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Primal-dual interior-point algorithm for semidefinite optimization based on a new kernel function with trigonometric barrier term

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Abstract In this paper we propose primal-dual interior-point algorithms for semidefinite optimization problems based on a new kernel function with a trigonometric barrier term. We show that the iteration bounds are $O(\sqrt{n}\log(\frac{n}{\epsilon}))$ for small-update methods and $O(n^{\frac{3}{4}}\log(\frac{n}{\epsilon}))$ for large-update, respectively. The resulting bound is better than the classical kernel function. For small-update, the iteration complexity is the best known bound for such methods.

Keywords Kernel function · Interior-point algorithm · Semidefinite optimization · Polynomial complexity · Primal-dual method

AMS 2000 Subject Classifications 90C22 · 90C51

1 Introduction

In this paper we deal with primal-dual interior-point methods (IPMs) for solving standard semidefinite optimization (SDO) problems which are the convex optimization problems over the intersection of an affine set with cone of the positive semidefinite matrices, i.e.;

$$\begin{aligned} \min \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\ & X \succeq 0, \end{aligned} \tag{P}$$

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where each $A_i \in \mathbf{S}^n$, $b = (b_1, b_2, \ldots, b_m)^T \in \mathbf{R}^m$ and $C \in \mathbf{S}^n$. Moreover, the matrices A_i are linearly independent. The dual problem of (P) is given by

$$\begin{aligned} \max \quad & b^T y \\ \text{s.t.} \quad & \sum_{i=1}^m y_i A_i + S = C, \\ & S \succeq 0, \end{aligned} \tag{D}$$

with $y \in \mathbf{R}^m$ and $S \in \mathbf{S}^n$. Here we use $\mathbf{S}^n$ to denote the set of all symmetric $n \times n$ matrices. The operator $\bullet$ denotes the standard inner product in $\mathbf{S}^n$, i.e., $C \bullet X := Tr(CX) = \sum_{i,j} C_{ij} X_{ij}$, and $X \succeq 0$ ($X \succ$) means that X is symmetric and positive semidfinite (symmetric and positive definite).

In 1984, Karmarkar [12] proposed a polynomial-time algorithm the so-called IPMs for solving linear optimization (LO) problems. This method is extended to SDO, which an important contribution in this field was made by Nesterov and Todd [15, 22]. For a comprehensive study, the reader is referred to [7, 10, 17, 25]. We assume that a strictly feasible pair ($X^0 \succ 0$, $S^0 \succ 0$) exists, i.e., there exists (X^0, y^0, S^0) such that

$$A_i \bullet X^0 = b_i,\ i = 1, 2, \ldots, m, \quad \sum_{i=1}^m y_i^0 A_i + S^0 = C, \quad X^0 \succ 0, \quad S^0 \succ 0.$$

This assumption is called the interior-point condition (IPC). The IPC ensures the existence of an optimal primal-dual pair (X^*, S^*) with zero duality gap:

$$C \bullet X^* - b^T y^* = X^* \bullet S^* = 0.$$

Here, y^* is uniquely determined by S^* due to the assumption that the matrices A_i are linearly independent. Thus, we can write the optimality conditions for (P) and (D) as follows:

$$\begin{aligned} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\ & \sum_{i=1}^m y_i A_i + S = C \\ & XS = 0, \quad X, S \succeq 0. \end{aligned} \tag{1}$$

The basic idea of primal-dual IPMs is to replace the third equation in (1), the so-called complementarity condition for (P) and (D), by the parameterized equation $XS = \mu E$ with $\mu > 0$; where E denotes the $n \times n$ identity matrix. Thus, one may consider

$$\begin{aligned} & A_i \bullet X = b_i, \quad i = 1, 2, \ldots, m \\ & \sum_{i=1}^m y_i A_i + S = C \\ & XS = \mu E, \quad X, S \succeq 0. \end{aligned} \tag{2}$$

For each $\mu > 0$, the parameterized system (2) has a unique solution $(X(\mu), y(\mu), S(\mu))$ (see [13, 16]), which is called a μ-center of (P) and (D). The set of μ-centers is said to be the central path of (P) and (D). The central path converges to the solution pair of (P) and (D) as μ reduces to zero [16].

The natural way to define a search direction is to follow the Newton approach and to linearize the third equation in (2) by replacing X, y and S with $X^+ = X + \Delta X$, $y^+ = y + \Delta y$ and $S^+ = S + \Delta S$ respectively. This leads to the following system:

$$\begin{aligned} &A_i \bullet \Delta X = 0, \quad i = 1, 2, \ldots, m, \\ &\sum_{i=1}^{m} \Delta y_i A_i + \Delta S = 0, \\ &\Delta X S + X \Delta S = \mu E - XS. \end{aligned} \tag{3}$$

The system (3) can be rewritten as

$$\begin{aligned} &A_i \bullet \triangle X = 0, \quad i = 1, 2, \ldots, m, \\ &\sum_{i=1}^{m} \triangle y_i A_i + \triangle S = 0, \\ &\triangle X + X \triangle S S^{-1} = \mu S^{-1} - X. \end{aligned} \tag{4}$$

It is clear that ΔS is symmetric due to the second equation in (4). However, a crucial observation is that $\triangle X$ is not necessarily symmetric because $X \triangle S S^{-1}$ may be not symmetric. Many researchers have proposed methods for symmetrizing the third equation in the Newton system (4) such that the resulting new system has a unique symmetric solution. In this paper, we consider the symmetrization scheme that yields NT-direction [22]. Let us define the matrix

$$P := X^{\frac{1}{2}} (X^{\frac{1}{2}} S X^{\frac{1}{2}})^{\frac{-1}{2}} X^{\frac{1}{2}} \left[= S^{\frac{-1}{2}} (S^{\frac{1}{2}} X S^{\frac{1}{2}})^{\frac{1}{2}} S^{\frac{-1}{2}} \right], \tag{5}$$

and also define $D = P^{\frac{1}{2}}$, where for any symmetric positive definite matrix G, the exponent $G^{\frac{1}{2}}$ denotes its symmetric square root. The matrix D can be used to scale X and S to the same matrix V defined by [18]

$$V := \frac{1}{\sqrt{\mu}} D^{-1} X D^{-1} = \frac{1}{\sqrt{\mu}} DSD = \frac{1}{\sqrt{\mu}} (D^{-1} XSD)^{\frac{1}{2}}. \tag{6}$$

Note that the matrices D and V are symmetric and positive definite. In the NT-scheme, we can get

$$\begin{aligned} &A_i \bullet \triangle X = 0, \quad i = 1, 2, \ldots, m, \\ &\sum_{i=1}^{m} \triangle y_i A_i + \triangle S = 0, \\ &\triangle X + P \triangle S P^T = \mu S^{-1} - X. \end{aligned} \tag{7}$$

Let us further define

$$\begin{aligned} \bar{A}_i &:= \frac{1}{\sqrt{\mu}} D A_i D,\ i = 1, 2, \ldots, m, \\ D_X &:= \frac{1}{\sqrt{\mu}} D^{-1} \triangle X D^{-1}, \\ D_S &:= \frac{1}{\sqrt{\mu}} D \triangle S D, \end{aligned} \tag{8}$$

then the NT search directions can be written as the solution of the following system:

$$\begin{aligned} &\bar{A}_i \bullet D_X = 0,\ \ i = 1, 2, \ldots, m, \\ &\sum_{i=1}^{m} \triangle y_i \bar{A}_i + D_S = 0, \\ &D_X + D_S = V^{-1} - V. \end{aligned} \tag{9}$$

The solution of this system is unique, and we can get the original directions via (8).

