Numer Algor (2013) 62:59–77
DOI 10.1007/s11075-012-9566-x

ORIGINAL PAPER

Modulus-based synchronous two-stage multisplitting iteration methods for linear complementarity problems

Zhong-Zhi Bai · Li-Li Zhang

Received: 5 January 2012 / Accepted: 19 March 2012 /
Published online: 31 March 2012

Abstract In order to solve large sparse linear complementarity problems on parallel multiprocessor systems, we construct modulus-based synchronous two-stage multisplitting iteration methods based on two-stage multisplittings of the system matrices. These iteration methods include the multisplitting relaxation methods such as Jacobi, Gauss–Seidel, SOR and AOR of the modulus type as special cases. We establish the convergence theory of these modulus-based synchronous two-stage multisplitting iteration methods and their relaxed variants when the system matrix is an H_+-matrix. Numerical results show that in terms of computing time the modulus-based synchronous two-stage multisplitting relaxation methods are more efficient than the modulus-based synchronous multisplitting relaxation methods in actual implementations.

Keywords Linear complementarity problem · Modulus method · Matrix multisplitting · Two-stage iteration · Successive relaxation · Convergence

Supported by The National Natural Science Foundation for Creative Research Groups (No. 11021101), The Hundred Talent Project of Chinese Academy of Sciences, The National Basic Research Program (No. 2011CB309703), and The National Natural Science Foundation (No. 91118001), People's Republic of China.

Z.-Z. Bai (✉) · L.-L. Zhang
State Key Laboratory of Scientific and Engineering Computing,
Institute of Computational Mathematics and Scientific/Engineering Computing,
Academy of Mathematics and Systems Science, Chinese Academy of Sciences,
P.O. Box 2719, Beijing 100190, People's Republic of China
e-mail: bzz@lsec.cc.ac.cn

L.-L. Zhang
e-mail: zhangll@lsec.cc.ac.cn

1 Introduction

Let $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ be the n-dimensional real vector space and the n-by-n real matrix space, respectively. For given matrix $A \in \mathbb{R}^{n\times n}$ and vector $q \in \mathbb{R}^n$, the linear complementarity problem, abbreviated as LCP(q, A), consists of finding a pair of vectors r, $z \in \mathbb{R}^n$ such that

$$r := Az + q \geq 0, \quad z \geq 0 \quad \text{and} \quad z^T(Az + q) = 0,$$

where z^T denotes the transpose of the vector z. We refer to [9] for detailed descriptions about this problem and its practical backgrounds.

To solve the LCP(q, A) fast and economically by iterative methods, Bai recently proposed in [2] a class of modulus-based splitting iteration methods, which provides a general framework for the modulus iteration [18, 19], the modified modulus iteration [13], and the nonstationary extrapolated modulus iteration [16, 17]. In order to suit computational requirements of the modern high-speed multiprocessor environments, Bai and Zhang further presented in [7] synchronous parallel counterparts for the modulus-based splitting iteration methods by making use of multiple splittings of the system matrix A [6, 20]. This class of *modulus-based synchronous multisplitting* (**MSM**) iteration methods only needs to solve, at each iteration step, sub-systems of linear equations rather than linear complementarity sub-problems [4, 5], and is also convergent when the system matrix $A \in \mathbb{R}^{n\times n}$ is an H_+-matrix.

To precisely describe the MSM iteration method, we first state the concept of matrix multisplitting. Let ℓ be a given positive integer with $\ell \leq n$, $A = M_k - N_k$, $k = 1, 2, \ldots, \ell$, be splittings of the system matrix $A \in \mathbb{R}^{n\times n}$, and $E_k \in \mathbb{R}^{n\times n}$, $k = 1, 2, \ldots, \ell$, be nonnegative diagonal matrices satisfying $\sum_{k=1}^{\ell} E_k = I$ (the identity matrix). Then the collection of triples (M_k, N_k, E_k) $(k = 1, 2, \ldots, \ell)$ is called a multisplitting of the matrix A. In addition, the matrices E_k $(k = 1, 2, \ldots, \ell)$ are called weighting matrices. Then the MSM iteration method established in [7] can be described as follows.

Method 1.1 (The MSM iteration method for LCP(q, A)) Let (M_k, N_k, E_k) $(k = 1, 2, \ldots, \ell)$ be a multisplitting of the system matrix $A \in \mathbb{R}^{n\times n}$, $\Omega \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix, and γ be a positive constant. Given an initial vector $x^{(0)} \in \mathbb{R}^n$, for $m = 0, 1, 2, \ldots$ until the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty} \subset \mathbb{R}^n$ is convergent, compute $z^{(m+1)} \in \mathbb{R}^n$ by

$$z^{(m+1)} = \frac{1}{\gamma}\left(\left|x^{(m+1)}\right| + x^{(m+1)}\right)$$

and $x^{(m+1)} \in \mathbb{R}^n$ according to

$$x^{(m+1)} = \sum_{k=1}^{\ell} E_k x^{(m,k)},$$

where $x^{(m,k)}$, $k = 1, 2, \ldots, \ell$, are obtained by solving the linear systems

$$(\Omega + M_k)x^{(m,k)} = N_k x^{(m)} + (\Omega - A)|x^{(m)}| - \gamma q, \qquad k = 1, 2, \ldots, \ell, \tag{1}$$

respectively.

In practical applications, however, the multiple splittings $A = M_k - N_k$, $k = 1, 2, \ldots, \ell$, may be determined by the inherent properties of the original problem, which probably results in unbalanced distribution of the tasks among the processors. Also, the sub-systems of linear equations defined by (1) may be still too large to be solved by direct method. To avoid these shortcomings of the MSM iteration method, in this paper we employ splitting iteration methods again to solve the sub-systems of linear equations in (1), resulting in a class of *modulus-based synchronous two-stage multisplitting* (**MSTM**) iteration methods for parallelly solving the LCP(q, A). With special choices of the two-stage multisplitting of the system matrix, we can obtain a sequence of modulus-based synchronous two-stage multisplitting relaxation methods, including Jacobi, Gauss–Seidel, SOR and AOR, respectively. When the system matrix $A \in \mathbb{R}^{n\times n}$ is an H_+-matrix [1], we prove the convergence of the MSTM iteration method and its relaxed variants. Numerical results show that in terms of computing time the modulus-based synchronous two-stage multisplitting Gauss–Seidel and SOR methods are more efficient than the MSM iteration method in actual implementations.

The outline of the paper is as follows. In Section 2 we present some necessary notations and useful lemmas. In Section 3 we establish the MSTM iteration method and its relaxed variants. The convergence of these modulus-based synchronous two-stage multisplitting iteration methods are proved in Section 4, and the numerical results are given in Section 5. Finally, in Section 6, we end the paper by a few concluding remarks.

2 Notations and lemmas

We denote by $\mathbb{N}_n = \{1, 2, \ldots, n\}$ the set of the first n positive integers. Given two real m-by-n matrices $A = (a_{ij})$ and $B = (b_{ij})$, we write $A \geq B$ (or $A > B$) if $a_{ij} \geq b_{ij}$ (or $a_{ij} > b_{ij}$) hold for all $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$. $A \in \mathbb{R}^{m\times n}$ is said to be a nonnegative (or positive) matrix if $a_{ij} \geq 0$ (or $a_{ij} > 0$) hold for all $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$. $|A| = (|a_{ij}|)$ stands for the absolute value of the matrix $A \in \mathbb{R}^{m\times n}$. Note that $|A|$ is a nonnegative matrix. We use A^T and $\rho(A)$ to represent the transpose and the spectral radius of the square matrix $A \in \mathbb{R}^{n\times n}$, respectively. These notations are easily specified to vectors in $\mathbb{R}^n$.

