Numer Algor (2012) 61:613–629
DOI 10.1007/s11075-012-9554-1

ORIGINAL PAPER

Control of error in the homotopy analysis of semi-linear elliptic boundary value problems

Robert A. Van Gorder

Received: 29 December 2011 / Accepted: 27 February 2012 /
Published online: 17 March 2012

Abstract In the present paper, we have considered three methods with which to control the error in the homotopy analysis of elliptic differential equations and related boundary value problems, namely, control of residual errors, minimization of error functionals, and optimal homotopy selection through appropriate choice of auxiliary function $H(x)$. After outlining the methods in general, we consider three applications. First, we apply the method of minimized residual error in order to determine optimal values of the convergence control parameter to obtain solutions exhibiting central symmetry for the Yamabe equation in three or more spatial dimensions. Secondly, we apply the method of minimizing error functionals in order to obtain optimal values of the convergnce control parameter for the homotopy analysis solutions to the Brinkman–Forchheimer equation. Finally, we carefully selected the auxiliary function $H(x)$ in order to obtain an optimal homotopy solution for Liouville's equation.

Keywords Elliptic boundary value problem · Yamabe equation · Brinkman–Forchheimer equation · Liouville's equation · Homotopy analysis method · Error analysis and control

1 Introduction

Consider the semi-linear elliptic boundary value problem

$$-\triangle u = F(u)\,, \quad x \in \Omega \subset \mathbb{R}^m \tag{1.1}$$

$$u|_{\partial\Omega} = g(y)\,, \quad y \in \partial\Omega\,. \tag{1.2}$$

For our interests, we shall restrict Ω to a compact subset of $\mathbb{R}^m$.

R. A. Van Gorder (✉)
Department of Mathematics, University of Central Florida, Orlando, FL 32816-1364, USA
e-mail: rav@knights.ucf.edu

Such boundary value problems arise in many areas of mathematics and science. In the two possible linear cases, when $F(u) = 0$ (1.1) corresponds to the Laplace equation whereas when $F(u) = C \neq 0$ (where C is a constant in u but may depend on x) (1.1) corresponds to the Poisson equation. When $F(u) = u$ (1.1) corresponds to the Helmholtz equation. When $F(u) = Au^2 + Bu + C$ (where A, B, C are constants) (1.1) corresponds to the Brinkman–Forchheimer equation. When $F(u) = e^{\lambda u}$, (1.1) is Liouville's equation. When $F(u)$ takes on the power law term $F(u) = h_1(x)u - \lambda h_2(x)u^{(m+2)/(m-2)}$ $(m \neq 2)$ then (1.1) is the Yamabe equation.

While (1.1) does admit exact solutions for a number of forms of $F(u)$, for highly nonlinear $F(u)$ solutions can become very challenging to obtain. For this reason, we are interested in applying the method of homotopy analysis to such equations, with the hope of obtaining approximate solutions to within a tolerable level of error. For many physical application, approximate solutions, while clearly less informative than exact solutions, are sufficient to describe the true solutions (assuming that the error is sufficiently small). The method of homotopy analysis [1–8] has recently been applied to the study of a number of non-trivial and traditionally hard to solve nonlinear differential equations, for instance nonlinear equations arising in heat transfer [9–12], fluid mechanics [13–20], solitons and integrable models [21–24], nanofluids [25, 26] and the Lane–Emden equation which appears in stellar astrophysics [27–30], to name a few areas.

While the homotopy analysis method provides a near algorithmic way to obtain approximate solutions in an iterative manner, often the biggest question is on the convergence of the method. Hence, in the present paper we devote considerable attention to the manner in which approximate solutions converge to give appropriate errors. As the problem (1.1) and (1.2) is elliptic, we shall be able to phrase the question of an optimal homotopy solution in terms of the minimization of an energy functional.

2 Iterative homotopy formulation of the boundary value problem

As is standard in the method of homotopy analysis, we construct a homotopy

$$\mathcal{H}[\phi(x,q)] = (1-q)L[\phi(x,q) - u_0(z)] - qhH(x)N[\phi(x,q)], \tag{2.1}$$

where $\mathcal{H}$ denotes the homotopy between a nonlinear operator N (which is the operator describing the nonlinear differential equation we wish to solve, namely $N[u] = \triangle u + F(u)$) and an auxiliary linear operator L. Here $q \in [0, 1]$ is the embedding parameter (when $q = 0, \mathcal{H} = L$ while when $q = 1, \mathcal{H} = N$), h is the convergence control parameter, and $H(x)$ is the auxiliary function. See, for instance, [1–8]. The linear operator should be chosen in order to permit the initial approximation taken. Note that when $q = 0$ we have the initial approximation, i.e. $\phi(x, 0) = u_0(x)$, whereas when $q = 1$ we see that $\phi(z, 1)$ is a solution to the nonlinear differential equation of interest, i.e. $N[\phi(x, 1)] = 0$.

It shall make sense to consider initial approximations u_0 satisfying the boundary conditions, i.e.

$$u|_{\partial\Omega} = g(y) \quad \text{implies} \quad u_0(x) = G(x) \quad \text{on} \quad \Omega\,, \tag{2.2}$$

where $G = g$ on the boundary of the domain Ω. Then, when selecting the auxiliary linear operator L, we need to ensure that, $L[G] = 0$. The linear operator L has often been seen to give the best convergence when properly related to the nonlinear operator N. To this end, we may take $L = \triangle$ and select $G(x)$ as the solution to the linear boundary value problem

$$\begin{cases} \triangle G = 0\,, \\ G(y) = g(y) \quad \text{for} \quad y \in \partial\Omega\,. \end{cases} \tag{2.3}$$

Note that (2.3) is the Laplace equation held subject to the boundary conditions relevant to the initial boundary value problem (1.1) and (1.2). Then, setting $u_0(x) = G(x)$ we obtain the order zero approximation (or, the initial guess to the true solution). We should remark that the choice of $L = \triangle$ is only one possible choice. For brevity, we shall employ this choice through the remainder of the present section, as well as in Section 3. In the examples we consider, we will sometimes deviate from this choice of L, in order to obtain better approximations.

