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DOI 10.1007/s11075-012-9548-z

ORIGINAL PAPER

# Unique and multiple PHAM series solutions of a class of nonlinear reactive transport model

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Received: 24 October 2011 / Accepted: 1 February 2012


**Abstract** The purpose of this paper is to visit a class of nonlinear reactive transport model in the case including advective and diffusive transport with the Michaelis-Menten reaction term. We apply the method so-called predictor homotopy analysis method (PHAM) which has been recently proposed to predict multiplicity of solutions of nonlinear BVPs. Consequently two consequential matters are indicated which confirms the authority of PHAM to identify multiple solutions: (i) The Talylor series solutions are improved by the so-called convergence-controller parameter (ii) The possibility of existence of multiple solutions is discovered in some cases for the model.

**Keywords** Predictor homotopy analysis method · Rule of multiplicity of solutions · Prescribed parameter · Reactive transport model

## 1 Introduction and problem formulation

Consider dimensionless steady state reactive transport model which is governed by [1]

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \mathrm{P}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\alpha u}{\beta + u} = 0, \quad 0 \leq x \leq 1, \tag{1}$$

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Published online: 21 February 2012

with boundary conditions

$$\frac{\mathrm{d}u}{\mathrm{d}x}(0) = 0, \quad u(1) = 1, \tag{2}$$

where $u(x)$ is dimensionless reactant concentration at $x$, P advective transport (Péclet number) and $\frac{\alpha u}{\beta + u}$ so-called Michaelis-Menten reaction term with $\alpha$ as characteristic reaction rate and $\beta$ as half saturation concentration.

The problem (1)–(2), recently introduced by Ellery and Simpson [1], is a kind of modification of the primer model so-called nonlinear reaction-diffusion model in porous catalysts which has been used to study porous catalyst pellets and more, it has been analyzed by different methods [7–9]. The model (1)–(2) involves advective and diffusive transport with the Michaelis-Menten reaction model that is routinely used to represent biochemical processes [2–4]. This model encodes a number of important engineering processes including several applications in chemical engineering [5, 6] and environmental engineering [3, 4]. The boundary value problem (1)–(2) contains nonlinear fractional term which makes it somewhat difficult to treat even by numerical methods. Ellery and Simpson [1] presented Taylor series solution for this model which truly is convergent on the condition that the Michaelis-Menten reaction term has bounded derivatives as they mentioned.

The aim of this paper is to go advance with this model by applying predictor homotopy analysis method (PHAM) [9–11] which is more general than HAM in some sense and can be applied to predict and calculate multiple solutions of BVPs simultaneously. The homotopy analysis method [12] has been successfully applied to several nonlinear problems such as the viscous flows of non- Newtonian fluids [13–19], the KdV-type equations [20], nano boundary layer flows [21], nonlinear heat transfer [22], finance problems [23], Riemann problems related to nonlinear shallow water equations [24], projectile motion [25], Glauert-jet flow [26], nonlinear water waves [27], ground water flows [28], BurgersHuxley equation [29], time-dependent Emden Fowler type equations [30], differential difference equation [31], Laplace equation with Dirichlet and Neumann boundary conditions [32], thermalhydraulic networks [33] and also readers are referred to see [34–43]. It is not unknown to anyone familiar with the analytical methods that HAM series is general Taylor series [12, 44] which uses the convergence-controller parameter to make convergence fast, so we use PHAM to get series solution more accurate than usual Taylor series solution. We consider nonlinear fractional term in equations (1)–(2) in some cases which have unbounded derivatives then it is revealed by PHAM that the problem admits multiple (dual) solutions in these cases, while the exact solution of this problem is unknown. we conclude that, for practical use in science and engineering, predictor homotopy analysis method might give new unfamiliar class of solutions which is of fundamental interest and furthermore, the proposed approach convinces to apply it on nonlinear equations by todays powerful software programs so that it does not need tedious stages of evaluation and can be used without studying the whole theory.

