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A class of split-step balanced methods for stiff stochastic differential equations

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Abstract In this paper we design a class of general split-step balanced methods for solving Itô stochastic differential systems with m-dimensional multiplicative noise, in which the drift or deterministic increment function can be taken from any chosen one-step ODE solver. We then give an analysis of their order of strong convergence in a general setting, but for the mean-square stability analysis, we confine our investigation to a special case in which the drift increment function of the methods is replaced by the one from the well known Rosenbrock method. The resulting class of stochastic differential equation (SDE) solvers will have more appropriate and useful mean-square stability properties for SDEs with stiffness in their drift and diffusion parts, compared to some other already reported split-step balanced methods. Finally, numerical results show the effectiveness of these methods.

Keywords Stochastic differential equations · split-step balanced methods · Mean-square stability · Stiff equations

1 Introduction

As we know, many models in physics, economics yield stochastic differential equations with multiplicative noise. Numerical methods are important tools for calculating approximation solutions of stochastic differential equations.

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In the recent years many numerical methods for $SDEs$ have been designed, for example see [10, 14, 16]. One of the important subjects that should be investigated for numerical methods, consists of inspecting their ability to preserve qualitative behavior of the solution of the original system that is going to be approximated. Like the drift coefficient, the diffusion coefficient also contributes to stiffness of $SDEs$. We can find several examples of this kind of equations in physics, for example, hydrology models, the *Langevin* equations of chemical physics and the models for laser emission [2, 12, 14]. It is often necessary to use some implicit methods to overcome this difficulty in the simulation of solution of stochastic stiff differential equations see [14, 17, 18, 20]. But as we know a straightforward formulation of a fully implicit method faces the problem of being stochastically unstable and divergent [14]. Some methods were constructed for solving this kind of equations, for example, Tian and Burrage [20] construct some implicit Taylor methods and Platen [7, 14] introduced some kind of predictor-corrector methods for solving $SDEs$ in the weak sense.

In this paper we consider numerical methods for strong solution of Itô stochastic differential equation

$$dX_t = f(t, X_t)dt + \sum_{j=1}^{m} g_j(t, X_t)dW_t^j, \quad X_{t_0} = x_0, \quad t \in [t_0, T], \tag{1.1}$$

where $f : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is drift and $g : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ with $g = [g_1, ..., g_m]$ is the diffusion and $W = \{W_t : t \geq 0\}$ is an m-dimensional Wiener process.

Assume that for some $r \in \mathbb{N}$ all of the initial moments $E(|X_{t_0}|^{2r}) < \infty$. We also assume that SDE (1.1) satisfies the required conditions of the existence and uniqueness theorem [14]. For simplicity in this paper we consider equation (1.1) in autonomous case and numerical methods on the given time interval $[0, T]$ will be employed with equidistant time discretization points $t_n = t_0 + nh, n = 0, 1, ..., N$ with step-size $h = \frac{T-t_0}{N}$, $N = 1, 2, ...$. Here we shall use the notation y_n to denote the value of the approximation of the exact solution X at time t_n.

Definition 1.1 ([14]) We say a discrete approximation $y_0, y_1, ..., y_n$ (based on step-size h) converges strongly with order $\gamma > 0$ to the solution $X = X_t$ as $h \to 0$ at time t_n if there exist constants $\delta_0 > 0$ and $K > 0$ (independent of h), such that for each $h \in (0, \delta_0)$ we have a mean global error,

$$E(|X_{t_n} - y_n|) \leq Kh^{\gamma}. \tag{1.2}$$

In this paper we design and analyze the strong convergence of a class of general split-step balanced methods for solving Itô stochastic differential systems with m-dimensional multiplicative noise, in which the drift increment function can be taken from any classic ODE solver of order of at least one. In Section 2, first we give an overview on numerical methods for solving ordinary differential systems with initial conditions, then a brief review of some previous split-step balanced methods and some assumptions that must be considered, will be brought and based on these information we propose this class of split-step balanced methods. In Section 3 we analyze convergence properties of this class of methods under *Lipschitz* conditions and after that in the Section 4 the mean-square stability properties of them will be investigated along with some useful illustrations. Finally, in Section 5 numerical experiments are given to complete our numerical investigation of the proposed methods.

