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Polynomial splines as examples of Chebyshevian splines

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Abstract We consider geometrically continuous polynomial splines defined on a given knot-vector by lower triangular connection matrices with positive diagonals. In order to find out which connection matrices make them suitable for design, we regard them as examples of geometrically continuous piecewise Chebyshevian splines. Indeed, in this larger context we recently achieved a simple characterisation of all suitable splines for design. Applying it to our initial polynomial splines will require us to treat polynomial spaces on given closed bounded intervals as instances of Extended Chebyshev spaces, so as to determine all possible systems of generalised derivatives which can be associated with them.

Keywords B-splines · Total positivity · Chebyshev spaces · Bernstein-type bases · Weight functions · Generalised derivatives · Blossoms

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1 Introduction

The polynomial splines we are concerned with are geometrically continuous: at each knot, a number of left/right derivatives are linked by a connection matrix, supposed to be lower triangular and to have a positive diagonal. A classical sufficient condition for such splines to be suitable for either approximation or geometric design is the total positivity of all the connection matrices [3]. Their

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entries can then serve as shape parameters. As an example, the parameters in question can be efficiently used in spline interpolation to make up for the Gibbs phenomenon.

For cubic splines, going beyond total positivity has proved to usefully increase the possibilities of shape effects [5]. This is why we are interested here in determining all sequences of connection matrices leading to suitable polynomial spline spaces of any degree. Even more interesting than the results themselves is the way we achieve them. They follow from considering geometrically continuous polynomial splines as examples of geometrically continuous piecewise Chebyshevian splines. This enables us to enjoy the use of results recently obtained for this more general class of splines. In particular it will require us to treat polynomial spaces on closed bounded intervals as special instances of Extended Chebyshev spaces and to associate with them tools specific to the Chebyshevian context.

We explain our exact problematic in Section 2 after a brief reminder about geometric continuity. In Section 3 we move to the Chebyshevian world, presenting only the main tools which this work relies on. We more specially insist on two crucial recent results concerning the generalised derivatives associated with a given Extended Chebyshev space [10] and their use to build all suitable geometrically continuous piecewise Chebyshevian splines [11]. We then apply the two results in question to build all sequences of connection matrices leading to convenient geometrically continuous polynomial splines. The general theoretical arguments are described in Section 4 and the problem is then completely solved for cubic and quartic splines in Section 5. In the first case, it gives a further proof of the conditions previously obtained in [5] via a geometrical approach. The quartic case gives a good understanding of the general method on which we comment in the final section.

2 Geometrically continuous polynomial splines

2.1 A brief reminder on geometric continuity

Geometric continuity has been a fashionable subject in geometric design in the 80’s: it was then realised that, when building piecewise parametric curves in $\mathbb{R}^d$, $d > 1$, joining the given parameterisations of two consecutive pieces up to some order had no special meaning for the curves in question. It has since become natural to directly integrate the presence of connection matrices in general settings, in order to give users the possibility of replacing parametric continuity by geometric continuity if needed. It is useful to give a brief reminder on this subject to explain our choice for the polynomial splines we will consider.

Let us start with two $(n+1)$-dimensional spaces $\mathbb{E}_i \subset C^n([t_i, t_{i+1}])$, $i = 0, 1$, with $t_0 < t_1 < t_2$, both assumed to contain constants. We additionally assume that, at the point t_1, any Taylor interpolation problem in $(n+1)$ data is unisolvent in either space. Given a fixed integer p, $1 \le p \le n$, and a fixed regular *connection matrix* M of order p, it is then meaningful to build a parametric curve by joining two given $F_0 \in \mathbb{E}_0{}^d$ and $F_1 \in \mathbb{E}_1{}^d$ via the relations

$$F_0(t_1) = F_1(t_1), \quad \left(F_1{}'(t_1), \ldots, F_1{}^{(p)}(t_1)\right)^T = M.\left(F_0{}'(t_1), \ldots, F_0{}^{(p)}(t_1)\right)^T.$$

Compared to parametric continuity of order p (obtained when M is the identity matrix of order p) the entries of M play the rôle of shape parameters which permit more flexibility for the resulting parametric curve defined by the continuous function $S : [t_0, t_2] \to \mathbb{R}^d$ whose restriction to $[t_i, t_{i+1}]$ coincides with F_i for $i = 0, 1$. However, for the latter curve to be considered geometrically continuous of order p it is necessary to assume M to satisfy some properties, which we list below from the weakest to the strongest requirements.

- *Frenet continuity of order p:* We can assume that, for $i = 0, 1$, F_i is a *mother function* in $\mathbb{E}_i$, in the sense that $\mathbb{E}_i$ is the set of all affine images of F_i. The Frenet frames of order p at t_1 of both functions (obtained by applying the Gram-Schmidt process to their first p derivatives at t_1) are then well-defined. Requiring them to coincide is the weakest possible sense of geometric continuity of order p. It is obtained if and only if the connection matrix M is lower triangular with positive diagonal entries. We then have at our disposal $p(p+1)/2$ free shape parameters.
- *Visual continuity of order p:* this corresponds to continuity both of the Frenet frames of order p and of the first $(p-1)$ curvatures at t_1. It is obtained by additionally requiring the positive diagonal of M to be of the form $(a, a^2, \ldots, a^p)$. The total number of free parameters is then $1 + p(p-1)/2$.
- *Arc-length continuity of order p:* this corresponds to C^p joint at t_1 up to reparameterisation, or, equivalently, to C^p joint of the arc-length parameterisations of the two pieces. In that case, only the p entries of the first column of M are free parameters, all others being deduced from them via the chain rule.

We would like to call the reader's attention on the fact that the expressions used to refer to the various kinds of geometric continuity are not universally fixed. For instance, the expression "Frenet continuity" is quite often used in the sense of what we call "visual continuity", while "visual continuity" sometimes has the meaning of "arc-length continuity", which in turn is often ambiguously referred to as "geometric continuity". We use the expression "visual continuity" with the idea that if the human eye were able to "see" in any dimension p, it would not distinguish between two curves having at a common point the same Frenet frames of order p along with the same corresponding meaningful curvatures.

In general, no specific name is allocated to what we most logically call "Frenet continuity", the other two geometric kinds of connection being more popular. Nevertheless, this is the type of connection we will work with, because it is the most general geometrical one, and also because we consider it essential to allow jumps in the curvatures for specific design purposes.

2.2 Splines

Throughout the rest of the paper, we consider a fixed positive integer n, a fixed bi-infinite sequence of *knots* t_k, $k \in \mathbb{Z}$, with $t_k < t_{k+1}$ for all k, and a fixed bi-infinite sequence of associated multiplicities m_k, $k \in \mathbb{Z}$, with

$$0 \leq m_k \leq n \quad \text{for all } k \in \mathbb{Z}, \quad \sum_{i\leq 0} m_i = \sum_{i\geq 0} m_i = +\infty. \tag{1}$$

The extended knot-vector $\mathbb{K} := \left({t_k}^{[m_k]}\right)_{k\in\mathbb{Z}}$ formed by the knots repeated with their multiplicities can then be written as a bi-infinite sequence

$$\mathbb{K} = \left(\xi_\ell\right)_{\ell\in\mathbb{Z}}, \quad \text{with } \xi_\ell \leq \xi_{\ell+1} \text{ and } \xi_\ell < \xi_{\ell+n} \text{ for all } \ell \in \mathbb{Z}.$$

Assuming $\mathbb{K}$ to be bi-infinite is not real limitation since it permits to treat the usual case of splines based on a finite partition of a closed bounded interval.

For each $k \in \mathbb{Z}$, let M_k be a lower triangular matrix of order $(n - m_k)$, with positive diagonal entries. Based on the bi-infinite sequence of connection matrices M_k, $k \in \mathbb{Z}$, a *geometrically continuous polynomial spline* (Frenet continuity) is a continuous function $S :]\inf_k t_k, \sup_k t_k[\to \mathbb{R}$ which satisfies the following two properties:

1. for each $k \in \mathbb{Z}$, there exists a polynomial $F_k \in \mathbb{P}_n$ which coincides with S on the interval $[t_k, t_{k+1}]$;
2. S satisfies the connection conditions:

$$\left(S'({t_k}^+), \dots, S^{(n-m_k)}({t_k}^+)\right)^T = M_k \,.\, \left(S'({t_k}^-), \dots, S^{(n-m_k)}({t_k}^-)\right)^T, \quad k \in \mathbb{Z}. \tag{2}$$

For the space $\mathbb{S}$ of all such splines to be considered suitable for design we expect it to offer the same possibilities as the ordinary polynomial spline space [2, 13]. In particular it must possess a B-spline basis, in the usual sense recalled below.