Considering a series expansion in q (treating q as a "small parameter"), we have

$$u(x) = u_0(x) + \sum_{n=1}^{\infty} u_n(x) q^n \tag{2.4}$$

where

$$u_n(x) = \frac{1}{n!} \frac{\partial^n \phi(x,q)}{\partial q^n}|_{q=0} \tag{2.5}$$

is the solution to (1.1) provided that the series for $\phi(x,q)$ converges at $q = 1$. A more useful recursive formula for the w_n's is given by the nth order deformation equations

$$L[u_n(x) - \chi_n u_{n-1}(x)] = hH(x) R_n(x,h) \tag{2.6}$$

where

$$R_n(x,h) = \triangle u_{n-1}(x) + \left[\frac{\partial^{n-1}}{\partial q^{n-1}} f\left(u_0(x) + \sum_{n=1}^{\infty} u_n(x) q^n \right) \right]_{q=0}, \tag{2.7}$$

and

$$\chi_n = \begin{cases} 0, & n = 0, 1, \\ 1, & n \geq 2. \end{cases} \tag{2.8}$$

The first few terms for $R_n(x,h)$ are

$$R_1(x,h) = \triangle u_0(x) + F(u_0)\,, \tag{2.9}$$

$$R_2(x,h) = \triangle u_1(x) + F'(u_0)u_1\,, \tag{2.10}$$

$$R_3(x,h) = \triangle u_2(x) + F'(u_0)u_2 + \frac{1}{2}F''(u_0)u_1^2\,, \tag{2.11}$$

$$R_4(x,h) = \triangle u_3(x) + F'(u_0)u_3 + F''(u_0)u_1u_2 + \frac{1}{6}F'''(u_0)u_1^2\,, \tag{2.12}$$

etcetera. From here,

$$\triangle u_n(x) = \chi_n \triangle u_{n-1}(x) + hH(x)R_n(x,h)\,. \tag{2.13}$$

The terms in the expansion (2.4) governed by (2.13) should be solved subject to homogeneous boundary conditions

$$u_n|_{\partial\Omega} = 0 \quad \text{for all} \quad n \geq 1\,, \tag{2.14}$$

as the initial approximation $u_0(x)$ already holds the relevant inhomogeneous boundary data.

In practice, we will truncate the series (2.4) to some desired number of terms. Thus, we shall be concerned with an approximate solution $\tilde{u}(x)$ with n^*+1 terms:

$$\tilde{u}(x) = u_0(x) + \sum_{n=1}^{n^*} u_n(x)\,. \tag{2.15}$$

With this we have outlined the general method of applying homotopy analysis to the boundary value problem (1.1) and (1.2). Up to this point the choice of the convergence control parameter h and the auxiliary function $H(x)$ have both been kept arbitrary. We shall now discuss the process of selecting these unknowns in the case of Gaussian initial approximations. Once $H(x)$ is selected, we can compute approximations of the form (2.15) for fixed n^*. We will then attempt to minimize the error in such an approximation by way of choosing the convergence control parameter, h, in an appropriate manner.

Note that we may proceed by inverting (2.13), viz.,

$$\begin{aligned} u_n(x) &= \chi_n u_{n-1}(x) + h\triangle^{-1}(H(x)R_n(x,h)) \\ &= \chi_n u_{n-1}(x) + h\triangle^{-1}(H(x)\triangle u_{n-1}(x)) \\ &\quad + h\triangle^{-1}(H(x)\mathcal{F}_{n-1}(u_0(x),u_1(x),\ldots,u_{n-1}(x)))\,, \end{aligned} \tag{2.16}$$

where

$$\mathcal{F}_{n-1}(u_0(x),u_1(x),\ldots,u_{n-1}(x)) = \left[\frac{\partial^{n-1}}{\partial q^{n-1}} f\left(u_0(x) + \sum_{n=1}^{\infty} u_n(x)q^n\right)\right]_{q=0}\,. \tag{2.17}$$

For such complicated expressions, a good plan of action is the minimize the error in the first order term $u_1(x)$, so that as few higher order terms as possible are needed. Note that

$$\begin{aligned} u_1(x) &= h\triangle^{-1}(H(x)\triangle u_0(x)) + h\triangle^{-1}(H(x)F(u_0(x))) \\ &= h\triangle^{-1}(H(x)(\triangle u_0(x) + F(u_0(x)))) \,. \end{aligned} \tag{2.18}$$

Recall that, for a general equation of the form $\triangle U(x) = a(x)$, the solution (subject to homogeneous boundary conditions on $\partial\Omega$) will read

$$U(x) = -\int_\Omega a(y)G(x, y)dy \tag{2.19}$$

where the Green's function $G(x, y)$ is defined by the boundary. Then, for $u_1(x)$, we obtain the relation

$$u_1(x) = -h\int_{y\in\Omega} H(y)N[u_0(y)]G(x, y)dy \,. \tag{2.20}$$

Continuing in this manner, we may obtain relations for the $u_n(x)$'s for $n \geq 2$. However, we still have h and $H(x)$ free to choose, and we shall choose these quantities so that the error in the homotopy analysis approximation is minimal. Such control of the error will be the focus of the next section.

3 Control of error and an energy functional approach

When applying analytical approximation methods for PDEs and related boundary value problems such as (1.1) and (1.2), a method to control the convergence and error of such solutions is essential. One benefit of applying the homotopy analysis method is that it gives us a way to adjust and control the convergence of solutions. However, it is at times difficult to determine error minimizing control parameters or auxiliary functions directly. Hence, in order to minimize the error of such approximate solutions, we shall need to discuss the relationship between such choices and the observed error from such approximate solutions.