1.1 The matrix functions

Using the concept of a matrix function [11], the definition of kernel function ψ can be extended to any diagonalizable matrix with positive eigenvalues. In particular, for a given eigen-decomposition

$$V = Q_V^{-1}\ \mathrm{diag}\ (\lambda_1(V), \ldots, \lambda_n(V)) Q_V,$$

of V with a nonsingular matrix Q_V, the matrix function $\psi(V)$ is defined by

$$\psi(V) = Q_V^{-1}\ \mathrm{diag}\ (\psi(\lambda_1(V)), \ldots, \psi(\lambda_n(V))) Q_V. \tag{10}$$

Then we can define a matrix barrier function $\Psi(V) : \mathbf{S}^n_{++} \to \mathbf{R}$ by

$$\Psi(V) := Tr(\psi(V)) = \sum_{i=1}^{n} \psi(\lambda_i(V)), \tag{11}$$

where $\mathbf{S}^n_{++}$ denotes the set of all symmetric positive definite $n \times n$ matrices.

As in the linear case, we can call $\psi(t)$ the kernel function for the matrix function $\psi(V)$ and $\Psi(V)$. Since the derivatives $\psi'(t)$ and $\psi''(t)$ are well defined, we can obtain the matrix functions $\psi'(V)$ and $\psi''(V)$ if $\psi(\lambda_i(V))$ in (10) is replaced by $\psi'(\lambda_i(V))$ and $\psi''(\lambda_i(V))$ for each i, respectively.

Definition 1 A matrix $M(t)$ is said to be a matrix of functions (or a matrix-valued function) if each entry of $M(t)$ is a function of t, that is, $M(t) = [M_{ij}(t)]$.

The usual concepts of continuity, differentiability and integrability can be naturally extended to matrix-valued functions, by interpreting them component-wise. Let $M(t)$ and $N(t)$ be two matrices of functions. Then, we have

$$\frac{d}{dt}M(t) = M'(t) \tag{12}$$

$$\frac{d}{dt}Tr(M(t)) = Tr(M'(t)) \tag{13}$$

$$\frac{d}{dt}Tr(\psi(M(t))) = Tr(\psi(M'(t))M'(t)) \tag{14}$$

$$\frac{d}{dt}(M(t)N(t)) = M'(t)N(t) + M(t)N'(t). \tag{15}$$

In fact, the right-hand side of the third equation in (9) is the negative gradient of the matrix barrier function $\Psi_c(V)$ with the classical kernel function $\psi_c(t) = \frac{t^2-1}{2} - \log(t)$, while $\psi_c(t)$ satisfies

$$\begin{aligned} &\psi_c'(1) = \psi_c(1) = 0, \\ &\psi_c''(t) > 0, \ t > 0, \\ &\lim_{t\to 0^+} \psi_c(t) = \lim_{t\to +\infty} \psi_c(t) = +\infty. \end{aligned} \tag{16}$$

We replace the right-hand-side of the third equation in (9) by $-\nabla\Psi(V)$, where $-\nabla\Psi(V)$ is the negative gradient of the matrix barrier function $\Psi(V)$ with the kernel function (18). Thus this system can be rewritten as

$$\begin{aligned} &\bar{A}_i \bullet D_X = 0, \quad i = 1, 2, \ldots, m, \\ &\sum_{i=1}^{m} \triangle y_i \bar{A}_i + D_S = 0, \\ &D_X + D_S = -\nabla\Psi(V). \end{aligned} \tag{17}$$

The new search direction $(D_X, \Delta y, D_S)$ is obtained by solving (17) so that $(\Delta X, \Delta y, \Delta S)$ is computed via (8).

Algorithm1 : *Primal − Dual Algorithm for SDO*

Input : Accuracy parameter $\epsilon > 0$;
  barrier update parameter θ, $0 < \theta < 1$;
  threshold parameter $\tau \geq 1$;
  $X^0 \succ 0, S^0 \succ 0$, and $\mu^0 = 1$ such that $\Psi(X^0, S^0, \mu^0) \leq \tau$
begin :
    $X := X^0, S := S^0, \mu := \mu^0$;
  while $n\mu \geq \epsilon$ **do**
  begin
    μ − update :
      $\mu := (1 - \theta)\mu$;
   while $\Psi(X, S, \mu) > \tau$ **do**
   begin
    Solve the system (17) and use (8) for $\Delta X, \Delta y, \Delta S$;
    Determine a step size α;
      $X := X + \alpha\Delta X$;
      $y := y + \alpha\Delta y$;
      $S := S + \alpha\Delta S$;
    end
   end
 end

Now, we explain our algorithm for the primal-dual IPM for the SDO. Assuming that a starting point in a certain neighborhood of the central path is available, we can set out from this point. Then, we will go to the outer iteration. If μ satisfies $n\mu \geq \epsilon$, then it is reduced by factor $1 - \theta$, where $\theta \in (0, 1)$. Then, we make use of inner iteration, and we repeat the procedure until we find iterates that are close to $(X(\mu), y(\mu), S(\mu))$, that is, the proximity $\Psi(V) \leq \tau$. Indeed, each outer iteration performs an update of the barrier parameter and a sequence of inner iterations. It is agreed that the total number of inner iterations required by algorithm is an appropriate measure for the efficiency of the algorithm. This number is called as the iteration complexity of the algorithm; it is usually described as a function of the dimension n of the problem and the accuracy parameter ϵ. The iteration complexity is bounded by multiplying the number of inner iteration bound K by the number of barrier parameter updates, which is bounded above by $\frac{1}{\theta} \log \frac{n}{\epsilon}$ (Lemma II-17 in [21]). A crucial question is that how to choose the parameters τ, θ and the step size α, that minimizes the iteration complexity of the algorithm. Figure 1 gives some examples of the kernel functions that have been analyzed already as well as the complexity results for the corresponding algorithms.