For $A, B \in \mathbb{R}^{n\times n}$ satisfying $|A| \leq B$, it holds that $\rho(A) \leq \rho(|A|) \leq \rho(B)$. For a nonnegative matrix $A \in \mathbb{R}^{n\times n}$, if there exists a positive vector $x \in \mathbb{R}^n$ such that $Ax < x$, then $\rho(A) < 1$.

For $A = (a_{ij}) \in \mathbb{R}^{n\times n}$, its comparison matrix $\langle A\rangle = (\langle a_{ij}\rangle) \in \mathbb{R}^{n\times n}$ is defined by

$$\langle a_{ij}\rangle = \begin{cases} |a_{ij}|, & \text{for} \quad i = j, \\ -|a_{ij}|, & \text{for} \quad i \neq j, \end{cases} \qquad i, j = 1, 2, \ldots, n.$$

A square matrix $A \in \mathbb{R}^{n\times n}$ is called a Z-matrix if its off-diagonal entries are non-positive. A nonsingular matrix $A \in \mathbb{R}^{n\times n}$ is called an M-matrix if it is a Z-matrix and $A^{-1} \geq 0$, an H-matrix if its comparison matrix $\langle A\rangle$ is an M-matrix, and an H_+-matrix if it is an H-matrix with positive diagonal entries; see [1, 8, 22]. Let $A, B \in \mathbb{R}^{n\times n}$ be M-matrices, $D \in \mathbb{R}^{n\times n}$ be a positive diagonal matrix, and $C \in \mathbb{R}^{n\times n}$. Then $A \leq B$ implies $B^{-1} \leq A^{-1}$, and $A \leq C \leq D$ implies that C is an M-matrix.

For a given matrix $A \in \mathbb{R}^{n\times n}$, let $M, N \in \mathbb{R}^{n\times n}$ be such that $A = M - N$. Then $A = M - N$ is called a splitting of the matrix A if M is nonsingular. The splitting $A = M - N$ is called a convergent splitting if $\rho(M^{-1}N) < 1$. It is called a weak regular splitting if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$; a regular splitting if $M^{-1} \geq 0$ and $N \geq 0$; an M-splitting if M is an M-matrix and $N \geq 0$; an H-splitting if $\langle M\rangle - |N|$ is an M-matrix; and an H-compatible splitting if $\langle A\rangle = \langle M\rangle - |N|$; see [1, 15]. Evidently, if $A = M - N$ is an H-splitting, then A and M are H-matrices and $\rho(M^{-1}N) \leq \rho(\langle M\rangle^{-1}|N|) < 1$; and if it is an H-compatible splitting and A is an H-matrix, then it is an H-splitting and thus convergent.

The following lemma presents basic and useful properties of an H-matrix.

Lemma 2.1 [8, 14] *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix, $D = \mathrm{diag}(A)$ be the diagonal matrix of A, and $B = D - A$. Then the following statements hold true:*

(i) *A is nonsingular;*
(ii) *$|A^{-1}| \leq \langle A\rangle^{-1}$;*
(iii) *$|D|$ is nonsingular and $\rho(|D|^{-1}|B|) < 1$.*

If the system matrix $A \in \mathbb{R}^{n\times n}$ is an H_+-matrix, the LCP(q, A) has a unique solution. This result was proved in [3] and is precisely described in the following lemma.

Lemma 2.2 [3–5] *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix. Then the LCP(q, A) has a unique solution for any $q \in \mathbb{R}^n$.*

By making use of modulus of a vector, the LCP(q, A) can be equivalently transformed into a system of fixed-point equations. This result is exactly stated below.

Lemma 2.3 [2] *Let $A = M - N$ be a splitting of the matrix $A \in \mathbb{R}^{n\times n}$, Ω be a positive diagonal matrix, and γ a positive constant. Then for the LCP(q, A) the following statements hold true:*

(i) *if (z, r) is a solution of the LCP(q, A), then $x = \frac{1}{2}\gamma(z - \Omega^{-1}r)$ satisfies the implicit fixed-point equation*

$$(\Omega + M)x = Nx + (\Omega - A)|x| - \gamma q; \tag{2}$$

(ii) *if x satisfies the implicit fixed-point equation (2), then*

$$z = \gamma^{-1}(|x| + x) \quad \text{and} \quad r = \gamma^{-1}\Omega(|x| - x)$$

is a solution of the LCP(q, A).

3 The MSTM iteration methods

Let (M_k, N_k, E_k) $(k = 1, 2, \ldots, \ell)$ be a multisplitting of the system matrix A of the LCP(q, A). For $k = 1, 2, \ldots, \ell$, let $M_k = F_k - G_k$ be splittings of the matrices $M_k \in \mathbb{R}^{n\times n}$. Then we call the collection $(M_k : F_k, G_k; N_k; E_k)$ $(k = 1, 2, \ldots, \ell)$ a two-stage multisplitting of the matrix A; see [6, 21].

Given a positive diagonal matrix Ω and a positive constant γ, from Lemma 2.3 we straightforwardly know that if x satisfies each of the ℓ implicit fixed-point equations

$$(\Omega + M_k)x = N_k x + (\Omega - A)|x| - \gamma q, \qquad k = 1, 2, \ldots, \ell, \tag{3}$$

then

$$z = \gamma^{-1}(|x| + x) \quad \text{and} \quad r = \gamma^{-1}\Omega(|x| - x) \tag{4}$$

is a solution of the LCP(q, A).

Based on the above concept and the equivalent formulation (3) and (4), we can establish the MSTM iteration method for solving the LCP(q, A) as follows.

Method 3.1 (The MSTM iteration method for LCP(q, A)) Let $(M_k : F_k, G_k; N_k; E_k)$ $(k = 1, 2, \ldots, \ell)$ be a two-stage multisplitting of the system matrix $A \in \mathbb{R}^{n\times n}$, and v_k $(k = 1, 2, \ldots, \ell)$ be prescribed positive integers. Given an initial vector $x^{(0)} \in \mathbb{R}^n$, for $m = 0, 1, 2, \ldots$ until the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty} \subset \mathbb{R}^n$ is convergent, compute $z^{(m+1)} \in \mathbb{R}^n$ by

$$z^{(m+1)} = \frac{1}{\gamma}(|x^{(m+1)}| + x^{(m+1)})$$

and $x^{(m+1)} \in \mathbb{R}^n$ according to

$$x^{(m+1)} = \sum_{k=1}^{\ell} E_k x^{(m,k,v_k)},$$

where $x^{(m,k,\nu_k)}$, $k = 1, 2, \ldots, \ell$, are obtained by solving the linear systems

$$\begin{cases} (\Omega + F_k)x^{(m,k,j+1)} = G_k x^{(m,k,j)} + b^{(m,k)}, \\ \qquad\qquad\qquad k = 1, 2, \ldots, \ell, \quad j = 0, 1, \ldots, \nu_k - 1, \end{cases} \tag{5}$$

respectively, with

$$b^{(m,k)} = N_k x^{(m)} + (\Omega - A)|x^{(m)}| - \gamma q$$

and $x^{(m,k,0)} = x^{(m)}$.