3.1 Primitive control of error via residuals

Let us assume that $\hat{u}$ is a homotopy solution to (1.1) and (1.2). In the absence of an exact solution, there is no way to directly compute error in terms of deviation from a solution. However, we can compute how poorly an approximate solution deviates from a solution in terms of residuals. Hence, one measure of the error in such an approximation is given by computing the residual errors at each point. That is, for $x_0 \in \Omega$, we compute

$$\text{Res}(x_0) = |\triangle u(x_0) + F(u(x_0))| \,, \tag{3.1}$$

and we may maximize over all $x_0 \in \Omega$ in order to determine the maximum extent of deviation from the true solution (in terms of the nonlinear PDE). Of

course, we can attempt to relate the two measures of error. Let us assume that $\hat{u} = u + \epsilon u^*$, where u is the true solution, ϵ is the magnitude of the maximum absolute error, and u^* is a function bounded like $-1 \leq u^* \leq 1$. Feeding this into the residual error we obtain

$$\text{Res}(x) = \left| \triangle u + \epsilon \triangle u^* + F(u + \epsilon u^*) \right| , \tag{3.2}$$

and, expanding $F(u + \epsilon u^*)$ as

$$F(u + \epsilon u^*) = F(u) + \epsilon F'(u) u^* + O(\epsilon^2) , \tag{3.3}$$

we have that

$$\begin{aligned} \text{Res}(x) &= \left| \triangle u + F(u) + \epsilon \triangle u^* + \epsilon F'(u) u^* + O(\epsilon^2) \right| \\ &= \left| \epsilon \triangle u^* + \epsilon F'(u) u^* + O(\epsilon^2) \right| \\ &\leq \epsilon \left| \triangle u^* + F'(u) u^* \right| + O(\epsilon^2) . \end{aligned} \tag{3.4}$$

Then, for sufficiently smooth $F(u)$, the residual error for this problem is expected to scale as the absolute error ϵ.

3.2 The energy functional approach

Note that the energy functional corresponding to (1.1) is given by

$$J[u] = \int_\Omega \left(|\nabla u|^2 - 2 f(u) \right) d\Omega = \int_\Omega \left(\sum_{n=1}^{N} \left(\frac{\partial u}{\partial x_n} \right)^2 - 2 f(u) \right) d\Omega , \tag{3.5}$$

where

$$f(u) = \int_a^u F(w) dw \tag{3.6}$$

and a is a root of $F(u)$, i.e. $F(a) = 0$. (The cases we consider, such a root will exist. For other cases, we shall simply retain a constant of integration, which is set to zero.)

Now, it is well-known that an optimal solution will minimize the functional J. For a HAM approximation $\hat{u}$, we define the function $\mathcal{J}(h) = J(\hat{u})$, as clearly the value of the functional will change depending on the value of h. Thus, an optimal HAM solution will correspond to a value of the convergence control parameter, h, so that $\mathcal{J}(h) = J(\hat{u})$ is minimal. In the following section, we apply this line of thinking in order to obtain accurate HAM solutions to the Brinkman–Forchheimer equation in a duct.

3.3 Optimal first order expansion

In the previous two methods for controlling error, note that the function $H(x)$ plays a passive role in the control of error, whereas the parameter h plays the dominant role. Indeed, in the former two cases, we were able to select h in such a way to minimize the respective measure of error. In the present

subsection, we shall demonstrate one method for using the auxiliary function $H(x)$ to control the residual error.

Recall that a two-term homotopy analysis solution take the form $\hat{u} = u_0 + u_1$. Then, the residual at any point in the domain Ω is given by

$$\begin{aligned} N[\hat{u}(x)] &= N[u_0(x) + u_1(x)] \\ &= \triangle(u_0(x) + u_1(x)) + F(u_0(x) + u_1(x)) \\ &= \triangle u_0(x) - hH(x)(N[u_0(x)]) \\ &\quad + F\left(u_0(x) - h\int_{y\in\Omega} H(y)N[u_0(y)]G(x, y)dy\right), \end{aligned} \tag{3.7}$$

where we have used the representation (2.20) for $u_1(x)$. This is a nonlinear relation for the residuals in terms of h and $H(x)$. In order to minimize the absolute values of the residuals, it is sufficient (and rather strong) to assume that the right hand side of (3.7) is zero. Under such an assumption,

$$\triangle u_0(x) - hH(x)(N[u_0(x)]) + F\left(u_0(x) - h\int_{y\in\Omega} H(y)N[u_0(y)]G(x, y)dy\right) = 0\,, \tag{3.8}$$

or, rewriting terms by adding and subtracting $F(u_0(x))$, we have

$$\begin{aligned} &N[u_0(x)](1 - hH(x)) + F\left(u_0(x) - h\int_{y\in\Omega} H(y)N[u_0(y)]G(x, y)dy\right) \\ &\quad - F(u_0(x)) = 0\,. \end{aligned} \tag{3.9}$$

Note that (3.9) is a nonlinear integral equation for the unknown function $\hat{H}(x) = hH(x)$. As both $u_0(x)$ and $F(u)$ are assumed known by this point in the problem, it only remains to solve $\hat{H}(x)$. The existence of a solution $\hat{H}(x)$ will depend on the properties of $F(u)$ and such questions are best left to the theory of integral equations for each individual case of $F(u)$. If, indeed, a solution $\hat{H}(x)$ to (3.9) exists, then $\hat{u} = u_0 + u_1$, where u_1 is defined as in (2.20), is the optimal first order expansion obtained via the homotopy analysis method.

Utilizing (3.3) to expand the nonlinearity, and assuming sufficiently well-behaved and smooth $F(u)$, we have

$$\begin{aligned} &F\left(u_0(x) - h\int_{y\in\Omega} H(y)N[u_0(y)]G(x, y)dy\right) \\ &\quad = F(u_0(x)) - hF'(u_0(x))\int_{y\in\Omega} H(y)N[u_0(y)]G(x, y)dy, \end{aligned} \tag{3.10}$$

we obtain the following approximation to the residual error:

$$\begin{aligned} N[\hat{u}(x)] = N[u_0(x)](1 - hH(x)) - hF'(u_0(x)) \\ \times\int_{y\in\Omega} H(y)N[u_0(y)]G(x, y)dy + O(h^2)\,. \end{aligned} \tag{3.11}$$

In the case where h can be taken to be small (subject to a scaling of $H(x)$), minimization of the residuals is attained when $H(x)$ satisfies the integral equation

$$hH(x) + h\frac{F'(u_0(x))}{N[u_0(x)]}\int_{y\in\Omega} H(y)N[u_0(y)]G(x,y)dy = 1\,, \tag{3.12}$$

or, introducing a new function $\hat{H}(x) = hH(x)$ (which is a scaling of $H(x)$),

$$\hat{H}(x) + \frac{F'(u_0(x))}{N[u_0(x)]}\int_{y\in\Omega} \hat{H}(y)N[u_0(y)]G(x,y)dy = 1\,. \tag{3.13}$$

Assuming h could be taken sufficiently small, the existence of a solution $\hat{H}(x)$ to the integral (3.13) implies the existence of a two-term optimal homotopy solution, in an approximate sense.