Kernel functions play an important role in the design and analysis of interior-point algorithms. They are not only used for determining the search directions but also for measuring the distance between the given iterate and

i	$Kernel\ functions\ \psi_i(t)$	$Large-update$	$Small-update$	Ref
1	$\frac{t^2-1}{2} - \log t$	$O(n \log \frac{n}{\epsilon})$	$O(\sqrt{n} \log \frac{n}{\epsilon})$	[21]
2	$\frac{1}{2}(t - \frac{1}{t})^2$	$O(n^{\frac{2}{3}} \log \frac{n}{\epsilon})$	$O(\sqrt{n} \log \frac{n}{\epsilon})$	[19]
3	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q-1}, q > 1$	$O(qn^{\frac{q+1}{2q}} \log \frac{n}{\epsilon})$	$O(q^2\sqrt{n} \log \frac{n}{\epsilon})$	[17]
4	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q(q-1)} - \frac{q-1}{q}(t-1), q > 1$	$O(qn^{\frac{q+1}{2q}} \log \frac{n}{\epsilon})$	$O(q^2\sqrt{n} \log \frac{n}{\epsilon})$	[17]
5	$\frac{t^2-1}{2} + \frac{e^{\frac{1}{t}}-e}{e}$	$O(\sqrt{n} \log^2 n \log \frac{n}{\epsilon})$	$O(\sqrt{n} \log \frac{n}{\epsilon})$	[1]
6	$\frac{t^2-1}{2} - \int_1^t e^{\frac{1}{\xi}-1} d\xi$	$O(\sqrt{n} \log^2 n \log \frac{n}{\epsilon})$	$O(\sqrt{n} \log \frac{n}{\epsilon})$	[1]
7	$\frac{t^2-1}{2} - \int_1^t e^{q(\frac{1}{\xi}-1)} d\xi, q \geq 1$	$O(q\sqrt{n}(1 + \frac{1}{q} \log n)^2 \log \frac{n}{\epsilon})$	$O(q\sqrt{qn} \log \frac{n}{\epsilon})$	[2]
8	$\frac{t^2-1}{2} + \frac{(e-1)^2}{e} \frac{1}{e^t-1} - \frac{e-1}{e}$	$O(n^{\frac{3}{4}} \log \frac{n}{\epsilon})$	$O(\sqrt{n} \log \frac{n}{\epsilon})$	[3]
9	$\frac{t^{p+1}-1}{p+1} + \frac{t^{1-q}-1}{q-1}, p \in [0,1], q > 1$	$O(qn^{\frac{p+q}{q(1+p)}} \log \frac{n}{\epsilon})$	$O(q^2\sqrt{n} \log \frac{n}{\epsilon})$	[4]
10	$\frac{t^2-1}{2} + \frac{1}{\sigma}\left(e^{\sigma(1-t)} - 1\right), \sigma \geq 1$	$O(\sigma\sqrt{n} \log \frac{n}{\epsilon})$	$O(\sqrt{n} \log \frac{n}{\epsilon})$	[5]
11	$\frac{t^2-1}{2} + \frac{6}{\pi} \tan(h(t)), h(t) = \frac{\pi(1-t)}{4t+2}$	$O(n^{\frac{3}{4}} \log \frac{n}{\epsilon})$	$O(\sqrt{n} \log \frac{n}{\epsilon})$	[8]
12	$t - 1 + \frac{t^{1-q}-1}{q-1},\ q > 1$	$O(qn \log \frac{n}{\epsilon})$	$O(q^2\sqrt{n} \log \frac{n}{\epsilon})$	[24]
13	$\frac{(t-1)^2}{2t} + \frac{(t-1)^2}{2}$	$O(n^{\frac{3}{4}} \log \frac{n}{\epsilon})$	$--$	[20]

Fig. 1 Iteration bounds for large- and small-update methods

the corresponding μ-center for the algorithms. Bai et al. [1] presented the approach of using kernel function to determine the search directions and to design primal-dual IPMs for solving LO problems. El Ghami et al. [6] extended the approach presented in [1] for LO, which is based on so-called eligible kernel functions, to SDO which yields a wide class of new methods for SDO. Some kernel functions introduced in Fig. 1, so-called self-regular kernel functions [17, 18] and some non-self-regular kernel functions [1, 3, 5, 24]. Recently, El Ghami et al. [8] introduced a new kernel function with a trigonometric barrier term, which is not logarithmic and not self-regular, and analyzed large- and small-update methods of the primal-dual interior-point algorithm for LO. Motivated by their work, in this paper we present a primal-dual interior-point algorithm for SDO based on the kernel function:

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{4}{\pi} \cot(h(t)), \quad \text{where } h(t) = \frac{\pi t}{1 + t}, \quad t > 0, \tag{18}$$

and derive the complexity analysis for algorithms with large- and small-update methods.

The paper is organized as follows: In Section 2, we derive some properties of $\psi(t)$ and $\Psi(V)$ based on the new kernel function (18). In Section 3, we propose an expression for the decrease of the proximity during an inner iteration, and derive a default value for the step size. The analysis is completed in Section 4 by deriving the iteration complexity. Finally, in the last section we conclude with some remarks.

2 Properties of the new proximity function

In this section, we study some properties of the kernel function (18). We start with three technically lemmas.

2.1 Some technical results

For ψ we have the first three derivatives as follows:

$$\psi'(t) = t - \frac{4}{\pi} h'(t) \left(1 + \cot^2(h(t))\right), \tag{19}$$

$$\psi''(t) = 1 - \frac{4}{\pi} \left(1 + \cot^2(h(t))\right) \left(h''(t) - 2h'(t)^2 \cot(h(t))\right), \tag{20}$$

$$\psi'''(t) = -\frac{4}{\pi} \left(1 + \cot^2(h(t))\right) g(t), \tag{21}$$

where

$$g(t) = -6h'(t)h''(t)\cot(h(t)) + h'''(t) + 2h'(t)^3 \left(1 + 3\cot^2(h(t))\right).$$

Lemma 2 *For the function $\psi(t)$ defined in (18), we have*

$$\lim_{t\to+\infty} \psi(t) = +\infty.$$

Proof Let $t = \frac{1}{x}$. Then,

$$\begin{aligned}
\lim_{t\to+\infty} \psi(t) &= \lim_{t\to+\infty} \left(\frac{t^2-1}{2} + \frac{4}{\pi}\cot(h(t))\right) \\
&= \lim_{x\to+0} \frac{1-x^2}{2x^2} + \frac{4}{\pi}\cot\left(\frac{\pi}{1+x}\right) \\
&= \lim_{x\to+0} \frac{\pi(1-x^2)\sin\left(\frac{\pi}{1+x}\right) + 8x^2\cos\left(\frac{\pi}{1+x}\right)}{2\pi x^2 \sin\left(\frac{\pi}{1+x}\right)}.
\end{aligned}$$