Definition 2.1 A sequence of splines $N_\ell \in \mathbb{S}$, $\ell \in \mathbb{Z}$, is said to be a *B-spline basis of* $\mathbb{S}$ if it meets the following requirements:

- ⋆ *Support property*: for each $\ell \in \mathbb{Z}$, N_ℓ has support $[\xi_\ell, \xi_{\ell+n+1}]$;
- ⋆ *Positivity property*: for each $\ell \in \mathbb{Z}$, N_ℓ is positive on $]\xi_\ell, \xi_{\ell+n+1}[$;
- ⋆ *Normalisation property*: $\sum_{\ell\in\mathbb{Z}} N_\ell(x) = 1$ for all $x \in]\inf_k t_k, \sup_k t_k[$;
- ⋆ *Endpoint property*: for each $\ell \in \mathbb{Z}$, N_ℓ vanishes exactly $(n - s + 1)$ times at ξ_ℓ and exactly $(n - s' + 1)$ at $\xi_{\ell+n+1}$, where $s := \sharp\{j \geq \ell \mid \xi_j = \xi_\ell\}$ and $s' := \sharp\{j \leq \ell + n + 1 \mid \xi_j = \xi_{\ell+n+1}\}$.

In such a B-spline basis, one can expand a spline $S \in \mathbb{S}^d$ as

$$S(x) = \sum_{\ell\in\mathbb{Z}} N_\ell(x) P_\ell, \quad x \in]\inf_k t_k, \sup_k t_k[,$$

where the points $P_\ell \in \mathbb{R}^d$, $\ell \in \mathbb{Z}$, are called the *poles* of S. The parametric curve defined by S is then located in the convex hull of its *control polygon*, with vertices the poles. For the curve to be considered a smooth version of the control polygon, it is essential to guarantee shape preserving properties. This is why a B-spline basis should also be *totally positive* like the ordinary polynomial B-spline basis, in the sense that, for any strictly increasing sequence $(x_1, \ldots, x_m)$ all minors of the collocation matrix $\big(N_i(x_j)\big)_{i\in\mathbb{Z}, 1\leq j\leq m}$ should be non-negative. The last crucial property expected from $\mathbb{S}$ is that it should permit the development of all the classical design algorithms (knot insertion, subdivision, ...).

In the general setting presented here, geometrically continuous polynomial splines were investigated by Dyn and Micchelli in [3]. They proved the following result:

Theorem 2.2 *Let us assume that each connection matrix M_k is totally positive (i.e., all its minors are non-negative). Then the spline space $\mathbb{S}$ possesses a totally positive B-spline basis.*

Prior to [3], the same result had been achieved by Goodman [4] in the special case of one-banded connection matrices. Later on, Seidel [14] used Theorem 2.2 to point out the existence of blossoms in the spline space $\mathbb{S}$ as soon as all matrices were totally positive. He then initiated a blossoming approach of design algorithms in $\mathbb{S}$ (conversion from poles to Bézier points, knot insertion, de Boor evaluation).

In spite of their crucial importance, the works mentioned above [3, 4, 14] all present the same inconvenience: they give only sufficient conditions for the space $\mathbb{S}$ to satisfy the properties commonly expected for design. We would like to try and determine all possible connection matrices ensuring the same results. This will be made possible by embedding the class of geometrically continuous polynomial splines in the much larger class of geometrically continuous piecewise Chebyshevian splines, and therefore, by considering polynomial spaces as special instances of Extended Chebyshev spaces. This justifies the next section.

3 Incursion into the Chebyshevian world

Extended Chebyshev spaces, Bernstein-type bases, generalised derivatives, and of course, geometrically continuous piecewise Chebyshevian splines are the tools which our approach of geometrically continuous polynomial splines relies on. We present them here as briefly as possible.

3.1 Extended Chebyshev spaces and Bernstein bases

Let I be a real interval with a non-empty interior and let $\mathbb{E} \subset C^n(I)$ be an $(n+1)$-dimensional linear space. Then, $\mathbb{E}$ is a W-space on I if the Wronskian of any basis of E does not vanish on I. It is an Extended Chebyshev space (for short, EC-space) on I if any non-zero function $F \in \mathbb{E}$ vanishes at most n times in I, multiplicities included up to $(n+1)$ [12, 15].

As classical examples, all null spaces of linear differential operators of order $(n+1)$ with constant coefficients of which the characteristic polynomials have only real roots are $(n+1)$-dimensional EC-space on $I = \mathbb{R}$. Of course, the polynomial space $\mathbb{P}_n$ of degree n falls into this category. It is well known that when the characteristic polynomial has at least one non-real root, then the null space—which is a W-space on $\mathbb{R}$—is not an EC-space on the whole of $\mathbb{R}$, but only on sufficiently small intervals (at least on any interval of length less than π/a, where a denotes the greatest imaginary part of all non-real roots of the characteristic polynomial).

As recalled in the two propositions below, EC-spaces possess remarkable bases which generalise the classical Bernstein polynomials. Given any $c, d \in I$, $c < d$, we say that a sequence $(V_0, \ldots, V_n)$ of functions in $C^n(I)$ is a *Bernstein-like basis relative to* (c, d) if it meets the following two requirements:

1. for $k = 0, \ldots, n$, V_k vanishes exactly k times at c, and exactly $(n-k)$ times at d;
2. for $k = 0, \ldots, n$, V_k is positive on $]c, d[$.

Proposition 3.1 [8] *An $(n+1)$-dimensional space $\mathbb{E} \subset C^n(I)$ is an EC-space on I if and only if it possesses a Bernstein-like basis relative to any $(c, d) \in I^2$, $c < d$.*

For geometric design, Bernstein-like bases are not sufficient, we need to have at our disposal Bernstein bases, given that $(B_0, \ldots, B_n)$ is *a Bernstein basis* relative to (c, d) if it is a Bernstein-like basis relative to (c, d) which is normalised, i.e., the functions $B_0, \ldots, B_n$ sum to the constant function $\mathbb{I}(x) := 1$ for all $x \in I$. Given points $P_0, \ldots, P_n \in \mathbb{R}^d$, the parametric curve defined on $[c, d]$ by $F(x) = \sum_{i=0}^{n} B_i(x) P_i$, is then automatically contained in the convex hull of the control polygon with vertices $P_0, \ldots, P_n$. Unlike Bernstein-like bases, if a space $\mathbb{E} \subset C^n(I)$ possesses a Bernstein basis relative to (c, d), it is unique. The following characterisations are essential for geometric design:

Theorem 3.2 [8] *Given any $(n+1)$-dimensional space $\mathbb{E} \subset C^n(I)$, supposed to be a W-space on I and to contain constants, the following properties are equivalent:*

1. *$\mathbb{E}$ possesses a Bernstein basis relative to any $(c, d) \in I^2$, $c < d$;*
2. *the space $D\mathbb{E} := \{DF := F' \mid F \in \mathbb{E}\}$ is an (n-dimensional) EC-space on I;*
3. *$\mathbb{E}$ possesses blossoms.*

We will not give the precise definition of blossoms, limiting ourselves to mentioning that, in the present context, it is now classical to introduce them as geometrical objects defined by means of intersections of osculating flats [8, 12]. When $\mathbb{E}$ possesses blossoms, each function $F \in \mathbb{E}$ is associated with a function $f : I^n \to \mathbb{R}$ (its blossom) satisfying three fundamental properties: it is symmetric on I^n; it gives F by restriction to the diagonal of I^n; it is *pseudoaffine* in each variable, in the sense that, for any $x_1, \ldots, x_{n-1}, c, d \in I$, with $c < d$, there exists a strictly increasing function $\beta(x_1, \ldots, x_{n-1}; c, d; .)$: $I \to \mathbb{R}$ (independent of F) such that

$$f(x_1, \ldots, x_{n-1}, x) = \left[1 - \beta(x_1, \ldots, x_{n-1}; c, d; x)\right] f(x_1, \ldots, x_{n-1}, c) \\ + \beta(x_1, \ldots, x_{n-1}; c, d; x) f(x_1, \ldots, x_{n-1}, d), \quad x \in I. \tag{3}$$

The latter three properties permit the development of all the classical geometric design algorithms. In particular, there exists a de Casteljau-type evaluation algorithm with respect to any given $(c, d) \in I^2, c < d$. Due to the pseudoaffinity property (3), this algorithm is corner cutting on $[c, d]$, which guarantees the total positivity on $[c, d]$ of the corresponding Bernstein basis. Let us mention that it is even the optimal normalised totally positive basis in the restriction of $\mathbb{E}$ to $[c, d]$, this resulting from the number of zeroes at its endpoints [6]. This brief reminder highly justifies the following terminology:

Definition 3.3 When the W-space $\mathbb{E}$ contains constants and when any of the three properties (1), (2), or (3) of Theorem 3.2 holds, we say that the space $\mathbb{E}$ is *good for design*.