4 Applications

In order to illustrate the more theoretical results of the previous sections, we shall now consider applications of homotopy analysis to various semi-linear differential equations.

4.1 A Yamabe equation on $\Omega = \mathcal{B}^m \subset \mathbb{R}^m$

The Yamabe equation is a nonlinear differential equation arising in geometry and related areas of physics [31–33]. In the present section, we apply the method of homotopy analysis to a Yamabe equation on the unit ball in $\mathbb{R}^m$ for $m \geq 3$. (That is, we take the domain $\Omega = \mathcal{B}^m \subset \mathbb{R}^m$, where $\mathcal{B}^m$ is the m-ball. Then, note that the boundary of Ω is $\partial\Omega = \mathcal{S}^{m-1}$, where $\mathcal{S}^{m-1}$ is the $m-1$ -sphere.) To this end, consider the boundary value problem

$$-\triangle u = u - \lambda u^{(m+2)/(m-2)} = u - \lambda u^{1+\frac{4}{m-2}} \tag{4.1}$$

$$u(y) = 1 \quad \text{for} \quad y \in \mathcal{S}^{m-1}\,. \tag{4.2}$$

Let us consider a solution with central symmetry about the origin. Then, we assume a solution of the form

$$u(x) = v(r) \quad \text{where} \quad r = \sqrt{\sum_{j=1}^{m} x_j^2}\,. \tag{4.3}$$

The Yamabe (4.1) is then reduced to

$$v'' + \frac{1}{r}v' - v + \lambda v^{\alpha} = 0\,, \tag{4.4}$$

where $\alpha = 1 + \frac{4}{m-2} \in (1, 5]$ for $m = 3, 4, 5, \ldots$. The relevant boundary conditions then become

$$v'(0) = 0 \quad \text{and} \quad v(1) = 1\,. \tag{4.5}$$

Equations (4.4) and (4.5) specify a nonlinear singular boundary value problem. Note that this boundary value problem is quite similar in form to Lane–Emden equations (see, for instance, [34–39]), which have previously been solved by the method of homotopy analysis [29].

In order to apply the method of homotopy analysis to the boundary value problem (4.4) and (4.5), we first specify the linear operator. Let us select

$$L = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - 1\,, \tag{4.6}$$

so that $L[v] = v'' + \frac{1}{r}v' - v$. Assuming that the zeroth order approximation, $v_0(r)$, satisfies $L[v_0(r)] = 0$, we find that $v_0(r)$ must take the form

$$v_0(r) = \frac{I_0(r)}{I_0(1)}\,, \tag{4.7}$$

where I_0 denotes the modified Bessel function of the first kind. From here, we may obtain the higher order terms in the HAM expansion which are governed by the higher order deformation equations. We take $N[v] = v'' + \frac{1}{r}v' - v + \lambda v^\alpha$ and set $H(r) = 1$. From the higher order deformation equations, we find that $v_1(r)$ satisfies

$$L[v_1] = h\lambda(v_0(r))^\alpha \qquad \text{subject to} \quad v_1'(0) = 0 \quad \text{and} \quad v_1(1) = 0\,. \tag{4.8}$$

Solving this equation for $v_1(r)$, we find that

$$\begin{aligned} v_1(r) = {} & \frac{h\lambda}{(I_0(1))^\alpha}\int_0^r \rho\,(K_0(\rho)I_0(r) - K_0(r)I_0(\rho))\,(I_0(\rho))^\alpha\,d\rho \\ & - \frac{h\lambda I_0(r)}{(I_0(1))^{\alpha-1}}\int_0^1 \rho\,(K_0(\rho)I_0(1) - K_0(1)I_0(\rho))\,(I_0(\rho))^\alpha\,d\rho\,, \end{aligned} \tag{4.9}$$

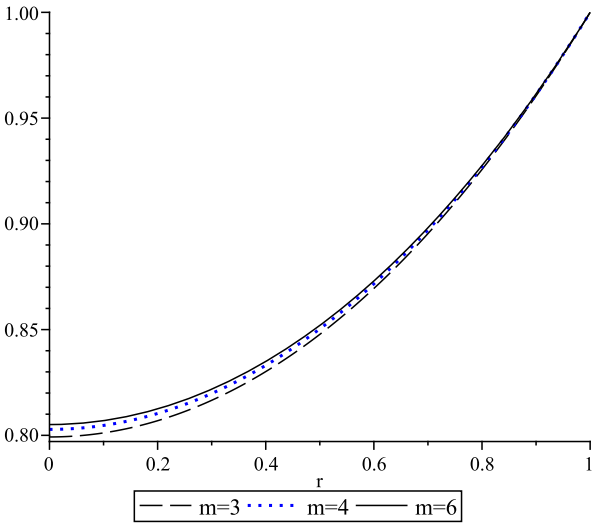

Fig. 1 We fix $\lambda = 0.1$ and give solutions to the Yamabe equation with central symmetry when $m = 3, 4, 6$

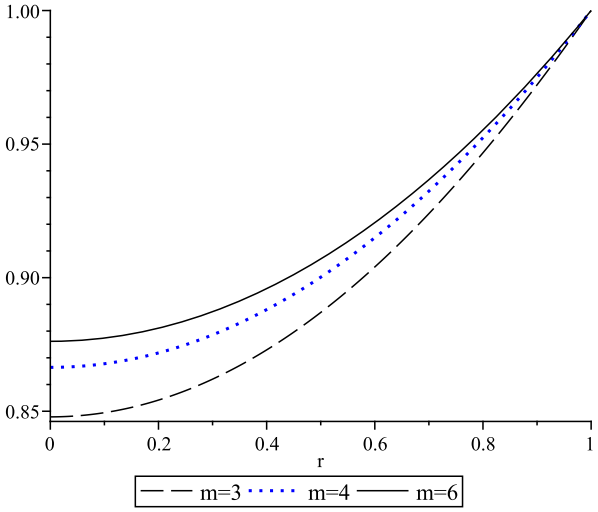

Fig. 2 We fix $\lambda = 0.5$ and give solutions to the Yamabe equation with central symmetry when $m = 3, 4, 6$

where K_0 denotes the modified Bessel function of the second kind. Continuing in this way, we obtain the higher order approximations.