This MSTM iteration method has quite good parallel computational properties. At every iteration step m, each sub-system of linear equations defined by (5) can be solved independently on one processor of the multiprocessor system and, hence, Method 3.1 can be implemented in parallel. The splitting matrices M_k and F_k ($k = 1, 2, \ldots, \ell$), the weighting matrices E_k ($k = 1, 2, \ldots, \ell$), and the positive integers ν_k ($k = 1, 2, \ldots, \ell$) can be chosen in such a way that the tasks distributed on the ℓ processors of the multiprocessor system are well balanced so that Method 3.1 achieves high parallel computing efficiency in actual implementations. Moreover, considerable saving on the computational workload is available, since the entries of $x^{(m,k,\nu_k)}$ corresponding to the zero-diagonal elements of the weighting matrix E_k need not be computed.

We remark that when $\nu_k = 1$ and $G_k = 0$, $k = 1, 2, \ldots, \ell$, the MSTM iteration method naturally reduces to the MSM iteration method.

Method 3.1 is an implicit one and, at every iterate, each processor needs to solve a linear sub-system of the form (5). This makes the MSTM iteration method less convenient in concrete applications. In the following, we will discuss several special explicit forms of Method 3.1, which are convenient for practical implementations.

To this end, we let $D = \mathrm{diag}(A)$ and, for $k = 1, 2, \ldots, \ell$, let $D_k^{(M)} = \mathrm{diag}(M_k)$, $L_k^{(M)}$ be strictly lower-triangular parts of M_k, and $U_k^{(M)}$ be zero-diagonal matrices such that $M_k = D_k^{(M)} - L_k^{(M)} - U_k^{(M)}$. Then the collection $(M_k : D_k^{(M)} - L_k^{(M)}, U_k^{(M)}; N_k; E_k)$ ($k = 1, 2, \ldots, \ell$) is called a two-stage triangular multisplitting of the matrix A. Analogously, when $A \in \mathbb{R}^{n\times n}$ is partitioned into blocks, we naturally admit that the matrices M_k, N_k, F_k and G_k, as well as $D_k^{(M)}$, $L_k^{(M)}$ and $U_k^{(M)}$ have the conformable block partitions, too. Take

$$\begin{cases} F_k = \dfrac{1}{\alpha}\left(D_k^{(M)} - \beta L_k^{(M)}\right), \\ G_k = \dfrac{1}{\alpha}\left((1-\alpha)D_k^{(M)} + (\alpha - \beta)L_k^{(M)} + \alpha U_k^{(M)}\right), \end{cases} \qquad k = 1, 2, \ldots, \ell,$$

in the secondary splittings $M_k = F_k - G_k$ ($k = 1, 2, \ldots, \ell$) in Method 3.1, where α and β are prescribed relaxation parameters. Then we obtain the following *modulus-based synchronous two-stage multisplitting accelerated over-relaxation* (**MSTMAOR**) iteration method.

Method 3.2 (The MSTMAOR iteration method for LCP(q, A)) Let $(M_k : D_k^{(M)} - L_k^{(M)}, U_k^{(M)}; N_k; E_k)$ $(k = 1, 2, \ldots, \ell)$ be a two-stage triangular multisplitting of the system matrix $A \in \mathbb{R}^{n\times n}$, and ν_k $(k = 1, 2, \ldots, \ell)$ be prescribed positive integers. Given an initial vector $x^{(0)} \in \mathbb{R}^n$, for $m = 0, 1, 2, \ldots$ until the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty} \subset \mathbb{R}^n$ is convergent, compute $z^{(m+1)} \in \mathbb{R}^n$ by

$$z^{(m+1)} = \frac{1}{\gamma}\left(\left|x^{(m+1)}\right| + x^{(m+1)}\right)$$

and $x^{(m+1)} \in \mathbb{R}^n$ according to

$$x^{(m+1)} = \sum_{k=1}^{\ell} E_k x^{(m,k,\nu_k)},$$

where $x^{(m,k,\nu_k)}$, $k = 1, 2, \ldots, \ell$, are obtained by solving the linear systems

$$\begin{cases} \left(\alpha\Omega + D_k^{(M)} - \beta L_k^{(M)}\right) x^{(m,k,j+1)} = \left[(1-\alpha) D_k^{(M)} + (\alpha - \beta) L_k^{(M)} + \alpha U_k^{(M)}\right] \\ \qquad\qquad \times x^{(m,k,j)} + \alpha b^{(m,k)}, \\ \qquad\qquad k = 1, 2, \ldots, \ell, \quad j = 0, 1, \ldots, \nu_k - 1, \end{cases}$$

respectively, with

$$b^{(m,k)} = N_k x^{(m)} + (\Omega - A)\left|x^{(m)}\right| - \gamma q$$

and $x^{(m,k,0)} = x^{(m)}$.

In particular, when we choose the parameter pairs (α, β) to be (α, α), $(1, 1)$ and $(1, 0)$, respectively, Method 3.2 gives the *modulus-based synchronous two-stage multisplitting successive overrelaxation* (**MSTMSOR**) iteration method, the *modulus-based synchronous two-stage multisplitting Gauss–Seidel* (**MSTMGS**) iteration method, and the *modulus-based synchronous two-stage multisplitting Jacobi* (**MSTMJ**) iteration method, correspondingly, for solving the LCP(q, A). Hence, Method 3.2 produces an extensive sequence of modulus-based synchronous two-stage multisplitting relaxation methods, which are quite practical and efficient for solving the large sparse linear complementarity problems on the high-speed multiprocessor systems. Moreover, the relaxation parameters (α, β) can be adjusted suitably so that the convergence properties of the modulus-based synchronous two-stage multisplitting AOR method can be improved substantially in actual applications.

4 Convergence theory

In this section, we are going to establish the convergence theory for the modulus-based synchronous two-stage multisplitting iteration method and its relaxed variants, described in Section 3, when the system matrix $A \in \mathbb{R}^{n\times n}$ of the LCP(q, A) is an H_+-matrix.

To this end, we assume that (z_*, r_*) is a solution of the LCP(q, A). Then by Lemma 2.3 we know that

$$x_* = \frac{1}{2}\gamma\left(z_* - \Omega^{-1} r_*\right)$$

satisfies the implicit fixed-point equation (2). For the iteration sequences $\{z^{(m)}\}_{m=0}^{\infty} \subset \mathbb{R}^n$ and $\{x^{(m)}\}_{m=0}^{\infty} \subset \mathbb{R}^n$ generated by Methods 3.1 and 3.2, respectively, we see that $\lim_{m\to\infty} z^{(m)} = z_*$ if and only if $\lim_{m\to\infty} x^{(m)} = x_*$. It follows that in order to prove the convergence of the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$, we only need to prove the convergence of the iteration sequence $\{x^{(m)}\}_{m=0}^{\infty}$ instead.

Because A is an H_+-matrix, with the notations $D = \mathrm{diag}(A)$ and $B = D - A$ we see that $|D| = D$. Moreover, from Lemma 2.1(iii) we know that $\rho(D^{-1}|B|) < 1$ holds true. Denote by $J = D^{-1}|B|$ and $J_\varepsilon = J + \varepsilon e e^T$, where $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ is the vector of entries being all equal to one and $\varepsilon > 0$ is an arbitrary small number such that $\rho_\varepsilon := \rho(J_\varepsilon) < 1$. Then $\langle A\rangle = D - |B| = D(I - J)$ and, by the Perron–Frobenius theorem [22], there exists a positive vector $v_\varepsilon \in \mathbb{R}^n$ such that $J_\varepsilon v_\varepsilon = \rho_\varepsilon v_\varepsilon$.