## 2 PHAM-unique solution of the model

The predictor homotopy analysis method (PHAM) has been fully discussed by Abbasbandy and Shivanain in [10]. Let us rewrite the (1)–(2) as follows:

$$(\beta+u)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-(\beta+u)\mathrm{P}\frac{\mathrm{d}u}{\mathrm{d}x}-\alpha u=0,\quad 0\leq x\leq 1, \tag{3}$$

or equivalently

$$\beta u''-\mathrm{P}\beta u'+uu''-\mathrm{P}uu'-\alpha u=0,\quad 0\leq x\leq 1. \tag{4}$$

The boundary conditions by prescribed parameter $\gamma$, as it is straightforward in PHAM, become

$$u(0)=\gamma,\quad u'(0)=0, \tag{5}$$

with the additional forcing condition

$$u(1)=1 \tag{6}$$

which plays essential role in determining multiplicity of solutions as it is described in PHAM. Now, we apply predictor homotopy analysis method on (4)–(5) where prescribed parameter $\gamma$, which is played important role to realize about multiplicity of solutions, will be obtained with the help of *rule of multiplicity of solutions*.

It is straightforward to use the set of base functions

$$\left\{x^n,\ n=0,1,2,...\right\}. \tag{7}$$

Under the rule of solution expression and according to the initial conditions (5), it is easy to choose

$$u_0(x,\gamma)=\gamma+x^2, \tag{8}$$

as initial guess of solution $u(x)$, $H(x)=1$ as auxiliary function, and to choose auxiliary linear operator

$$\mathcal{L}\left[\phi(x,\gamma;p)\right]=\frac{\partial^2\phi(x,\gamma;p)}{\partial x^2}, \tag{9}$$

with the property

$$\mathcal{L}\left[c_1+c_2x\right]=0. \tag{10}$$

Therefore, after two subsequent integrations, the $M$-th order deformation equation of PHAM yields for $M\geq 1$

$$u_m(x,\gamma)=\chi_m u_{m-1}(x,\gamma)+\hbar\int_0^x\int_0^\eta R_m(\vec{u}_{m-1},\tau,\gamma)\mathrm{d}\tau\mathrm{d}\eta+c_1+c_2x, \tag{11}$$

where from (4)

$$R_m(\vec{u}_{m-1},\tau,\gamma)=\beta u''_{m-1}(\tau,\gamma)-\mathrm{P}\beta u'_{m-1}(\tau,\gamma)+\sum_{j=0}^{m-1}u_j(\tau,\gamma)u''_{m-1-j}(\tau,\gamma)$$
$$-\mathrm{P}\sum_{j=0}^{m-1}u_j(\tau,\gamma)u'_{m-1-j}(\tau,\gamma)-\alpha u_{m-1}(\tau), \tag{12}$$

and integration constants $c_1$ and $c_2$ are obtained by the conditions

$$u_m(0,\gamma)=u'_m(0,\gamma)=0. \tag{13}$$

In this way we obtain the functions $u_m(x,\gamma)$ for $m=1,2,3,\ldots$ from (11) successively. Finally, we can obtain $M$-th order approximate solution

$$U_M(x,\gamma,\hbar)=\sum_{m=0}^{M}u_m(x,\gamma), \tag{14}$$

we give below the PHAM series solution (14) from the order $M=1$ until the order $M=2$ in its form valid for any $\alpha$, $\beta$ and Péclet number:

$$U_1(x,\gamma,\hbar)=-\frac{1}{10}\hbar\mathrm{P}x^5-\frac{1}{3}\hbar\mathrm{P}x^3\beta-\frac{1}{3}h\mathrm{P}x^3\gamma-\frac{1}{12}\hbar x^4\alpha+\frac{\hbar x^4}{6}-\frac{1}{2}\hbar x^2\alpha\gamma$$
$$+\hbar x^2\beta+\hbar x^2\gamma+x^2+\gamma \tag{15}$$