In Fig. 1, we fix $\lambda = 0.1$ give solutions to the Yamabe equation with central symmetry when $m = 3, 4, 6$ in order to demonstrate the method. In Fig. 2, we do the same for $\lambda = 0.5$. For each value of m, we select a sufficient number of terms in the HAM expansion in order to obtain residual errors of less than 10^{-4}. In Table 1, we list the number of terms required for each value of m, as well as the optimal value of h, the convergence control parameter, taken in each case. We expect that, due to similarities between the transformed Yamabe equation and the Lane–Emden equation of the first kind, we should see similar rates of convergence in the HAM solutions for each.

Table 1 Number of terms required for the HAM expansion for the Yamabe equation to have an error within 10^{-4}

λ	m	Required number of terms in the HAM expansion	h value
0.1	3	4	−0.15
0.1	4	4	−0.15
0.1	6	6	−0.15
0.5	3	5	−0.18
0.5	4	6	−0.18
0.5	6	8	−0.25

Optimal values of h, the convergence control parameter, are also given. Note that optimal values of h are similar to those found in [29] for Lane–Emden equations of first kind, which are closely related to the form of the Yamabe equation we've considered here

4.2 Brinkman–Forchheimer equation for a rectangular duct

Hooman [40] considered the analog to the Brinkman–Forchheimer equation over only one variable and obtained a perturbation solution for forced convection in a saturated porous duct. Hooman and Merrikh [41] studied a linear form of the partial differential equation. Recently, in [42], the Brinkman-Forchheimer equation for the unidirectional flow in the x_3 direction of a rectangular duct was studied via a variational approximation method. The physical equation reads

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{C_f \rho}{\tilde{\mu}\sqrt{K}} u^2 - \frac{\mu}{\tilde{\mu} K} u + \frac{G}{\tilde{\mu}} = 0\,, \tag{4.10}$$

where u is the velocity in the x_3 direction, C_f is the inertial coefficient, ρ is the density, μ is the viscosity of the fluid, $\tilde{\mu}$ is the effective viscosity, K is the permeability, and G is the adverse applied pressure gradient. Natural boundary conditions for the problem are

$$u(0, x_2) = u(1, x_2) = u(x_1, 0) = u(x_1, 1) = 0\,. \tag{4.11}$$

For simplicity, we define constants A, B, C as in CITE:

$$A = \frac{C_f \rho}{\tilde{\mu}\sqrt{K}}\,, \quad B = \frac{\mu}{\tilde{\mu} K}\,, \quad C = \frac{G}{\tilde{\mu}}\,. \tag{4.12}$$

Then, we have the equation

$$-\triangle u = -Au^2 - Bu + C \tag{4.13}$$

defined on the square $(x_1, x_2) \in \Omega = [0, 1] \times [0, 1]$. The associated energy functional for this equation is given by

$$J[u] = \int_\Omega \left(\left(\frac{\partial u}{\partial x_1} \right)^2 + \left(\frac{\partial u}{\partial x_2} \right)^2 + \frac{2A}{3} u^3 + Bu^2 - 2Cu \right) dx_1 dx_2\,. \tag{4.14}$$

Pursuant to this choice of energy functional, we seek approximate solutions which minimize J. Such minimizing solutions will naturally depend on h, the convergence control parameter. Hence, for a family of approximate solutions $\hat{u}(x_1, x_2, h)$, note that $J[\hat{u}(x_1, x_2, h)] = \mathcal{J}(h)$, a function of h. Then, selecting h in order to minimize $\mathcal{J}$, we have an error minimizing choice of h. This follows from the fact that $J[u]$ is one possible representation for the aggregate residual error over the problem domain.

In order to better demonstrate the method, let us consider the cross-section where one variable is fixed, say $x_2 =$ constant. Dropping the remaining subscript, we have $x_1 = x$, and then $u_{xx} = Au^2 + Bu - C$. We take $u_0 = 0$, $L = u_{xx}$ and $N[u] = u_{xx} - Au^2 - Bu + C = u_{xx} + F(u)$. We then apply the higher order deformation equations. The first approximation is governed by the equation

$$u_{1,xx} = hC \qquad \text{subject to} \quad u_1(0) = 0 \quad \text{and} \quad u_1(1) = 0\,, \tag{4.15}$$

and hence

$$u_1(x) = -\frac{hC}{2}x(1-x)\,. \tag{4.16}$$

The second order term is governed by the equation

$$u_{2,xx} = (1+h)hC + hF'(0)u_1 \qquad \text{subject to} \quad u_1(0) = 0 \quad \text{and} \quad u_1(1) = 0\,, \tag{4.17}$$

and hence

$$u_2(x) = \frac{h^2BC}{24}(2-x)x^3 + \frac{(1+h)hC}{2}x^2 - \frac{hC}{24}\left(Bh + 12(1+h)\right)x\,. \tag{4.18}$$

Continuing in this manner, we may successively obtain the higher order terms in the HAM expansion.

For sake of demonstration, we take a four term approximation of the form $\hat{u}(x) = u_1(x) + u_2(x) + u_3(x) + u_4(x)$. In Fig. 3, we plot the function $\mathcal{J}(h)$ in order to demonstrate the existence of an error minimizing value of h. We list values of h which minimize $\mathcal{J}(h)$ in Table 2, for various values of the model parameters A, B, and C. In Fig. 4, we plot the residual errors for the optimal choices of h obtained from minimizing the function $\mathcal{J}(h)$. In the residual plots, we see that we have been able to control the residual error to within 10^{-3} using only a four term approximation.