Let $(M_k : F_k, G_k; N_k; E_k)$ $(k = 1, 2, \ldots, \ell)$ and $(M_k : D_k^{(M)} - L_k^{(M)}, U_k^{(M)}; N_k; E_k)$ $(k = 1, 2, \ldots, \ell)$ be a two-stage multisplitting and a two-stage triangular multisplitting of the matrix A, respectively. Then we have the following facts:

(a) For the MSTM iteration method, it holds that

$$(\Omega + F_k)x_* = G_k x_* + N_k x_* + (\Omega - A)|x_*| - \gamma q, \qquad k = 1, 2, \ldots, \ell; \tag{6}$$

(b) For the MSTMAOR iteration method, it holds that

$$\begin{aligned}\left(\alpha\Omega + D_k^{(M)} - \beta L_k^{(M)}\right) x_* = &\left[(1-\alpha) D_k^{(M)} + (\alpha - \beta) L_k^{(M)} + \alpha U_k^{(M)}\right] x_* \\ &+ \alpha(N_k x_* + (\Omega - A)|x_*| - \gamma q), \\ &k = 1, 2, \ldots, \ell. \end{aligned} \tag{7}$$

Based on the above preparation, now we demonstrate the convergence of the MSTM and the MSTMAOR iteration methods, i.e., Methods 3.1 and 3.2.

Theorem 4.1 *Let $A \in \mathbb{R}^{n\times n}$ be an H_+-matrix, with $D = \mathrm{diag}(A)$ and $B = D - A$, and let $(M_k : F_k, G_k; N_k; E_k)$ $(k = 1, 2, \ldots, \ell)$ and $(M_k : D_k^{(M)} - L_k^{(M)}, U_k^{(M)}; N_k; E_k)$ $(k = 1, 2, \ldots, \ell)$ be a two-stage multisplitting and a two-stage triangular multisplitting of the matrix A, respectively. Assume that $\gamma > 0$ and the positive diagonal matrix Ω satisfies $\Omega \geq D$.*

(i) *If $A = M_k - N_k$ and $M_k = F_k - G_k$, $k = 1, 2, \ldots, \ell$, are H-compatible splittings, then the iteration sequence $\{z^{(m)}\}_{m=0}^{\infty}$ generated by Method 3.1 converges to the unique solution z_* of the LCP(q, A) for any initial vector $x^{(0)} \in \mathbb{R}^n$ and any positive integers v_k $(k = 1, 2, \ldots, \ell)$ not less than 1.*

(ii) *If, for* $k = 1, 2, \ldots, \ell$, $A = M_k - N_k$ *are H-compatible splittings and* $M_k = D - L_k^{(M)} - U_k^{(M)}$ *satisfy* $\langle M_k \rangle = D - |L_k^{(M)}| - |U_k^{(M)}|$ *with* $\mathrm{diag}(M_k) = D$, *then the iteration sequence* $\{z^{(m)}\}_{m=0}^{\infty}$ *generated by Method* 3.2 *converges to the unique solution* z_* *of the LCP(q, A) for any initial vector* $x^{(0)} \in \mathbb{R}^n$ *and any positive integers* ν_k $(k = 1, 2, \ldots, \ell)$ *not less than* 1, *provided the relaxation parameters* α *and* β *satisfy*

$$0 < \beta \le \alpha < \frac{1}{\rho(D^{-1}|B|)}.$$

Proof From the definition of Method 3.1 we have

$$x^{(m+1)} = \sum_{k=1}^{\ell} E_k \left[\left((\Omega + F_k)^{-1} G_k \right)^{\nu_k} x^{(m)} + \sum_{i=0}^{\nu_k - 1} \left((\Omega + F_k)^{-1} G_k \right)^i \cdot (\Omega + F_k)^{-1} \left(N_k x^{(m)} + (\Omega - A)|x^{(m)}| - \gamma q \right) \right], \tag{8}$$

and from the system of fixed-point equation (6) we obtain

$$x_* = \sum_{k=1}^{\ell} E_k \left[\left((\Omega + F_k)^{-1} G_k \right)^{\nu_k} x_* + \sum_{i=0}^{\nu_k - 1} \left((\Omega + F_k)^{-1} G_k \right)^i \cdot (\Omega + F_k)^{-1} \left(N_k x_* + (\Omega - A)|x_*| - \gamma q \right) \right]. \tag{9}$$

Subtracting (9) from (8) results in the error about the MSTM iteration method as follows:

$$x^{(m+1)} - x_* = \sum_{k=1}^{\ell} E_k \left[\left((\Omega + F_k)^{-1} G_k \right)^{\nu_k} (x^{(m)} - x_*) + \sum_{i=0}^{\nu_k - 1} \left((\Omega + F_k)^{-1} G_k \right)^i \cdot (\Omega + F_k)^{-1} \left(N_k (x^{(m)} - x_*) + (\Omega - A)(|x^{(m)}| - |x_*|) \right) \right]. \tag{10}$$

Analogously, based on the definition of Method 3.2 and the system of fixed-point equations (7) we can obtain the error about the MSTMAOR iteration method as follows:

$$
\begin{aligned}
&x^{(m+1)}-x_* \\
&= \sum_{k=1}^{\ell} E_k \Bigg[\left(\left(\alpha\Omega + D - \beta L_k^{(M)} \right)^{-1} \left[(1-\alpha) D + (\alpha-\beta) L_k^{(M)} + \alpha U_k^{(M)} \right] \right)^{\nu_k} \\
&\qquad \times \left(x^{(m)} - x_* \right) \\
&\qquad + \sum_{i=0}^{\nu_k - 1} \left(\left(\alpha\Omega + D - \beta L_k^{(M)} \right)^{-1} \left[(1-\alpha) D + (\alpha-\beta) L_k^{(M)} + \alpha U_k^{(M)} \right] \right)^{i} \\
&\qquad \cdot \alpha \left(\alpha\Omega + D - \beta L_k^{(M)} \right)^{-1} \\
&\qquad \times \left(N_k (x^{(m)} - x_*) + (\Omega - A)(|x^{(m)}| - |x_*|) \right) \Bigg]. \qquad (11)
\end{aligned}
$$

The error relationships (10) and (11) are the bases for proving the convergence of Methods 3.1 and 3.2, respectively.

We first demonstrate the validity of (i). As $A = M_k - N_k$ $(k = 1, 2, \ldots, \ell)$ are H-compatible splittings of the H_+-matrix A, we know that M_k $(k = 1, 2, \ldots, \ell)$ are H_+-matrices. Similarly, as $M_k = F_k - G_k$ $(k = 1, 2, \ldots, \ell)$ are H-compatible splittings of the H_+-matrices M_k and Ω is a positive diagonal matrix, we see that F_k and, hence, $\Omega + F_k$, $k = 1, 2, \ldots, \ell$, are H_+-matrices. Therefore, from Lemma 2.1(ii) we have

$$
\left| (\Omega + F_k)^{-1} \right| \le \langle \Omega + F_k \rangle^{-1} = (\Omega + \langle F_k \rangle)^{-1}, \qquad k = 1, 2, \ldots, \ell. \qquad (12)
$$

