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The error norm of quadrature formulae

Sotirios E. Notaris

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Abstract In certain spaces of analytic functions the error term of a quadrature formula is a bounded linear functional. We give a survey of the methods used in order to compute explicitly, or in some cases estimate, the norm of the error functional. The results, some of which are fairly recent, cover Gauss, Gauss–Lobatto, Gauss–Radau, Gauss–Kronrod and Fejér type rules.

Keywords Quadrature formulae · Error norm

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1 Introduction

The most common method for estimating the error of a quadrature formula is by means of a high-order derivative of the function involved. The disadvantages of such an approach are too well-known to be analyzed here. Derivative-free error estimates can be obtained by either contour integration or Hilbert space methods.

Contour integration techniques appeared for the first time in a 1932 paper by Fock (cf. [9]) for estimating the error of Gaussian quadrature rules, although the same idea had been previously used by Hermite in 1878 (cf. [21]), and Heine in 1881 (cf. [20, p. 16]), for estimating the error in polynomial interpolation; the latter, when integrated, leads to error estimates for

S. E. Notaris (✉)
Department of Mathematics, University of Athens, Panepistemiopolis,
15784 Athens, Greece
e-mail: notaris@math.uoa.gr

interpolatory quadrature rules. The method is fairly straightforward; using Cauchy's theorem, the error term $R_n(f)$ of a quadrature formula can be estimated by

$$|R_n(f)| \leq \frac{l(\Gamma)}{2\pi} \max_{z\in\Gamma} |K_n(z)| \max_{z\in\Gamma} |f(z)|,$$

assuming that f is analytic in a domain D, Γ is a contour in D surrounding the interval of integration, $l(\Gamma)$ is the length of Γ, and $K_n(z)$ is the so-called kernel (for further details see the end of Section 2). An apparent advantage of this method is that the above estimate can be optimized by appropriately choosing the contour Γ; contours most frequently used are either concentric circles or confocal ellipses; the latter shrink to the interval of integration as the sum of the semiaxes approaches 1, and this makes them a preferred choice for functions having a pole in the vicinity of the interval of integration. All this gave rise to a rich literature; for the earlier work on the subject one can look at [8, Section 4.6], [10, Section 4.1.1] and [13], while more recent results can be found in [23–39, 46, 48–56].

Hilbert space methods were first proposed by Davis in 1953 (see [6]), who considered the error term $R_n(f)$ of a quadrature formula as a bounded linear functional in an appropriate Hilbert space H of analytic functions f. Then one immediately obtains

$$|R_n(f)| \leq \|R_n\| \, \|f\|, \tag{1.1}$$

where $\|R_n\|$ is the norm of the error functional R_n and $\|f\|$ is the norm of f in the Hilbert space H. This approach has a number of advantages. For one thing, it is quite sharp as equality can be obtained in (1.1) for some $f \in H$. Moreover, the nice separation of (1.1) between the influence of the quadrature formula (expressed by $\|R_n\|$) and of the function to which it is applied (expressed by $\|f\|$) allows to compare estimates coming from different quadrature rules. Of course, all these depend on the ability to compute not only the norm of f, but also the norm of R_n. Regarding the latter, if $\{p_k\}$ is a complete orthonormal system in H, then

$$\|R_n\|^2 = \sum_{k=0}^{\infty} |R_n(p_k)|^2 .$$

The idea of Davis was soon followed by work of Davis and Rabinowitz [7], Yanagiwara [57] and Hämmerlin [16–18] with an aim towards obtaining bounds of $\|R_n\|$ for specific quadrature rules and Hilbert spaces H. This inspired many authors to try minimizing $\|R_n\|$ over certain classes of quadrature rules. Extensive reviews on the subject can be found in [8, Section 4.7], [10, Section 4.1.2] and [13].

It is obvious that estimate (1.1) is sharper whenever $\|R_n\|$ can be computed explicitly in an appropriate Hilbert space H. The inspiration was given by Hämmerlin in [19], where he defined a seminormed linear space in order to estimate the error of the Gauss formula for the Legendre weight function. In the present paper, we review error estimates of type (1.1) where $\|R_n\|$ can be computed explicitly. Our estimates concern Gauss, Gauss–Lobatto, Gauss–Radau, Gauss–Kronrod and interpolatory quadrature rules.

2 The norm of the error functional

We consider the quadrature formula

$$\int_{-1}^{1} f(t)w(t)dt = \sum_{\nu=1}^{n} w_\nu f(\tau_\nu) + R_n(f), \tag{2.1}$$

where w is a nonnegative weight function, assumed to be integrable over $[-1, 1]$, the τ_ν are certain distinct nodes in $[-1, 1]$, ordered decreasingly, and the w_ν are the corresponding weights.

An interesting method for obtaining derivative-free error estimates was suggested by Hämmerlin in [19]. Let f be a holomorphic function in $C_r = \{z \in \mathbb{C} : |z| < r\},\ r > 1$. Then f can be written as

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in C_r. \tag{2.2}$$

Define

$$|f|_r = \sup\left\{|a_k| r^k : k \in \mathbb{N}_0 \text{ and } R_n\left(t^k\right) \neq 0\right\}. \tag{2.3}$$

Then $|\cdot|_r$ is a seminorm in the space

$$X_r = \{f : f \text{ holomorphic in } C_r \text{ and } |f|_r < \infty\}.$$

As $f \in C[-1, 1]$, we have, from (2.1),

$$|R_n(f)| \leq \left(\|w\|_1 + \sum_{\nu=1}^{n} |w_\nu|\right) \|f\|_\infty, \tag{2.4}$$

where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the L_1 and L_∞ norm of a function, respectively; hence, R_n is a bounded and, equivalently, continuous linear functional on $(C[-1, 1],\ \|\cdot\|_\infty)$. The continuity of R_n, together with the uniform convergence of the series in (2.2) on $[-1, 1]$, implies

$$R_n(f) = \sum_{k=0}^{\infty} a_k R_n\left(t^k\right),$$

which, by virtue of (2.3), gives

$$|R_n(f)| \leq \left[\sum_{k=0}^{\infty} \frac{\left|R_n\left(t^k\right)\right|}{r^k}\right] |f|_r. \tag{2.5}$$

As, from (2.4), $|R_n(t^k)| \leq \|w\|_1 + \sum_{\nu=1}^{n} |w_\nu|$, the series in (2.5) is converging, and therefore R_n is a bounded linear functional on $(X_r, |\cdot|_r)$ with norm $\|R_n\|$. Consequently,

$$|R_n(f)| \leq \|R_n\| |f|_r, \tag{2.6}$$

where, from (2.5),

$$\|R_n\| \leq \sum_{k=0}^{\infty} \frac{|R_n(t^k)|}{r^k}. \tag{2.7}$$

Furthermore, for $\phi(z) = \sum_{k=0}^{\infty} \operatorname{sign}(R_n(t^k)) \dfrac{z^k}{r^k}$, hence, $|\phi|_r = 1$, we have

$$R_n(\phi) = \sum_{k=0}^{\infty} \frac{\operatorname{sign}\left(R_n\left(t^k\right)\right) R_n\left(t^k\right)}{r^k},$$

that is,

$$|R_n(\phi)| = \left[\sum_{k=0}^{\infty} \frac{|R_n(t^k)|}{r^k}\right] |\phi|_r,$$

which, combined with (2.7), implies

$$\|R_n\| = \sum_{k=0}^{\infty} \frac{|R_n(t^k)|}{r^k}. \tag{2.8}$$

The computation of the seminorm $|f|_r$ requires the knowledge of the coefficients a_k, $k \geq 0$ (cf. (2.2)), which are not always available, hence $|f|_r$ often has to be estimated. If f belongs to the Hardy space H_2,

