left(t_l\right)\right)\| + \left\| \delta_{j+1}' \right\| \right) \right\} \\
\leq{}& \sum_{j=1}^{n} C_3 h \|e_j\| + C_4 h^{k+1},
\end{aligned} \tag{26}$$

where $\|g(t_l, u_l) - g(t_l, u(t_l))\| \leq C_0 \|e_l\|$ are used.

From the computation process of (26) it can be observed that C_3 only depends on k and T whereas C_4 only depends on k, T, $\sup\limits_{0 \leq t \leq t_{n+1}} \| f^{(k)}(t)\|$ and $\sup\limits_{0 \leq t \leq t_{n+1}} \| f^{(k+1)}(t)\|$.

When h is sufficiently small, it follows from (26) that

$$\|e_n\| = \sum_{j=1}^{n-1} \frac{C_3}{1 - C_3 h} h \|e_j\| + \frac{C_4}{1 - C_3 h} h^{k+1} \leq \sum_{j=1}^{n-1} C_5 h \|e_j\| + C_6 h^{k+1}, \quad n \geq k. \tag{27}$$

The conditions $\|e_i\| \leq c_0 h^{k+1},\ i = 0, 1, \ldots, k-1$ give

$$\|e_n\| \leq \sum_{j=1}^{n-1} h\|e_j\| + c_0 h^{k+1}, \quad 0 \leq n \leq k-1. \tag{28}$$

Then from (27) and (28), we obtain

$$\|e_n\| \leq \sum_{j=1}^{n-1} C_7 h\|e_j\| + C_8 h^{k+1}. \tag{29}$$

By Lemma 3.1, we set

$$\phi_n = \|e_n\|, \qquad \psi_n = 0, \qquad \chi_n = C_8 h^{k+1}, \qquad \alpha_n = C_7. \tag{30}$$

Therefore, as long as the error e_n remains in a neighborhood of 0, the following result holds:

$$h\sum_{j=0}^{n-1} \alpha_j = h\sum_{j=0}^{n-1} C_7 \leq \hat{C}. \tag{31}$$

The application of the discrete Gronwall lemma to (29) gives

$$\|e_n\| \leq C_8 h^{k+1} e^{\hat{C}nh} \leq Ch^{k+1}. \tag{32}$$

where C only depends on k, T, $\sup_{0\leq t\leq t_{n+1}} \|f^{(k)}(t)\|$ and $\sup_{0\leq t\leq t_{n+1}} \|f^{(k+1)}(t)\|$ and is independent of $\|M\|$, n and h.

Similarly,

$$\begin{aligned}
\left\|e'_n\right\| &\leq (T-t_0)\,\|M\|\,\|e_{k-1}\| + \left\|e'_{k-1}\right\| \\
&\quad + \sum_{j=k-1}^{n-1} \Big\{ (T-t_0)\,\|M\| \left(h^2 \left\|p_j\left(t_j+zh\right) - \tilde{p}_j\left(t_j+zh\right)\right\| + \left\|\delta_{j+1}\right\|\right) \\
&\qquad + \left(h\left\|p^*_j\left(t_j+zh\right) - \tilde{p}^*_j\left(t_j+zh\right)\right\| + \left\|\delta'_{j+1}\right\|\right)\Big\} \\
&\leq (T-t_0)\,\|M\|\,\|e_{k-1}\| + \left\|e'_{k-1}\right\| \\
&\quad + \sum_{j=k-1}^{n-1} \left\{ (T-t_0)\,\|M\| \left(h^2 C_2 \sum_{l=j-k+1}^{j} \|g\left(t_l, u_l\right) - g\left(t_l, u\left(t_l\right)\right)\| + \left\|\delta_{j+1}\right\|\right)\right. \\
&\qquad \left. + \left(hC'_2 \sum_{l=j-k+1}^{j+1} \|g\left(t_l, u_l\right) - g\left(t_l, u\left(t_l\right)\right)\| + \left\|\delta'_{j+1}\right\|\right)\right\} \\
&\leq \sum_{j=k}^{n} C'_3 h \left\|e_j\right\| + C'_4 h^{k+1},
\end{aligned} \tag{33}$$

where C_3' depends on k and $\|M\|$, h whereas C_4' depends on k, T, $\|M\|$, $\sup\limits_{0\le t\le t_{n+1}} \|f^{(k)}(t)\|$ and $\sup\limits_{0\le t\le t_{n+1}} \|f^{(k+1)}(t)\|$. The bound of e_n' can be obtained from $\|e_n\| \le Ch^{k+1}$. □

4 Stability

Firstly, we study zero-stability of the adapted Falkner-type method (13). Since the symmetric positive semi-definite matrix M can be diagonalized as a diagonal matrix, we only need to consider one dimensional equation $u''(t) + w^2u(t) = g(t, u(t))$. In this case, it is clear that $V = w^2h^2$, $\phi_0(V) = \cos(wh)$ and $\phi_1(V) = \sin(wh)/(wh)$. We will write the k-step adapted Falkner-type method (13) as a one-step method with high dimension and bound the product of the resulting matrices in an adequate norm.