4.3 Liouville's equation in one variable

As our last application, we consider the Liouville's equation (which plays a prominent role in differential geometry) in one variable. While the method we outline here is applicable in m variables, computations are most straightfor-

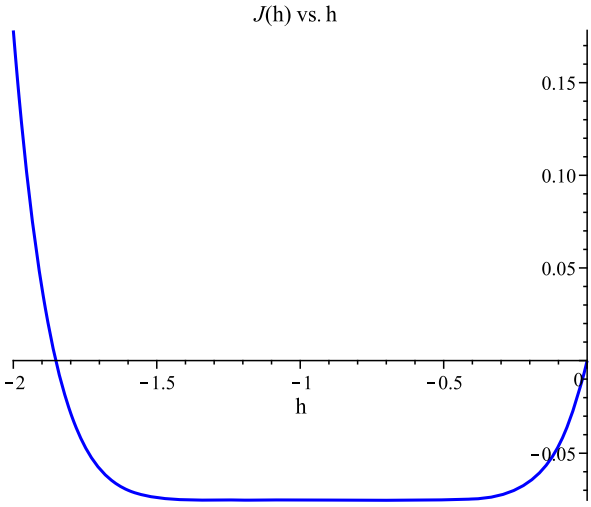

Fig. 3 Plot of the function $\mathcal{J}(h)$ in order to demonstrate the existence of an error minimizing value of h in the case of a Brinkman–Forchheimer equation when $(A, B, C) = (1, 1, 1)$

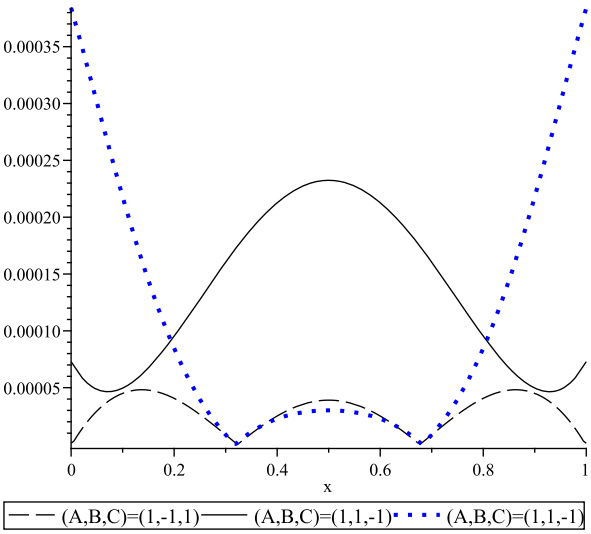

Fig. 4 Plot of the residual errors for the Brinkman–Forchheimer equation given various values of A, B, C. We see that, using only the first four terms in the HAM expansion, the error can be controlled to within 10^{-3} to even as low as 10^{-4} in some cases

ward in one variable, and this simplicity will help us to illustrate the methods more clearly.

The Liouville equation in one variable reads

$$-u_{xx} = e^{\lambda u}\,, \tag{4.19}$$

where λ is a parameter. In order to construct solutions, we prescribe the boundary conditions

$$u(0) = 0 \quad \text{and} \quad u(1) = 1\,. \tag{4.20}$$

Taking the nonlinear operator to be $N[u] = u_{xx} + e^{\lambda u}$ and the linear operator to be $L[u] = u_{xx}$, we have that $u_0(x) = x$ is the first approximation to the

Table 2 Table of error minimizing values of the convergence control parameter h for various parameter values (A, B, C) for the Brinkman–Forchheimer equation

(A, B, C)	h value
$(1, 1, 1)$	-0.8568
$(1, 1, -1)$	-0.9076
$(1, -1, 1)$	-1.0325
$(1, -1, -1)$	-1.1133
$(-1, 1, 1)$	-0.9076
$(-1, 1, -1)$	-0.8568
$(-1, -1, 1)$	-1.1133
$(-1, -1, -1)$	-1.0325

Certain values repeat due to symmetries between the $A = 1$ or $A = -1$ and the $C = 1$ or $C = -1$ possibilities. Here we have always taken the four term HAM expansion

solution. The second term in the homotopy approximation, $u_1(x)$, is governed by the boundary value problem

$$L[u_1] = \hat{H}(x)e^{\lambda x}\,, \quad \text{subject to} \quad u_1(0) = u_1(1) = 0\,. \tag{4.21}$$

Here we have again taken $\hat{H}(x) = hH(x)$. Solving for $u_1(x)$, we find that

$$u_1(x) = -\int_0^1 G(x,\xi)\hat{H}(\xi)e^{\lambda\xi}d\xi\,, \tag{4.22}$$

where the Green's function $G(x,\xi)$ is defined by

$$G(x,\xi) = \begin{cases} \xi(1-x)\,, & \xi < x\,, \\ x(1-\xi)\,, & \xi > x\,. \end{cases} \tag{4.23}$$

Employing the formula for the residuals given by (3.7), we have that

$$N[u_0+u_1] = \hat{H}(x)e^{\lambda x} + \exp\left[\lambda x - \lambda\int_0^1 G(x,\xi)\hat{H}(\xi)e^{\lambda\xi}d\xi\right]. \tag{4.24}$$