$$H_2 = \left\{ f : f \text{ holomorphic in } C_r \text{ and } \|f\|_{2,r} = \left(\int_{|z|=r} |f(z)|^2 |dz| \right)^{1/2} < \infty \right\},$$

then the polynomials $p_k(z) = \dfrac{z^k}{r^k \sqrt{2\pi r}}$, $k = 0, 1, 2, \ldots,$ form a complete orthonormal system in H_2, thus, from Parseval's identity, we have

$$\|f\|_{2,r} = \sqrt{2\pi r} \left(\sum_{k=0}^{\infty} |a_k|^2 r^{2k} \right)^{1/2},$$

and given that

$$\begin{aligned} \sum_{k=0}^{\infty} |a_k|^2 r^{2k} &\geq \sup\left\{|a_k|^2 r^{2k} : k \in \mathbb{N}_0\right\} \\ &\geq \sup\left\{|a_k|^2 r^{2k} : k \in \mathbb{N}_0 \text{ and } R_n\left(t^k\right) \neq 0\right\} = |f|_r^2, \end{aligned}$$

we get

$$|f|_r \le \frac{1}{\sqrt{2\pi r}} \|f\|_{2,r}.$$

Also, from the latter and the definition of $\|f\|_{2,r}$, there follows that

$$|f|_r \le \max_{|z|=r} |f(z)| \tag{2.9}$$

(cf. [19, Section 4] or [1, Section 1.1]).

Although formula (2.8) is useful for obtaining an estimate for $\|R_n\|$ (see [19, 41]), it cannot be used for computing $\|R_n\|$ explicitly. A practical representation for $\|R_n\|$ can be derived if we have some information on the sign of $R_n(t^k)$, $k \ge 0$; and the representation becomes particularly useful if the quadrature formula (2.1) is of interpolatory type. All this was presented for the first time by Akrivis in [1, Section 1.2] and is summarized in Theorem 2.1.

Formula (2.1) is called interpolatory if it integrates exactly all polynomials of degree up to (at least) $n-1$, i.e., $R_n(f) = 0$ for all $f \in \mathbb{P}_{n-1}$. Many well-known formulae are of interpolatory type, among them the Gauss, Gauss–Lobatto, Gauss–Radau and Gauss–Kronrod rules as well as the Fejér rule of the first or second kind (also known as Pólya and Filippi rule, respectively), and the Basu and Clenshaw–Curtis rules.

Theorem 2.1 *Consider the quadrature formula* (2.1)*. Let* $\pi_n(t) = \Pi_{\nu=1}^n (t - \tau_\nu)$ *and* $\epsilon \in \{-1, 1\}$.

(a) *If* $\epsilon R_n(t^k) \ge 0$, $k \ge 0$*, then*

$$\|R_n\| = r \left| R_n\left(\frac{1}{r-t}\right)\right|. \tag{2.10}$$

If, in addition, formula (2.1) *is interpolatory, then*

$$\|R_n\| = r \left| \frac{1}{\pi_n(r)} \int_{-1}^{1} \frac{\pi_n(t)}{r-t} w(t)dt \right|. \tag{2.11}$$

(b) *If* $\epsilon(-1)^k R_n(t^k) \ge 0$, $k \ge 0$*, then*

$$\|R_n\| = r \left| R_n\left(\frac{1}{r+t}\right)\right|. \tag{2.12}$$

If, in addition, formula (2.1) *is interpolatory, then*

$$\|R_n\| = r \left| \frac{1}{\pi_n(-r)} \int_{-1}^{1} \frac{\pi_n(t)}{r+t} w(t)dt \right|. \tag{2.13}$$

Proof

(a) From (2.8), we have, in view of $\epsilon R_n(t^k) \geq 0,\ k \geq 0$, and the continuity of R_n on $(C[-1, 1], \|\cdot\|_\infty)$,

$$\|R_n\| = \sum_{k=0}^{\infty} \frac{|\epsilon R_n(t^k)|}{|\epsilon| r^k} = \sum_{k=0}^{\infty} \frac{\epsilon R_n(t^k)}{r^k}$$

$$= \left|\sum_{k=0}^{\infty} \frac{\epsilon R_n(t^k)}{r^k}\right| = |\epsilon| \left|\sum_{k=0}^{\infty} \frac{R_n(t^k)}{r^k}\right| = \left|\sum_{k=0}^{\infty} R_n\left(\left(\frac{t}{r}\right)^k\right)\right|$$

$$= \left|R_n\left(\sum_{k=0}^{\infty}\left(\frac{t}{r}\right)^k\right)\right| = \left|R_n\left(\frac{1}{1-t/r}\right)\right| = r\left|R_n\left(\frac{1}{r-t}\right)\right|.$$

If formula (2.1) is interpolatory, then, letting p_{n-1} to be the polynomial of degree at most $n-1$ interpolating the function $1/(r-t)$ at the points $\tau_1, \tau_2, \ldots, \tau_n$, we have

$$\frac{1}{r-t} - p_{n-1}(t) = \frac{1-(r-t)p_{n-1}(t)}{r-t}. \tag{2.14}$$

Now, as the left-hand side vanishes at the interpolating points, the $\tau_1, \tau_2, \ldots, \tau_n$ must be zeros of the numerator on the right-hand side, and as this is a polynomial of degree at most n, we get

$$1-(r-t)p_{n-1}(t) = c_n \pi_n(t). \tag{2.15}$$

If we write p_{n-1} in the Lagrange form of the interpolating polynomial and then equalize the coefficients of t^n on both sides of (2.15) or set $t = r$ in (2.15), we find

$$c_n = \frac{1}{\pi_n(r)},$$

which, inserted into (2.15), gives, together with (2.14),

$$\frac{1}{r-t} - p_{n-1}(t) = \frac{1}{\pi_n(r)} \frac{\pi_n(t)}{r-t}. \tag{2.16}$$

Now, if we integrate (2.16) with respect to the weight function w on $[-1, 1]$, we get

$$R_n\left(\frac{1}{r-t}\right) = \frac{1}{\pi_n(r)} \int_{-1}^{1} \frac{\pi_n(t)}{r-t} w(t)dt,$$

which, inserted into (2.10), implies (2.11).

(b) The proof is similar to that of part (a) using the condition $\epsilon(-1)^k R_n(t^k) \geq 0,\ k \geq 0$, and the function $1/(r+t)$ instead of $\epsilon R_n(t^k) \geq 0,\ k \geq 0$, and $1/(r-t)$, respectively. □

Incidentally, estimate (2.6) with $|f|_r$ bounded by (2.9) can be obtained by a contour integration method (cf. [13]). If f is a single-valued holomorphic function in a domain D containing $[-1, 1]$ in its interior, and Γ is a contour in D surrounding $[-1, 1]$, then applying the error term $R_n(f)$, viewed as a linear functional, on Cauchy's formula

$$f(t) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{z-t} dz, \quad t \in [-1, 1],$$

we get the representation

$$R_n(f) = \frac{1}{2\pi i} \oint_\Gamma K_n(z) f(z) dz, \tag{2.17}$$

where the function $K_n(z)$, referred to as the kernel, is given by

$$K_n(z) = R_n\left(\frac{1}{z-\cdot}\right).$$

From (2.17), there immediately follows

$$|R_n(f)| \leq \frac{l(\Gamma)}{2\pi} \max_{z\in\Gamma} |K_n(z)| \max_{z\in\Gamma} |f(z)|, \tag{2.18}$$

where $l(\Gamma)$ denotes the length of Γ. Now, taking $\Gamma = \partial C_r = \{z \in \mathbb{C} : |z| = r\}$, $r > 1$, it can be shown, using arguments similar to those of Theorem 2.1, that