In order to obtain a one-step recurrence, we give variable $v_{n+1} = (u_{n+1} - \cos(wh)u_n)/h$ and two $k+1$-vectors

$$V_n = \left(u_n, u_n', v_n, \ldots, v_{n-(k-2)}\right)^T, \quad E_n = (h\Psi_n, \Phi_n, \Psi_n, 0, \ldots, 0)^T, \tag{34}$$

where $\Psi_n = \sum\limits_{j=0}^{k-1} \beta_j(w^2h^2)\nabla^j g_n$ and $\Phi_n = \sum\limits_{j=0}^{k} \gamma_j^*(w^2h^2)\nabla^j g_{n+1}$. Thus, the adapted Falkner-type method (13) can be rewritten in the form

$$V_{n+1} = LV_n + hE_n, \tag{35}$$

where L is a $(k+1)\times(k+1)$ matrix given by

$$L = \begin{pmatrix} \cos(wh) & \sin(wh)/w & 0 \ldots 0\; 0 \\ -w\sin(wh) & \cos(wh) & 0 \ldots 0\; 0 \\ 0 & \sin(wh)/(wh) & 0 \ldots 0\; 0 \\ 0 & 0 & 1 \ldots 0\; 0 \\ \vdots & \vdots & \vdots \ddots \vdots\; \vdots \\ 0 & 0 & 0 \ldots 1\; 0 \end{pmatrix}_{(k+1)\times(k+1)} \tag{36}$$

In what follows we restate the the stability definition in terms of a bounded product of matrices [14].

Definition 4.1 The method given by (35) is stable if

$$\left\|L^j\right\| \le R, \quad \text{for} \quad k-1 \le j \le N,$$

where $N = \frac{T-t_0}{h}$ and $R > 0$ is a real number.

In order to calculate L^j simply, we let

$$A = \begin{pmatrix} \cos(wh) & \sin(wh)/w \\ -w\sin(wh) & \cos(wh) \end{pmatrix}_{2\times 2}, \quad B = \begin{pmatrix} 0 \dots 0\ 0 \\ 0 \dots 0\ 0 \end{pmatrix}_{2\times(k-1)},$$

$$C = \begin{pmatrix} 0 & \sin(wh)/(wh) \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}_{(k-1)\times 2}, \quad D = \begin{pmatrix} 0 & \dots & 0 & 0 \\ 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}_{(k-1)\times(k-1)}$$

and have

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{37}$$

It can be verified that L^m has the following form

$$L^m = \begin{pmatrix} A^m & O \\ {}_mC^* & D^m \end{pmatrix}, \tag{38}$$

where ${}_mC^*$ will be determined in the following two propositions. In the following analysis, we use ${}_mc^*_{ij}$, $i = 1, \dots, k-1$, $j = 1, 2$ to denote the entries of ${}_mC^*$.

Proposition 4.1 *When $m \le k-1$, the entries of ${}_mC^*$ are given by*

$$\begin{aligned} {}_mc^*_{i1} &= \frac{\cos((m+1-i)wh) - \cos((m-1-i)wh)}{2h}, \quad 1 \le i \le m, \\ {}_mc^*_{i2} &= \frac{\sin((m+1-i)wh) - \sin((m-1-i)wh)}{2wh}, \quad 1 \le i \le m, \\ {}_mc^*_{i1} &= {}_mc^*_{i2} = 0, \quad i \ge m+1. \end{aligned} \tag{39}$$

Proof We prove this proposition by induction. For $m = 1$, we have

$${}_1c^*_{11} = 0, \quad {}_1c^*_{12} = \frac{\sin(wh)}{wh}, \quad {}_1c^*_{i1} = {}_1c^*_{i2} = 0, \quad i \ge 2$$

which satisfy (39). Assuming that the conditions in (39) hold for $m = q$, we can obtain

$$\begin{aligned} {}_{q+1}c^*_{i1} &= \left[\frac{\cos((q+1-i)wh) - \cos((q-1-i)wh)}{2h}\right]\cos(wh) \\ &\quad + \left[\frac{\sin((q+1-i)wh) - \sin((q-1-i)wh)}{2wh}\right](-w\sin(wh)) \\ &= \frac{\cos((q+2-i)wh) - \cos((q-i)wh)}{2h}, \quad 1 \le i \le q, \end{aligned} \tag{40}$$

where ${}_{q+1}C^* = {}_qC^*A + D^qC$ is used. In a similar way, we have

$$ {}_{q+1}c^*_{i2} = \frac{\sin((q+2-i)wh) - \sin((q-i)wh)}{2wh}, \quad 1 \le i \le q. \tag{41} $$

The equation ${}_{q+1}C^* = {}_qC^*A + D^qC$ also gives

$$ \begin{aligned} {}_{q+1}c^*_{q+1,1} &= 0 = \frac{\cos((q+2-i)wh) - \cos((q-i)wh)}{2h}, \quad i = q+1 \\ {}_{q+1}c^*_{q+1,2} &= \frac{\sin(wh)}{wh} = \frac{\sin((q+2-i)wh) - \sin((q-i)wh)}{2wh}, \quad i = q+1, \\ {}_{q+1}c^*_{i,1} &= {}_{q+1}c^*_{i,2} = 0, \quad i \ge q+2. \end{aligned} \tag{42} $$

It follows that the conditions in (39) hold for $m = q+1$ from (40), (41) and (42). □

Proposition 4.2 *When $m \ge k-1$, the entries of ${}_mC^*$ satisfy*

$$ \begin{aligned} {}_mc^*_{i1} &= \frac{\cos((m+1-i)wh) - \cos((m-1-i)wh)}{2h}, \\ {}_mc^*_{i2} &= \frac{\sin((m+1-i)wh) - \sin((m-1-i)wh)}{2wh}, \end{aligned} \tag{43} $$

where $i = 1, \ldots, k-1$.