Since

$$\lim_{x\to+0} \pi(1-x^2)\sin\left(\frac{\pi}{1+x}\right) + 8x^2\cos\left(\frac{\pi}{1+x}\right) = 0,$$

and

$$\lim_{x\to+0} 2\pi x^2 \sin\left(\frac{\pi}{1+x}\right) = 0,$$

we can apply L'Hospital's role:

$$\lim_{t\to+\infty}\psi(t)$$
$$=\lim_{x\to+0}\frac{-2\pi x\sin\left(\frac{\pi}{1+x}\right)-\frac{\pi^2(1-x^2)}{(1+x)^2}\cos\left(\frac{\pi}{1+x}\right)+16x\cos\left(\frac{\pi}{1+x}\right)+\frac{8\pi x^2}{(1+x)^2}\sin\left(\frac{\pi}{1+x}\right)}{4\pi x\sin\left(\frac{\pi}{1+x}\right)-\frac{2\pi^2x^2}{(1+x)^2}\cos\left(\frac{\pi}{1+x}\right)}$$
$$=+\infty.$$

□

Lemma 3 *For the function $h(t)$ defined in (18), we have*

$$1+t+\pi\cot(h(t))>0,\quad t>1.$$

Proof For $t>1$, $\frac{\pi}{2}<h(t)\le\pi$. Define $f(t):=1+t+\pi\cot(h(t))$. Since $\frac{\sin(x)}{x}>\frac{\pi-x}{x}$, for $0\le x\le\pi$ [14], we have

$$f'(t)=1-\pi h'(t)\left(1+\cot^2(h(t))\right)=\frac{\sin^2(h(t))-\pi h'(t)}{\sin^2(h(t))}$$

$$=\frac{(1+t)^2\sin^2(h(t))-\pi^2}{(1+t)^2\sin^2(h(t))}=\frac{(1+t)^2\left(\frac{\sin(h(t))}{h(t)}\right)^2-\left(\frac{\pi}{h(t)}\right)^2}{(1+t)^2\left(\frac{\sin(h(t))}{h(t)}\right)^2}$$

$$>\frac{(1+t)^2\left(\frac{\pi-h(t)}{h(t)}\right)^2-\left(\frac{\pi}{h(t)}\right)^2}{(1+t)^2\left(\frac{\sin(h(t))}{h(t)}\right)^2}=\frac{(1+t)^2\left(\frac{1}{t}\right)^2-\left(\frac{1+t}{t}\right)^2}{(1+t)^2\left(\frac{\sin(h(t))}{h(t)}\right)^2}=0.$$

Thus $f(t)$ is increasing for $t>1$, and hence $f(t)\ge f(1)=2>0$. This implies the lemma. □

Lemma 4 *For the function $h(t)$ defined in (18), one has*

$$h''(t)-2h'(t)^2\cot(h(t))<0,\quad t>0. \tag{22}$$

Proof Since $h(t)$ is increasing, we have $0<h(t)\le\pi$, for $t>0$. Now, we consider two cases:

Case 1 Assume that $t\in(0,1]$. Then $0<h(t)\le\frac{\pi}{2}$ and so $\cot(h(t))\ge0$. Since $h''(t)=\frac{-2\pi}{(1+t)^3}<0$ for $t>0$, we obtain

$$h''(t)-2h'(t)^2\cot(h(t))<0,$$

which shows that (22) holds for all $t\in(0,1]$.

Case 2 Assume that $t \in (1, \infty)$. By Lemma 3, we have $\frac{1+t}{\pi} > -\cot(h(t))$. Using the first two derivatives $h(t)$ for all $t \geq 1$, we get

$$h''(t) - 2h'(t)^2 \cot(h(t)) < h''(t) + 2h'(t)^2 \frac{1+t}{\pi} = -\frac{2\pi}{(1+t)^3} + \frac{2\pi^2(1+t)}{\pi(1+t)^4} = 0.$$

This completes the proof. □

The next lemma shows that the new kernel function (18) is eligible.

Lemma 5 *Let* $\psi(t)$ *be as defined in* (18) *and* $t > 0$. *Then*

$$\psi''(t) > 1, \tag{23}$$

$$t\psi''(t) + \psi'(t) > 0, \tag{24}$$

$$t\psi''(t) - \psi'(t) > 0, \tag{25}$$

$$\psi'''(t) < 0. \tag{26}$$

Proof Clearly, Lemma 4 and the second derivative of $\psi(t)$ follow (23).

By using (19), (20) and $h'(t)$ and $h''(t)$ the first two derivatives of $h(t)$, we have

$$\begin{aligned}
&t\psi''(t) + \psi'(t) \\
&= 2t - \frac{4}{\pi}\left(1 + \cot^2(h(t))\right)\left(th''(t) - 2th'(t)^2\cot(h(t)) + h'(t)\right) \\
&= 2t - \frac{4}{\pi}\left(1 + \cot^2(h(t))\right)\left(\frac{-2\pi t}{(1+t)^3} - \frac{2\pi^2 t}{(1+t)^4}\cot(h(t)) + \frac{\pi}{(1+t)^2}\right) \\
&= 2t + \left(\frac{2}{(1+t)^2\sin(h(t))}\right)^2\left(t^2 - 1 + 2\pi t\cot(h(t))\right).
\end{aligned} \tag{27}$$

Consider two cases:

Case 1 Assume that $0 < t < 1$.

Define

$$k(t) = t^2 - 1 + 2\pi t\cot(h(t)) = 2\pi t\left(\cot(h(t)) + \frac{t}{2\pi} - \frac{1}{2\pi t}\right)$$

and

$$f(t) = \cot(h(t)) + \frac{t}{2\pi} - \frac{1}{2\pi t}.$$

We have

$$\begin{aligned} f'(t) &= -h'(t)\left(1+\cot^2(h(t))\right)+\frac{1}{2\pi}+\frac{1}{2\pi t^2} \\ &= \frac{-h'(t)}{\sin^2(h(t))}+\frac{1}{2\pi}+\frac{1}{2\pi t^2} \\ &= \frac{-\pi}{(1+t)^2\sin^2(h(t))}+\frac{1}{2\pi}+\frac{1}{2\pi t^2} \\ &\le \frac{-\pi}{(1+t)^2 h(t)^2}+\frac{1}{2\pi}+\frac{1}{2\pi t^2} \\ &= \frac{-1}{\pi t^2}+\frac{1}{2\pi}+\frac{1}{2\pi t^2}=\frac{t^2-1}{2\pi t^2}<0. \end{aligned}$$

This implies that $f(t)$ is strictly decreasing and hence $f(t) > f(1) = 0$. Since $2\pi t > 0$ for each $t \in (0, 1)$, we obtain $k(t) > 0$ for each $t \in (0, 1)$. Thus the right-hand-side of (27) is positive which proves (24), for all $t \in (0, 1)$.