By taking absolute values on both sides of the equality (10), making use of the estimates (12) and the inequality $||x^{(m)}| - |x_*|| \le |x^{(m)} - x_*|$, and arranging similar terms together, we obtain

$$
\begin{aligned}
\left| x^{(m+1)} - x_* \right| &\le \sum_{k=1}^{\ell} E_k \Bigg[\left(\left| (\Omega + F_k)^{-1} \right| |G_k| \right)^{\nu_k} + \sum_{i=0}^{\nu_k - 1} \left(\left| (\Omega + F_k)^{-1} \right| |G_k| \right)^{i} \\
&\qquad \cdot \left| (\Omega + F_k)^{-1} \right| (|N_k| + |\Omega - A|) \Bigg] \left| x^{(m)} - x_* \right| \\
&\le \sum_{k=1}^{\ell} E_k \Bigg[\left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{\nu_k} + \sum_{i=0}^{\nu_k - 1} \left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{i} \\
&\qquad \cdot (\Omega + \langle F_k \rangle)^{-1} (|N_k| + |\Omega - A|) \Bigg] \left| x^{(m)} - x_* \right| \\
&:= \mathcal{L}_{\mathrm{MSTM}} \left| x^{(m)} - x_* \right|, \qquad (13)
\end{aligned}
$$

where

$$\mathcal{L}_{\mathrm{MSTM}} = \sum_{k=1}^{\ell} E_k \left[\left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{\nu_k} + \sum_{i=0}^{\nu_k - 1} \left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{i} \cdot (\Omega + \langle F_k \rangle)^{-1} \left(|N_k| + |\Omega - A| \right) \right]. \tag{14}$$

Again, as $A = M_k - N_k$ and $M_k = F_k - G_k$, $k = 1, 2, \ldots, \ell$, are H-compatible splittings, i.e.,

$$\langle A \rangle = \langle M_k \rangle - |N_k| \quad \text{and} \quad \langle M_k \rangle = \langle F_k \rangle - |G_k|, \qquad k = 1, 2, \ldots, \ell,$$

it holds for $k = 1, 2, \ldots, \ell$ that

$$|N_k| = \langle F_k \rangle - |G_k| - \langle A \rangle = \langle F_k \rangle - |G_k| - D + |B|. \tag{15}$$

Denote by $D_k^{(F)} = \mathrm{diag}(F_k)$, $k = 1, 2, \ldots, \ell$. Then $D_k^{(F)}$ $(k = 1, 2, \ldots, \ell)$ are positive diagonal matrices. As Ω is also a positive diagonal matrix, we immediately see that $\Omega + \mathrm{D}_k^F (k = 1, 2, \ldots, \ell)$ are also positive diagonal matrices and

$$(\Omega + \langle F_k \rangle)^{-1} \geq (\Omega + D_k^{(F)})^{-1}, \qquad k = 1, 2, \ldots, \ell. \tag{16}$$

Therefore, from (14) and (15) we have

$$\begin{aligned}
\mathcal{L}_{\mathrm{MSTM}} &= \sum_{k=1}^{\ell} E_k \left[\left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{\nu_k} + \sum_{i=0}^{\nu_k - 1} \left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{i} \right. \\
&\qquad \left. \cdot (\Omega + \langle F_k \rangle)^{-1} \left((\Omega + \langle F_k \rangle) - |G_k| - 2(D - |B|) \right) \right] \\
&= \sum_{k=1}^{\ell} E_k \left[\left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{\nu_k} + \sum_{i=0}^{\nu_k - 1} \left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{i} \right. \\
&\qquad \left. \cdot \left(I - (\Omega + \langle F_k \rangle)^{-1} (|G_k| + 2(D - |B|)) \right) \right] \\
&= I - 2 \sum_{k=1}^{\ell} E_k \sum_{i=0}^{\nu_k - 1} \left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{i} (\Omega + \langle F_k \rangle)^{-1} (D - |B|) \\
&\leq I - 2 \sum_{k=1}^{\ell} E_k \sum_{i=0}^{\nu_k - 1} \left((\Omega + \langle F_k \rangle)^{-1} |G_k| \right)^{i} (\Omega + \langle F_k \rangle)^{-1} D (I - J_\varepsilon).
\end{aligned}$$

It then follows that

$$\begin{aligned}\mathcal{L}_{\mathrm{MSTM}} v_\varepsilon &\le v_\varepsilon - 2\sum_{k=1}^{\ell} E_k \sum_{i=0}^{\nu_k-1} \left((\Omega+\langle F_k\rangle)^{-1}|G_k|\right)^i (\Omega+\langle F_k\rangle)^{-1} D(I-J_\varepsilon)v_\varepsilon \\ &= v_\varepsilon - 2(1-\rho_\varepsilon)\sum_{k=1}^{\ell} E_k \sum_{i=0}^{\nu_k-1} \left((\Omega+\langle F_k\rangle)^{-1}|G_k|\right)^i (\Omega+\langle F_k\rangle)^{-1} Dv_\varepsilon. \end{aligned} \tag{17}$$

After substituting (16) into (17) and noticing $\rho_\varepsilon < 1$, we can further obtain

$$\begin{aligned}\mathcal{L}_{\mathrm{MSTM}} v_\varepsilon &\le v_\varepsilon - 2(1-\rho_\varepsilon)\sum_{k=1}^{\ell} E_k(\Omega+\langle F_k\rangle)^{-1} Dv_\varepsilon \\ &\le v_\varepsilon - 2(1-\rho_\varepsilon)\sum_{k=1}^{\ell} E_k(\Omega+D_k^{(F)})^{-1} Dv_\varepsilon \\ &< v_\varepsilon. \end{aligned}$$

Therefore, $\rho(\mathcal{L}_{\mathrm{MSTM}}) < 1$ and, according to (13) and (14), we immediately know that the iteration sequence $\{x^{(m)}\}_{m=0}^{\infty}$ and, thereby, $\{z^{(m)}\}_{m=0}^{\infty}$, generated by Method 3.1, converges to the unique solution z_* of the LCP(q, A) for any initial vector $x^{(0)} \in \mathbb{R}^n$ and any positive integers ν_k ($k = 1, 2, \ldots, \ell$) not less than 1.

Now, we demonstrate the validity of (ii). By taking absolute values on both sides of the equality (11), making use of Lemma 2.1(ii) and the estimate $||x^{(m)}| - |x_*|| \le |x^{(m)} - x_*|$, and arranging similar terms together, we obtain

$$|x^{(m+1)} - x_*| \le \mathcal{L}_{\mathrm{MSTMAOR}}(\alpha,\beta)|x^{(m)} - x_*|, \tag{18}$$

where

$$\begin{aligned}&\mathcal{L}_{\mathrm{MSTMAOR}}(\alpha,\beta) \\ &= \sum_{k=1}^{\ell} E_k \left[\left(\left(\alpha\Omega + D - \beta\left|L_k^{(M)}\right|\right)^{-1}\left[|1-\alpha|D + (\alpha-\beta)|L_k^{(M)}| + \alpha|U_k^{(M)}|\right]\right)^{\nu_k}\right. \\ &\quad + \sum_{i=0}^{\nu_k-1}\left(\left(\alpha\Omega + D - \beta|L_k^{(M)}|\right)^{-1}\left[|1-\alpha|D + (\alpha-\beta)|L_k^{(M)}| + \alpha|U_k^{(M)}|\right]\right)^i \\ &\quad \left.\cdot\, \alpha(\alpha\Omega + D - \beta|L_k^{(M)}|)^{-1}(|N_k| + |\Omega - A|)\right]. \end{aligned} \tag{19}$$