$$U_2(x,\gamma,\hbar)=\frac{1}{80}\hbar^2\mathrm{P}^2x^8+\frac{13}{180}\hbar^2\mathrm{P}^2x^6\beta+\frac{13}{180}\hbar^2\mathrm{P}^2x^6\gamma+\frac{1}{12}\hbar^2\mathrm{P}^2x^4\beta^2$$
$$+\frac{1}{6}\hbar^2\mathrm{P}^2x^4\beta\gamma+\frac{1}{12}\hbar^2\mathrm{P}^2x^4\gamma^2+\frac{1}{70}\hbar^2\mathrm{P}x^7\alpha-\frac{8}{105}\hbar^2\mathrm{P}x^7$$
$$+\frac{1}{30}\hbar^2\mathrm{P}x^5\alpha\beta+\frac{2}{15}\hbar^2\mathrm{P}x^5\alpha\gamma-\frac{7}{15}\hbar^2\mathrm{P}x^5\beta-\frac{7}{15}\hbar^2\mathrm{P}x^5\gamma$$
$$+\frac{1}{6}\hbar^2\mathrm{P}x^3\alpha\beta\gamma+\frac{1}{6}\hbar^2\mathrm{P}x^3\alpha\gamma^2-\frac{2}{3}\hbar^2\mathrm{P}x^3\beta^2-\frac{4}{3}\hbar^2\mathrm{P}x^3\beta\gamma$$
$$-\frac{2}{3}\hbar^2\mathrm{P}x^3\gamma^2+\frac{1}{360}\hbar^2x^6\alpha^2-\frac{2}{45}\hbar^2x^6\alpha+\frac{7\hbar^2x^6}{90}+\frac{1}{24}\hbar^2x^4\alpha^2\gamma$$
$$-\frac{1}{6}\hbar^2x^4\alpha\beta-\frac{1}{3}\hbar^2x^4\alpha\gamma+\frac{1}{2}\hbar^2x^4\beta+\frac{1}{2}\hbar^2x^4\gamma-\frac{1}{2}\hbar^2x^2\alpha\beta\gamma$$
$$-\frac{1}{2}\hbar^2x^2\alpha\gamma^2+\hbar^2x^2\beta^2+2\hbar^2x^2\beta\gamma+\hbar^2x^2\gamma^2-\frac{1}{5}\hbar\mathrm{P}x^5$$
$$-\frac{2}{3}\hbar\mathrm{P}x^3\beta-\frac{2}{3}\hbar\mathrm{P}x^3\gamma-\frac{1}{6}\hbar x^4\alpha+\frac{\hbar x^4}{3}-\hbar x^2\alpha\gamma$$
$$+2\hbar x^2\beta+2\hbar x^2\gamma+x^2+\gamma \tag{16}$$

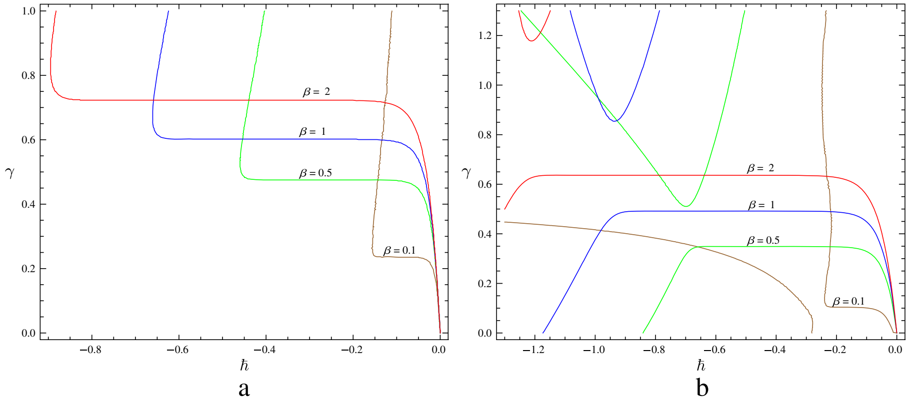


**Fig. 1** Prescribed parameter $\gamma$ via convergence-controller parameter $\hbar$ in according to (17) for different $\beta$: **a** $M = 25$, $\mathrm{P} = 0$, $\alpha = 2$ **b** $M = 20$, $\mathrm{P} = 1$, $\alpha = 2$