Case 2 Assume that $t \geq 1$.

Then $\psi'(1) = 0$ and using (23), we see that $\psi'(t)$ is strictly increasing. Hence

$$t\psi''(t)+\psi'(t) > 1+\psi'(t) > 1+\psi'(1) = 1.$$

The two cases together prove (24).

To prove (25), considering the first two derivatives of $\psi(t)$ we have

$$\begin{aligned} t\psi''(t)-\psi'(t) &= -\frac{4}{\pi}\left(1+\cot^2(h(t))\right)\left(th''(t)-2th'(t)^2\cot(h(t))-h'(t)\right) \\ &= -\frac{4}{\pi}\left(1+\cot^2(h(t))\right)\left(t\left(h''(t)-2h'(t)^2\cot(h(t))\right)-h'(t)\right). \end{aligned} \quad (28)$$

From Lemma 4 and $-h'(t) = -\frac{\pi}{(1+t)^2} < 0$, we have $t\left(h''(t) - 2h'(t)^2\cot(h(t))\right) - h'(t) < 0$. Therefore, the right-hand-side of (28) is positive, which proves (25).

The third derivative of $\psi(t)$ is given in (21). Since $-\frac{4}{\pi}\left(1+\cot^2(h(t))\right) < 0$, for all $t > 0$, thus for prove (26) suffers that $g(t) > 0$. By substitution of $h'(t)$ and $h''(t)$ in $g(t)$, we obtain

$$\begin{aligned} g(t) &= -6\left(\frac{\pi}{(1+t)^2}\right)\left(\frac{-2\pi}{(1+t)^3}\right)\cot(h(t)) \\ &\quad +\frac{6\pi}{(1+t)^4}+2\left(\frac{\pi}{(1+t)^2}\right)^3\left(1+3\cot^2(h(t))\right) \\ &= \frac{6\pi}{(1+t)^5}\left(t+1+2\pi\cot(h(t))\right)+\frac{2\pi^3}{(1+t)^6}\left(1+3\cot^2(h(t))\right) \\ &= \frac{6\pi}{(1+t)^6}\left(\frac{\pi^2}{3}+\left((t+1)+(\pi\cot(h(t)))\right)^2\right) > 0. \end{aligned} \quad (29)$$

This completes the proof. □

Lemma 2 shows that the new kernel function (18) satisfies

$$\lim_{t\to 0^+}\psi(t)=\lim_{t\to +\infty}\psi(t)=+\infty.$$

Note that $\psi'(1)=\psi(1)=0$. Then $\psi(t)$ is determined by

$$\psi(t)=\int_1^t\int_1^\xi \psi''(\zeta)d\zeta d\xi. \tag{30}$$

The next lemma is very useful in the analysis of interior-point algorithms based on the kernel functions (see for example [1, 17]).

Lemma 6 (Lemma 2.1.2 in [18]) *Let $\psi(t)$ be a twice differentiable function for $t>0$. Then the following three properties are equivalent:*

(i) $\psi(\sqrt{t_1t_2})\le\frac{1}{2}(\psi(t_1)+\psi(t_2))$ *for* $t_1,t_2>0$.
(ii) $\psi'(t)+t\psi''(t)\ge 0,\ t>0$.
(iii) $\psi(e^\xi)$ *is convex.*

Following [18], the property described in Lemma 6 is called exponential convexity, or shortly *e*-convexity. Therefore, Lemma 6 and (24) show that the our new kernel function (18) is *e*-convex for $t>0$.

Lemma 7 *If $t\ge 1$, then*

$$\frac{\psi'(t)}{2}(t-1)\le\psi(t)\le(t-1)^2.$$

Proof If $f(t)=2\psi(t)-(t-1)\psi'(t)$, then $f'(t)=\psi'(t)-(t-1)\psi''(t)$, $f''(t)=-(t-1)\psi'''(t)$ and $f(1)=f'(1)=0$. Since $\psi'''(t)<0$, we deduce that $f''(t)\ge 0$ which implies that f' is increasing. Thus $f'(t)\ge 0$ for $t\ge 1$. Similarly, $f(t)\ge 0$ for $t\ge 0$. This proves left inequality.

To prove right inequality, by Taylor's expansion and the fact $\psi(1)=\psi'(1)=0$, we obtain

$$\begin{aligned}\psi(t)&=\psi(1)+\psi'(1)(t-1)+\frac{1}{2}\psi''(1)(t-1)^2+\frac{1}{6}\psi'''(\eta)(t-1)^3,\\&=\frac{1}{2}\psi''(1)(t-1)^2+\frac{1}{6}\psi'''(\eta)(t-1)^3,\end{aligned}$$

where $1<\eta<t$. Since $\psi'''(t)<0$ and $\psi''(1)=2$, we have

$$\psi(t)\le(t-1)^2.$$

This completes the proof. □

The proof of next lemma is essentially similar to the proof of Lemma 4.2 in [9].

Lemma 8 *For $t \geq 1$, one has*

$$\psi'(t) \geq \frac{\psi(t)}{t}.$$

Proof We define $f(t) = t\psi'(t) - \psi(t)$. It is clear $f(1) = 0$ and $f'(t) = t\psi''(t) \geq 1 > 0$. That is, $f(t)$ is monotone increasing function, $f(t) \geq f(1) = 0$, and this completes the proof. □

The proof of following lemma is identical to the proof of Lemma 2.1 in [5].

Lemma 9 *For $\psi(t)$, as defined in (18), we have*

$$\frac{1}{2}(t-1)^2 \leq \psi(t) \leq \frac{1}{2}\psi'(t)^2.$$

Proof Using $\psi''(t) \geq 1$ and (30), we have

$$\begin{aligned} \psi(t) &= \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi \\ &\geq \int_1^t \int_1^\xi d\zeta d\xi \\ &= \int_1^t (\xi - 1) d\xi = \frac{1}{2}(t-1)^2, \end{aligned}$$

this proves the first inequality. The second inequality is obtained as follows:

$$\begin{aligned} \psi(t) &= \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi \leq \int_1^t \int_1^\xi \psi''(\zeta)\psi''(\xi) d\zeta d\xi \\ &= \int_1^t \psi''(\xi)\psi'(\xi) d\xi = \int_1^t \psi'(\xi) d(\psi'(\xi)) = \frac{1}{2}\psi'(t)^2. \end{aligned}$$

This complete the proof. □

Lemma 10 *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t)$ for $t \geq 1$. Then*

(a) $\varrho(s) \geq \sqrt{1 + 2s}$,
(b) $\varrho(s) \leq 3\sqrt{s}, \quad s \geq 1$.