Because $|\Omega - A| = (\Omega - D) + |B|$ and

$$\langle A\rangle = \langle M_k\rangle - |N_k|, \quad \langle M_k\rangle = D - |L_k^{(M)}| - |U_k^{(M)}|, \qquad k = 1, 2, \ldots, \ell,$$

it holds for $k = 1, 2, \ldots, \ell$ that

$$\alpha\,(|N_k| + |\Omega - A|) = \left(\alpha\Omega + D - \beta|L_k^{(M)}|\right) - \left[(1+\alpha)D + (\alpha-\beta)|L_k^{(M)}| + \alpha|U_k^{(M)}| - 2\alpha|B|\right]$$

and

$$\begin{aligned}&\alpha(\alpha\Omega + D - \beta|L_k^{(M)}|)^{-1}(|N_k| + |\Omega - A|)\\ &= I - (\alpha\Omega + D - \beta|L_k^{(M)}|)^{-1}\\ &\quad\cdot\left[(|1-\alpha|D + (\alpha-\beta)|L_k^{(M)}| + \alpha|U_k^{(M)}|) + ((1+\alpha-|1-\alpha|)D - 2\alpha|B|)\right].\end{aligned}$$

It then follows from (19) that

$$\begin{aligned}&\mathcal{L}_{\mathrm{MSTMAOR}}(\alpha,\beta)v_\varepsilon\\ &= v_\varepsilon - \sum_{k=1}^{\ell} E_k \sum_{i=0}^{v_k-1}\left(\left(\alpha\Omega + D - \beta\left|L_k^{(M)}\right|\right)^{-1}\right.\\ &\qquad\left.\times\left[|1-\alpha|D + (\alpha-\beta)\left|L_k^{(M)}\right| + \alpha\left|U_k^{(M)}\right|\right]\right)^i\\ &\qquad\cdot\left(\alpha\Omega + D - \beta\left|L_k^{(M)}\right|\right)^{-1}\left[(1+\alpha-|1-\alpha|)D - 2\alpha|B|\right]v_\varepsilon\\ &\le v_\varepsilon - \sum_{k=1}^{\ell} E_k \sum_{i=0}^{v_k-1}\left(\left(\alpha\Omega + D - \beta\left|L_k^{(M)}\right|\right)^{-1}\right.\\ &\qquad\left.\times\left[|1-\alpha|D + (\alpha-\beta)\left|L_k^{(M)}\right| + \alpha\left|U_k^{(M)}\right|\right]\right)^i\\ &\qquad\cdot\left(\alpha\Omega + D - \beta\left|L_k^{(M)}\right|\right)^{-1} D\left[(1+\alpha-|1-\alpha|)I - 2\alpha J_\varepsilon\right]v_\varepsilon\\ &= v_\varepsilon - \theta_\varepsilon \sum_{k=1}^{\ell} E_k \sum_{i=0}^{v_k-1}\left(\left(\alpha\Omega + D - \beta\left|L_k^{(M)}\right|\right)^{-1}\right.\\ &\qquad\left.\times\left[|1-\alpha|D + (\alpha-\beta)\left|L_k^{(M)}\right| + \alpha\left|U_k^{(M)}\right|\right]\right)^i\\ &\qquad\cdot\left(\alpha\Omega + D - \beta\left|L_k^{(M)}\right|\right)^{-1} Dv_\varepsilon,\end{aligned} \tag{20}$$

where

$$\theta_\varepsilon = 1 + \alpha - |1-\alpha| - 2\alpha\rho_\varepsilon.$$

As $0 < \alpha < \frac{1}{\rho(D^{-1}|B|)}$ implies

$$\theta := 1 + \alpha - |1-\alpha| - 2\alpha\rho(J) > 0,$$

by continuity of the spectral radius of a matrix we can take ε sufficiently small such that $\theta_\varepsilon > 0$, too. Thereby, from (20) we can further obtain the estimate

$$\begin{aligned}\mathcal{L}_{\mathrm{MSTMAOR}}(\alpha,\beta)v_\varepsilon &\le v_\varepsilon - \theta_\varepsilon \sum_{k=1}^{\ell} E_k \left(\alpha\Omega + D - \beta\left|L_k^{(M)}\right|\right)^{-1} Dv_\varepsilon \\ &\le v_\varepsilon - \theta_\varepsilon \sum_{k=1}^{\ell} E_k(\alpha\Omega + D)^{-1} Dv_\varepsilon \\ &= v_\varepsilon - \theta_\varepsilon(\alpha\Omega + D)^{-1} Dv_\varepsilon \\ &< v_\varepsilon.\end{aligned}$$

Therefore, $\rho(\mathcal{L}_{\mathrm{MSTMAOR}}(\alpha,\beta)) < 1$ and, according to (18) and (19), we immediately know that the iteration sequence $\{x^{(m)}\}_{m=0}^{\infty}$ and, thereby, $\{z^{(m)}\}_{m=0}^{\infty}$, generated by Method 3.2, converges to the unique solution z_* of the LCP(q, A) for any initial vector $x^{(0)} \in \mathbb{R}^n$ and any positive integers v_k ($k = 1, 2, \ldots, \ell$) not less than 1. □

We remark that under the conditions of Theorem 4.1, the MSTMGS and the MSTMJ iteration methods are convergent, and the MSTMSOR iteration method is convergent if the iteration parameter α satisfies $0 < \alpha < \frac{1}{\rho(D^{-1}|B|)}$.

5 Numerical results

In this section, we use two numerical examples to examine the parallel computing efficiency, defined by $\mathcal{E}_\ell = T_1/(\ell T_\ell)$, of the modulus-based synchronous two-stage multisplitting relaxation methods, where T_1 and T_ℓ denote the elapsed wall times in seconds for solving the LCP(q, A) on sequential and parallel (with ℓ processors) computers, respectively.

We write the codes in C and MPICH2, perform the experiments on the PC clusters in which each computing node consists of two X5550 CPUs (four processors and 2.67 GHz for each CPU) with 24 GB of memory, and employ local communication to exchange messages among the processors. In actual implementations, all iterations are started from the initial vector $x^{(0)} = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ and terminated when the current residuals satisfy

$$\mathrm{RES}(z^{(m)}) := \left\|\min\left\{Az^{(m)} + q, \quad z^{(m)}\right\}\right\|_2 < 10^{-5},$$

where the minimum is taken componentwisely. The multisplitting (M_k, N_k, E_k) ($k = 1, 2, \ldots, \ell$) of the system matrix $A \in \mathbb{R}^{n\times n}$ of the LCP(q, A) is taken analogously to that in [7]. More specifically, $M_k = D - E_k B E_k$, $k = 1, 2, \ldots, \ell$, where $D = \mathrm{diag}(A)$ and $B = D - A$. Then the two-stage triangular multisplitting ($M_k : D_k^{(M)} - L_k^{(M)}, U_k^{(M)}; N_k; E_k$) ($k = 1, 2, \ldots, \ell$) of the matrix A is chosen such that $D_k^{(M)} = D$, and $L_k^{(M)}$ and $U_k^{(M)}$ are, respectively, the strictly lower-triangular and the strictly upper-triangular matrices of M_k for

$k = 1, 2, \ldots, \ell$. In addition, we set $\gamma = 2$ and $\Omega = D$, and choose the iteration parameters α in MSTMSOR such that the number of total iteration steps is minimized; see Tables 1 and 3. We also choose $\mu_k = 2$ $(k = 1, 2, \ldots, \ell)$, which are the experimentally found best numbers of inner iteration steps.