So additional forcing condition (6), becomes

$$U_M(1, \gamma, \hbar) \approx 1. \tag{17}$$

Now, to be specific, we consider two cases consist of ($\mathrm{P} = 0, \alpha = 2$) and ($\mathrm{P} = 1$, $\alpha = 2$) for different positive $\beta$. In Fig. 1, according to the above equation $\gamma$ (prescribed parameter) as a function of convergence-controller parameter $\hbar$, has been plotted implicitly. One $\gamma$-plateau (horizontal line) for each case can be identified in these figures, namely $\gamma = 0.23581$ for [–0.15,–0.05] of $\hbar$, $\gamma = 0.47568$ for [–0.45,–0.1] of $\hbar$, $\gamma = 0.60154$ for [–0.65, –0.1] of $\hbar$, $\gamma = 0.72284$ for [–0.85, –0.15] of $\hbar$ in Fig. 1a for $\beta = 0.1, 0.5, 1, 2$, respectively and $\gamma = 0.10355$ for [–0.22,–0.1] of $\hbar$, $\gamma = 0.34880$ for [–0.65,–0.15] of $\hbar$,

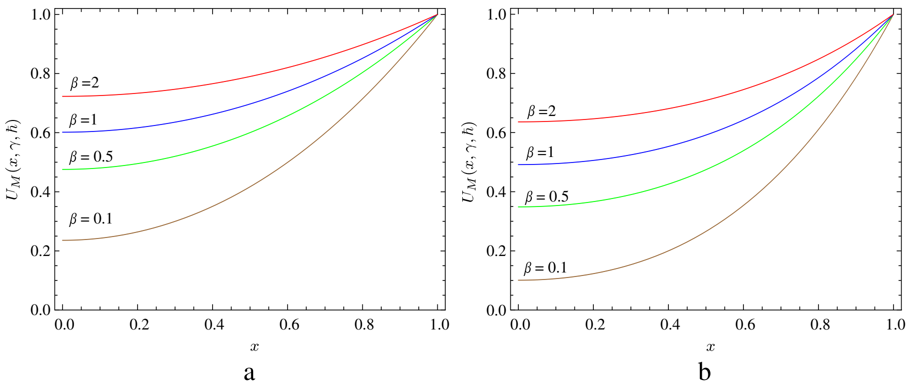


**Fig. 2** Dimensionless reactant concentration profiles **a** $M = 25$, $\mathrm{P} = 0$, $\alpha = 2$ by $\hbar = -0.1$ (*brown*), $\hbar = -0.3$ (*green*), $\hbar = -0.4$ (*blue*), $\hbar = -0.5$ (*red*) **b** $M = 20$, $\mathrm{P} = 1$, $\alpha = 2$ by $\hbar = -0.15$ (*brown*), $\hbar = -0.4$ (*green*), $\hbar = -0.6$ (*blue*), $\hbar = -0.8$ (*red*)

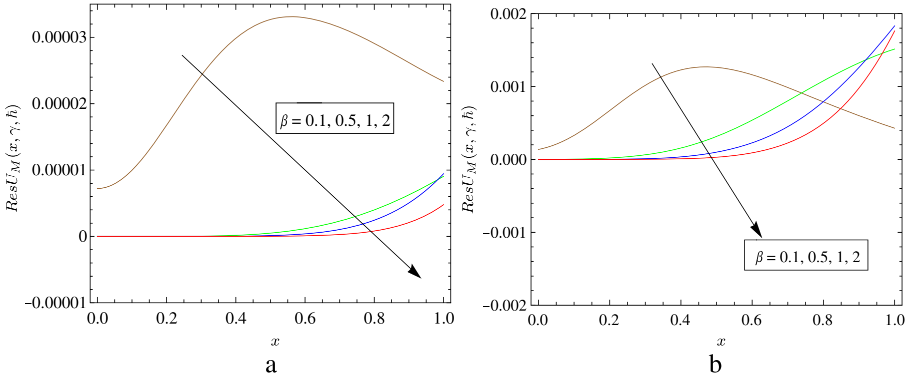


**Fig. 3** Residual error defined by (4) **a** $M = 25$, $\mathrm{P} = 0$, $\alpha = 2$ by $\hbar = -0.1$ (*brown*), $\hbar = -0.3$ (*green*), $\hbar = -0.4$ (*blue*), $\hbar = -0.5$ (*red*) **b** $M = 20$, $\mathrm{P} = 1$, $\alpha = 2$ by $\hbar = -0.15$ (*brown*), $\hbar = -0.4$ (*green*), $\hbar = -0.6$ (*blue*), $\hbar = -0.8$ (*red*)