Proof The proof is similar to that of the above proposition. □

About A^m, we have the following results

$$ A^m = \begin{pmatrix} \cos(mwh) & \sin(mwh)/w \\ -w\sin(mwh) & \cos(mwh) \end{pmatrix} \tag{44} $$

which can be proved by induction.

Now we consider another norm denoted $\|\cdot\|_1$ and given by

$$ \|L\|_1 = \max_{1 \le j \le k+1} \sum_{i=1}^{k+1} |l_{ij}|, \tag{45} $$

where $l_{ij}, i, j = 1, \ldots, k+1$ denote the entries of L. For $k-1 \le m \le N$,

$$ \begin{aligned} \|L^m\|_1 = \max \Bigg\{ & |\cos(mwh)| + |-w\sin(mwh)| + \sum_{i=1}^{k-1} \left|{}_mc^*_{i1}\right|, \\ & \left|\frac{\sin(mwh)}{w}\right| + |\cos(mwh)| + \sum_{i=1}^{k-1} \left|{}_mc^*_{i2}\right| \Bigg\}, \end{aligned} \tag{46} $$

where $D^m = O_{(k-1)\times(k-1)}$ with $m \ge k-1$ are used.

From Proposition 4.2, we have

$$
\begin{aligned}
&|\cos(mwh)| + |-w\sin(mwh)| + \sum_{i=1}^{k-1} \left|{}_m c_{i1}^*\right| \\
&= |\cos(mwh)| + |-w\sin(mwh)| \\
&\quad + \sum_{i=1}^{k-1} \left| \frac{\cos((m+1-i)wh) - \cos((m-1-i)wh)}{2h} \right| \\
&= |\cos(mwh)| + |-w\sin(mwh)| + \sum_{i=1}^{k-1} \left| \frac{-\sin((m-i)wh)\sin(wh)}{h} \right| \\
&\leq 1 + w + \frac{1}{2} w^2 h \sum_{i=1}^{k-1} (m-i) \leq \bar{C}_1,
\end{aligned}
\tag{47}
$$

where $\bar{C}_1$ depends on w. Similarly, it follows that

$$
|\cos(wh)| + |\sin(mwh)/w| + \sum_{i=1}^{k-1} |{}_m c_{i2}^*| \leq \bar{C}_2. \tag{48}
$$

Due to (46)–(48), we have for $k-1 \leq m \leq N$

$$
\left\| L^m \right\|_1 \leq \bar{C}, \quad \text{with} \quad \bar{C} = \max\{\bar{C}_1, \bar{C}_2\}
$$

and hence, according to Definition 4.1, the method is stable.

In order to determine whether a numerical method will produce reasonable results with a given step length $h > 0$, we need another notion of stability that is different from zero-stability. In order to analyze the stability and phase property of the new methods in this paper, we consider the modified test equation [15]

$$
y''(t) + w^2 y(t) = -\epsilon y(t), \quad w^2 + \epsilon > 0, \tag{49}
$$

where w represents an estimate of the dominant frequency λ, and $\epsilon = \lambda^2 - w^2$ is the error of estimation.

The k-step adapted Falkner-type method (13) can also be expressed in the following form

$$
\begin{aligned}
u_{n+1} &= \phi_0(V) u_n + h\phi_1(V) u_n' \\
&\qquad + h^2 \big(\bar{b}_1(V) g_n + \bar{b}_2(V) g_{n-1} + \cdots + \bar{b}_k(V) g_{n-(k-1)}\big), \\
u_{n+1}' &= -hM\phi_1(V) u_n + \phi_0(V) u_n' \\
&\qquad + h\big(b_0(V) g_{n+1} + b_1(V) g_n + \cdots + b_k(V) g_{n-(k-1)}\big),
\end{aligned}
\tag{50}
$$

where $\bar{b}_i(V)$, $i = 1, \ldots, k$ and $b_i(V)$, $i = 0, \ldots, k$ depend on k. They can be formulated as follows

$$\bar{b}_i(V) = \sum_{j=i-1}^{k-1} \binom{j}{i-1} \beta_j(V)(-1)^{i-1}, \quad b_i(V) = \sum_{j=i}^{k} \binom{j}{i} \gamma_j^*(V)(-1)^i \tag{51}$$

which are shown in the Appendix C.

Letting $U_n = (u_{n+1}, u_n, \ldots, u_{n-(k-2)}, hu'_{n+1})^T$, the application of scheme (50) to (49) yields

$$A(V, z)U_n = B(V, z)U_{n-1},$$

where $V = w^2h^2$, $z = \epsilon h^2$,

$$A(V, z) = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ zb_0(V) & & & & 1 \end{pmatrix} \tag{52}$$

and

$$B(V, z) = \begin{pmatrix} \phi_0(V) - z\bar{b}_1(V) & -z\bar{b}_2(V) & \cdots & -z\bar{b}_{k-1}(V) & -z\bar{b}_k(V) & \phi_1(V) \\ 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & & 1 & \\ -V\phi_1(V) - zb_1(V) & -zb_2(V) & \cdots & -zb_{k-1}(V) & -zb_k(V) & \phi_0(V) \end{pmatrix}. \tag{53}$$

The matrix $M(V, z) = A(V, z)^{-1}B(V, z)$ is called the stability matrix. The behavior of the numerical solution will depend on the eigenvalues $r_i(V, z)$, $i = 1, \ldots, k+1$ of the stability matrix, and the stability property of the method will be characterized by the spectral radius $\rho(M)$, respectively.