Now, let us assume that we may pick $\hat{H}(x)$ in order to minimize the residuals. A sufficient condition for the minimization of residuals is simply $N[u_0+u_1]=0$ (although in some cases this exact satisfaction of the residual condition is not possible). Assuming that such a condition is indeed possible, after some simplifications we obtain the relation

$$\hat{H}(x) + \exp\left[-\lambda\int_0^1 G(x,\xi)\hat{H}(\xi)e^{\lambda\xi}d\xi\right] = 0\,, \tag{4.25}$$

which is a nonlinear integral equation for $\hat{H}(x)$. Such an equation is very hard to solve under the standard ansatz. Hence, we are left to attempt an approximation method in order to determine $\hat{H}(x)$. Consider the iterative process

$$I_n(x) = -\exp\left[-\lambda\int_0^1 G(x,\xi)I_{n-1}(\xi)e^{\lambda\xi}d\xi\right], \tag{4.26}$$

$$I_0(x) = 1\,, \tag{4.27}$$

for $n \geq 1$. Clearly, if the iterative process converges as $n \to +\infty$, then $\hat{H}(x) = \lim_{n\to+\infty} I_n(x)$. We compute

$$I_1(x) = -\exp\left[\frac{e^\lambda - 1}{\lambda}(1-x)\right], \tag{4.28}$$

$$I_2(x) = -\exp\left[-\frac{e^\lambda - 1}{\lambda}(1-x)\right], \tag{4.29}$$

$$I_3(x) = -\exp\left[-\frac{e^\lambda - 1}{\lambda}(1-x)\right], \tag{4.30}$$

and hence, as $I_3(x) = I_2(x)$, we have found a fixed point of the map (4.26). Therefore, we conclude that

$$\hat{H}(x) = -\exp\left[-\frac{e^{\lambda}-1}{\lambda}(1-x)\right] \tag{4.31}$$

is an error minimizing choice for $\hat{H}(x)$. Placing this choice of $\hat{H}(x)$ back into (4.22), we have that the optimal second term is given by

$$u_1(x) = \int_0^1 G(x,\xi)\exp\left(\lambda\xi - \left(\frac{e^{\lambda}-1}{\lambda}\right)(1-\xi)\right)d\xi\,. \tag{4.32}$$

The optimal two-term homotopy solution is then given by

$$\hat{u}(x) = x + \int_0^1 G(x,\xi)\exp\left(\lambda\xi - \left(\frac{e^{\lambda}-1}{\lambda}\right)(1-\xi)\right)d\xi\,. \tag{4.33}$$

Note that, while we were able to obtain an accurate solution to the present problem by use of the two-term homotopy solution, in most cases we will not be able to solve the integral equation governing $H(x)$. In such cases, approximate solutions to $H(x)$ can still be obtained, and such auxiliary functions can still be used to decreases the residual errors. The results in this section demonstrate yet again that the auxiliary function $H(x)$ can prove to be a useful tool when applying the homotopy analysis method.

In Fig. 5, we plot the optimal first order HAM solution for various values of λ.

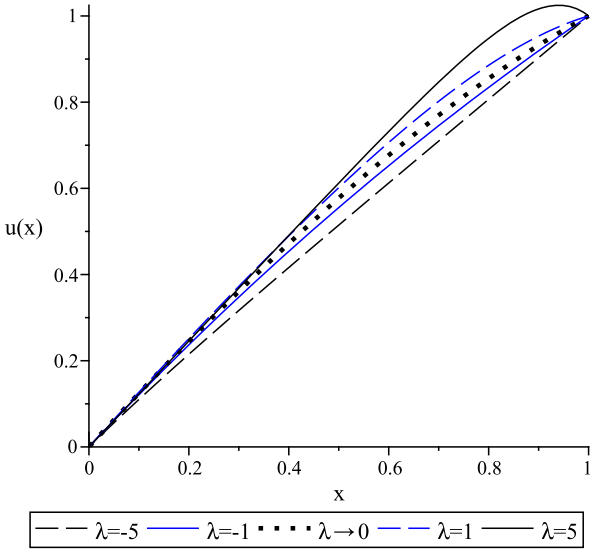

Fig. 5 Plot of the optimal homotopy solutions to Liouville's equation for various λ. In the limit $\lambda \to 0$, the solution form we have selected becomes singular. So, at $\lambda = 0$, we have simply plotted the exact solution for comparison purposes. We see that both this exact solution and the HAM solutions agree nicely

5 Conclusions

In the present paper, we have considered three methods with which to control the error in the homotopy analysis of elliptic differential equations, namely, control of residual errors, minimization of error functionals, and optimal homotopy selection through appropriate choice of auxiliary function $H(x)$. After outlining the methods in general, we consider three applications. First, we apply the method of minimized residual error in order to determine optimal values of the convergence control parameter to obtain solutions exhibiting central symmetry for the Yamabe equation in three or more spatial dimensions. Secondly, we apply the method of minimizing error functionals in order to obtain optimal values of the convergnce control parameter for the homotopy analysis solutions to the Brinkman–Forchheimer equation. Finally, we carefully selected the auxiliary function $H(x)$ in order to obtain an optimal homotopy solution for Liouville's equation.

References

1. Liao, S.J.: On the Proposed Homotopy Analysis Techniques for Nonlinear Problems and its Application. Ph.D. dissertation. Shanghai Jiao Tong University (1992)
2. Liao, S.J.: Beyond Perturbation: Introduction to the Homotopy Analysis Method. Chapman & Hall/CRC Press, Boca Raton (2003)
3. Liao, S.J.: An explicit, totally analytic approximation of Blasius viscous flow problems. Internat. J. Non-Linear Mech. **34**, 759–778 (1999)
4. Liao, S.J.: On the homotopy analysis method for nonlinear problems. Appl. Math. Comput. **147**, 499–513 (2004)
5. Liao, S.J., Tan, Y.: A general approach to obtain series solutions of nonlinear differential equations. Stud. Appl. Math. **119**, 297–354 (2007)
6. Liao, S.J.: Notes on the homotopy analysis method: some definitions and theorems. Commun. Nonlinear Sci. Numer. Simul. **14**, 983–997 (2009)
7. Van Gorder, R.A., Vajravelu, K.: On the selection of auxiliary functions, operators, and convergence control parameters in the application of the Homotopy Analysis Method to nonlinear differential equations: A general approach. Commun. Nonlinear Sci. Numer. Simul. **14**, 4078–4089 (2009)
8. Liao, S.: An optimal homotopy-analysis approach for strongly nonlinear differential equations. Commun. Nonlinear Sci. Numer. Simul. **15**, 2315–2332 (2010)
9. Abbasbandy, S.: The application of homotopy analysis method to nonlinear equations arising in heat transfer. Phys. Lett. A **360**, 109–113 (2006)
10. Abbasbandy, S.: Homotopy analysis method for heat radiation equations. Int. Comm. Heat Mass Transfer **34**, 380–387 (2007)
11. Liao, S.J., Su, J., Chwang, A.T.: Series solutions for a nonlinear model of combined convective and radiative cooling of a spherical body. Int. J. Heat Mass Transfer **49**, 2437–2445 (2006)
12. Liao, S.J., Campo, A.: Analytic solutions of the temperature distribution in Blasius viscous flow problems. J. Fluid Mech. **453**, 411–425 (2002)
13. Liao, S.J.: An explicit, totally analytic approximation of Blasius viscous flow problems. Internat. J. Non-Linear Mech. **34**, 759–778 (1999)
14. Liao, S.J.: A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate. J. Fluid Mech. **385**, 101–128 (1999)
15. Liao, S.J.: On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet. J. Fluid Mech. **488**, 189–212 (2003)