$\gamma = 0.49169$ for $[-0.9, -0.2]$ of $\hbar$, $\gamma = 0.63616$ for $[-1.2, -0.25]$ of $\hbar$ in Fig. 1b for $\beta = 0.1, 0.5, 1, 2$, respectively. Consequently, we conclude that the PHAM furnishes unique solution in these cases which are shown in Fig. 2, in a full agreement with those obtained in [1]. To show the accuracy of those approximate solutions plotted in Fig. 2, we have shown the residual error for these solution in Fig. 3 i.e. following function which is obtained from (4).

$$ResU_M(x,\gamma,\hbar) = \beta U''_M(x,\gamma,\hbar) - \mathrm{P}\beta U'_M(x,\gamma,\hbar) + U_M(x,\gamma,\hbar)U''_M(x,\gamma,\hbar) - \mathrm{P}U_M(x,\gamma,\hbar)U'_M(x,\gamma,\hbar) - \alpha U_M(x,\gamma,\hbar) \quad (18)$$

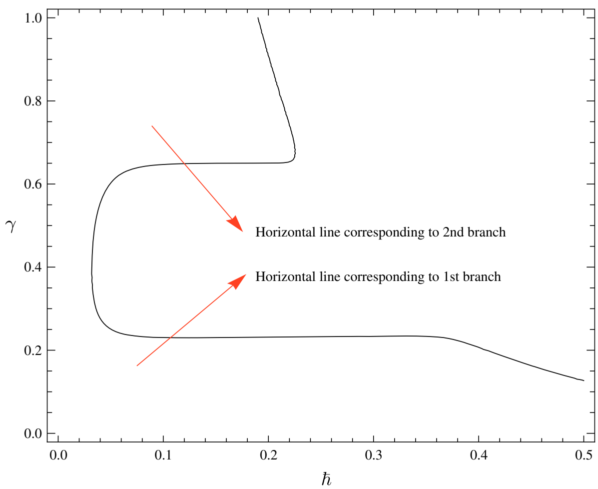


**Fig. 4** Prescribed parameter $\gamma$ via convergence-controller parameter $\hbar$ in according to (17) with $M = 25$

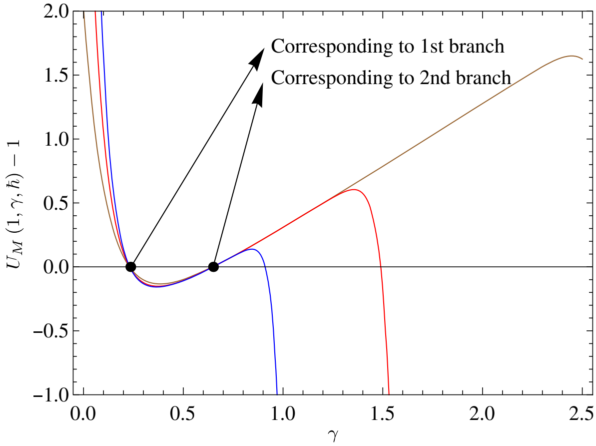


**Fig. 5** The residual of (17) i.e. $U_M(1,\gamma,\hbar)-1$ with different values of $\hbar$ when $M=25$ for P $=0$, $\alpha=0.5$ and $\beta=-0.2$; brown color: $\hbar=0.1$; red color: $\hbar=0.15$; blue color: $\hbar=0.2$

## 3 PHAM-multiple solutions of the model

Ellery and Simpson [1] showed that Taylor series solution of the problem (1)–(2) is convergent when $\frac{\alpha u}{\beta+u}$ has bounded derivatives by applying the ratio test to this series. So, if we consider negative value for $\beta$ then it is possible the Taylor series solution be divergent. In this section, not only we get convergent PHAM series solution but also we discover that the existence of multiple solutions are possible. To be specific, assume the case consist of (P $=0$, $\alpha=0.5$ and $\beta=-0.2$) then according to the equation (17) $\gamma$ as a function of convergence-controller parameter $\hbar$ has been plotted in the $\hbar$-range [0, 0.5] implicitly in Fig. 4. Two $\gamma$-plateaus can be identified in this figure, namely $\gamma=0.23147$ in the range [0.1,0.36] of $\hbar$ and $\gamma=0.65029$ in the