Proof The inverse function of $\psi(t)$ for $t \geq 1$ is obtained by solving t from

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{4}{\pi} \cot\left(\frac{\pi t}{1+t}\right) = s, \quad t \geq 1.$$

Defining $w(t) = \frac{4}{\pi} \cot\left(\frac{\pi t}{1+t}\right)$, one has

$$w'(t) = -\left(\frac{2}{(1+t)\sin(h(t))}\right)^2 < 0.$$

Therefore, $w(t)$ is decreasing for $t \geq 1$ and since $w(1) = 0$, we get

$$\frac{t^2 - 1}{2} \geq s,$$

this implies that $t = \varrho(s) \geq \sqrt{1 + 2s}$. This proves the first inequality. For the proof of second inequality, by $s \geq 1$ and Lemma 9, we have

$$s = \psi(t) \geq \frac{1}{2}(t - 1)^2,$$

whence

$$t = \varrho(s) \leq 1 + \sqrt{2s},$$

and therefore, by $s \geq 1$, we get

$$t = \varrho(s) \leq \sqrt{s} + \sqrt{2s} \leq 3\sqrt{s}.$$

This completes the proof. □

In the analysis of the algorithm, we also use the norm-based proximity measure defined by

$$\delta := \delta(V) = \frac{1}{2}\|\nabla\Psi(V)\| = \frac{1}{2}\sqrt{Tr(\psi'(V)^2)}. \tag{31}$$

The next theorem gives a lower bound on the norm-based proximity measure $\delta(V)$, as defined by (31), in terms of $\Psi(V)$, which is an extension of Theorem 4.9 in [1] to positive definite matrices.

Theorem 11 (Theorem 3.2 in [6]) *Let ϱ be the inverse function of $\psi(t)$ for $t \geq 1$. Then*

$$\delta(V) \geq \frac{1}{2}\psi'\left(\varrho(\Psi(V))\right).$$

Lemma 12 *If $V \in \mathbf{S}^n_{++}$ and $\Psi(V) \geq \tau \geq 1$, then*

$$\delta(V) \geq \frac{1}{6}\sqrt{\Psi(V)}.$$

Proof By using Theorem 11 and Lemma 8, we obtain

$$\delta(V) \geq \frac{1}{2}\psi'\left(\varrho(\Psi(V))\right) \geq \frac{1}{2}\frac{\psi\left(\varrho(\Psi(V))\right)}{\varrho(\Psi(V))} = \frac{\Psi(V)}{2\varrho(\Psi(V))}.$$

Now, by the second inequality of the Lemma 10, we have

$$\varrho(\Psi(V)) \leq 3\sqrt{\Psi(V)}.$$

Therefore,

$$\delta(V) \geq \frac{\Psi(V)}{6\sqrt{\Psi(V)}} = \frac{1}{6}\sqrt{\Psi(V)}.$$

This completes the proof of the lemma. □

At the start of each outer iteration, just before the update of μ with the factor $1-\theta$, we have $\Psi(V) \leq \tau$. Due to the update of μ the matrix V, defined by (6), is divided by the factor $\sqrt{1-\theta}$, with $0 < \theta < 1$, which leads to an increasing in the value of $\Psi(V)$. Then, during the inner iterations, $\Psi(V)$ decreases until it passes the threshold τ again. Hence, during the course of the algorithm the largest values of $\Psi(V)$ occur just after the updates of μ. In the rest this section, we derive an estimate for the effect of a μ-update on the value of $\Psi(V)$.

The next theorem is an extension of Theorem 3.2 in [1] to positive definite matrices.

Theorem 13 (Theorem 3.1 in [6]) *Let ϱ be the inverse function of $\psi(t)$ for $t \geq 1$. Then for any positive definite matrix V, and any $\beta \geq 1$,*

$$\Psi(\beta V) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(V)}{n}\right)\right).$$

Corollary 14 *Let $0 \leq \theta < 1$ and $V_+ = \frac{V}{\sqrt{1-\theta}}$. If $\Psi(V) \leq \tau$, then*

$$\Psi(V_+) \leq n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right) \leq n\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2.$$

Proof Since $\frac{1}{\sqrt{1-\theta}} \geq 1$ and $\varrho\left(\frac{\Psi(V)}{n}\right) \geq 1$, we have $\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(V)}{n}\right) \geq 1$. Using Theorem 13 with $\beta = \frac{1}{\sqrt{1-\theta}}$ and the function ϱ is monotonically increasing since $\psi(t)$ for $t \geq 1$ is monotonically increasing because of its definition, we have

$$\Psi(V_+) \leq n\psi\left(\frac{1}{\sqrt{1-\theta}}\varrho\left(\frac{\Psi(V)}{n}\right)\right) \leq n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right).$$

This proves the first inequality. The second inequality follows from Lemma 7. □

3 Analysis of the algorithm

In this section, we determine a default step size and obtain an upper bound to the decrease of the barrier function $\Psi(V)$ during an inner iteration.

3.1 Decrease the value of $\psi(V)$ and choose a default step size α

In each iteration the search directions ΔX, Δy and ΔS are obtained by solving the system (17) and via (8). After a step with size α, the new iterate is given by

$$X_+ = X + \alpha\Delta X, \quad y_+ = y + \alpha\Delta y, \quad S_+ = S + \alpha\Delta S.$$

Due to (8), we may write

$$pX_+ = X + \alpha\Delta X = X + \alpha\sqrt{\mu}DD_XD = \sqrt{\mu}D(V + \alpha D_X)D,$$

and

$$S_+ = S + \alpha\Delta S = S + \alpha\sqrt{\mu}D^{-1}D_SD^{-1} = \sqrt{\mu}D^{-1}(V + \alpha D_S)D^{-1}.$$

According to (6), we have

$$V_+ = \frac{1}{\sqrt{\mu}}\left[D^{-1}X_+S_+D\right]^{\frac{1}{2}}.$$

Therefore, V_+^2 is similar to the matrix $\frac{1}{\mu}X_+S_+ = \frac{1}{\mu}X_+^{\frac{1}{2}}S_+X_+^{\frac{1}{2}}$ and thus to $(V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}}$. Consequently, the eigenvalues of the matrix V_+ are the same as those of $\left[(V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}}\right]^{\frac{1}{2}}$. Since the proximity after one step is defined by $\Psi(V_+)$, it follows from (11) that

$$\Psi(V_+) = \Psi\left(\left[(V + \alpha D_X)^{\frac{1}{2}}(V + \alpha D_S)(V + \alpha D_X)^{\frac{1}{2}}\right]^{\frac{1}{2}}\right).$$