$$\max_{|z|=r} |K_n(z)| = \begin{cases} |K_n(r)| & \text{if } \epsilon R_n(t^k) \geq 0, \ k \geq 0, \\ |K_n(-r)| & \text{if } \epsilon(-1)^k R_n(t^k) \geq 0, \ k \geq 0 \end{cases}$$

$$= \sum_{k=0}^{\infty} \frac{|R_n(t^k)|}{r^{k+1}} = \frac{\|R_n\|}{r}$$

(cf. (2.8)), and given that $l(\partial C_r) = 2\pi r$, (2.18) gives

$$|R_n(f)| \leq \|R_n\| \max_{|z|=r} |f(z)|. \tag{2.19}$$

3 Computation of the error norm

The estimation or computation of $\|R_n\|$ in $(X_r, |\cdot|_r)$ by means of formula (2.8) or Theorem 2.1 began with the work of Hämmerlin in 1972 (cf. [19]) for estimating $\|R_n\|$ in the case of the Gauss formula for the Legendre weight function $w(t) = 1$, $-1 \leq t \leq 1$. Starting from (2.8) and estimating effectively $|R_n(t^{2k})|$, $k \geq n$ (by symmetry, $R_n(t^{2k-1}) = 0$, $k \geq 1$), he obtained

$$\|R_n\| \leq \frac{2^{2n-1}(n!)^4}{n(2n+1)[(2n)!]^2} \left[\frac{r}{(r-1)^{2n}} - \frac{r}{(r+1)^{2n}}\right].$$

This was continued a few years later by Akrivis (cf. [1, 2]). He obtained estimates for and examined the asymptotic behavior as $r \to 1^+$ of $\|R_n\|$ for symmetric quadrature formulae, such as the Gauss and Gauss–Lobatto rules for the Gegenbauer weight function $w(t) = (1 - t^2)^\alpha,\ \alpha > -1,\ -1 < t < 1$, paying particular attention to the special cases $\alpha = 0$ (Legendre weight) and $\alpha = \mp 1/2$ (Chebyshev weights of the first or second kind), or the Filippi and Clenshaw–Curtis rules. Subsequently, all this was extended (cf. [1, 5]) to nonsymmetric quadrature formulae, such as the Gauss rule for the Jacobi weight function $w(t) = (1 - t)^\alpha (1 + t)^\beta,\ \alpha, \beta > -1,\ -1 < t < 1$, paying particular attention to the case $\alpha = -\beta = \mp 1/2$ (Chebyshev weights of the third or fourth kind), or the Gauss–Radau rule for the Legendre weight function. The methods used are based on expressions of the form $\|R_n\| = r|R_n(\phi)|$ with an appropriate ϕ (cf. (2.10) and (2.12)), and then utilizing either the best approximation of ϕ in $\mathbb{P}_d$, where d is the degree of exactness of the quadrature formula in question, or the expansion of ϕ in terms of Chebyshev polynomials of any one of the four kinds.

In what follows, we mainly concentrate on quadrature formulae for which $\|R_n\|$ can be computed explicitly by means of (2.10)–(2.13).

3.1 The error norm of Gaussian rules

If the quadrature formula (2.1) is the Gauss rule for the weight function w on $[-1, 1]$, i.e., τ_ν are the zeros of the nth-degree (monic) orthogonal polynomial $\pi_n(\cdot; w)$, then there exists the following important result of Gautschi (see [11] or [13, Lemma 4.1]).

Lemma 3.1

(a) *If* $w(t)/w(-t)$ *is nondecreasing on* $(-1, 1)$*, then* $R_n(t^k) \geq 0,\ k \geq 0$.
(b) *If* $w(t)/w(-t)$ *is nonincreasing on* $(-1, 1)$*, then* $(-1)^k R_n(t^k) \geq 0,\ k \geq 0$.

The proof of Lemma 3.1 makes use of an interesting result of Hunter (cf. [22]).

In case that $w(t)/w(-t)$ is constant, then $w(t)/w(-t) = 1$, i.e., w is an even function, and, by symmetry, $R_n(t^k) = 0$ for all k odd, hence, both cases of the lemma hold simultaneously.

Obviously, Lemma 3.1 can be used in conjunction with Theorem 2.1 in order to compute $\|R_n\|$. First of all, for the Jacobi weight function $w(t) = (1 - t)^\alpha (1 + t)^\beta,\ \alpha, \beta > -1,\ -1 < t < 1$, we have

$$\frac{w(t)}{w(-t)} = \left(\frac{1+t}{1-t}\right)^{\beta - \alpha},$$

which, as it can easily be seen, is increasing on $(-1, 1)$ if $\alpha < \beta$ and decreasing if $\alpha > \beta$.

A special case of the Jacobi weight function are the Chebyshev weights of any one of the four kinds

$$w^{(1)}(t) = (1-t^2)^{-1/2}, \quad w^{(2)}(t) = (1-t^2)^{1/2}, \quad -1 < t < 1, \tag{3.1}$$

$$w^{(3)}(t) = (1-t)^{-1/2}(1+t)^{1/2}, \quad w^{(4)}(t) = (1-t)^{1/2}(1+t)^{-1/2}, \quad -1 < t < 1. \tag{3.2}$$

The first two are even functions, so, according to what was said before, $w^{(1)}$, $w^{(2)}$ and $w^{(3)}$ satisfy part (a) of Lemma 3.1, while $w^{(4)}$ satisfies part (b). Then $\|R_n\|$ can be computed by means of (2.11) and (2.13), respectively. The following results appeared first in [1, 2, 5].

Theorem 3.2 *Consider the Gauss formula (2.1), and let* $\tau = r - \sqrt{r^2-1}$.

(a) *For* $w = w^{(1)}$, *we have*

$$\|R_n^{(1)}\| = \frac{2\pi r \tau^{2n}}{(1+\tau^{2n})\sqrt{r^2-1}}, \quad n \geq 1. \tag{3.3}$$

(b) *For* $w = w^{(2)}$, *we have*

$$\|R_n^{(2)}\| = \frac{2\pi r \tau^{2n+2}\sqrt{r^2-1}}{1-\tau^{2n+2}}, \quad n \geq 1. \tag{3.4}$$

(c) *For* $w = w^{(3)}$ *or* $w = w^{(4)}$, *we have*

$$\|R_n^{(3)}\| = \frac{2\pi r \tau^{2n+1}}{1+\tau^{2n+1}}\sqrt{\frac{r+1}{r-1}}, \quad n \geq 1, \tag{3.5}$$

and $\|R_n^{(4)}\|$ *is given by the same formula (3.5).*

Proof

(a) Applying (2.11) with $w = w^{(1)}$, we have

$$\|R_n^{(1)}\| = \frac{r}{T_n(r)} \int_{-1}^{1} \frac{T_n(t)}{r-t}(1-t^2)^{-1/2}dt, \tag{3.6}$$

where T_n is the nth-degree Chebyshev polynomial of the first kind. Setting $t = \cos\theta$ in the integral on the right-hand side of (3.6), and using the well-known trigonometric representation $T_n(\cos\theta) = \cos n\theta$, we get

$$\int_{-1}^{1} \frac{T_n(t)}{r-t}(1-t^2)^{-1/2}dt = \int_0^{\pi} \frac{\cos n\theta}{r-\cos\theta}d\theta$$

$$= \frac{\pi\tau^n}{\sqrt{r^2-1}}, \quad n = 0, 1, 2, \ldots \tag{3.7}$$

(cf. [15, Equation 3.613.1 with $a = -1/r$]). Also,

$$T_n(r) = \frac{(r-\sqrt{r^2-1})^n + (r+\sqrt{r^2-1})^n}{2}$$
$$= \frac{1+\tau^{2n}}{2\tau^n}, \quad n = 0, 1, 2, \ldots . \tag{3.8}$$

(cf. [47, p. 5]), which inserted, together with (3.7), into (3.6), yields (3.3).