Due to (2) $\Leftrightarrow$ (3) in Theorem 3.2 we can state:

Corollary 3.4 *A W-space which is good for design on I is an EC-space on I.*

3.2 EC-spaces and generalised derivatives

As is well known, it is possible to build as many instances of EC-spaces as we want by means of weight functions. We recall this essential fact below.

A sequence $(w_0, \ldots, w_n)$ is said to be *a system of weight functions on I* if, for $i = 0, \ldots, n$, the function w_i is positive and C^{n-i} on the interval I. With such a system it is classical to associate a sequence $L_0, \ldots, L_n$ of linear differential operators on $C^n(I)$ obtained by alternating division by a weight function and ordinary differentiation as follows:

$$L_0 F := \frac{F}{w_0}, \quad L_i F := \frac{D L_{i-1} F}{w_i}, \; i = 1, \ldots, n. \tag{4}$$

For each $i \leq n$, the operator L_i is of order i. These operators $L_0, \ldots, L_n$ are often referred to as *the generalised derivatives* associated with the system $(w_0, \ldots, w_n)$. The space $\mathbb{E}$ composed of all functions $F \in C^n(I)$ for which the last generalised derivative $L_n F$ is constant on I is an $(n+1)$-dimensional EC-space on I. Let us briefly remind the two reasons explaining this classical result:

– Multiplication of an $(n+1)$-dimensional EC-space on I by a non-vanishing C^n function transforms it into another EC-space on I.
– By application of Rolle's theorem, integration of an $(n+1)$-dimensional EC-space on I gives an $(n+2)$-dimensional EC-space on I which contains constants.

The $(n+1)$-dimensional EC-space $\mathbb{E}$ in question is called the EC-space associated with the system $(w_0, \ldots, w_n)$. We denote it as $\mathbb{E} = EC(w_0, \ldots, w_n)$. Take $I = \mathbb{R}$, and $w_i = \mathbb{I}$ for $i = 0, \ldots, n$. Then, the associated generalised derivatives $L_0, \ldots, L_n$ are simply the ordinary derivatives $D^0 = \mathrm{Id}, D, D^2, \ldots, D^n$, and the space $EC(w_0, \ldots, w_n)$ is thus the polynomial space $\mathbb{P}_n$.

Without any requirement on the interval I it is not at all guaranteed that a given $(n+1)$-dimensional EC-space on I can be associated with a system of weight functions on I.

Theorem 3.5 [8, 12] *Suppose that the interval I is* **closed and bounded**. *Then, if $\mathbb{E}$ is an $(n+1)$-dimensional EC-space on I, there exist systems $(w_0, \ldots, w_n)$ of weight functions on I such that $\mathbb{E} = EC(w_0, \ldots, w_n)$.*

As a consequence of Definition 3.3 and of Theorems 3.2 and 3.5, we can state:

Corollary 3.6 *Suppose that the interval I is* **closed and bounded**. *Then, an $(n+1)$-dimensional W-space $\mathbb{E}$ on I is good for design if and only if there exist systems $(w_1, \ldots, w_n)$ of weight functions on I such that $\mathbb{E} = EC(\mathbb{I}, w_1, \ldots, w_n)$.*

We recently showed that all systems of Theorem 3.5 are obtained by iteration of the theorem below. This result is one of the two key results on which the present work relies.

Theorem 3.7 [10] *Let $\mathbb{E}$ be an $(n+1)$-dimensional EC-space on a given* **closed bounded interval** *$I = [a, b]$. Then, given any $w_0 \in \mathbb{E}$, the following properties are equivalent*

1. *the coordinates of w_0 in a given Bernstein-like basis relative to (a, b) are all positive;*
2. *w_0 is positive on $[a, b]$ and, setting $L_0 V := V/w_0$ for all functions V defined on I, the space $DL_0\mathbb{E}$ is an EC-space on I.*

3.3 Geometrically continuous piecewise Chebyshevian splines

Along with the earlier sequence of connection matrices M_k, $k \in \mathbb{Z}$, we now additionally consider a bi-infinite sequence of *section spaces* $\mathbb{E}_k$, $k \in \mathbb{Z}$: each $\mathbb{E}_k$ is an $(n+1)$-dimensional EC-space on $[t_k, t_{k+1}]$ which is good for design (see Theorem 3.2). Based on these data, a *geometrically continuous piecewise Chebyshevian spline* is a continuous function $S : \mathbb{R} \to \mathbb{R}$ such that:

1. for each $k \in \mathbb{Z}$, there exists a function $F_k \in \mathbb{E}_k$ which coincides with S on the interval $[t_k, t_{k+1}]$;
2. S satisfies the connection conditions (2).

The expression "piecewise Chebyshevian splines" is used to stress the fact that the pieces are taken from different EC-spaces. Subsequently we denote by $\mathbb{S}$ the set all such splines.

Let k be any integer. The space $\mathbb{E}_k$ being good for design, we can choose a system $(w_1^k, \ldots, w_n^k)$ of weight functions on $[t_k, t_{k+1}]$ such that $\mathbb{E}_k = EC(\mathbb{I}_k, w_1^k, \ldots, w_n^k)$ (see Corollary 3.6). Let $L_0^k = \text{Id}$, $L_1^k, \ldots, L_n^k$ denote the generalised derivatives associated with $(\mathbb{I}_k, w_1^k, \ldots, w_n^k)$ via (4). For functions $F \in C^n([t_k, t_{k+1}])$, ordinary and generalised derivatives are linked by

$$\left(F'(x), \ldots, F^{(n)}(x)\right)^T = \Gamma_n^k\left(w_1^k, \ldots, w_n^k; x\right) . \left(L_1^k F(x), \ldots, L_n^k F(x)\right)^T, \quad x \in \left[t_k, t_{k+1}\right],$$

where $\Gamma_n^k(w_1^k, \ldots, w_n^k; x)$ is a lower triangular matrix depending on the values at x of the system $(w_1^k, \ldots, w_n^k)$ and of its derivatives. For instance, the entries on the diagonal are the numbers $\prod_{i=1}^p w_i(x)$, $1 \le p \le n$, while, in its first column we find $w_1{}^{(p-1)}(x)$, $1 \le p \le n$. One can then replace the connection conditions (2) by the equivalent ones

$$\left(L_1^k S\left(t_k^+\right), \ldots, L_{n-m_k}^k S\left(t_k^+\right)\right)^T = R_k . \left(L_1^{k-1} S\left(t_k^-\right), \ldots, L_{n-m_k}^{k-1} S\left(t_k^-\right)\right)^T, \quad k \in \mathbb{Z},$$

where the matrices R_k's, of the same nature as the M_k's (that is, lower triangular with positive diagonal entries), are given by

$$R_k = \Gamma_{n-m_k}^k\left(w_1^k, \ldots, w_{n-m_k}^k; t_k\right)^{-1} . M_k . \Gamma_{n-m_k}^{k-1}\left(w_1^{k-1}, \ldots, w_n^{k-1}; t_k\right), \quad k \in \mathbb{Z}, \tag{5}$$

the matrix $\Gamma_p^k(w_1^k, \ldots, w_p^k; x)$ being obtained for $p < n$ by deletion of the last $(n-p)$ rows and columns in $\Gamma_n^k(w_1^k, \ldots, w_n^k; x)$.