Hence, by Lemma 5,

$$\Psi(V_+) \leq \frac{1}{2}\left(\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right).$$

Let us denote the difference between the proximity before and after one step by a function of the step size, that is,

$$f(\alpha) := \Psi(V_+) - \Psi(V).$$

Then $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \frac{1}{2}\left(\Psi(V + \alpha D_X) + \Psi(V + \alpha D_S)\right) - \Psi(V).$$

Obviously

$$f(0) = f_1(0) = 0.$$

Taking the derivative with respect to α, by using (12)–(15), we obtain

$$f_1'(\alpha) = \frac{1}{2}Tr\left(\psi'(V + \alpha D_X)D_X + \psi'(V + \alpha D_S)D_S\right), \tag{32}$$

and

$$f_1''(\alpha) = \frac{1}{2}Tr\left(\psi''(V + \alpha D_X)D_X^2 + \psi''(V + \alpha D_S)D_S^2\right). \tag{33}$$

Hence, using (31) and the third equation of (17), we obtain

$$\begin{aligned} f_1'(0) &= \frac{1}{2} Tr\left(\psi'(V)D_X + \psi'(V)D_S\right) \\ &= \frac{1}{2} Tr\left(\psi'(V)(D_X + D_S)\right) \\ &= \frac{1}{2} Tr\left(\psi'(V)(-\psi'(V))\right) \\ &= \frac{1}{2} Tr\left(-\psi'(V)^2\right) \\ &= -2\delta(V)^2. \end{aligned} \tag{34}$$

In what follows, we use the short notation $\delta := \delta(V)$ and state some important results without proofs.

Lemma 15 (Lemma 4.2 in [23]) *One has*

$$f_1''(\alpha) \leq 2\delta^2 \psi''\left(\lambda_{\min}(V) - 2\alpha\delta\right).$$

Lemma 16 (Lemma 4.2 in [1]) *If the step size α satisfies*

$$-\psi'\left(\lambda_{\min}(V) - 2\alpha\delta\right) + \psi'\left(\lambda_{\min}(V)\right) \leq 2\delta, \tag{35}$$

then $f_1'(\alpha) \leq 0$.

Lemma 17 (Lemma 4.3 in [1]) *Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ on the interval $(0, 1]$, then the largest possible value of the step size of α satisfying (35) is given by*

$$\bar{\alpha} := \frac{1}{2\delta}(\rho(\delta) - \rho(2\delta)).$$

Lemma 18 (Lemma 4.4 in [1]) *Let ρ and $\bar{\alpha}$ be the same as defined in Lemma* 17. *Then*

$$\bar{\alpha} \geq \frac{1}{\psi''(\rho(2\delta))}.$$

For the purpose of finding an upper bound for $f(\alpha)$, we need a default step size $\tilde{\alpha}$ that is the lower bound of the $\bar{\alpha}$ and consists of δ.

Lemma 19 *Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of the restriction of $-\frac{1}{2}\psi'(t)$ on the interval $(0, 1]$ and $\Psi(V) \geq \tau \geq 1$. Then*

$$\frac{1}{\psi''(\rho(2\delta))} \geq \frac{1}{\left(6^{\frac{3}{2}} + 80(2\pi + \sqrt{6})\right)\delta^{\frac{3}{2}}}.$$

Proof To obtain the inverse function $t = \rho(s)$ of $-\frac{1}{2}\psi'(t)$ for $t \in (0, 1]$, we need to solve the equation

$$-\psi'(t) = -t + \frac{4}{\pi} h'(t) \left(1 + \cot^2(h(t))\right) = 2s.$$

But this is hard to solve, so we should derive a lower bound for $\rho(s)$. To do this, the above equation implies

$$1 + \cot^2(h(t)) = \frac{\pi}{4h'(t)}(2s + t) = \frac{(1+t)^2}{4}(2s + t) \le 2s + 1. \tag{36}$$

Letting $-\psi'(t) = 2s$, we can say that $t = \rho(s)$. By setting $t = \rho(2\delta)$, we have

$$-\psi'(t) = 4\delta.$$

Hence,

$$\cot(h(t)) \le 2\sqrt{\delta}. \tag{37}$$

Since $h'(t) = \frac{\pi}{(1+t)^2} \le \pi$ and $h''(t) = \frac{-2\pi}{(1+t)^3} \ge -2\pi$ for all $0 \le t \le 1$, it follows from (20) and (37) that

$$\begin{aligned}
\frac{1}{\psi''(\rho(2\delta))} &= \frac{1}{1 - \frac{4}{\pi}\left(1 + \cot^2(h(t))\right)\left(h''(t) - 2h'(t)^2 \cot(h(t))\right)} \\
&\ge \frac{1}{1 - \frac{4}{\pi}(4\delta + 1)\left(h''(t) - 4h'(t)^2\sqrt{\delta}\right)} \\
&\ge \frac{1}{1 + \frac{4}{\pi}(4\delta + 1)(4\pi^2\sqrt{\delta} + 2\pi)} \\
&= \frac{1}{1 + 8(4\delta + 1)(2\pi\sqrt{\delta} + 1)}.
\end{aligned}$$

By Lemma 12, we get

$$\frac{1}{\psi''(\rho(2\delta))} \ge \frac{1}{(6\delta)^{\frac{3}{2}} + 8(4\delta + 6\delta)\left(2\pi\sqrt{\delta} + \sqrt{6\delta}\right)} = \frac{1}{\left(6^{\frac{3}{2}} + 80\left(2\pi + \sqrt{6}\right)\right)\delta^{\frac{3}{2}}}.$$

□

In the sequel, we use the notation

$$\tilde{\alpha} = \frac{1}{\left(6^{\frac{3}{2}} + 80(2\pi + \sqrt{6})\right)\delta^{\frac{3}{2}}}, \tag{38}$$

and we will use $\tilde{\alpha}$ as the default step size. By Lemma 18, $\bar{\alpha} \ge \tilde{\alpha}$.