(b)–(c) The proof of (3.4) and (3.5) follows similarly to that of (3.3), using the trigonometric representations for the nth-degree Chebyshev polynomials of the second and third kind U_n and V_n, formula

$$\int_0^\pi \frac{\cos n\theta}{r+\cos\theta} d\theta = \frac{(-1)^n \pi \tau^n}{\sqrt{r^2-1}}, \quad n = 0, 1, 2, \ldots$$

(cf. (3.7) and set $\pi - \theta$ in place of θ), and expressions analogous to (3.8) for U_n and V_n.

Also, as $w^{(4)}(t) = w^{(3)}(-t)$, from (2.1) and (2.8), it is easy to see that $\|R_n^{(4)}\| = \|R_n^{(3)}\|$. □

The class of Gauss formulae (2.1), for which one can explicitly compute $\|R_n\|$ has been substantially extended in [3, 4, 40], where the underlying weight function is of Bernstein–Szegö type. These are weight functions consisting of any one of the four Chebyshev weights divided by an arbitrary polynomial which remains positive on $[-1, 1]$, i.e.,

$$w_\rho^{(1)}(t) = \frac{(1-t^2)^{-1/2}}{\rho(t)}, \quad w_\rho^{(2)}(t) = \frac{(1-t^2)^{1/2}}{\rho(t)}, \quad -1 < t < 1, \tag{3.9}$$

$$w_\rho^{(3)}(t) = \frac{(1-t)^{-1/2}(1+t)^{1/2}}{\rho(t)}, \quad w_\rho^{(4)}(t) = \frac{(1-t)^{1/2}(1+t)^{-1/2}}{\rho(t)}, \quad -1 < t < 1, \tag{3.10}$$

where $\rho(t) > 0$ on $[-1, 1]$. In [3], there were considered the two cases

$$\rho_a(t) = 1 + a^2 + 2at \quad \text{and} \quad \rho_{b_1}(t) = (2b_1+1)t^2 + b_1^2, \quad b_1 > 0,$$

in [4], the case

$$\rho_{b_2}(t) = b_2^2 - (2b_2 - 1)t^2, \quad b_2 > 1,$$

while [40] is concerned with the case of an arbitrary quadratic polynomial

$$\rho(t) = \rho(t; \alpha, \beta, \delta) = \beta(\beta - 2\alpha)t^2 + 2\delta(\beta-\alpha)t + \alpha^2 + \delta^2,$$
$$0 < \alpha < \beta, \beta \neq 2\alpha, |\delta| < \beta - \alpha \tag{3.11}$$

(cf. [14, Proposition 2.1]). Clearly,

$$\rho_a(t) = \rho(t; 1, 2, a), \quad \rho_{b_1}(t) = \rho(t; b_1, 2b_1 + 1, 0) \text{ and}$$
$$\rho_{b_2}(t) = \rho(t; b_2, 2b_2 - 1, 0),$$

hence, we present here only the results for the general case (3.11). There should be noted that when $\rho = \rho_a$ the conditions on α, β, δ in (3.11) impose $|a| < 1$; however, this is not restrictive, as for $|a| > 1$, we have $\rho_a(t) = a^2\left(\frac{1}{a^2} + 1 + \frac{2}{a}t\right) = a^2(1 + d^2 + 2dt)$, where $d = 1/a$, $|d| < 1$, so, apart from a constant factor, this case falls into the previous one; on the other hand, for $|a| = 1$, the resulting weight function, assuming it is integrable, is one of the Chebyshev weights; for a detailed analysis of all this the reader is referred to [3, Section 2a]. First, we begin with

Lemma 3.3

(a) *Consider the weight functions* $w_\rho^{(1)}$ *and* $w_\rho^{(2)}$, *with* ρ *given by* (3.11). *Then* $w_\rho^{(1)}(t)/w_\rho^{(1)}(-t)$ *and* $w_\rho^{(2)}(t)/w_\rho^{(2)}(-t)$ *are strictly increasing on* $(-1, 1)$ *if*

$$\beta - 2\alpha > 0, \ \beta(\beta - 2\alpha) \leq \alpha^2 + \delta^2, \ \delta < 0, \tag{3.12$_1$}$$

or

$$\beta - 2\alpha < 0, \ \delta < 0, \tag{3.12$_2$}$$

equal to 1 if $\delta = 0$, *and strictly decreasing on* $(-1, 1)$ *if*

$$\beta - 2\alpha > 0, \ \beta(\beta - 2\alpha) \leq \alpha^2 + \delta^2, \ \delta > 0, \tag{3.13$_1$}$$

or

$$\beta - 2\alpha < 0, \ \delta > 0. \tag{3.13$_2$}$$

(b) *Consider the weight functions* $w_\rho^{(3)}$ *and* $w_\rho^{(4)}$, *with* ρ *given by* (3.11). *Then* $w_\rho^{(3)}(t)/w_\rho^{(3)}(-t)$ *is strictly increasing on* $(-1, 1)$ *if either* (3.12$_1$) *or* (3.12$_2$) *holds, or* $\delta = 0$, *and* $w_\rho^{(4)}(t)/w_\rho^{(4)}(-t)$ *is strictly decreasing on* $(-1, 1)$ *if either* (3.13$_1$) *or* (3.13$_2$) *holds, or* $\delta = 0$.

The proof of Lemma 3.3 is rather straightforward and is given in [40, Lemma 2.2].

Now, Lemma 3.3, together with Lemma 3.1, allow us to use Theorem 2.1 in order to compute $\|R_n\|$ for each of the weight functions (3.9) and (3.10), with ρ given by (3.11). Of paramount importance in this computation is that the corresponding (monic) orthogonal polynomials are given in terms of the respective Chebyshev polynomials, that is,

$$\pi_{n,\rho}^{(1)}(t) = \frac{1}{2^{n-1}}\left[T_n(t) + \frac{2\delta}{\beta}T_{n-1}(t) + \left(1 - \frac{2\alpha}{\beta}\right)T_{n-2}(t)\right], \quad n \geq 2,$$

$$\pi_{1,\rho}^{(1)}(t) = t + \frac{\delta}{\beta - \alpha},$$

$$\pi_{n,\rho}^{(2)}(t) = \frac{1}{2^n}\left[U_n(t) + \frac{2\delta}{\beta}U_{n-1}(t) + \left(1 - \frac{2\alpha}{\beta}\right)U_{n-2}(t)\right], \quad n \geq 1,$$

$$\pi_{n,\rho}^{(3)}(t) = \frac{1}{2^n}\left[V_n(t) + \frac{2\delta}{\beta}V_{n-1}(t) + \left(1 - \frac{2\alpha}{\beta}\right)V_{n-2}(t)\right], \quad n \geq 2,$$

$$\pi_{1,\rho}^{(3)}(t) = t - \frac{\alpha - \delta}{\beta},$$

$$\pi_{n,\rho}^{(4)}(t;\alpha,\beta,\delta) = (-1)^n \pi_{n,\rho}^{(3)}(-t;\alpha,\beta,-\delta), \quad n \geq 1$$

(cf. [14, Equations (3.8), (3.8)^1, (3.9), (3.10), (3.10)^1, (3.11)]). Then following the steps in the proof of Theorem 3.2, we obtain (cf. [40, Theorems 3.1–3.3])

Theorem 3.4 *Consider the Gauss formula (2.1), and let* $\tau = r - \sqrt{r^2 - 1}$.