2 A class of general split-step balanced methods

2.1 One-step methods for solving ordinary differential equations

Consider ODE system $X'_t = f(t, X_t)$ with initial condition $X_{t_0} = X_0$. The general form of a one-step method for solving this system with step-size h is as follows:

$$Y_{n+1} = Y_n + h\Phi(t_n, h, Y_n), \quad n = 0, 1, ..., N-1, \tag{2.1}$$

with initial value given by $Y_0 = X_0$. The function Φ is called the increment function of the approximation method (2.1). This method has order $p > 0$ if for sufficiently smooth function $f \in F_p(x_0, +\infty)$

$$X(t_n + h) = X(t_n) + h\Phi(t_n, h, X(t_n)) + \mathcal{O}(h^{p+1}), \tag{2.2}$$

where $X(t)$ represents the exact solution of given ODE [19].

2.2 Balanced implicit methods

As mentioned before, it is often necessary to use some implicit methods in the simulation of solution of stochastic stiff differential equations. But as we know a straightforward formulation of a fully implicit method faces the drawback of being stochastically unstable and divergent [14]. Some research have been done in this direction, for example, some high-order explicit methods considered and tried to introduce implicitness there that contains deterministic terms [17]. Here, we review the significant developments in some implicit methods for numerically integrating $SDEs$ that have stiffness in both drift and diffusion parts:

1. Platen et al. in [17] proposed a balanced implicit method on a uniform mesh over the simulation interval $[t_0, T]$ by adding the term $C_n(y_n - y_{n+1})$ to the simple Euler method, where $C_n = c_0(t_n, y_n)h + \sum_{j=1}^{m} c_j(t_n, y_n)|\Delta W_n^{(j)}|$, $\Delta W_n^{(j)} = W_{t_{n+1}}^{(j)} - W_{t_n}^{(j)}$, c_0 and c_j, $j = 1, ..., m$ represent $d \times d$ matrix valued functions.
   In this method the functions c_0 and c_j, $j = 1, ..., m$ are called control functions.

The control functions must satisfy some conditions that we express them here as Assumption 1.

Assumption 1 For any sequence of real numbers α_i, $i = 1, ..., m$ with $\alpha_0 \in [0, \bar{\alpha}]$, $\alpha_1 \geq 0$,..., $\alpha_m \geq 0$ and $\bar{\alpha} \geq h$ for all time

$$M(t, x) := \mathcal{I} + \alpha_0 c_0(t, x) + \sum_{j=1}^{m} \alpha_j c_j(t, x),$$

where $\mathcal{I}$ is the $d \times d$ identity matrix, has an inverse and satisfies the condition $\|M(t, x)^{-1}\| \leq K < \infty$;

2. Alcock and Burrage in [1] have analyzed asymptotic and mean-square stability for several implementations of the balanced method and have given a generalized result for the mean-square stability region of any balanced method that we can also see some similar results here in our investigation, as will be mentioned whenever needed;
3. Kahl and Schurz in [11] represent a class of linear-implicit methods with some qualitative improvement on the balanced implicit methods of [6];
4. More recently the modified split-step backward balanced *Milstien* methods have been described in [21] for a single noise system under the assumptions in [17], in which the control function is defined as

$$C_n = c_0(t_n, y_n)h + c_2(t_n, y_n)\left[(\Delta W_n)^2 - h\right] \tag{2.3}$$

with $c_2 > 0$ and $c_0 - c_2 > 0$.