Example 5.1 The LCP(q, A) is given by

$$A = \begin{bmatrix} S & -I & -I & & \\ & S & -I & \ddots & \\ & & S & \ddots & -I \\ & & & \ddots & -I \\ & & & & S \end{bmatrix} \in \mathbb{R}^{n \times n}$$

and $q = (-1, 1, -1, 1, \ldots) \in \mathbb{R}^n$, where $n = n_o^2$, with n_o being a positive integer, and $S = \text{tridiag}(-1, 4, -1) \in \mathbb{R}^{n_o \times n_o}$ is a tridiagonal matrix with diagonal entries being equal to 4 and off-diagonal entries being equal to -1.

The experimentally found optimal parameters α for MSTMSOR for Example 5.1 are shown in Table 1. From this table we see that for each n_o the range of this parameter changes with respect to the number of processors.

The elapsed wall time and the parallel computing efficiency of MSM, MSTMGS and MSTMSOR iteration methods for solving the LCP(q, A) given by Example 5.1 are listed in Table 2.

From this table we have observed that for almost all values of n_o and ℓ MSM costs much more computing time than MSTMGS and MSTMSOR, and MSTMGS costs slightly more computing time than MSTMSOR. And for a fixed value of n_o the computing time of MSM is decreasing with a reduction factor more than $\frac{1}{2}$, while those of MSTMGS and MSTMSOR are decreasing with a reduction factor a little bit less than $\frac{1}{2}$, when the number ℓ of processors doubles.

The parallel computing efficiency of almost all tested cases exceeds 0.6, and some of them even exceed 1.0. For a fixed value of n_o, the parallel computing efficiency of MSM is increasing with the number ℓ of processors. For each n_o, the parallel computing efficiency of MSTMGS and MSTMSOR is not monotone with respect to the number of processors due to the memory systems of the PC clusters, as well as different communication ways, data sizes, and workload overlaps with respect to different processor numbers. The highest

Table 1 The experimentally found optimal parameters α for MSTMSOR for Example 5.1

n_o	ℓ							
	1	2	4	8	16	32	64	128
512	[2.8, 2.9]	[2.6, 3.1]	[2.5, 3.1]	[2.5, 3.1]	2.7	[2.3, 2.9]	[2.2, 2.6]	[2.1, 2.2]
1,024	[2.9, 3.3]	[3.0, 3.1]	[3.0, 3.1]	[2.9, 3.1]	[2.8, 3.1]	[2.7, 3.2]	[2.7, 2.8]	[2.3, 2.7]
2,048	[3.1, 3.5]	[3.2, 3.4]	[3.1, 3.4]	[3.1, 3.4]	[3.1, 3.4]	[3.0, 3.4]	[2.9, 3.2]	2.9

Table 2 Numerical results for MSM, MSTMGS and MSTMSOR methods for Example 5.1

n_o	Method	ℓ	1	2	4	8	16	32	64	128
512	MSM	T_ℓ	89.77	43.97	20.24	8.74	3.64	1.28	0.49	0.23
		$\mathcal{E}_\ell$	–	1.02	1.11	1.28	1.54	2.19	2.86	3.05
	MSTMGS	T_ℓ	8.84	5.21	3.47	1.73	0.56	0.25	0.14	0.10
		$\mathcal{E}_\ell$	–	0.85	0.64	0.64	0.99	1.11	0.99	0.69
	MSTMSOR	T_ℓ	8.58	4.87	3.04	1.52	0.56	0.26	0.14	0.10
		$\mathcal{E}_\ell$	–	0.88	0.71	0.71	0.96	1.03	0.96	0.67
1,024	MSM	T_ℓ	752.73	364.68	176.34	80.63	36.74	15.29	6.41	2.24
		$\mathcal{E}_\ell$	–	1.03	1.07	1.17	1.28	1.54	1.83	2.63
	MSTMGS	T_ℓ	67.06	39.45	26.29	13.63	6.88	3.34	1.14	0.57
		$\mathcal{E}_\ell$	–	0.85	0.64	0.62	0.61	0.63	0.92	0.92
	MSTMSOR	T_ℓ	64.50	36.43	22.74	11.75	5.95	2.92	1.11	0.57
		$\mathcal{E}_\ell$	–	0.89	0.71	0.69	0.68	0.69	0.91	0.88
2,048	MSM	T_ℓ	10,725.01	5,065.48	1,890.34	842.14	345.74	151.88	66.52	27.56
		$\mathcal{E}_\ell$	–	1.06	1.42	1.59	1.94	2.21	2.52	3.04
	MSTMGS	T_ℓ	515.81	303.62	202.57	105.17	53.10	26.62	13.49	6.69
		$\mathcal{E}_\ell$	–	0.85	0.64	0.61	0.61	0.61	0.60	0.60
	MSTMSOR	T_ℓ	491.96	277.28	173.27	89.98	45.34	22.81	11.59	5.82
		$\mathcal{E}_\ell$	–	0.89	0.71	0.68	0.68	0.67	0.66	0.66

parallel computing efficiency of MSTMGS and MSTMSOR are attained at $\ell = 32, 64$ and 2 when $n_o = 512, 1{,}024$ and $2{,}048$, respectively.

Example 5.2 The LCP(q, A) is yielded from the nine-point finite difference approximation on a uniform grid with the mesh size $h = 16/2^\tau$ of the following free boundary problem modeling the flow of water through a porous dam: Find $\mathbf{u}$ on the rectangle domain $[0, 16] \times [0, 24]$ such that in the domain

$$\mathbf{u} \geq 0, \quad \mathbf{u}_{ss} + \mathbf{u}_{tt} \leq 1 \quad \text{and} \quad \mathbf{u}(\mathbf{u}_{ss} + \mathbf{u}_{tt} - 1) = 0,$$

and on the boundary

$$\mathbf{u} = \begin{cases} (24 - t)^2/2, & \text{for} \quad s = 0, \quad 0 < t \leq 24, \\ (4 - t)^2/2, & \text{for} \quad s = 16, \quad 0 < t \leq 4, \\ (24^2(16 - s) + 4^2 s)/32, & \text{for} \quad 0 \leq s \leq 16, \ t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

See [7] for the concrete structure of the system matrix A.

The experimentally found optimal parameters α for MSTMSOR for Example 5.2 are shown in Table 3. From this table we see that for each τ this

Table 3 The experimentally found optimal parameters α for MSTMSOR for Example 5.2

τ	Grid	ℓ							
		1	2	4	8	16	32	64	128
6	65×97	2.0	2.0	2.0	1.9	[1.8, 1.9]	1.7	–	–
7	129×193	2.0	2.0	2.0	2.0	1.9	1.9	1.7	–
8	257×385	2.0	2.0	2.0	2.0	2.0	1.9	1.9	1.7