Because the $M(V, z)$ depend on the variables V and z, geometrically, the characterization of stability becomes a two-dimensional region in the (V, z)-plane for an adapted Falkner-type method. According to the terminology introduced by Coleman and Ixaru [16], we have the following definitions of stability for a k step adapted Falkner-type integrator.

- $R_s = \{V > 0, z > 0 |\ |r_i(V, z)| < 1,\ i = 1, \ldots, k+1\}$ is called the *region of stability of a k-step adapted Falkner-type method.*
- $R_p = \{V > 0, z > 0 |, r_1(V, z) = e^{i\theta(V,z)}, r_2(V, z) = e^{-i\theta(V,z)}, |r_i(V, z)| \leq 1,\ i = 3, \ldots, k+1\}$ is called the *region of periodicity of a k-step adapted Falkner-type method.*

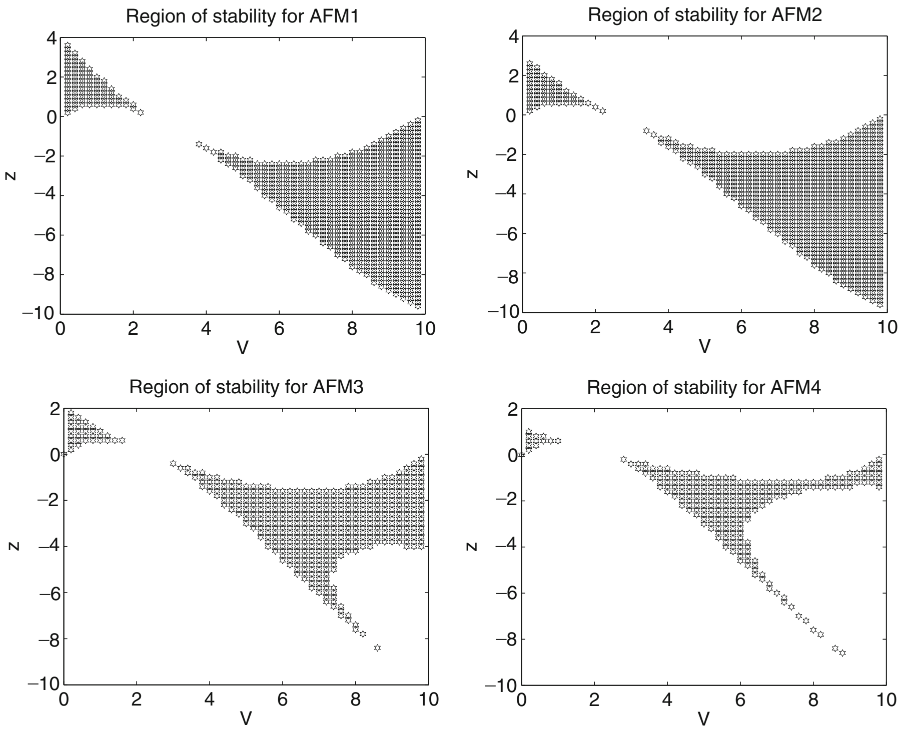


Fig. 1 The regions of stability of k-step adapted Falkner-type methods (AFMk) with $k = 1, 2, 3, 4$

The regions of stability for the k-step adapted Falkner-type methods with $k = 1, 2, 3, 4$ are depicted in Fig. 1.

Remark 4.1 We can observe that there exists a nonempty region of stability for 4-step adapted Falkner-type method (AFM4) whereas both the interval of stability and the interval of periodicity are empty for 4-step reformed Falkner method (RFM4) given in (6). This can explain why the global error obtained by RFM4 is very large when h is large to some extent in figures of Section 5. The paper [8] contains detailed explanation about this phenomenon.

5 Numerical experiments

In this section, we will illustrate our new methods with four model problems. The methods chosen for comparison are:

- RFMk: the k-step reformed Falkner methods (6) with $k = 1, 2, 3, 4$ given in [8];
- AFMk: the k-step adapted Falkner-type methods with $k = 1, 2, 3, 4$ given in this paper.

Problem 1 Two coupled oscillators with different frequencies, studied by Vigo-Aguiar et al. [17]

$$\begin{cases} y_1'' + y_1 = 2\varepsilon y_1 y_2, & y_1(0) = 1, \quad y_1'(0) = 0, \\ y_2'' + 2y_2 = \varepsilon y_1^2 + 4\varepsilon y_2^3, & y_2(0) = 1, \quad y_2'(0) = 0, \quad t \in [0, 1000]. \end{cases}$$

In our numerical test we choose $\varepsilon = 10^{-3}$. The system is integrated with the step-sizes $h = 1/(20j)$, $j = 1, 2, 3, 4$ to show the accuracy of the different numerical methods. The numerical results are presented in Fig. 2.