In this general framework, when they exist, blossoms are defined by means of intersections osculating flats as in the non-spline case. However, the major difference is that their natural domain of definition is a restricted set $\mathbb{A}_n(\mathbb{K})$ of n-tuples, said to be admissible (with respect to the extended knot-vector $\mathbb{K}$): given $x_1, \ldots, x_n \in]\inf_k t_k, \sup_k t_k[$, the n-tuple $(x_1, \ldots, x_n)$ is admissible if, whenever a knot t_k satisfies $\min(x_1, \ldots, x_n) < t_k < \max(x_1, \ldots, x_n)$, at least m_k copies of t_k are present in the sequence $(x_1, \ldots, x_n)$. When blossoms exist, they satisfy the same three fundamental properties as earlier, but of course only on $\mathbb{A}_n(\mathbb{K})$. As a matter of fact, existence of blossoms is the proper theoretical requirement for considering a space $\mathbb{S}$ of geometrically continuous piecewise Chebyshevian splines to be *good for design*. We can give three main justifications for this statement:

- existence of blossoms is equivalent to existence a B-spline basis both in $\mathbb{S}$ and in any spline space derived from it by knot insertion [7, 9];
- under existence of blossoms, all the classical design algorithms for splines (evaluation, knot insertion, subdivision, ...) can be developed in $\mathbb{S}$ and they all are corner-cutting [7, 9];

- as a consequence of the evaluation algorithms being corner-cutting, existence of blossoms guarantees that the B-spline basis of $\mathbb{S}$ is totally positive [6].

The following recent practical characterisation of existence of blossoms is the second key result for the present work:

Theorem 3.8 [11] *The following two properties are equivalent:*

1. *the piecewise Chebyshevian spline space $\mathbb{S}$ is good for design (i.e., it possesses blossoms);*
2. *among all bi-infinite sequences $(w_1^k, \dots, w_n^k)$, $k \in \mathbb{Z}$,—where, for each k, $(w_1^k, \dots, w_n^k)$ is a system of weight functions on $[t_k, t_{k+1}]$ ensuring that $D\mathbb{E}_k = EC(w_1^k, \dots, w_n^k)$,– one can find one such that*

$$\text{for each } k \in \mathbb{Z}, \text{ the matrix } R_k \text{ defined in (5) is the identity matrix of order } (n - m_k).$$

Remark 3.9 In [1], P.J. Barry considered splines with section-spaces defined by fixed weight functions. Under the additional condition that $w_i^k \in C^{i-1}([t_k, t_{k+1}])$, $1 \leq i \leq n$, $k \in \mathbb{Z}$, and that all matrices R_k were totally positive, he proved the existence of a B-spline basis. He also showed that it was then possible to develop some classical algorithms such as knot insertion. His approach was based on de Boor-Fix-type dual linear functionals. This was a crucial step in the study of piecewise Chebyshevian splines. However, in comparison, it should be observed that, when varying the systems of weight functions, Theorem 3.8 now yields exactly the same class of spline spaces using only identity matrices, and no extra differentiabily requirement.

4 Back to geometrically continuous polynomial splines

We now consider the space $\mathbb{S}$ of geometrically continuous piecewise Chebyshevian splines obtained when the section spaces are the restrictions of $\mathbb{P}_n$ to the intervals $[t_k, t_{k+1}]$, i.e.,

$$\mathbb{E}_k := \mathbb{P}_{n|[t_k, t_{k+1}]}, \quad k \in \mathbb{Z}.$$

The space $\mathbb{S}$ is then simply the space of geometrically continuous polynomial splines presented in Section 2.2. By application of Theorem 3.8 we will try to determine when it is good for design. The theoretical principles will be addressed in the present section and illustrated in the next one.

4.1 Weight functions for polynomial spaces

A preliminary task consists in determining all systems of weight functions on a given closed bounded interval I associated with the restriction of $\mathbb{P}_n$ to I. To simplify, we take $I := [0, 1]$ and we set $\mathbb{E} := \mathbb{P}_{n|[0,1]}$. Let $(V_0^n, \ldots, V_n^n)$ stand for a polynomial Bernstein-like basis relative to (0, 1), e.g.,

$$V_i^n(x) := x^i(1-x)^{n-i}, \quad x \in [0, 1], \quad i = 0, \ldots, n. \tag{6}$$

In accordance with Theorem 3.5, select any positive numbers $\alpha_{n,0}, \ldots, \alpha_{n,n}$, and set

$$w_0 := \sum_{i=0}^{n} \alpha_{n,i} V_i^n. \tag{7}$$

This function is positive on [0, 1]. Division of both sides of (7) by w_0 yields $\mathbb{I} = \sum_{i=0}^n B_i^n$, where the functions $B_0^n, \ldots, B_n^n$ are defined on [0, 1] by

$$B_i^n(x) := \frac{\alpha_{n,i} V_i^n(x)}{w_0(x)}, \quad x \in [0, 1], \quad 0 \le i \le n.$$

Clearly, $(B_0^n, \ldots, B_n^n)$ is the Bernstein basis relative to (0, 1) in the $(n+1)$-dimensional space $L_0\mathbb{E} := \{L_0 F := F/w_0 \mid F \in \mathbb{E}\}$ which is an EC-space on [0, 1] and which contains constants. We want to draw the reader's attention on the fact that, in general, the space $L_0\mathbb{E}$ is not a polynomial space, but a space of rational functions, and it is a priori defined only on the interval [0, 1]. The space $\mathbb{E}^{\{1\}} := DL_0\mathbb{E} \subset C^{n-1}([0, 1])$ obtained by differentiation is an n-dimensional space. According to Theorem 3.7, it is an EC-space on [0, 1], in which we can consider the functions

$$V_i^{n-1} := \sum_{k=i+1}^{n} DB_i^n = -\sum_{k=0}^{i} DB_i^n, \quad 0 \le i \le n-1.$$

For $0 \le i \le n-1$, V_i^{n-1} vanishes exactly i times at 0 and exactly $(n-1-i)$ times at 1 and its ith derivative at 0 is positive due to the positivity property satisfied by the Bernstein basis $(B_0^n, \ldots, B_n^n)$. Accordingly, we can say that $(V_0^{n-1} \ldots, V_{n-1}^{n-1})$ is a Bernstein-like basis relative to (0, 1) in the space $\mathbb{E}^{\{1\}}$ (see [10]).

We can thus iterate the process. Selecting any positive numbers $\alpha_{n-1,0}, \ldots, \alpha_{n-1,n-1}$, and setting

$$w_1 := \sum_{i=0}^{n-1} \alpha_{n-1,i} V_i^{n-1}, \tag{8}$$

we obtain a positive function on [0, 1], which permits division by w_1, thus leading to

$$\mathbb{I} = \sum_{i=0}^{n-1} B_i^{n-1}, \quad \text{with } B_i^{n-1} := \frac{\alpha_{n-1,i} V_i^{n-1}}{w_1} \text{ for } 0 \le i \le n-1. \tag{9}$$

The functions

$$V_i^{n-2} := \sum_{k=i+1}^{n-1} DB_i^{n-1} = -\sum_{k=0}^{i} DB_i^{n-1}, \quad 0 \le i \le n-2,$$

form a Bernstein-like basis relative to $(0, 1)$ in the space $\mathbb{E}^{\{2\}} := DL_1\mathbb{E} \subset C^{n-2}([0, 1])$ (with $L_1V := (DL_0V)/w_1$) which is an $(n-1)$-dimensional EC-space on $[0, 1]$. Continuing the same way provides us with all systems $(w_0, \ldots, w_n)$ of weight functions on $[0, 1]$ such that $\mathbb{E} = EC(w_0, \ldots, w_n)$, and therefore, all sequences of generalised derivatives which can be associated with $\mathbb{P}_{n\,|[0,1]}$.

Remark 4.1 The process we have described is the same for any $(n+1)$-dimensional EC-space $\mathbb{E}$ on any closed bounded interval I. Two observations will be essential for the next subsection.