(a) *For* $w = w_\rho^{(1)}$, *with* ρ *given by (3.11), we have*

$$\|R_{n,\rho}^{(1)}\| = \frac{8\pi r \tau^{2n}}{[(\beta - 2\alpha)\tau^2 + 2\delta\tau + \beta][(\beta - 2\alpha)\tau^2(1 + \tau^{2n-4}) + 2\delta\tau(1 + \tau^{2n-2}) + \beta(1 + \tau^{2n})]\sqrt{r^2 - 1}}, \quad n \geq 1, \tag{3.14}$$

if either (3.12$_1$) *or* (3.12$_2$) *holds, or* $\delta = 0$, *and the same formula* (3.14), *with* δ *replaced by* $-\delta$, *if either* (3.13$_1$) *or* (3.13$_2$) *holds.*

(b) *For* $w = w_\rho^{(2)}$, *with* ρ *given by (3.11), we have*

$$\|R_{n,\rho}^{(2)}\| = \frac{8\pi r \tau^{2n+2}\sqrt{r^2 - 1}}{[(\beta - 2\alpha)\tau^2 + 2\delta\tau + \beta][(\beta - 2\alpha)\tau^2(1 - \tau^{2n-2}) + 2\delta\tau(1 - \tau^{2n}) + \beta(1 - \tau^{2n+2})]}, \quad n \geq 1, \tag{3.15}$$

if either (3.12$_1$) *or* (3.12$_2$) *holds, or* $\delta = 0$, *and the same formula* (3.15), *with* δ *replaced by* $-\delta$, *if either* (3.13$_1$) *or* (3.13$_2$) *holds.*

(c) *For* $w = w_\rho^{(3)}$ *or* $w = w_\rho^{(4)}$*, with* ρ *given by* (3.11)*, we have*

$$\|R_{n,\rho}^{(3)}\| = \frac{8\pi r \tau^{2n+1}}{[(\beta-2\alpha)\tau^2+2\delta\tau+\beta][(\beta-2\alpha)\tau^2(1+\tau^{2n-3})+2\delta\tau(1+\tau^{2n-1})+\beta(1+\tau^{2n+1})]} \times \sqrt{\frac{r+1}{r-1}}, \quad n \geq 1, \tag{3.16}$$

if either (3.12$_1$) *or* (3.12$_2$) *holds, or* $\delta = 0$*, and* $\|R_{n,\rho}^{(4)}\|$ *is given by the same formula* (3.16)*, with* δ *replaced by* $-\delta$*, if either* (3.13$_1$) *or* (3.13$_2$) *holds, or* $\delta = 0$.

Obviously, Theorem 3.4 includes, as a special case, Theorem 3.2; indeed, for $\alpha = 1,\ \beta = 2,\ \delta = 0$ in (3.11), we have $\rho(t) = \rho(t; 1, 2, 0) = 1$, hence, $w_\rho^{(i)} = w^{(i)},\ i = 1, 2, 3, 4$.

Estimates (2.19) for the Gauss formula (2.1) relative to the weight functions $w = w_\rho^{(i)},\ i = 1, 2, 3, 4$, with ρ given by (3.11), are quite sharp. This has been attested not only in [40, Section 4], but also fairly recently when these estimates were compared to bounds derived by contour integration on elliptic contours, which are known to be more efficient than circular ones (cf. [46, Section 3], [52, Section 3], [54, Section 3] and [56, Section 5]); in most cases, the estimates (2.19) were as good as, and in some cases even better than, the bounds obtained by contour integration.

3.2 The error norm of Gauss–Lobatto rules

Assume that the quadrature formula (2.1) is the Gauss–Lobatto rule for the weight function w on $[-1, 1]$,

$$\int_{-1}^{1} f(t)w(t)dt = w_0^L f(1) + \sum_{\nu=1}^{n} w_\nu^L f(\tau_\nu^L) + w_{n+1}^L f(-1) + R_n^L(f), \tag{3.17}$$

where τ_ν^L are the zeros of the nth-degree (monic) orthogonal polynomial $\pi_n^L(\cdot) = \pi_n(\cdot; w^L)$ relative to the weight function $w^L(t) = (1-t^2)w(t)$. Then the following theorem holds.

Theorem 3.5 *Consider the Gauss–Lobatto formula* (3.17) *for the weight function* w *on the interval* $[-1, 1]$.

(a) *If* $w(t)/w(-t)$ *is nondecreasing on* $(-1, 1)$*, then*

$$\|R_n^L\| = \frac{r}{(r^2-1)\pi_n^L(r)} \int_{-1}^{1} \frac{\pi_n^L(t)}{r-t} w^L(t)dt. \tag{3.18}$$

(b) *If* $w(t)/w(-t)$ *is nonincreasing on* $(-1, 1)$*, then*

$$\|R_n^L\| = \frac{r}{(r^2-1)\pi_n^L(-r)} \int_{-1}^{1} \frac{\pi_n^L(t)}{r+t} w^L(t)dt. \tag{3.19}$$

The proof is based on the comparison between formula (3.17) and the Gauss formula for the weight function w^L (cf. [42, Theorem 2.1]).

The first case for which $\|R_n^L\|$ has been computed explicitly is that of the Chebyshev weight function of the first kind $w^{(1)}$. It has been done by Akrivis in his doctoral dissertation (cf. [1, Section 1.5b]). By using (2.10) and expressing $1/(r-t)$ in terms of the Chebyshev polynomials of the first kind, he derived

$$\|R_n^{L(1)}\| = \frac{2\pi r\tau^{2n+2}}{(1-\tau^{2n+2})\sqrt{r^2-1}}, \quad n \geq 1, \tag{3.20}$$

where $\tau = r - \sqrt{r^2-1}$.

Much more can be obtained if we apply Theorem 3.5 to the weight functions (3.9) and (3.10), with ρ given by (3.11), and use Lemma 3.3.

Theorem 3.6 *Consider the Gauss–Lobatto formula* (3.17)*, and let* $\tau = r - \sqrt{r^2-1}$.

(a) *For* $w = w_\rho^{(1)}$*, with* ρ *given by* (3.11)*, we have*

$$\|R_{n,\rho}^{L(1)}\| = \frac{8\pi r\tau^{2n+2}}{[(\beta-2\alpha)\tau^2+2\delta\tau+\beta][(\beta-2\alpha)\tau^2(1-\tau^{2n-2})+2\delta\tau(1-\tau^{2n})+\beta(1-\tau^{2n+2})]\sqrt{r^2-1}}, \quad n \geq 1, \tag{3.21}$$

if either (3.12_1) *or* (3.12_2) *holds, or* $\delta = 0$*, and the same formula* (3.21)*, with* δ *replaced by* $-\delta$*, if either* (3.13_1) *or* (3.13_2) *holds.*

(b) *For* $w = w_\rho^{(2)}$*, with* ρ *given by* (3.11)*, we have*

$$\|R_{n,\rho}^{L(2)}\| = \frac{8\pi r\tau^{2n+4}(\tau^2-2\gamma_1\tau-4\gamma_2)\sqrt{r^2-1}}{[(\beta-2\alpha)\tau^2+2\delta\tau+\beta](4\gamma_2\tau^2\omega_n+2\gamma_1\tau\omega_{n+1}-\omega_{n+2})}, \quad n \geq 1, \tag{3.22}$$

where

$$\begin{aligned} \gamma_1 &= \frac{\alpha\delta}{[(\beta-\alpha)^2-\delta^2]n^2+(\beta^2-\alpha^2-\delta^2)n+\alpha\beta}, \\ \gamma_2 &= \frac{[(\beta-\alpha)^2-\delta^2](n+1)^2+(\beta^2-\alpha^2-\delta^2)(n+1)+\alpha\beta}{4\{[(\beta-\alpha)^2-\delta^2]n^2+(\beta^2-\alpha^2-\delta^2)n+\alpha\beta\}}, \\ \omega_n &= (\beta-2\alpha)\tau^2(1-\tau^{2n-2})+2\delta\tau(1-\tau^{2n})+\beta(1-\tau^{2n+2}), \end{aligned} \tag{3.23}$$

if either (3.12_1) *or* (3.12_2) *holds, or* $\delta = 0$*, and the same formulae* (3.22)*,* (3.23)*, with* δ *replaced by* $-\delta$*, if either* (3.13_1) *or* (3.13_2) *holds.*