Problem 2 Consider a nonlinear wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{1}{5}u^3, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, \quad u(x,0) = \dfrac{\sin(\pi x)}{2}, \quad u_t(x,0) = 0. \end{cases}$$

By using second-order symmetric differences, this problem is converted into a system of ODEs in time

$$\begin{cases} \dfrac{d^2 u_i}{dt^2} - \dfrac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = -\dfrac{1}{5}u_i^3, \quad 0 < t \le t_{end}, \\ u_i(0) = \dfrac{\sin(\pi x_i)}{2}, \quad u_i'(0) = 0, \quad i = 1, \ldots, N-1, \end{cases}$$

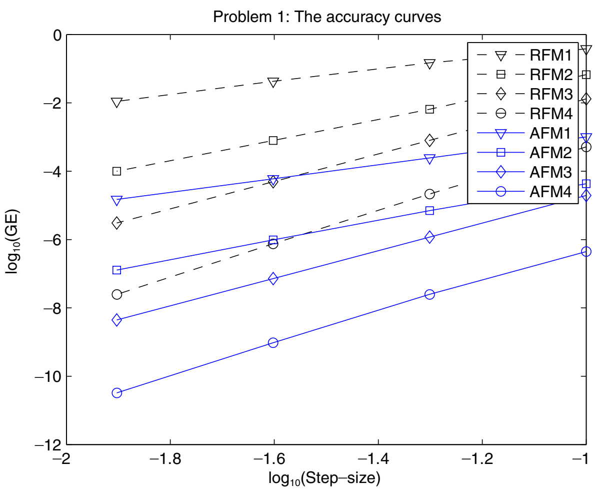


Fig. 2 Accuracy curves for Problem 1

where $\Delta x = 1/N$ is the spatial mesh step and $x_i = i\Delta x$. This semi-discrete oscillatory system has the form

$$\begin{cases} \dfrac{d^2U}{dt^2} + MU = F(t, U), \quad 0 < t \le t_{end}. \\ U(0) = \left(\dfrac{\sin(\pi x_1)}{2}, \ldots, \dfrac{\sin(\pi x_{N-1})}{2}\right)^T, \quad U'(0) = \mathbf{0}, \end{cases}$$

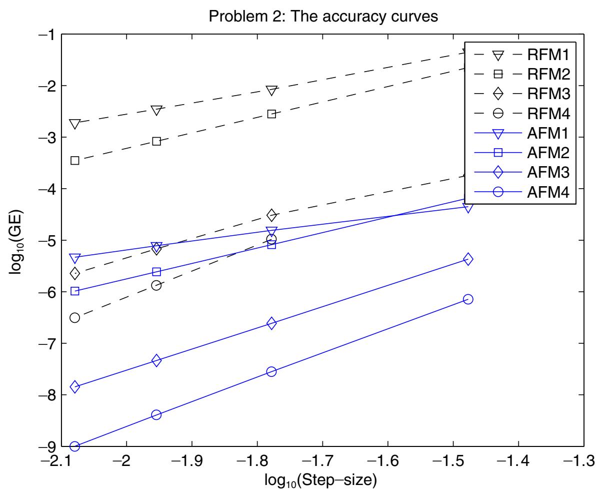
where $U(t) = (u_1(t), \ldots, u_{N-1}(t))^T$ with $u_i(t) \approx u(x_i, t),\ i = 1, \ldots, N-1$, and

$$M = \frac{1}{\Delta x^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}, \tag{54}$$

$$F(t, U) = F(t, U) = \left(-\frac{1}{5}u_1^3, \ldots, -\frac{1}{5}u_{N-1}^3\right)^T.$$

The system is integrated in the interval $t \in [0, 150]$ with $N = 20$ and the integration step sizes $h = 1/(30j),\ j = 1, 2, 3, 4$. The numerical results are presented in Fig. 3. In this experiment we note that the error $\log_{10}(GE)$ is very large for RFM4 with $h = 1/30$, hence we do not plot the points in Fig. 3. The same situations will be encountered in the following problems and we deal with them in a similar way. This is because both the interval of stability and the interval of periodicity are empty. Ramos explain this phenomenon with another example in [8].

Fig. 3 Accuracy curves for Problem 2



Problem 3 Consider the sine-Gordon equation with periodic boundary conditions

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} - \sin u, \quad -1 < x < 1, \ \ t > 0, \\ u(-1,t) = u(1,t). \end{cases}$$

We carry out a semi-discretization on the spatial variable by using second-order symmetric differences and obtain the following system of second-order ODEs in time

$$\frac{d^2U}{dt^2} + MU = F(t,U), \quad 0 < t \le t_{end},$$

where $U(t) = (u_1(t), \dots, u_N(t))^T$ with $u_i(t) \approx u(x_i, t), \ i = 1, \dots, N,$

$$M = \frac{1}{\Delta x^2} \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ -1 & & & -1 & 2 \end{pmatrix}$$

with $\Delta x = 2/N$ and $x_i = -1 + i\Delta x$, and $F(t,U) = -\sin(U) = -\big(\sin u_1, \dots, \sin u_N\big)^T$. We take the initial conditions as

$$U(0) = (\pi)_{i=1}^N, \quad U_t(0) = \sqrt{N}\left(0.01 + \sin\left(\frac{2\pi i}{N}\right)\right)_{i=1}^N \quad \text{with} \ \ N = 64.$$

The problem is integrated in the interval $[0, 10]$ with the step-sizes $h = 1/100, 1/150, 1/200, 1/250$ for RFM4 and $h = 1/50, 1/100, 1/150, 1/200$ for the other methods. Figure 4 shows that for the same step-size h, our methods are more accurate than the methods proposed in [8].