1. At each step p, $1 \leq p \leq n$, the weight function w_p is defined by means of positive coefficients $\alpha_{n-p,0}, \ldots, \alpha_{n-p,n-p}$ as $w_p := \sum_{i=0}^{n-p} \alpha_{n-p,i} V_i^{n-p}$, where the Bernstein-like basis $(V_0^{n-p}, \ldots, V_{n-p}^{n-p})$ relative to $(0, 1)$ is completely determined by the coefficients $\alpha_{j,q}$, $0 \leq q \leq j$, $n-p+1 \leq j \leq n$, and by the initial selected Bernstein-like basis $(V_0^n, \ldots, V_n^n)$. As a consequence, for any $j \leq n-p$, ${w_p}^{(j)}(0)$ and ${w_p}^{(j)}(1)$ are completely determined by the latter coefficients and the derivatives at 0 (resp. 1) of $V_0^n, \ldots, V_n^n$ of order less than or equal to $j+p$.
2. Once the initial Bernstein-like basis $(V_0^n, \ldots, V_n^n)$ selected, a system of weight functions on I such that $\mathbb{E} = EC(w_0, \ldots, w_n)$ is thus completely determined by a sequence $\boldsymbol{\alpha} = \left(\alpha_{p,q}\right)_{0 \leq q \leq p \leq n}$ of $(n+1)(n+2)/2$ positive numbers.

4.2 Geometrically continuous polynomial splines for design

For each $k \in \mathbb{Z}$, we assume that a Bernstein-like basis $(V_{k,0}^{n-1}, \ldots, V_{k,n-1}^{n-1})$ relative to (t_k, t_{k+1}) has been chosen once and for all in the space $D\mathbb{E}_k = \mathbb{P}_{n-1\,|[t_k,t_{k+1}]}$. Theorem 3.8 can then be applied in two directions.

4.2.1 Determining all "good" spaces of geometrically continuous polynomial splines

We actually have to determine all suitable sequences of connection matrices M_k, $k \in \mathbb{Z}$. According to Theorem 3.8, this consists in the following steps.

- Select any bi-infinite sequence $\boldsymbol{\alpha}_k$, $k \in \mathbb{Z}$, where, for each k, $\boldsymbol{\alpha}_k = \left(\alpha_{p,q}^k\right)_{0 \leq q \leq p \leq n-1}$ is any sequence of $n(n+1)/2$ positive numbers.
- For each $k \in \mathbb{Z}$, apply to $\mathbb{P}_{n-1}$ the procedure explained in the previous section using the selected sequence $\boldsymbol{\alpha}_k$ and the selected Bernstein-like basis $(V_{k,0}^{n-1}, \ldots, V_{k,n-1}^{n-1})$. This yields a system $(w_1^k, \ldots, w_n^k)$ of weight functions on $[t_k, t_{k+1}]$ such that

$$\mathbb{P}_{n-1\,|[t_k,t_{k+1}]} = EC\left(w_1^k, \ldots, w_n^k\right), \quad \text{i.e., } \mathbb{E}_k = EC\left(\mathbb{I}_k, w_1^k, \ldots, w_n^k\right). \quad (10)$$

- Take the connection matrices

$$M_k := \Gamma^k_{n-m_k}\left(w^k_1, \dots, w^k_{n-m_k}; t_k\right) . \Gamma^{k-1}_{n-m_k}\left(w^{k-1}_1, \dots, w^{k-1}_n; t_k\right)^{-1}, \quad k \in \mathbb{Z}. \tag{11}$$

  This provides us with all sequences of connection matrices M_k, $k \in \mathbb{Z}$, for which the corresponding space $\mathbb{S}$ of geometrically continuous polynomial splines is good for design.

4.2.2 *Is a given spline space good for design?*

A space $\mathbb{S}$ of geometrically continuous polynomial splines being given, that is, a sequence M_k, $k \in \mathbb{Z}$, of connection matrices being given, can we answer the question: *is $\mathbb{S}$ good for design?* From Theorem 3.8 we know that this amounts to answering the following one: is it possible to find sequences $(w^k_1, \dots, w^k_n)$, $k \in \mathbb{Z}$, where $(w^k_1, \dots, w^k_n)$ is a system of weight functions on $[t_k, t_{k+1}]$ ensuring both (10) and (11) for all k? In other words, can we determine positive numbers $\alpha^k_{p,q}$, $0 \le q \le p \le n-1$, $k \in \mathbb{Z}$, such that the associated bi-infinite sequence $(w^k_1, \dots, w^k_n)$, $k \in \mathbb{Z}$, of systems of weight functions determined as previously satisfy

$$\Gamma^k_{n-m_k}\left(w^k_1, \dots, w^k_{n-m_k}; t_k\right) = M_k . \Gamma^{k-1}_{n-m_k}\left(w^{k-1}_1, \dots, w^{k-1}_n; t_k\right), \quad k \in \mathbb{Z}, \tag{12}$$

Let an integer $k \in \mathbb{Z}$ be given. Assume that we know the sequence $\boldsymbol{\alpha}_{k-1} = \left(\alpha^{k-1}_{p,q}\right)_{0 \le q \le p \le n-1}$ of positive numbers determining the system $(w^{k-1}_1, \dots, w^{k-1}_n)$ of weight functions on $[t_{k-1}, t_k]$. The first question to address is: is it possible to find a sequence $\boldsymbol{\alpha}_k = \left(\alpha^k_{p,q}\right)_{0 \le q \le p \le n-1}$ of positive numbers such that the associated system $(w^k_1, \dots, w^k_n)$ of weight functions on $[t_k, t_{k+1}]$ will satisfy the corresponding equality (12)? Due to the zeroes of the successive Bernstein-like bases involved in the construction of the weight functions, only the numbers $\alpha^k_{n-j,q}$, $0 \le q \le n-j-m_k$, $1 \le j \le n-m_k$ are involved in (12). Therefore, we should regard (12) as a (non-linear) system of $(n-m_k)(n-m_k+1)/2$ equations in the $(n-m_k)(n-m_k+1)/2$ unknowns $\alpha^k_{n-j,q}$, $0 \le q \le n-j-m_k$, $1 \le j \le n-m_k$. The problem consists in proving the existence of a positive solution, in the sense that all concerned $\alpha^k_{j,q}$ should be positive. The second crucial issue is the compatibility between the conditions making all such systems "positively" solvable. The two issues will clearly appear in the example of degree four splines investigated in next section.

5 Examples

The problem we address here is the one treated in Section 4.2.2 in special cases with $n = 3$ and $n = 4$. Let us first observe that any affine change of variable on either side of a knot t_k results in multiplication of the connection matrix M_k by a diagonal matrix of the form $(a, a^2, \dots, a^{n-m_k})$, where a is a positive number.

Accordingly, with no loss of generality we can assume the knots to be equally spaced with

$$t_k = k \quad \text{for all } k \in \mathbb{Z}.$$

In that case, everything in the restriction of $\mathbb{P}_{n-1}$ to $[t_k, t_{k+1}]$ can be obtained by translation from the restriction of $\mathbb{P}_{n-1}$ to $[t_0, t_1] = [0, 1]$. For instance, $(w_1^k, \ldots, w_n^k)$ is a system of weight functions on $[t_k, t_{k+1}]$ if and only if $\left(w_1^k(. - k), \ldots, w_n^k(. - k)\right)$ is a system of weight functions on [0, 1], and

$$\Gamma_n^k\left(w_1^k, \ldots, w_n^k; x\right) = \Gamma_n^0\left(w_1^k(. - k), \ldots, w_n^k(. - k); x - k\right), \quad x \in \left[t_k, t_{k+1}\right].$$

Moreover, the initial bases in the restrictions of $\mathbb{P}_{n-1}$ to all intervals $[k, k+1]$ will systematically be the integer shifts of the Bernstein-like basis $(V_0^{n-1}, \ldots, V_{n-1}^{n-1})$ relative to (0, 1) defined in accordance with (6). As a consequence, for $x = t_k$ or $x = t_{k+1}$ all these matrices will be expressed as functions of the sequence $\boldsymbol{\alpha}_k$, independent of k. These functions are completely determined by the values of the initial Bernstein-like basis $(V_0^{n-1}, \ldots, V_{n-1}^{n-1})$ and its derivatives at 0 and 1.

5.1 Geometrically continuous cubic splines

Let us first observe that any system (w_1, w_2, w_3) of weight functions on any interval I and the generalised derivatives $L_0 = \mathrm{Id}$, L_1, L_2, L_3 associated with $(\mathbb{I}, w_1, w_2, w_3)$ satisfy

$$\begin{bmatrix} F' \\ F'' \\ F''' \end{bmatrix} = \begin{bmatrix} w_1 & 0 & 0 \\ w_1{}' & w_1 w_2 & 0 \\ w_1{}'' & 2w_1{}' w_2 + w_1 w_2{}' & w_1 w_2 w_3 \end{bmatrix} \begin{bmatrix} L_1 F \\ L_2 F \\ L_3 F \end{bmatrix}, \tag{13}$$

for any sufficiently differentiable function F on I.