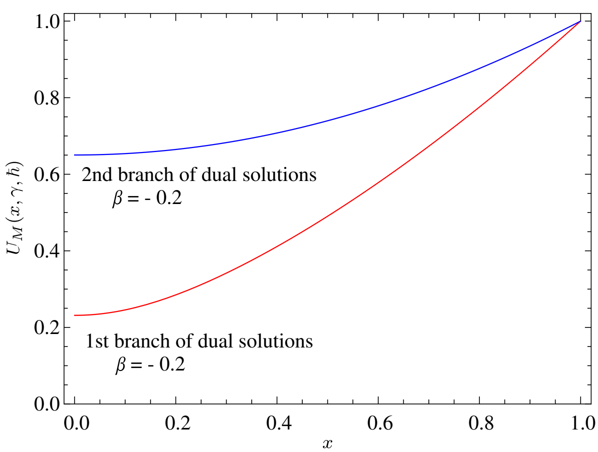


**Fig. 6** Dual approximate PHAM solutions with $M=25$: $U_{25}(x,0.23147,0.2)$ for the first branch (*red color*) and $U_{25}(x,0.65029,0.2)$ for the second branch (*blue color*)

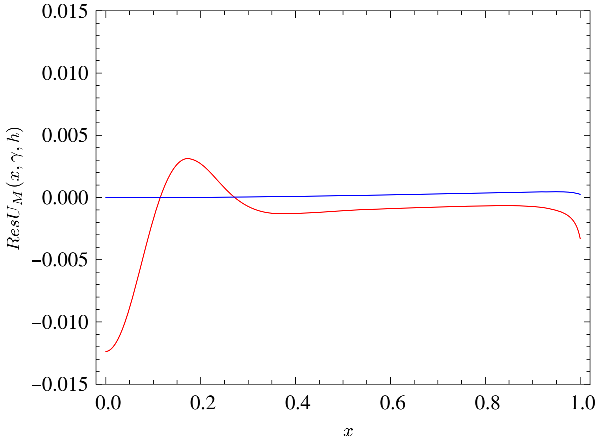


**Fig. 7** The residual error (18) with $M = 25$; red color: $\hbar = 0.30$ and $\gamma = 0.23147$; blue color: $\hbar = 0.18$ and $\gamma = 0.65029$

range [0.1,0.22] of $\hbar$. Consequently, we conclude that the PHAM furnishes dual solutions. It is worth mentioning here that Fig. 4 indicates existence of two solutions, $u(0) = \gamma = 0.23147$ for the first branch solution and $u(0) = \gamma = 0.65029$ for the second branch solution. In predictor homotoy analysis method, another technique to find out that how many solutions the nonlinear problem (1)–(2) admits is to count the number of cross points by horizontal axis which do not change with the variation of $\hbar$ in the graph of Residual error i.e. $U_M(1, \gamma, \hbar) - 1$ as function of $\gamma$. Figure 5 shows that there are two crosses with horizontal axis which dose not vary with change of $\hbar$ so we turn out that there exist dual solutions again.

We remark here that both the first branch and second branch of solutions are calculated at the same time only by (14) with different $\gamma$ and $\hbar$ which are specified from Fig. 4 or Fig. 5. Furthermore, we emphasize that there is no need to use more than one initial approximation guess, one auxiliary linear operator, and one auxiliary function that is in a sharp contrast to all approximation methods which are used to converge to one solution. In the plot shown in Fig. 6, correspond to $\gamma = 0.23147$ and $\gamma = 0.65029$, the approximate dual PHAM solutions $U_{25}(x, 0.23147, 0.2)$ and $U_{25}(x, 0.65029, 0.2)$ given by (14) have been plotted. To show the accuracy of these dual approximate solutions, we have shown the residual error (18) for these solution in Fig. 7.

## 4 Conclusions

It is very important not to lose any solution of nonlinear differential equations with boundary conditions in engineering and physical sciences. In this regard, the present paper has revisited the nonlinear reactive transport model and applied predictor homotopy analysis method (PHAM) to this problem. we have shown that not only we can get convergent series solution but also we can talk about multiplicity of solutions. In specific case, the dual approximate solutions have been obtained and also the accuracy of these solutions by showing the residual error of original equation has been observed.

**Acknowledgements** Authors acknowledge financial support from the Islamshahr branch, Islamic Azad University.

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