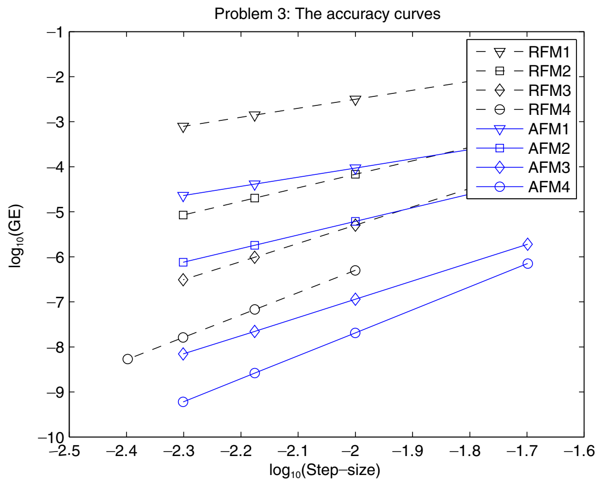
Problem 4 Consider a nonlinear wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} = u^5 - u^3 - 10u, \quad 0 < x < 1, \ \ t > 0, \\ u(0,t) = u(1,t) = 0, \quad u(x,0) = \dfrac{x(1-x)}{100}, \quad u_t(x,0) = 0. \end{cases}$$

By using second-order symmetric differences, this problem was converted into a system of ODEs in time

$$\begin{cases} \dfrac{d^2 u_i}{dt^2} - \dfrac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} = u_i^5 - u_i^3 - 10u_i, \quad 0 < t \le t_{end}, \\ u_i(0) = \dfrac{x_i(1-x_i)}{100}, \quad u_i'(0) = 0, \quad i = 1, \dots, N-1, \end{cases}$$

Fig. 4 Accuracy curves for Problem 3



where $\Delta x = 1/N$ is the spatial mesh step and $x_i = i\Delta x$. This semi-discrete oscillatory system has the form

$$\begin{cases} \dfrac{d^2U}{dt^2} + MU = F(t, U), \quad 0 < t \leq t_{end}. \\ U(0) = \left(\dfrac{x_1(1-x_1)}{100}, \ldots, \dfrac{x_{N-1}(1-x_{N-1})}{100}\right)^T, \quad U'(0) = \mathbf{0}, \end{cases}$$

where $U(t) = (u_1(t), \ldots, u_{N-1}(t))^T$ with $u_i(t) \approx u(x_i, t), i = 1, \ldots, N-1$, M is given by

$$M = \frac{1}{\Delta x^2}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \tag{55}$$

and

$$F(t, U) = \left(u_1^5 - u_1^3 - 10u_1, \ldots, u_{N-1}^5 - u_{N-1}^3 - 10u_{N-1})\right)^T.$$

The system is integrated in the interval $t \in [0, 100]$ with $N = 20$ and the step-sizes $h = 1/(30j)$, $j = 1, 2, 3, 4$. The numerical results are presented in Fig. 5.

The four figures show that for the same step-size h, our methods are more accurate than the methods proposed in [8]. About this point, some comments are listed below.

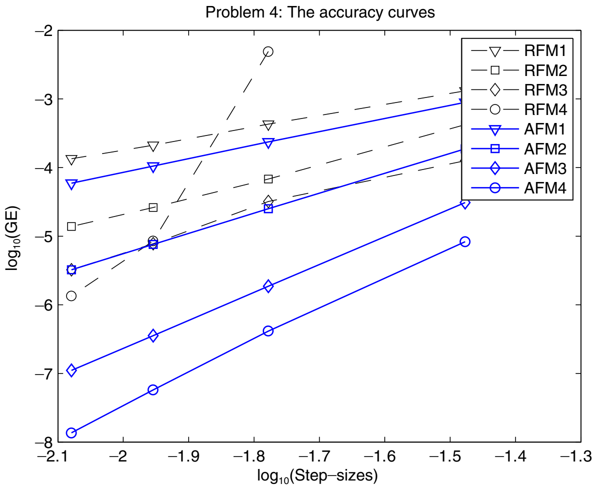


Fig. 5 Accuracy curves for Problem 4

Remark 5.1 For adapted Falkner methods, at the beginning of the implementation, the matrices $\phi_0(V)$, $\phi_1(V)$, $\beta_j(V)$ and $\gamma_j^*(V)$ need to be evaluated with Horner's method then can be used repeatedly in the sequel calculations.

Remark 5.2 At each time step, it can be observed that for adapted Falkner methods, the evaluations of matrix $\times$ vector multiplication are more than the reformed Falkner methods. Therefore we show the accuracy versus the CPU time to show that our methods are more efficient than the methods proposed in [8]. Let's take RFM4 and AFM4 as examples. Applying them to Problem 1, we present the numerical results in Table 1. Table 1 shows that AFM4 needs less CPU time than RFM4 for achieving the same accuracy and so AFM4 is more efficient than RFM4. For the other problems, it is noted that the matrices by using second-order symmetric differences based on the method of lines are sparse, in fact, tridiagonal, and we obtain the same conclusions.