Here, we assume that $n = 3$, that all knots are simple (i.e., $m_k = 1$ for all $k \in \mathbb{Z}$), and that the connection matrices are defined by

$$M_k = \begin{bmatrix} a_k & 0 \\ b_k & c_k \end{bmatrix}, \qquad \text{with } a_k, c_k > 0 \text{ for all } k \in \mathbb{Z}. \tag{14}$$

For the sake of simplicity we will now call $\alpha_0^k, \alpha_1^k, \alpha_2^k, \beta_0^k, \beta_1^k, \gamma_0^k$, the six positive coefficients producing w_1^k, w_2^k, w_3^k, respectively, starting from the integer shifts of the Bernstein-like basis

$$V_0^2(x) = (1 - x)^2, \quad V_1^2(x) = x(1 - x), \quad V_2^2(x) = x^2, \quad x \in [0, 1],$$

via the procedure explained in Section 4.1. Taking account of the values of the latter basis and of its derivatives at 0, 1, this leads to

$$\Gamma_3^k\left(w_1^k, w_2^k, w_3^k; t_k\right) = \begin{bmatrix} \alpha_0^k & 0 & 0 \\ \alpha_1^k - \alpha_0^k & \alpha_1^k\beta_0^k & 0 \\ 2\alpha_0^k - \alpha_1^k + \alpha_2^k & 2\beta_0^k(\alpha_2^k - \alpha_1^k) + 2\beta_1^k\alpha_2^k & 2\alpha_2^k\beta_1^k\gamma_0^k \end{bmatrix},$$

$$\Gamma_3^{k-1}\left(w_1^{k-1}, w_2^{k-1}, w_3^{k-1}; t_k\right)$$
$$= \begin{bmatrix} \alpha_2^{k-1} & 0 & 0 \\ 2\alpha_2^{k-1} - \alpha_1^{k-1} & \alpha_1^{k-1}\beta_1^{k-1} & 0 \\ 2\alpha_2^{k-1} - \alpha_1^{k-1} + 2\alpha_0^{k-1} & 2\beta_1^{k-1}(\alpha_1^{k-1} - \alpha_0^{k-1}) - \beta_0^{k-1}\alpha_0^{k-1} & 2\alpha_0^{k-1}\beta_0^{k-1}\gamma_0^{k-1} \end{bmatrix}.$$

In particular, for any given $k \in \mathbb{Z}$, supposing that the positive coefficients producing the system $(w_1^{k-1}, w_2^{k-1}, w_3^{k-1})$ are known, the corresponding system (12) we have to consider is

$$\begin{bmatrix} \alpha_0^k & 0 \\ \alpha_1^k - \alpha_0^k & \alpha_1^k\beta_0^k \end{bmatrix} = \begin{bmatrix} a_k & 0 \\ b_k & c_k \end{bmatrix} \begin{bmatrix} \alpha_2^{k-1} & 0 \\ 2\alpha_2^{k-1} - \alpha_1^{k-1} & \alpha_1^{k-1}\beta_1^{k-1} \end{bmatrix}.$$

Under which conditions on the entries a_k, b_k, c_k of the connection matrix M_k can we guarantee the existence of a positive solution? As a matter of fact, this amounts to finding conditions to guarantee the positivity of the number α_1^k defined by:

$$\alpha_1^k := \alpha_2^{k-1}(b_k + 2a_k + 2c_k) - c_k\alpha_1^{k-1}.$$

Clearly, if $\alpha_1^k > 0$, then

$$b_k + 2a_k + 2c_k = \frac{\alpha_1^k + c_k\alpha_1^{k-1}}{\alpha_2^{k-1}} > 0.$$

Conversely, assume that $b_k + 2a_k + 2c_k > 0$. The number α_2^{k-1} is not involved in the connection at any knot t_j with $j < k$. Accordingly, the positivity of the quantity $b_k + 2a_k + 2c_k$ enables us to choose α_2^{k-1} so as to satisfy $\alpha_2^{k-1}(b_k + 2a_k + 2c_k) - c_k\alpha_1^{k-1} > 0$, that is, $\alpha_1^k > 0$. We can thus state:

Theorem 5.1 *The space of all geometrically continuous cubic splines with regularly spaced simple knots and connection matrices* (14) *is good for design if and only if*

$$b_k + 2a_k + 2c_k > 0 \quad \textit{for all } k \in \mathbb{Z}. \tag{15}$$

Remark 5.2 Condition (15) was already achieved in [5] via intersection of osculating flats. However we would to stress that the proof of Theorem 5.1 is of remarkable simplicity compared to [5].

5.2 Geometrically continuous quartic splines

Here, $n = 4$, the knots are simple, and the connection matrices are given by

$$M_k = \begin{bmatrix} a_k & 0 & 0 \\ b_k & c_k & 0 \\ d_k & e_k & f_k \end{bmatrix}, \qquad \text{with } a_k, c_k, f_k > 0 \text{ for all } k \in \mathbb{Z}. \tag{16}$$

We shall achieve the following result.

Theorem 5.3 *The space of all geometrically continuous quartic splines with regularly spaced simple knots and connection matrices (16) is good for design if and only if the connection matrices satisfy the following conditions:*

$$B_k > 0, \quad D_k > 0, \quad 4B_k B_{k+1} f_{k+1} < D_k(E_{k+1}B_{k+1} - c_{k+1}D_{k+1}) \quad \textit{for all } k \in \mathbb{Z}, \tag{17}$$

where, for each $k \in \mathbb{Z}$, the numbers B_k, E_k, D_k are defined by

$$B_k := b_k + 3a_k + 3c_k, \quad E_k := e_k + 4c_k + 4f_k,$$
$$D_k := d_k + 3e_k + 4b_k + 6(a_k + 2c_k + f_k). \tag{18}$$

Proof We have to find conditions on the sequence M_k, $k \in \mathbb{Z}$, of connection matrices making it possible to find, for each $k \in \mathbb{Z}$, a system (w_1^k, w_2^k, w_3^k) of weight functions on $[t_k, t_{k+1}]$, so that

$$\mathbb{P}_{3|[t_k,t_{k+1}]} = EC\left(w_1^k, w_2^k, w_3^k\right),$$
$$\Gamma_3^k\left(w_1^k, w_2^k, w_3^k; t_k\right) = M_k \,.\, \Gamma_3^{k-1}\left(w_1^{k-1}, w_2^{k-1}, w_3^{k-1}; t_k\right), \quad k \in \mathbb{Z}. \tag{19}$$

As suggested earlier we will use the integer shifts of the Bernstein-like basis

$$V_0(x) = (1-x)^3, \quad V_1(x) = x(1-x)^2, \quad V_2(x) = x^2(1-x), \quad V_3(x) = x^3, \tag{20}$$

to apply the procedure explained in Section 4.1 for determining all systems (w_1^k, w_2^k, w_3^k) ensuring the first line in (19). In the kth section this involves nine coefficients which we now denote by $\alpha_0^k, \alpha_1^k, \alpha_2^k, \alpha_3^k, \beta_0^k, \beta_1^k, \beta_2^k, \gamma_0^k, \gamma_1^k$. On account of the values of the basis (20) and of its derivatives at 0, 1, formula (13) now leads to

$$\Gamma_3^k\left(w_1^k, w_2^k, w_3^k; t_k\right) = \begin{bmatrix} \alpha_0^k & 0 & 0 \\ \alpha_1^k - 3\alpha_0^k & \alpha_1^k\beta_0^k & 0 \\ 6\alpha_0^k - 4\alpha_1^k + 2\alpha_2^k & 2\beta_0^k(\alpha_2^k - \alpha_1^k) + 2\beta_1^k\alpha_2^k & 2\alpha_2^k\beta_1^k\gamma_0^k \end{bmatrix},$$

$$\Gamma_3^{k-1}\left(w_1^{k-1}, w_2^{k-1}, w_3^{k-1}; t_k\right)$$
$$= \begin{bmatrix} \alpha_3^{k-1} & 0 & 0 \\ 3\alpha_3^{k-1} - \alpha_2^{k-1} & \alpha_2^{k-1}\beta_2^{k-1} & 0 \\ 6\alpha_3^{k-1} - 4\alpha_2^{k-1} + 2\alpha_1^{k-1} & 2\beta_2^{k-1}(2\alpha_2^{k-1} - \alpha_1^{k-1}) - \beta_1^{k-1}\alpha_1^{k-1} & 2\alpha_1^{k-1}\beta_1^{k-1}\gamma_1^{k-1} \end{bmatrix}.$$