(c) *For* $w = w_\rho^{(3)}$ *or* $w = w_\rho^{(4)}$*, with* ρ *given by* (3.11)*, we have*

$$\|R_{n,\rho}^{L(3)}\| = \frac{8\pi r\tau^{2n+3}(\tau+2\gamma)}{[(\beta-2\alpha)\tau^2+2\delta\tau+\beta](2\gamma\tau\omega_n+\omega_{n+1})}\sqrt{\frac{r+1}{r-1}}, \quad n \geq 1, \tag{3.24}$$

where

$$\gamma = \frac{(\beta - \alpha - \delta)(n+1) + \alpha}{2[(\beta - \alpha - \delta)n + \alpha]}, \tag{3.25}$$

and ω_n *as in* (3.23), *if either* (3.12_1) *or* (3.12_2) *holds, or* $\delta = 0$*; and* $\| R_{n,\rho}^{L(4)} \|$ *is given by the same formulae* (3.24), (3.25) *and* (3.23), *with* δ *replaced by* $-\delta$, *if either* (3.13_1) *or* (3.13_2) *holds, or* $\delta = 0$.

Proof

(a) As $w_\rho^{L(1)}(t) = (1-t^2)w_\rho^{(1)}(t) = w_\rho^{(2)}(t)$, from (3.18) and (3.19), we get

$$\| R_{n,\rho}^{L(1)} \| = \frac{\| R_{n,\rho}^{(2)} \|}{r^2 - 1},$$

where $R_{n,\rho}^{(2)}$ is the error term of the Gauss formula for the weight function $w_\rho^{(2)}$, and the result follows from Theorem 3.4(b).

(b)–(c) The proof is substantially more complicated, and the reader is referred to [42, Theorems 3.4 and 3.5]. □

Obviously, (3.20) is a special case of (3.21) with $\alpha = 1,\ \beta = 2,\ \delta = 0$.

3.3 The error norm of Gauss–Radau rules

Assume that the quadrature formula (2.1) is the Gauss–Radau rule for the weight function w on $[-1, 1]$ and additional node at -1 or 1,

$$\int_{-1}^{1} f(t)w(t)dt = \sum_{\nu=1}^{n} w_\nu^{R(-)} f(\tau_\nu^{R(-)}) + w_{n+1}^{R(-)} f(-1) + R_n^{R(-)}(f), \tag{3.26}$$

or

$$\int_{-1}^{1} f(t)w(t)dt = w_0^{R(+)} f(1) + \sum_{\nu=1}^{n} w_\nu^{R(+)} f(\tau_\nu^{R(+)}) + R_n^{R(+)}(f), \tag{3.27}$$

where $\tau_\nu^{R(-)}$ are the zeros of the nth-degree (monic) orthogonal polynomial $\pi_n^{R(-)}(\cdot)=\pi_n(\cdot; w^{R(-)})$ relative to the weight function $w^{R(-)}(t) = (1+t)w(t)$, and $\tau_\nu^{R(+)}$ are the zeros of the nth-degree (monic) orthogonal polynomial $\pi_n^{R(+)}(\cdot) = \pi_n(\cdot; w^{R(+)})$ relative to the weight function $w^{R(+)}(t) = (1-t)w(t)$. Unfortunately, here we don't have a general result like Theorem 3.5 in the Gauss–Lobatto case, hence each weight function has to be treated separately. However, it is not difficult to show that

$$\| R_n^{R(+)}(\cdot; w(t)) \| = \| R_n^{R(-)}(\cdot; w(-t)) \|$$

(cf. [45, Eq. (1.17)]), which saves some computations; in particular, if w is an even weight function, then $\| R_n^{R(+)} \| = \| R_n^{R(-)} \|$.

Akrivis, in his doctoral dissertation (cf. [1, Section 1.7]), using (2.10) and expressing $1/(r-t)$ in terms of the Chebyshev polynomials of the first or second kind, obtained estimates for $\|R_n^{R(-)}\|$ and $\|R_n^{R(+)}\|$ in the case of the Legendre weight function $w(t)=1,\ -1\le t\le 1$.

Only recently, we succeeded to compute $\|R_n^{R(-)}\|$ and $\|R_n^{R(+)}\|$ explicitly for most of the Chebyshev weights (3.1) and (3.2). First of all, we showed

Lemma 3.7

(a) *The error term of the Gauss–Radau formula* (3.26) *with* $w=w^{(1)}$ *satisfies*

$$(-1)^{k-1}R_n^{R(-)(1)}(t^k)\ge 0,\quad k\ge 0. \tag{3.28}$$

(b) *The error term of the Gauss–Radau formula* (3.26) *with* $w=w^{(2)}$ *satisfies*

$$(-1)^{k-1}R_n^{R(-)(2)}(t^k)\ge 0,\quad k\ge 0,\ 1\le n\le 40. \tag{3.29}$$

(c) *The error term of the Gauss–Radau formula* (3.26) *with* $w=w^{(4)}$ *satisfies*

$$(-1)^{k-1}R_n^{R(-)(4)}(t^k)\ge 0,\quad k\ge 0. \tag{3.30}$$

The proof of (3.28) and (3.30) is based on the comparison of formula (3.26) with $w=w^{(1)}$ or $w=w^{(4)}$ with the $(2n)$-point Gauss–Lobatto formula for the weight $w^{(1)}$ or the Gauss formula for the weight $w^{(2)}$, respectively. For part (b), we proved (3.29) for $k(\text{even})\ge 2k_n^{R(-)(2)}$ and verified it numerically for $k(\text{even})<2k_n^{R(-)(2)},\ 1\le n\le 40$, where $k_n^{R(-)(2)}$ are certain integers tabulated in [45, Table 1]; for k odd, the validity of (3.29) follows from the definiteness of the Gauss–Radau formula (cf. [45, Propositions 2.1, 2.2, 2.11 and Lemma 2.4]).

Unfortunately, $R_n^{R(-)(3)}(t^k)$ does not keep a constant sign for all k even (cf. [45, Lemmas 2.7 and 2.8]).

Remark 3.1 In [45, Conjecture 2.5], it is conjectured that (3.29) is true for all $n\ge 1$.

Now, from (2.13), in view of Lemma 3.7, we get

Theorem 3.8 *Consider the Gauss–Radau formula* (3.26)*, and let* $\tau=r-\sqrt{r^2-1}$.