Table 4 Numerical results for MSM, MSTMGS and MSTMSOR methods for Example 5.2

τ	Method	ℓ	1	2	4	8	16	32	64	128
6	MSM	T_ℓ	13.95	6.91	3.33	1.51	0.64	0.43	–	–
		$\mathcal{E}_\ell$	–	1.01	1.05	1.15	1.36	1.01	–	–
	MSTMGS	T_ℓ	3.14	1.68	0.86	0.46	0.28	0.23	–	–
		$\mathcal{E}_\ell$	–	0.93	0.91	0.85	0.70	0.43	–	–
	MSTMSOR	T_ℓ	2.79	1.50	0.77	0.42	0.26	0.21	–	–
		$\mathcal{E}_\ell$	–	0.93	0.91	0.83	0.67	0.42	–	–
7	MSM	T_ℓ	352.58	170.74	83.48	30.63	13.92	5.49	3.20	–
		$\mathcal{E}_\ell$	–	1.03	1.06	1.44	1.58	2.01	1.72	–
	MSTMGS	T_ℓ	49.35	25.91	13.13	6.75	3.64	2.12	1.62	–
		$\mathcal{E}_\ell$	–	0.95	0.94	0.91	0.85	0.73	0.48	–
	MSTMSOR	T_ℓ	43.88	23.08	11.72	6.05	3.29	1.94	1.49	–
		$\mathcal{E}_\ell$	–	0.95	0.94	0.91	0.83	0.71	0.46	–
8	MSM	T_ℓ	7,620.86	3,792.76	2,024.22	846.90	325.29	128.74	41.87	22.08
		$\mathcal{E}_\ell$	–	1.00	0.94	1.12	1.46	1.85	2.84	2.70
	MSTMGS	T_ℓ	952.51	572.42	377.23	135.01	52.84	30.16	15.21	10.82
		$\mathcal{E}_\ell$	–	0.83	0.63	0.88	1.13	0.99	0.98	0.69
	MSTMSOR	T_ℓ	846.77	509.41	336.09	120.53	47.33	27.32	13.89	10.11
		$\mathcal{E}_\ell$	–	0.83	0.63	0.88	1.12	0.97	0.95	0.65

parameter first remains unchanged and then decreases with respect to the number of processors.

The elapsed wall time and the parallel computing efficiency of MSM, MSTMGS and MSTMSOR iteration methods for solving the LCP(q, A) given by Example 5.2 are listed in Table 4.

From this table we have observed that for almost all values of τ and ℓ MSM costs much more computing time than MSTMGS and MSTMSOR, and MSTMGS costs slightly more computing time than MSTMSOR. And for a fixed value of τ the computing time of MSM is almost always decreasing with a reduction factor more than $\frac{1}{2}$, while those of MSTMGS and MSTMSOR are decreasing with a reduction factor a little bit less than $\frac{1}{2}$, when the number ℓ of processors doubles.

The parallel computing efficiency of almost all tested cases exceeds 0.5, and some of them even exceed 1.0. For a fixed value of τ, the parallel computing efficiency of MSM is almost increasing with the number ℓ of processors. When $\tau = 6$ and 7, the parallel computing efficiency of MSTMGS and MSTMSOR is decreasing and, when $\tau = 8$, it is not monotone, with respect to the number ℓ of processors. When $\tau = 8$, the highest parallel computing efficiency of MSTMGS and MSTMSOR attained at $\ell = 16$ is 1.13 and 1.12, respectively.

6 Concluding remarks

We have established modulus-based synchronous two-stage multisplitting iteration methods for solving the large sparse linear complementarity problems. This class of multisplitting iteration methods includes a sequence of modulus-based synchronous two-stage multisplitting relaxation methods as

special cases, and is suitable for implementing on the modern high-speed multiprocessor systems. Moreover, these modulus-based synchronous multisplitting iteration methods and their relaxed variants are convergent when the system matrix of the linear complementarity problem is an H_+-matrix. The established theoretical conclusions equally hold true for linear complementarity problems with strictly or irreducibly diagonally dominant system matrices [10–12]. Numerical results have shown that in terms of computing time the modulus-based synchronous two-stage multisplitting Gauss–Seidel and SOR methods are more efficient than the modulus-based synchronous multisplitting iteration method in actual implementations.

Acknowledgements The authors are very much indebted to the referees for providing very useful comments and suggestions, which greatly improved the original manuscript of this paper.

References

1. Bai, Z.-Z.: On the convergence of the multisplitting methods for the linear complementarity problem. SIAM J. Matrix Anal. Appl. **21**, 67–78 (1999)
2. Bai, Z.-Z.: Modulus-based matrix splitting iteration methods for linear complementarity problems. Numer. Linear Algebra Appl. **17**, 917–933 (2010)
3. Bai, Z.-Z., Evans, D.J.: Matrix multisplitting relaxation methods for linear complementarity problems. Int. J. Comput. Math. **63**, 309–326 (1997)
4. Bai, Z.-Z., Evans, D.J.: Matrix multisplitting methods with applications to linear complementarity problems: parallel synchronous and chaotic methods. Réseaux et Systèmes Répartis: Calculateurs Parallelès **13**, 125–154 (2001)
5. Bai, Z.-Z., Evans, D.J.: Matrix multisplitting methods with applications to linear complementarity problems: parallel asynchronous methods. Int. J. Comput. Math. **79**, 205–232 (2002)
6. Bai, Z.-Z., Sun, J.-C., Wang, D.-R.: A unified framework for the construction of various matrix multisplitting iterative methods for large sparse system of linear equations. Comput. Math. Appl. **32**, 51–76 (1996)
7. Bai, Z.-Z., Zhang, L.-L.: Modulus-based synchronous multisplitting iteration methods for linear complementarity problems. Numer. Linear Algebra Appl. (2012). doi:10.1002/nla.1835
8. Berman, A., Plemmons, R.J.: Nonnegative Matrices in the Mathematical Sciences. Academic Press, New York (1979)
9. Cottle, R.W., Pang, J.-S., Stone, R.E.: The Linear Complementarity Problem. Academic Press, San Diego (1992)
10. Cvetković, Lj.: Some convergence conditions for a class of parallel decomposition-type linear relaxation methods. Appl. Numer. Math. **41**, 81–87 (2002)
11. Cvetković, Lj., Kostić, V.: New subclasses of block H-matrices with applications to parallel decomposition-type relaxation methods. Numer. Algorithms **42**, 325–334 (2006)
12. Cvetković, Lj. , Peña, J.M.: Minimal sets alternative to minimal Geršgorin sets. Appl. Numer. Math. **60**, 442–451 (2010)
13. Dong, J.-L., Jiang, M.-Q.: A modified modulus method for symmetric positive-definite linear complementarity problems. Numer. Linear Algebra Appl. **16**, 129–143 (2009)
14. Frommer, A., Mayer, G.: Convergence of relaxed parallel multisplitting methods. Linear Algebra Appl. **119**, 141–152 (1989)
15. Frommer, A., Szyld, D.B.: H-splittings and two-stage iterative methods. Numer. Math. **63**, 345–356 (1992)
16. Hadjidimos, A., Tzoumas, M.: The principle of extrapolation and the Cayley transform. Linear Algebra Appl. **429**, 2465–2480 (2008)
17. Hadjidimos, A., Tzoumas, M.: Nonstationary extrapolated modulus algorithms for the solution of the linear complementarity problem. Linear Algebra Appl. **431**, 197–210 (2009)

18. Leenaerts, D.M.W., van Bokhoven, W.M.G.: Piecewise Linear Modeling and Analysis. Kluwer Academic, Dordrecht (1998)
19. Murty, K.G.: Linear Complementarity, Linear and Nonlinear Programming. Heldermann, Berlin (1988)
20. O'Leary, D.P., White, R.E.: Multi-splittings of matrices and parallel solution of linear systems. SIAM J. Algebraic Discrete Methods **6**, 630–640 (1985)
21. Szyld, D.B., Jones, M.T.: Two-stage and multisplitting methods for the parallel solution of linear systems. SIAM J. Matrix Anal. Appl. **13**, 671–679 (1992)
22. Varga, R.S.: Matrix Iterative Analysis. Springer, Berlin (2000)