2.3 Formulation of the new split-step balanced methods

In the following we will suppose that Φ is the increment function of a one-step numerical ODE solver of order at least one and the control function

$$C_n = c_0(t_n, y_n)h + \sum_{j=1}^{m} c_j(t_n, y_n)\left[\left(\Delta W_n^j\right)^2 - h\right]. \tag{2.4}$$

Now based on the two previous subsections and the selected increment function, we introduce a class of general split-step balanced numerical methods for solving the SDE (1.1) as below:

$$Y_n = y_n + h\Phi(h, y_n, f) - \frac{1}{2}h\sum_{j=1}^{m} L^j g_j(Y_n),$$

$$\begin{aligned} y_{n+1} = Y_n &+ \sum_{j=1}^{m} g_j(Y_n)I_{(j)} + \frac{1}{2}\sum_{j=1}^{m} L^j g_j(Y_n)\left[\Delta W_n^j\right]^2 \\ &+ \sum_{j_1=1}^{m}\sum_{\substack{j_2=1 \\ j_2\neq j_1}}^{m} L^{j_1} g_{j_2}(Y_n)I_{(j_1,j_2)} + C_n\left(Y_n - y_{n+1}\right), \end{aligned} \tag{2.5}$$

with the differential operators

$$L^j = \sum_{k=1}^{m} g_j^k \frac{\partial}{\partial x^k}, \tag{2.6}$$

for $j = 1, ..., m$ and the stochastic Itô Integrals

$$I_{(j)} = \int_{t_n}^{t_{n+1}} dW_s^j = \Delta W_n^j,\ I_{(j_1, j_2)} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s_1} dW_{s_2}^{j_1} dW_{s_1}^{j_2}.$$

In the subsequent section we will analyze and investigate the convergence properties of this kind of methods and show the superiority in accuracy and stability in other sections.

Remark 2.1 It is obvious that if we consider $\Phi = f(t, x)$, the simple *Euler* method, this method for the case $m = 1$, reduces to the special split-step backward balanced *Milstien* introduced in [21].

3 Convergence properties

In this section the strong convergence order of (2.5) is analyzed. Following two lemmas and a theorem we show that if Φ is the increment function of a selected one-step ODE solver of order at least one, then the associated split-step balanced method defined as (2.5) has order one in strong sense.

Motivated by previous work in this area, we also make the following assumption on SDE (1.1) see [17, 21].

Assumption 2 The functions f, g_j and $L^{j_1} g_{j_2}$ for $j, j_1, j_2 = 1, ..., m$ in (1.1) and (2.5), for all $x_0, y_0 \in \mathbb{R}^d$, and all $t \in [t_0, T]$ satisfy the *Lipschitz* condition for constant $K > 0$ and linear growth bound, as follows:

$$\begin{aligned} |f(t, x_0) - f(t, y_0)| + |g_j(t, x_0) - g_j(t, y_0)| + |L^{j_1} g_{j_2}(t, x_0) - L^{j_1} g_{j_2}(t, y_0)| \\ \le K|x_0 - y_0|, \\ |f(t, x_0)|^2 + |g_j(t, x_0)|^2 + |L^{j_1} g_{j_2}(t, x_0)|^2 \le K^2(1 + |x_0|^2). \end{aligned}$$

To prove Theorem 3.4 in this section, we recall the following theorem concerning the order of convergence (see [15]).

Theorem 3.1 *Assume for a one-step discrete time approximation y, the local mean and mean-square errors for all* $N = 1, 2, ...,$ *and* $n = 0, 1, ..., N - 1$ *satisfy the inequalities*

$$\left| E\Big[\big(y_{n+1} - y(t_{n+1})\big) \big| y_n = y(t_n) \Big] \right| \le K(1 + |y_n|^2)^{\frac{1}{2}} \times h^{p_1}, \tag{3.1}$$

$$\left(E\Big[\big(y_{n+1} - y(t_{n+1})\big)^2 \big| y_n = y(t_n) \Big] \right)^{\frac{1}{2}} \le K(1 + |y_n|^2)^{\frac{1}{2}} \times h^{p_2}, \tag{3.2}$$

with $p_2 \geq \frac{1}{2}$ and $p_1 \geq p_2 + \frac{1}{2}$. Then,

$$\left(E\Big[\big(y_k - y(t_k)\big)^2 \big| y_0 = y(t_0)\Big]\right)^{\frac{1}{2}} \leq K(1+|y_0|^2)^{\frac{1}{2}} \times h^{p_2-\frac{1}{2}}, \tag{3.3}$$

holds for each $k = 0, 1, 2, ..., N$. Here K is independent of h, but it is dependent on the length of the time interval $T - t_0$.