Lemma 20 (Lemma 4.5 in [1]) *If the step size α is such that $\alpha \leq \tilde{\alpha}$, then*

$$f(\alpha) \leq -\alpha\delta^2.$$

Theorem 21 *If $\tilde{\alpha}$ is the default step size as given by (38), then*

$$f(\tilde{\alpha}) \leq -\frac{\Psi(V)^{\frac{1}{4}}}{\sqrt{6}\left(6^{\frac{3}{2}} + 80\left(2\pi + \sqrt{6}\right)\right)}.$$

Proof Using Lemma 20 with $\alpha = \tilde{\alpha}$ and (38), we have

$$f(\tilde{\alpha}) \leq -\tilde{\alpha}\delta^2 \leq -\frac{\delta^2}{\left(6^{\frac{3}{2}} + 80\left(2\pi + \sqrt{6}\right)\right)\delta^{\frac{3}{2}}} = -\frac{\delta^{\frac{1}{2}}}{6^{\frac{3}{2}} + 80\left(2\pi + \sqrt{6}\right)}.$$

Using Lemma 12, we obtain

$$f\left(\tilde{\alpha}\right) \leq -\frac{\Psi(V)^{\frac{1}{4}}}{\sqrt{6}\left(6^{\frac{3}{2}} + 80\left(2\pi + \sqrt{6}\right)\right)}.$$

This proves the theorem. □

4 Iteration bound

In this section, we derive the complexity bounds for large and small-update methods.

4.1 Upper bound for the total number of inner iterations

By the assumption $\Psi(V) \leq \tau$, after the update of μ to $(1-\theta)\mu$, by Corollary 14, we have

$$\Psi(V_+) := L \leq n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right).$$

We need to count how many inner iterations are required to return to the situation where $\Psi(V) \leq \tau$ after a μ-update. We denote the value of $\Psi(V)$ after the μ-update by Ψ_0; the subsequent values in the same outer iteration are denoted as Ψ_k, $k = 1, 2, \ldots, K$, where K denotes the total number of inner iterations in the outer iteration.

According to decrease of $f(\tilde{\alpha})$, for $k = 1, 2, \ldots, K-1$, we obtain

$$\Psi_{k+1} \leq \Psi_k - \frac{\Psi_k^{\frac{1}{4}}}{\sqrt{6}\left(6^{\frac{3}{2}} + 80\left(2\pi + \sqrt{6}\right)\right)}. \tag{39}$$

Lemma 22 (Lemma 14 in [17]) *Suppose $t_0, t_1, \ldots, t_k$ be a sequence of positive numbers such that*

$$t_{k+1} \leq t_k - \beta t_k^{1-\gamma}, \ k = 0, 1, \ldots, K-1,$$

where $\beta > 0$ and $0 < \gamma \leq 1$. Then $K \leq \lceil \frac{t_0^\gamma}{\beta\gamma} \rceil$.

Letting $t_k = \Psi_k$, $\beta = \frac{1}{\sqrt{6}\left(6^{\frac{3}{2}}+80(2\pi+\sqrt{6})\right)}$ and $\gamma = \frac{3}{4}$, we can get the following theorem from Lemma 22.

Theorem 23 *Let K be the total number of inner iterations in the outer iteration. Then*

$$K \leq \frac{4\sqrt{6}\left(6^{\frac{3}{2}} + 80\left(2\pi + \sqrt{6}\right)\right)}{3} \Psi_0^{\frac{3}{4}},$$

where, Ψ_0 is the value of $\Psi(V)$ after the μ-update in outer iteration.

4.2 Large-update methods

It is clear that $\psi(t) \leq \frac{t^2}{2}$ when $t \geq 1$. Applying Corollary 14 and Lemma 10, we obtain

$$\Psi_0 \leq n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right) \leq n\psi\left(\frac{3\left(\frac{\tau}{n}\right)^{\frac{1}{2}}}{\sqrt{1-\theta}}\right) \leq n\frac{(3)^2\frac{\tau}{n}}{2(1-\theta)}.$$

The number of outer iterations is bounded above by $\frac{1}{\theta}\log(\frac{n}{\epsilon})$ (Lemma II.17 in [21]). By multiplying the number of outer iterations and the number of inner iterations we get an upper bound for the total number of iterations, namely,

$$\begin{aligned}\frac{K}{\theta}\log\left(\frac{n}{\epsilon}\right) &\leq \frac{\frac{4\sqrt{6}\left(6^{\frac{3}{2}}+80(2\pi+\sqrt{6})\right)}{3}\Psi_0^{\frac{3}{4}}}{\theta}\log\left(\frac{n}{\epsilon}\right)\\ &\leq \frac{4\sqrt{6}\left(6^{\frac{3}{2}} + 80\left(2\pi+\sqrt{6}\right)\right)9^{\frac{3}{4}}}{3\theta(2(1-\theta))^{\frac{3}{4}}}\tau^{\frac{3}{4}}\log\left(\frac{n}{\epsilon}\right).\end{aligned}$$

Large-update methods use $\theta = \Theta(1)$ and $\tau = O(n)$. The iteration bound then becomes

$$O\left(n^{\frac{3}{4}}\log\left(\frac{n}{\epsilon}\right)\right).$$

4.3 Small-update methods

It is not hard to show that if the aforementioned analysis were used for small-update methods the iteration bound would not be as good as it can be for these types of methods. For small-update methods one has $\theta = \Theta(\frac{1}{\sqrt{n}})$ and $\tau = O(1)$. For get the improved iteration bound, we use Corollary 14 and obtain

$$\Psi_0 \leq n\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2. \tag{40}$$

For estimate this, we need an upper bound for the inverse function ϱ of ψ for $t \geq 1$. Hence, from the proof Lemma 10 we have

$$t = \varrho(s) \leq 1 + \sqrt{2s}. \tag{41}$$

Therefore, (40) and (41) follow that

$$\Psi_0 \leq n\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}} - 1\right)^2. \tag{42}$$

Using $1 - \sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, the above inequality can be simplified to

$$\Psi_0 \leq \frac{1}{1-\theta}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2. \tag{43}$$

Theorem 23 implies that the total number of iterations is bounded above by

$$\frac{K}{\theta}\log\left(\frac{n}{\epsilon}\right) \leq \frac{4\sqrt{6}\left(6^{\frac{3}{2}} + 80\left(2\pi + \sqrt{6}\right)\right)}{3\theta(1-\theta)^{\frac{3}{4}}}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^{\frac{3}{2}}\log\left(\frac{n}{\epsilon}\right).$$

Small-update methods use $\theta = \Theta(\frac{1}{\sqrt{n}})$ and $\tau = O(1)$. Therefore, the iteration bound becomes

$$O\left(\sqrt{n}\log\left(\frac{n}{\epsilon}\right)\right).$$

5 Conclusion

In this paper we have analyzed large and small-update methods of primal-dual interior-point algorithm based on a new kernel function with trigonometric barrier term. We proved that the iteration bound of a large-update interior-point method is $O(n^{\frac{3}{4}}\log(\frac{n}{\epsilon}))$, which improves the classical iteration complexity with a factor $n^{\frac{1}{4}}$. For small-update methods coincides to the best know iteration bound.

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