(a) *For* $w=w^{(1)}$*, we have*

$$\|R_n^{R(-)(1)}\|=\frac{2\pi r\tau^{2n+1}}{(1-\tau^{2n+1})\sqrt{r^2-1}},\quad n\ge 1. \tag{3.31}$$

(b) *For $w = w^{(2)}$, we have*

$$\| R_n^{R(-)(2)} \| = \frac{2\pi r \tau^{2n+3}\left(\frac{n+2}{n+1} - \tau\right)\sqrt{r^2-1}}{1 - \tau^{2n+4} - \frac{n+2}{n+1}\tau(1-\tau^{2n+2})}, \quad 1 \le n \le 40. \tag{3.32}$$

(c) *For $w = w^{(4)}$, we have*

$$\| R_n^{R(-)(4)} \| = \frac{2\pi r \tau^{2n+2}}{1 - \tau^{2n+2}}\sqrt{\frac{r+1}{r-1}}, \quad n \ge 1. \tag{3.33}$$

On the other hand, for $w = w^{(3)}$, we have

$$\| R_n^{R(-)(3)} \| < \frac{2\pi r \tau^{2n+2}\left(\tau + \frac{2n+3}{2n+1}\right)}{1 + \tau^{2n+3} + \frac{2n+3}{2n+1}\tau(1+\tau^{2n+1})}\sqrt{\frac{r+1}{r-1}} + \frac{2\pi}{r^{2k_n^{R(-)(3)}-2}(r^2-1)}, \quad n \ge 1, \tag{3.34}$$

where $k_n^{R(-)(3)}$ are integers tabulated in [45, Table 2] for $1 \le n \le 10$, but they can also be computed for any n. An estimate slightly sharper but substantially more complicated than (3.34) is given in [45, Eq. (2.58)], although for all practical purposes both estimates give the same results, and this is also the case even if one uses just the first term on the right-hand side of (3.34) (cf. [45, Example 3.2]).

Moreover, for the Gauss–Radau formula (3.27) with $w = w^{(i)}$, $i = 1, 2, 3$, $\| R_n^{R(+)(1)} \|$, $\| R_n^{R(+)(2)} \|$ for $1 \le n \le 40$, and $\| R_n^{R(+)(3)} \|$ are given by the same formulae (3.31), (3.32) and (3.33), respectively, while for $\| R_n^{R(+)(4)} \|$ the estimates (3.34) and [45, Eq. (2.58)] hold; for further details, the reader is referred to [45, Theorems 2.3, 2.6, 2.9, 2.10, 2.13 and 2.14].

Theorem 3.8 and estimates (3.34) and [45, Eq. (2.58)] shed some light on conjectures of Gautschi (cf. [12, Section 4.2]) regarding remainder estimates based on contour integration of the Gauss–Radau formulae in question (cf. [45, end of each of Subsections 2.1–2.4]).

3.4 The error norm of Gauss–Kronrod rules

Assume that the quadrature formula (2.1) is the Gauss–Kronrod rule for the weight function w on $[-1, 1]$,

$$\int_{-1}^{1} f(t)w(t)dt = \sum_{\nu=1}^{n} \sigma_\nu^K f(\tau_\nu) + \sum_{\mu=1}^{n+1} \sigma_\mu^{*K} f(\tau_\mu^{*K}) + R_n^K(f), \tag{3.35}$$

where τ_ν are the Gauss points, i.e., the zeros of the (monic) orthogonal polynomial $\pi_n(\cdot; w)$, and the $\tau_\mu^{*K}, \sigma_\nu^K, \sigma_\mu^{*K}$ are chosen such that (3.35) has maximum degree of exactness (at least) $3n + 1$, i.e., $R_n^K(f) = 0$ for all $f \in \mathbb{P}_{3n+1}$. Unfortunately, general results, like those in the Gauss and Gauss–Lobatto cases (cf. Theorems 2.1 and 3.5), are not possible here, therefore each weight function has to be treated as a separate case.

First of all, for the Legendre weight function $w(t) = 1,\ -1 \le t \le 1$, following the idea of Hämmerlin in [19] (cf. the beginning of Section 3), we showed that

$$\|R_n^K\| < \frac{(n!)^2(d_n - i_n + 1)!}{2^{n-2}(2n)!(d_n + 1)!}\left[\frac{r}{(r-1)^{d_n - i_n + 2}} + (-1)^{i_n}\frac{r}{(r+1)^{d_n - i_n + 2}}\right], \quad 2 \le n \le 30,$$

where d_n is the degree of exactness of the quadrature formula in question, $d_n = 3n + 1$ for n even and $d_n = 3n + 2$ for n odd, and i_n is an appropriate constant, which is tabulated for $2 \le n \le 30$ (cf. [41, Table 1]), but it can also be computed for any $n > 30$.

The next result concerns the Bernstein–Szegö weight functions (3.9) and (3.10), with ρ given by

$$\rho_\gamma(t) = \rho(t; 1, 2/(1+\gamma), 0) = 1 - \frac{4\gamma}{(1+\gamma)^2}t^2,\ -1 < \gamma \le 0. \tag{3.36}$$

First, we proved (cf. [44, Proposition 3.2])

Lemma 3.9

(a) *The error term of the Gauss–Kronrod formula* (3.35) *for* $w = w_{\rho_\gamma}^{(1)}$, *with* ρ_γ *given by* (3.36), *satisfies*

$$R_{n,\rho_\gamma}^{K(1)}(t^k) \le 0, \quad k \ge 0,\ n \ge 4. \tag{3.37}$$

(b) *The error term of the Gauss–Kronrod formula* (3.35) *for* $w = w_{\rho_\gamma}^{(2)}$, *with* ρ_γ *given by* (3.36), *satisfies*

$$R_{n,\rho_\gamma}^{K(2)}(t^k) \ge 0, \quad k \ge 0,\ n \ge 2. \tag{3.38}$$

(c) The error term of the Gauss–Kronrod formula (3.35) *for* $w = w_{\rho_\gamma}^{(4)}$, *with* ρ_γ *given by* (3.36), *satisfies*

$$(-1)^k R_{n,\rho_\gamma}^{K(4)}(t^k) \le 0, \quad k \ge 0,\ n \ge 3. \tag{3.39}$$

To prove (3.38) and (3.39), we compared the quadrature formulae in question with the $(2n + 1)$ and the $(2n)$-point Gauss formula for the Bernstein–Szegö weight function $\hat{w}_{\rho_\gamma}^{(2)}(t) = (1 - t^2)^{1/2}/[\rho_\gamma(t)]^2$, and then we used Lemmas 3.1 and 3.3; while, for (3.37), the quadrature formula under consideration was compared with the $(2n - 1)$-point Gauss–Kronrod formula for the weight function $w_{\rho_\gamma}^{(2)}$.

Now, from (2.11) and (2.13), in view of Lemma 3.9, we get (cf. [44, Theorems 3.8–3.10])

Theorem 3.10 *Consider the Gauss–Kronrod formula* (3.35)*, and let* $\tau = r - \sqrt{r^2-1}$.

(a) *For* $w = w^{(1)}_{\rho_\gamma}$*, with* ρ_γ *given by* (3.36)*, we have*

$$\|R^{K(1)}_{n,\rho_\gamma}\| = \frac{2\pi(1+\gamma)^2 r\tau^{4n-2}(\tau^2-\gamma)}{(1-\gamma\tau^2)[1-\tau^{4n}-2\gamma\tau^2(1-\tau^{4n-4})+\gamma^2\tau^4(1-\tau^{4n-8})]\sqrt{r^2-1}}, \quad n \ge 4,\ -1<\gamma\le 0,$$

$$\|R^{K(1)}_{1,\rho_\gamma}\| = \frac{2\pi(1+\gamma)^2 r\tau^6}{(1-\gamma\tau^2)[1+\tau^6-\gamma\tau^2(1+\tau^2)]\sqrt{r^2-1}}, \quad -1<\gamma<1,$$

$$\|R^{K(1)}_{2,\rho_\gamma}\| = \frac{2\pi(1+\gamma)^2 r\tau^8[\tau^2-(1+2\gamma)]}{(1-\gamma\tau^2)[1+\tau^{10}-(1+3\gamma)\tau^2(1+\tau^6)+\gamma(1+2\gamma)\tau^4(1+\tau^2)]\sqrt{r^2-1}}, \quad -1<\gamma\le -1/2.$$