Accordingly, the matricial relation in (19) will be satisfied if and only if the two consecutive families of coefficients are linked by the six relations

$$\alpha_0^k = a_k \alpha_3^{k-1}, \tag{21}$$

$$\alpha_1^k \beta_0^k = c_k \alpha_2^{k-1} \beta_2^{k-1}, \tag{22}$$

$$\alpha_2^k \beta_1^k \gamma_0^k = f_k \alpha_1^{k-1} \beta_1^{k-1} \gamma_1^{k-1}, \tag{23}$$

$$\alpha_1^k - 3\alpha_0^k = b_k \alpha_3^{k-1} + c_k (3\alpha_3^{k-1} - \alpha_2^{k-1}), \tag{24}$$

$$\begin{aligned} 6\alpha_0^k - 4\alpha_1^k + 2\alpha_2^k = d_k \alpha_3^{k-1} + e_k (3\alpha_3^{k-1} - \alpha_2^{k-1}) \\ + f_k (6\alpha_3^{k-1} - 4\alpha_2^{k-1} + 2\alpha_1^{k-1}), \end{aligned} \tag{25}$$

$$\begin{aligned} 2\beta_0^k (\alpha_2^k - \alpha_1^k) + 2\beta_1^k \alpha_2^k = e_k \alpha_2^{k-1} \beta_2^{k-1} \\ + f_k \left[2\beta_2^{k-1} (2\alpha_2^{k-1} - \alpha_1^{k-1}) - \beta_1^{k-1} \alpha_1^{k-1} \right]. \end{aligned} \tag{26}$$

Propositions 5.4 and 5.5 below investigate the previous six equalities as a system in the six unknowns $\alpha_0^k, \alpha_1^k, \alpha_2^k, \beta_0^k, \beta_1^k, \gamma_0^k$. They are preliminary results towards the proof of the present theorem which will be concluded later on. □

Proposition 5.4 *Given $k \in \mathbb{Z}$, let $\alpha_1^{k-1}, \alpha_2^{k-1}, \alpha_3^{k-1}$, $\beta_1^{k-1}, \beta_2^{k-1}$, γ_1^{k-1} be any positive numbers. Then, there exist (unique) positive numbers α_0^k, α_1^k, α_2^k, β_0^k, β_1^k, γ_0^k satisfying all six conditions (21) to (26) if and only if all properties below hold:*

1. *the connection matrix M_k satisfies, with the notations introduced in Theorem 5.3,*

$$B_k > 0, \quad D_k > 0, \quad E_k B_k - c_k D_k > 0; \tag{27}$$

2. *the coefficients $\alpha_1^{k-1}, \alpha_2^{k-1}, \alpha_3^{k-1}, \beta_2^{k-1}$ are chosen so that*

$$\begin{aligned} \alpha_2^{k-1} > \frac{2\alpha_1^{k-1} B_k f_k}{B_k E_k - c_k D_k}, \quad \alpha_3^{k-1} > \frac{E_k \alpha_2^{k-1} - 2 f_k \alpha_1^{k-1}}{D_k}, \\ \beta_2^{k-1} > \frac{2 f_k \alpha_1^{k-1} \beta_1^{k-1} (B_k \alpha_3^k - c_k \alpha_2^{k-1})}{\alpha_3^{k-1} \left[\alpha_2^{k-1} (B_k E_k - c_k D_k) - 2 B_k f_k \alpha_1^{k-1} \right]}. \end{aligned} \tag{28}$$

Proof Clearly, if the previous system has a solution $\alpha_0^k, \alpha_1^k, \alpha_2^k$, β_0^k, β_1^k, γ_0^k, then it is unique. Moreover, α_0^k is automatically positive, and the positivity of $\alpha_1^k, \alpha_2^k, \beta_1^k$ automatically implies that of β_0^k, γ_0^k. Therefore, we are only concerned with the positivity of the three coefficients $\alpha_1^k, \alpha_2^k, \beta_1^k$.

On account of relations (21) and (22) we can respectively replace (24), (25), (26) by

$$\alpha_1^k = \alpha_3^{k-1} B_k - c_k \alpha_2^{k-1}, \tag{29}$$

$$2\alpha_2^k = \alpha_3^{k-1} D_k - \alpha_2^{k-1} E_k + 2 f_k \alpha_1^{k-1}, \tag{30}$$

$$2\alpha_2^k \beta_1^k = \beta_2^{k-1} (\alpha_2^{k-1} E_k - 2 f_k \alpha_1^{k-1}) - 2 f_k \alpha_1^{k-1} \beta_1^{k-1} - 2\alpha_2^k \beta_0^k. \tag{31}$$

Due to (22), (29) and (30), equality (31) can also equivalently be written as follows

$$\begin{aligned} 2\alpha_1^k \alpha_2^k \beta_1^k &= \alpha_2^{k-1} \beta_2^{k-1} \left(\alpha_1^k E_k - 2c_k \alpha_2^k\right) - 2 f_k \alpha_1^{k-1} \alpha_1^k \left(\beta_1^{k-1} + \beta_2^{k-1}\right), \\ &= \alpha_3^{k-1} \beta_2^{k-1} \left[\alpha_2^{k-1} (B_k E_k - c_k D_k) - 2 B_k f_k \alpha_1^{k-1}\right] \\ &\quad -2 f_k \alpha_1^{k-1} \beta_1^{k-1} \left[B_k \alpha_3^{k-1} - c_k \alpha_2^{k-1}\right]. \end{aligned} \tag{32}$$

Assume that $\alpha_1^k, \alpha_2^k, \beta_1^k$ are positive. Then, from (31) we can deduce that the quantity $\alpha_2^{k-1} E_k - 2 f_k \alpha_1^{k-1}$ is positive, and accordingly, due to (30), D_k is positive too. From (29) we can see that $B_k > 0$. Moreover, the positivity of α_1^k and α_2^k also clearly implies that

$$\alpha_3^{k-1} > \max \left(\frac{c_k \alpha_2^{k-1}}{B_k}, \frac{E_k \alpha_2^{k-1} - 2 f_k \alpha_1^{k-1}}{D_k} \right). \tag{33}$$

Via (32), the positivity of $B_k \alpha_3^{k-1} - c_k \alpha_2^{k-1}$ proves the positivity of the quantities $E_k B_k - c_k D_k$ and $\alpha_2^{k-1} (B_k E_k - c_k D_k) - 2 B_k f_k \alpha_1^{k-1}$, along with the last two missing inequalities in (28).

Conversely, assume all inequalities in (27) and (28) to hold. The positivity of α_2^k, β_1^k readily follows from (30) and (32). That $\alpha_1^k > 0$ results from (29) after observing that the positivity of $E_k B_k - c_k D_k$ and the first inequality in (28) guarantee that $c_k \alpha_2^{k-1} / B_k < (E_k \alpha_2^{k-1} - 2 f_k \alpha_1^{k-1}) / D_k$. □

Proposition 5.5 *The data and notations are the same as in Proposition* 5.4, *and we assume that both* (27) *and* (28) *hold. Then the positive numbers* α_1^k, α_2^k *produced by the system* (21)–(26) *satisfy*

$$\alpha_2^k > A \alpha_1^k, \tag{34}$$

where A is a given positive number, if and only if we have both

$$2 A B_k < D_k \quad \textit{and} \quad \alpha_3^{k-1} > L := \frac{\alpha_2^{k-1} (E_k - 2c_k A) - 2\alpha_1^{k-1} f_k}{D_k - 2 A B_k}. \tag{35}$$

Proof On account of (29) and (30), condition (34) is satisfied if and only if

$$\alpha_3^{k-1}(D_k - 2AB_k) > \alpha_2^{k-1}(E_k - 2c_k A) - 2 f_k \alpha_1^{k-1}. \tag{36}$$

Let us consider the various possible situations.