In the following, with the help of Theorem 3.1 and Assumptions 1 and 2, we prove two useful lemmas.

Lemma 3.2 *Let y_k^A be the numerical approximation to $y(t_k)$ at the time T after k steps with step-size $h = T/N$ for $N = 1, 2, ...,$ that is generated by*

$$\begin{aligned} Y_n^A &= y_n + hf(y_n) - \frac{1}{2}h\sum_{j=1}^{m} L^j g_j\left(Y_n^A\right), \\ y_{n+1}^A &= Y_n^A + \sum_{j=1}^{m} g_j(Y_n^A) I_{(j)} + \frac{1}{2}\sum_{j=1}^{m} L^j g_j\left(Y_n^A\right)\left[\Delta W_n^j\right]^2 \\ &\quad + \sum_{j_1=1}^{m}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{m} L^{j_1} g_{j_2}\left(Y_n^A\right) I_{(j_1,j_2)}, \end{aligned} \tag{3.4}$$

then for all $k = 0, 1, ..., N$ we have

$$\left(E\Big[\big(y_k^A - y(t_k)\big)^2 \big| y_0 = y(t_0)\Big]\right)^{\frac{1}{2}} = \mathcal{O}(h).$$

Proof We try to estimate mean and mean-square errors (3.1) and (3.2), respectively for approximation y_k^A. Suppose $y_n = y(t_n)$ (local analysis assumption) and consider y_{n+1}^M, local Milstein approximation of $y(t_{n+1})$, that is defined as below

$$y_{n+1}^M = y_n + hf(y_n) + \sum_{j=1}^{m} g_j(y_n) I_{(j)} + \sum_{j_1=1}^{m}\sum_{j_2=1}^{m} L^{j_1} g_{j_2}(y_n) I_{(j_1,j_2)},$$

then we can write

$$\begin{aligned} \Big|E\Big[y(t_{n+1}) - y_{n+1}^A \big| y_n = y(t_n)\Big]\Big| &\leq \Big|E\Big[y(t_{n+1}) - y_{n+1}^M \big| y_n = y(t_n)\Big]\Big| \\ &\quad + \Big|E\Big[y_{n+1}^M - y_{n+1}^A \big| y_n = y(t_n)\Big]\Big| \\ &\leq K\left(1+|y_n|^2\right)^{\frac{1}{2}} h^2, \end{aligned}$$

because it is obviously seen that

$$\begin{aligned}
&\left|E\left[y^M_{n+1}-y^A_{n+1}\middle|y_n=y(t_n)\right]\right| \\
&=\left|E\left[\sum_{j=1}^{m}\left[g_j(Y^A_n)-g_j(y_n)\right]I_{(j)}\right.\right. \\
&\qquad\left.\left.+\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\left[L^{j_1}g_{j_2}(Y^A_n)-L^{j_1}g_{j_2}(y_n)\right]I_{(j_1,j_2)}\right]\right|=0.
\end{aligned}$$

Hence the first conclusion of Theorem 3.1 holds for $p_1=2$. On the other hand, because of the Assumptions 1 and 2 we can write

$$|y^M_{n+1}-y^A_{n+1}|\leq K|Y^A_n-y_n|\left[\sum_{j=1}^{m}|I_{(j)}|+\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}|I_{(j_1,j_2)}|\right].$$