Table 1 Numerical results of RFM4 and AFM4 for the Problem 1 in [0, 1000] with different step sizes $h = 1/(10 \cdot j)$

j	RFM4		AFM4	
	Max-error	CPU-time (s)	Max-error	CPU-time (s)
1	5.1453×10^{-4}	0.296	4.4302×10^{-7}	0.641
2	2.1465×10^{-5}	0.625	2.5085×10^{-8}	1.297
3	3.0554×10^{-6}	0.922	3.7912×10^{-9}	1.921
4	7.5206×10^{-7}	1.219	9.5870×10^{-10}	2.594
5	2.5173×10^{-7}	1.531	3.2614×10^{-10}	3.235
6	1.0258×10^{-7}	1.860	1.3558×10^{-10}	3.859
7	4.7926×10^{-8}	2.140	6.3775×10^{-11}	4.500
8	2.4760×10^{-8}	2.453	3.3389×10^{-11}	5.156

The numerical results in Table 1 are executed on the computer lenovo M6600 (Inter(R) Pentium(R) CPU 3.00 GHz, 0.99 G).

In our numerical experiments, for each initial value problem, we take the numerical solution obtained by the classical four-stage RKN method of order five [18] with small step size as the exact solution.

6 Conclusions

The oscillatory problems (1) have constituted a very important category of differential equations in scientific computing. New approaches to dealing with (1) have been proposed in recent years, such as exponential fitting modified Runge–Kutta–Nyström schemes, ARKN methods and ERKN integrators, we refer the reader to [19–23]. In this paper, we study adapted Falkner-type methods and give a rigorous error analysis. For a k-step adapted Falkner-type method, we give the error bounds $\|u_n - u(t_n)\| \leq Ch^{k+1}$ and $\left\|u'_n - u'(t_n)\right\| \leq C'h^{k+1}$, where C is independent of $\|M\|$, h and n whereas C' is independent of h and n. We also give a stability analysis and present the regions of stability for our new methods. Numerical examples confirm that the bounds are realistic and the new methods are more efficient than the reformed Falkner methods given in the paper [8].

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Appendix A

The expression (14) can be derived as follows:

$$
\begin{aligned}
G_\beta(t, V) &= \sum_{j=0}^{\infty} \beta_j(V) t^j \\
&= \sum_{j=0}^{\infty} (-1)^j \int_0^1 (1-z)\phi_1((1-z)^2 V) \binom{-z}{j} dz \cdot t^j \\
&= \int_0^1 (1-z)\phi_1((1-z)^2 V) \sum_{j=0}^{\infty} (-t)^j \binom{-z}{j} dz \\
&= \int_0^1 (1-z)\phi_1((1-z)^2 V)(1-t)^{-z} dz \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k V^k}{(2k+1)!} \int_0^1 (1-z)^{2k+1}(1-t)^{-z} dz.
\end{aligned}
$$

Using integration by parts gives

$$\int_0^1 (1-z)(1-t)^{-z}dz = \frac{1}{\ln(1-t)} + \frac{t}{(1-t)\ln^2(1-t)}. \tag{56}$$

Furthermore, for $k \geq 1$ we have

$$\int_0^1 (1-z)^{2k+1}(1-t)^{-z}dz = \frac{1}{\ln(1-t)} - \frac{2k+1}{\ln^2(1-t)} + \frac{(2k+1)2k}{\ln^2(1-t)} \int_0^1 (1-z)^{2k-1}(1-t)^{-z}dz. \tag{57}$$

The above analysis results in

$$\begin{aligned} G_\beta(t,V) &= \sum_{k=0}^{\infty} \frac{(-1)^k V^k}{(2k+1)!} \int_0^1 (1-z)^{2k+1}(1-t)^{-z}dz \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k V^k}{(2k+1)!} \cdot \frac{1}{\ln(1-t)} \\ &\quad + \left(\frac{t}{(1-t)\ln^2(1-t)} I + \sum_{k=1}^{\infty} \frac{(-1)^k V^k}{(2k+1)!} \left(-\frac{2k+1}{\ln^2(1-t)} \right) \right) \\ &\quad + \left(\sum_{k=1}^{\infty} \frac{(-1)^k V^k}{(2k+1)!} \frac{(2k+1)2k}{\ln^2(1-t)} \int_0^1 (1-z)^{2k-1}(1-t)^{-z}dz \right), \end{aligned} \tag{58}$$

where

$$\sum_{k=0}^{\infty} \frac{(-1)^k V^k}{(2k+1)!} \cdot \frac{1}{\ln(1-t)} = \frac{1}{\ln(1-t)} \phi_1(V), \tag{59}$$

$$\begin{aligned} &\frac{t}{(1-t)\ln^2(1-t)} I + \sum_{k=1}^{\infty} \frac{(-1)^k V^k}{(2k+1)!} \left(-\frac{2k+1}{\ln^2(1-t)} \right) \\ &\qquad = \frac{t}{(1-t)\ln^2(1-t)} I + \sum_{k=0}^{\infty} \frac{(-1)^k V^k}{(2k+1)!} \left(-\frac{2k+1}{\ln^2(1-t)} \right) \\ &\qquad\quad + \frac{1}{\ln^2(1-t)} I \\ &\qquad = \frac{1}{(1-t)\ln^2(1-t)} I - \frac{\phi_0(V)}{\ln^2(1-t)}, \end{aligned} \tag{60}$$