16. Akyildiz, F.T., Vajravelu, K., Mohapatra, R.N., Sweet, E., Van Gorder, R.A.: Implicit differential equation arising in the steady flow of a sisko fluid. Appl. Math. Comput. **210**, 189–196 (2009)
17. Hang, X., Lin, Z.L., Liao, S.J., Wu, J.Z., Majdalani, J.: Homotopy based solutions of the Navier-Stokes equations for a porous channel with orthogonally moving walls. Phys. Fluids **22**, 053601 (2010)
18. Sajid, M., Hayat, T., Asghar, S.: Comparison between the HAM and HPM solutions of thin film flows of non-Newtonian fluids on a moving belt. Nonlinear Dynam. **50**, 27–35 (2007)
19. Hayat, T., Sajid, M.: On analytic solution for thin film flow of a fourth grade fluid down a vertical cylinder. Phys. Lett. A **361**, 316–322 (2007)
20. Turkyilmazoglu, M.: Purely analytic solutions of the compressible boundary layer flow due to a porous rotating disk with heat transfer. Phys. Fluids **21**, 106104 (2009)
21. Abbasbandy, S., Zakaria, F.S.: Soliton solutions for the fifth-order KdV equation with the homotopy analysis method. Nonlinear Dynam. **51**, 83–87 (2008)
22. Wu, W., Liao, S.J.: Solving solitary waves with discontinuity by means of the homotopy analysis method. Chaos, Solitons & Fractals **26**, 177–185 (2005)
23. Sweet, E., Van Gorder, R.A.: Analytical solutions to a generalized Drinfel'd - Sokolov equation related to DSSH and KdV6. Appl. Math. Comput. **216**, 2783–2791 (2010)
24. Wu, Y., Wang, C., Liao, S.J.: Solving the one-loop soliton solution of the Vakhnenko equation by means of the homotopy analysis method. Chaos, Solitons & Fractals **23**, 1733–1740 (2005)
25. Cheng, J., Liao, S.J., Mohapatra, R.N., Vajravelu, K.: Series solutions of Nano-boundary-layer flows by means of the homotopy analysis method. J. Math. Anal. Appl. **343**, 233–245 (2008)
26. Van Gorder, R.A., Sweet, E., Vajravelu, K.: Nano boundary layers over stretching surfaces. Commun. Nonlinear Sci. Numer. Simul. **15**, 1494–1500 (2010)
27. Bataineh, A.S., Noorani, M.S.M., Hashim, I.: Solutions of time-dependent Emden-Fowler type equations by homotopy analysis method. Phys. Lett. A **371**, 72–82 (2007)
28. Bataineh, A.S., Noorani, M.S.M., Hashim, I.: Homotopy analysis method for singular IVPs of Emden-Fowler type. Commun. Nonlinear Sci. Numer. Simul. **14**, 1121–1131 (2009)
29. Van Gorder, R.A., Vajravelu, K.: Analytic and numerical solutions to the Lane-Emden equation. Phys. Lett. A **372**, 6060–6065 (2008)
30. Liao, S.: A new analytic algorithm of Lane-Emden type equations. Appl. Math. Comput. **142**, 1–16 (2003)
31. Andersson, L.: Chruściel, P.T., Friedrich, H.: On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations. Comm. Math. Phys. **149**, 587–612 (1992)
32. Brendle, S.: Blow-up phenomena for the Yamabe equation. J. Amer. Math. Soc. **21**, 951–980 (2008)
33. Alice Chang, S.Y., Han, Z.C., Yang, P.: Classification of singular radial solutions to the [sigma] k Yamabe equation on annular domains. J. Differential Equations **216**, 482–501 (2005)
34. Van Gorder, R.A.: Analytical solutions to a quasilinear differential equation related to the Lane-Emden equation of the second kind. Celest. Mech. Dynam. Astron. **109**, 137–145 (2011)
35. Van Gorder, R.A.: An elegant perturbation solution for the Lane-Emden equation of the second kind. New Astron. **16**, 65–67 (2011)
36. Van Gorder, R.A.: Exact first integrals for a Lane–Emden equation of the second kind modeling a thermal explosion in a rectangular slab. New Astron. **16**, 492–497 (2011)
37. Momoniat, E., Harley, C.: Approximate implicit solution of a Lane–Emden equation. New Astron. **11**, 520–526 (2006)
38. Wazwaz, A.M.: A new algorithm for solving differential equations of Lane–Emden type. Appl. Math. Comp. **118**, 287–310 (2001)
39. Chandrasekhar, S.: An Introduction to the Study of Stellar Structure, p. 90. Dover, New York (1967)
40. Hooman, K.: A perturbation solution for forced convection in a porous-saturated duct. J. Comput. Appl. Math. **211**, 57–66 (2008)
41. Hooman, K., Merrikh, A.A.: Analytical solution of forced convection in a duct of rectangular cross section saturated by a porous medium. Heat Transfer **128**, 596–600 (2006)
42. Van Gorder, R.A., Vajravelu, K., Akyildiz, F.T.; Solutions to the Brinkman-Forchheimer momentum equation for a unidirectional flow over a rectangular domain. Int. J. Fluid Mech. Res. **36**, 552–565 (2009)