1. Assume that $D_k - 2AB_k < 0$. Then (36) holds if and only $\alpha_3^{k-1} < L$. The first inequality in (28) implies that

$$L < \frac{c_k \alpha_2^{k-1}}{B_k} < \frac{E_k \alpha_2^{k-1} - 2 f_k \alpha_1^{k-1}}{D_k}.$$

   Accordingly, the condition $\alpha_3^{k-1} < L$ is not compatible with (28).
2. Assume that $D_k = 2AB_k$. Then, (36) yields

$$\alpha_2^{k-1}(E_k - 2c_k A)B_k = \alpha_2^{k-1}(E_k B_k - c_k D_k) < 2\alpha_1^{k-1} B_k f_k,$$

   which contradicts (28).
3. Assume that $D_k - 2AB_k > 0$. Then, (36) holds if and only $\alpha_3^{k-1} > L$. In that case, due to the first inequality in (28),

$$L > \frac{E_k \alpha_2^{k-1} - 2 f_k \alpha_1^{k-1}}{D_k} > \frac{c_k \alpha_2^{k-1}}{B_k}. \tag{37}$$

This completes the proof. □

Back to proof of Theorem 5.3

First note that, in (17), on account of the other two requirements, condition "$D_k > 0$ for all k" can equivalently be replaced by "$B_k E_k - c_k D_k > 0$ for all k".

The space $\mathbb{S}$ is good for design if and only if we are able to determine bi-infinite positive sequences $\boldsymbol{\alpha}_k := (\alpha_0^k, \alpha_1^k, \alpha_2^k, \alpha_3^k)$, $\boldsymbol{\beta}_k := (\beta_0^k, \beta_1^k, \beta_2^k)$, $\boldsymbol{\gamma}_k := (\gamma_0^k, \gamma_1^k)$, $k \in \mathbb{Z}$, satisfying the equalities (21) to (26) for all k. The crucial point is that, given $k \in \mathbb{Z}$, the three numbers $\alpha_3^{k-1}, \beta_2^{k-1}, \gamma_1^{k-1}$ are not involved in the connections at knots preceding t_k. We are therefore searching for conditions on the connection matrix M_k enabling us to choose the free parameters $\alpha_3^{k-1}, \beta_2^{k-1}, \gamma_1^{k-1}$ so as to ensure the existence of a positive solution to the system (21)–(26). Such conditions are provided by Proposition 5.4. However, we also have to be able to propagate (28) to the next interval. There is no problem for last two inequalities in (28) since the parameters $\alpha_3^{k-1}, \beta_2^{k-1}$ are free. In contrast, the first inequality in (28) concerns the non-free parameter α_2^{k-1}. Accordingly, we do have to see if it is possible to achieve convenient necessary and sufficient conditions ensuring that the obtained positive numbers α_1^k, α_2^k will satisfy the similar inequality, that is

$$\alpha_2^k(B_{k+1}E_{k+1} - c_{k+1}D_{k+1}) > 2\alpha_1^k B_{k+1} f_{k+1}. \tag{38}$$

This will be achieved via Proposition 5.5. Indeed, with $A := 2B_{k+1}f_{k+1}/(B_{k+1}E_{k+1} - c_{k+1}D_{k+1})$, the latter proposition yields the additional requirement

$$4B_kB_{k+1}f_{k+1} < D_k(E_{k+1}B_{k+1} - c_{k+1}D_{k+1})$$

on the two consecutive matrices M_k, M_{k+1}. Under it, there is no problem ensuring condition (38): it suffices to choose the free parameter $\alpha_3^{k-1} > L$, where L is the corresponding limit defined in (35). Due to (37) the latter inequality automatically ensures the previous weaker requirement $\alpha_3^{k-1} > (\alpha_2^{k-1}E_k - 2\alpha_1^{k-1})/D_k$ provided by (28). There is no difficulty propagating these results to greater integers. □

Remark 5.3 The total positivity sufficient condition of [3] corresponds to $b_k, e_k \geq 0, 0 \leq c_kd_k \leq b_ke_k$ for all $k \in \mathbb{Z}$. One can check that this implies (17). To compare the two situations, consider the elementary example $a_k = c_k = f_k = 1$, $b_k = e_k = 0$, for all k. We thus obtain a class of C^2 quartic splines depending on the sequence d_k, $k \in \mathbb{Z}$, of shape parameters. By contrast, in that case, total positivity requires $d_k = 0$ for all k, that is, we have no shape parameter. To prove the efficiency of our shape parameters, let us assume that $d_k = 0$ for all $k \neq k_0$. Then, our condition (17) says that we can choose any $d_{k_0} \in]-18, 18[$. This is illustrated in Fig. 1 for $k_0 = 4$: we show the four B-splines which change with d_4 (i.e., N_0, N_1, N_2, N_3) for various values of d_4. Opposite values give symmetric graphs, which is consistent with the fact that, here, taking the inverse of M_4, simply consists in changing d_4 into $-d_4$. The two B-splines which are most affected by d_4 are N_1 (which goes to 0 when $d_4 \to 18$)

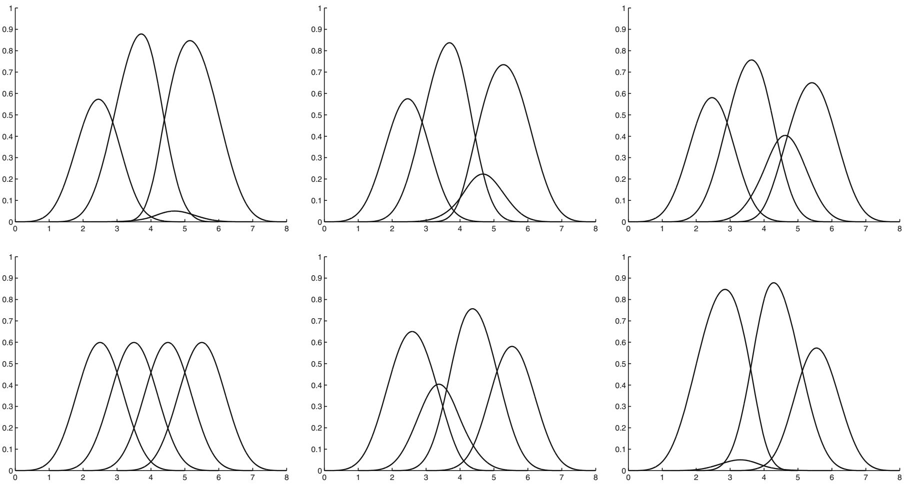

Fig. 1 Geometrically continuous quartic splines with, for each k, $a_k = c_k = f_k = 1$, $b_k = e_k = 0$, and $d_k = 0$ for each $k \neq 4$. *Top* (from *left* to *right*): $d_4 = -17.5$; -15; -10. *Bottom* (from *left* to *right*): $d_4 = 0$; 10; 17.5

and N_2 (which goes to 0 when $d_4 \to -18$). One can compare with the ordinary quartic B-splines ($d_4 = 0$).

6 Final comments

Due to page limitation we will not produce more illustrations, preferring to conclude with a few comments. Readers who would like to check the interest of going “beyond total positivity” are referred to [5].

1. Obviously, the greater n is (and the lower multiplicities are), the more difficult solving “positively” (12) is. As an instance, for $n = 3$, condition (15) concerns each matrix M_k separately, while for $n = 4$, part of (17) links two consecutive matrices. This is consistent with the blossoming approach. Indeed, for simple knots, existence of blossoms in $\mathbb{S}$ involves consecutive intervals two by two if $n = 3$, three by three if $n = 4$, four by four if $n = 5$, ... For $n = 5$, the condition ensuring that the spline space is good for design should thus concern any three consecutive connection matrices,...
2. EC-spaces being natural generalisations of polynomial spaces, a logical approach consists in going from the polynomial world to the larger Chebyshevian world to try and extend to the latter results known in the former. Such extensions require the development of specific difficult techniques which are a priori unjustified in the “easy” polynomial world: why should we worry about generalised derivatives when we have at our disposal the so easy-to-handle ordinary ones to decrease the dimension? The complicated techniques in question enabled us to achieve results a priori specific to the Chebyshevian world. This is why the approach used in this work goes the reverse way: starting from the Chebyshevian world we come back to the polynomial world, thus deducing new results in the latter from results recently obtained in the former. This is not the least interest of this article.

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