$$\sum_{k=1}^{\infty} \frac{(-1)^k V^k}{(2k+1)!} \frac{(2k+1)2k}{\ln^2(1-t)} \int_0^1 (1-z)^{2k-1}(1-t)^{-z} dz$$

$$= \frac{-V}{\ln^2(1-t)} \int_0^1 (1-z)\phi_1((1-z)^2 V)(1-t)^{-z} dz$$

$$= \frac{-V}{\ln^2(1-t)} G_\beta(t, V). \tag{61}$$

Inserting the expressions (59)–(61) into (58) yields

$$G_\beta(t, V) = \frac{1}{\ln(1-t)}\phi_1(V) + \frac{1}{(1-t)\ln^2(1-t)} I - \frac{\phi_0(V)}{\ln^2(1-t)}$$

$$+ \frac{-V}{\ln^2(1-t)} G_\beta(t, V). \tag{62}$$

The generating functions $G_\beta(t, V)$ can be derived from formula (62) straightforwardly.

The expression (15) can be obtained in a similar way.

Appendix B

We derive (25) by induction. $m = 1$ is trivial. Assuming that (25) holds for $m = k$, then we obtain

$$Q^{k+1} = Q^k Q = \begin{pmatrix} \phi_0(k^2 V) & kh\phi_1(k^2 V) \\ -khM\phi_1(k^2 V) & \phi_0(k^2 V) \end{pmatrix} \begin{pmatrix} \phi_0(V) & h\phi_1(V) \\ -hM\phi_1(V) & \phi_0(V) \end{pmatrix}$$

$$= \begin{pmatrix} \phi_0((k+1)^2 V) & (k+1)h\phi_1((k+1)^2 V) \\ -(k+1)hM\phi_1((k+1)^2 V) & \phi_0((k+1)^2 V) \end{pmatrix}.$$

The last equality is obtained by using the following propositions:

$$c\phi_1(c^2 V)\phi_0(V) + \phi_1(V)\phi_0(c^2 V) = (1+c)\phi_1((1+c)^2 V),$$

$$\phi_0(c^2 V)\phi_0(V) - cV\phi_1(V)\phi_1(c^2 V) = \phi_0((1+c)^2 V). \tag{63}$$

Now we prove the above propositions as follows

$$
\begin{aligned}
& c\phi_1(c^2V)\phi_0(V)+\phi_1(V)\phi_0(c^2V)\\
&= c\sum_{k=0}^{\infty}\frac{c^{2k}(-1)^kV^k}{(2k+1)!}\sum_{j=0}^{\infty}\frac{(-1)^jV^j}{(2j)!}+\sum_{k=0}^{\infty}\frac{(-1)^kV^k}{(2k+1)!}\sum_{j=0}^{\infty}\frac{c^{2j}(-1)^jV^j}{(2j)!}\\
&= \sum_{p=0}^{\infty}\Bigg(\sum_{k=0}^{p}\frac{c^{2k+1}(-1)^kV^k}{(2k+1)!}\cdot\frac{(-1)^{p-k}V^{p-k}}{(2(p-k))!}\\
&\qquad +\sum_{k=0}^{p}\frac{(-1)^kV^k}{(2k+1)!}\cdot\frac{c^{2(p-k)}(-1)^{p-k}V^{p-k}}{(2(p-k))!}\Bigg)\\
&= \sum_{p=0}^{\infty}\left(\sum_{k=0}^{p}\frac{c^{2k+1}}{(2k+1)!(2p-2k)!}+\sum_{k=0}^{p}\frac{c^{2(p-k)}}{(2k+1)!(2p-2k)!}\right)(-1)^pV^p\\
&= \sum_{p=0}^{\infty}\left(\sum_{k=0}^{p}\frac{c^{2k+1}}{(2k+1)!(2p-2k)!}+\sum_{q=0}^{p}\frac{c^{2q}}{(2p-2q+1)!(2q)!}\right)(-1)^pV^p\\
&= \sum_{p=0}^{\infty}\left(\sum_{k=0}^{2p+1}\frac{c^k}{k!(2p+1-k)!}\right)(-1)^pV^p=\sum_{p=0}^{\infty}\frac{(1+c)^{2p+1}}{(2p+1)!}(-1)^pV^p\\
&= (1+c)\phi_1((1+c)^2V).
\end{aligned}
$$

The second formula in (63) can be obtained in a similar way.

Appendix C

The proof of (51)

$$\sum_{l=0}^{k-1}\bar{b}_{l+1}(V)f_{n-l}=\sum_{j=0}^{k-1}\beta_j(V)\nabla^jf_n=\sum_{j=0}^{k-1}\beta_j(V)\sum_{l=0}^{j}\binom{j}{l}(-1)^lf_{n-l}$$

results in

$$\bar{b}_{l+1}=\sum_{j=l}^{k-1}\beta_j(V)\binom{j}{l}(-1)^l.$$

Replacing l with $i-1$, we obtain the first formula of (51). The second one can be obtained in a similar way. □

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