(b) *For* $w = w^{(2)}_{\rho_\gamma}$*, with* ρ_γ *given by* (3.36)*, we have*

$$\|R^{K(2)}_{n,\rho_\gamma}\| = \frac{2\pi(1+\gamma)^2 r\tau^{4n+2}(\tau^2-\gamma)\sqrt{r^2-1}}{(1-\gamma\tau^2)[1-\tau^{4n+4}-2\gamma\tau^2(1-\tau^{4n})+\gamma^2\tau^4(1-\tau^{4n-4})]}, \quad n \ge 2,\ -1<\gamma\le 0,$$

$$\|R^{K(2)}_{1,\rho_\gamma}\| = \frac{2\pi(1+\gamma)^2 r\tau^8\sqrt{r^2-1}}{(1-\gamma\tau^2)[1-\tau^8-\gamma\tau^2(1-\tau^4)]}, \quad -1<\gamma<1.$$

(c) *For* $w = w^{(4)}_{\rho_\gamma}$ *or* $w = w^{(3)}_{\rho_\gamma}$*, with* ρ_γ *given by* (3.36)*, we have*

$$\|R^{K(4)}_{n,\rho_\gamma}\| = \frac{2\pi(1+\gamma)^2 r\tau^{4n}(\tau^2-\gamma)}{(1-\gamma\tau^2)[1-\tau^{4n+2}-2\gamma\tau^2(1-\tau^{4n-2})+\gamma^2\tau^4(1-\tau^{4n-6})]}\sqrt{\frac{r+1}{r-1}}, \quad n \ge 3,\ -1<\gamma\le 0, \tag{3.40}$$

and $\|R^{K(3)}_{n,\rho_\gamma}\|$ *is given by the same formula* (3.40).

Remark 3.2 Unfortunately, the error term of the Gauss–Kronrod formula (3.35) for each of the weight functions $w_{\rho_\gamma}^{(1)}$, with $n = 3$, and $w_{\rho_\gamma}^{(4)}$ or $w_{\rho_\gamma}^{(3)}$, with $n = 1,\ 2$, does not satisfy a condition of type (3.37)–(3.39), which is necessary for applying Theorem 2.1.

3.5 The error norm of interpolatory rules

Let's assume that the quadrature formula (2.1) is the interpolatory rule with $w(t) = 1$,

$$\int_{-1}^{1} f(t)dt = \sum_{\nu=1}^{n} w_\nu^F f(\tau_\nu^F) + R_n^F(f), \tag{3.41}$$

where τ_ν^F are the zeros of the nth-degree Chebyshev polynomial of any one of the second, third or fourth kind, i.e.,

$$\tau_\nu^{F(2)} = \cos \frac{\nu}{n+1}\pi, \quad \nu = 1, 2, \cdots, n,$$

$$\tau_\nu^{F(3)} = \cos \frac{2\nu - 1}{2n+1}\pi, \quad \nu = 1, 2, \cdots, n,$$

$$\tau_\nu^{F(4)} = \cos \frac{2\nu}{2n+1}\pi, \quad \nu = 1, 2, \cdots, n,$$

respectively. For $\tau_\nu^F = \tau_\nu^{F(2)}$ formula (3.41) is known as the Fejér rule of the second kind or Filippi rule. In order to be able to compute $\|R_n^{F(i)}\|,\ i = 2, 3, 4$, we need (cf. [43, Section 3])

Lemma 3.11

(a) *The error term of the interpolatory formula* (3.41) *with* $\tau_\nu^F = \tau_\nu^{F(2)}$ *satisfies*

$$R_n^{F(2)}(t^k) \geq 0, \quad k \geq 0. \tag{3.42}$$

(b) *The error term of the interpolatory formula* (3.41) *with* $\tau_\nu^F = \tau_\nu^{F(3)}$ *satisfies*

$$(-1)^k R_n^{F(3)}(t^k) \geq 0, \quad k \geq 0,\ 1 \leq n \leq 20. \tag{3.43}$$

(c) *The error term of the interpolatory formula* (3.41) *with* $\tau_\nu^F = \tau_\nu^{F(4)}$ *satisfies*

$$R_n^{F(4)}(t^k) \geq 0, \quad k \geq 0,\ 1 \leq n \leq 20. \tag{3.44}$$

The proof of (3.42) is based on the definiteness of the respective quadrature formula, while for parts (b) and (c), we follow the technique used in Lemma 3.7(b) for k even, except that here it is applied to all k even or odd.

Remark 3.3 In [43, Conjecture 3.2], it is conjectured that (3.43) and (3.44) are true for all $n \geq 1$.

Now, in addition to Lemma 3.11, the computation of $\|R_n^{F(i)}\|$, $i = 2, 3, 4$, would require a formula for $\int_{-1}^{1} \frac{\pi_n(t)}{r \mp t} dt$, $r > 1$, with π_n being any one of the nth-degree Chebyshev polynomials of the second, third or fourth kind. Such formulae were obtained, even for the nth-degree Chebyshev polynomial of the first kind $\pi_n = T_n$, in [43, Proposition 2.2] for all $|r| > 1$; for example, if $\pi_n = U_n$, $\pi_n = V_n$ or $\pi_n = W_n$, then

$$\int_{-1}^{1} \frac{\pi_n(t)}{r \mp t} dt = \pi_n(\pm r) \ln\left(\frac{r+1}{r-1}\right) \mp 4 \sum_{k=1}^{[(n+1)/2]} \frac{\pi_{n-2k+1}(\pm r)}{2k-1}, \; n \geq 1, \tag{3.45}$$

where $[\cdot]$ denotes the integer part of a real number.

Analogous formulae were obtained for $|r| < 1$ in the Cauchy Principal Value sense (cf. [43, Proposition 2.3]).

Based on (2.11), (2.13) and (3.45), one derives, in view of (3.42)–(3.44) (cf. [43, Proposition 3.3]),

Theorem 3.12 *Consider the interpolatory formula* (3.41).

(a) *For $\tau_\nu^F = \tau_\nu^{F(2)}$, we have*

$$\|R_n^{F(2)}\| = r \ln\left(\frac{r+1}{r-1}\right) - \frac{4r}{U_n(r)} \sum_{k=1}^{[(n+1)/2]} \frac{U_{n-2k+1}(r)}{2k-1}, \quad n \geq 1. \tag{3.46}$$

(b) *For $\tau_\nu^F = \tau_\nu^{F(3)}$, we have*

$$\|R_n^{F(3)}\| = r \ln\left(\frac{r+1}{r-1}\right) + \frac{4r}{V_n(-r)} \sum_{k=1}^{[(n+1)/2]} \frac{V_{n-2k+1}(-r)}{2k-1}, \quad 1 \leq n \leq 20.$$

(c) *For $\tau_\nu^F = \tau_\nu^{F(4)}$, we have*

$$\|R_n^{F(4)}\| = r \ln\left(\frac{r+1}{r-1}\right) - \frac{4r}{W_n(r)} \sum_{k=1}^{[(n+1)/2]} \frac{W_{n-2k+1}(r)}{2k-1}, \quad 1 \leq n \leq 20.$$

Remark 3.4 Unfortunately, the error term of the interpolatory formula (3.41) with τ_ν^F the zeros of the nth-degree Chebyshev polynomial of the first kind, $\tau_\nu^{F(1)} = \cos \frac{2\nu - 1}{2n}\pi$, $\nu = 1, 2, \cdots, n$, known as the Fejér rule of the first kind or Pólya rule, does not satisfy a condition of type (3.42)–(3.44), which is necessary for applying Theorem 2.1.

Akrivis, in his doctoral dissertation (cf. [1, Eqs. (1.7.5) and (1.7.6)] and the correction in [43, p. 1227]), has obtained a bound for $\|R_n^{F(2)}\|$, which is pretty close to the actual value (3.46), so our result for $\|R_n^{F(2)}\|$ is a refinement of Akrivis's result.

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