Because of $Y^A_n-y_n=hf(y_n)-\frac{1}{2}h\sum_{j=1}^{m}L^jg_j(Y^A_n)$, we have $|Y^A_n-y_n|^2=\mathcal{O}(h^2)$. Now because of the inequality

$$(\alpha_1+...+\alpha_t)^2\leq t\left(\alpha_1^2+...+\alpha_t^2\right) \tag{3.5}$$

for any $\alpha_i>0$, for $i=1,...,t$, for any integer $t>0$ we can write

$$\begin{aligned}
\left|y^M_{n+1}-y^A_{n+1}\right|^2 &\leq 2K^2\left|Y^A_n-y^M_n\right|^2\left[\left[\sum_{j=1}^{m}\left|I_{(j)}\right|\right]^2+\left[\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\left|I_{(j_1,j_2)}\right|\right]^2\right] \\
&\leq K'\left|Y^A_n-y^M_n\right|^2\left[\sum_{j=1}^{m}I^2_{(j)}+\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}I^2_{(j_1,j_2)}\right],
\end{aligned} \tag{3.6}$$

for a constant K'. In the other hand, from Lemma 5.7.2 in [14] we have $E\big(I^2_{(j)}|y_n=y(t_n)\big)\leq\mathcal{O}(h)$ and $E\big(I^2_{(j_1,j_2)}|y_n=y(t_n)\big)\leq\mathcal{O}(h^2)$, therefore from (3.6) we have

$$\left(E\Big[(y^M_{n+1}-y^A_{n+1})^2\big|y_n=y(t_n)\Big]\right)^{\frac{1}{2}}=\mathcal{O}\left(h^{\frac{3}{2}}\right),$$

and based on the second conclusion of Theorem 3.1 the proof is established.

□

Lemma 3.3 *Let y_k^B be the numerical approximation of $y(t_k)$ at time T after k steps with step-size $h = T/N$ for $N = 1, 2, ...,$ that is generated by difference equation*

$$Y_n^B = y_n + h\Phi(h, y_n, f) - \frac{1}{2}h\sum_{j=1}^{m} L^j g_j\left(Y_n^B\right),$$

$$y_{n+1}^B = Y_n^B + \sum_{j=1}^{m} g_j(Y_n^B)I_{(j)} + \frac{1}{2}\sum_{j=1}^{m} L^j g_j\left(Y_n^B\right)\left[\Delta W_n^j\right]^2$$
$$+ \sum_{j_1=1}^{m}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{m} L^{j_1} g_{j_2}\left(Y_n^B\right) I_{(j_1,j_2)}, \tag{3.7}$$

where Φ is the increment function of the selected one-step ODE solver as mentioned before. Then for all $k = 0, 1, ..., N$ we have

$$\left(E\Big[\left(y_k^B - y(t_k)\right)^2 \big| y_0 = y(t_0)\Big]\right)^{\frac{1}{2}} = \mathcal{O}(h).$$

Proof Similar to Lemma 3.2, we try to estimate mean and mean-square errors for approximation of y_k^B. With assumption $y_n = y(t_n)$ and the definition of y_n^A we can write

$$y_{n+1}^B - y_{n+1}^A = \Big(y_n + h\Phi(h, y_n, f) - y_n - hf(y_n)\Big)$$
$$+ \sum_{j=1}^{m}\left[g_j\left(Y_n^B\right) - g_j\left(Y_n^A\right)\right]I_{(j)}$$
$$+ \sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\left[L^{j_1} g_{j_2}\left(Y_n^B\right) - L^{j_1} g_{j_2}\left(Y_n^A\right)\right]I_{(j_1,j_2)}.$$

Now because $y_n + h\Phi(h, y_n, f)$ and $y_n + hf(y_n)$ are two numerical methods of order one for ODE system $x(t)' = f(x(t))$ with initial condition $x(t_n) = y_n$, so

$$|y_n + \Phi(h, y_n, f) - y_n - hf(y_n)| \le \mathcal{O}(h^2),$$

see (2.2), then from Lemma 3.2 we have

$$\left|E\Big[y(t_{n+1}) - y_{n+1}^B | y_n = y(t_n)\Big]\right| \le \left|E\Big[y(t_{n+1}) - y_{n+1}^A | y_n = y(t_n)\Big]\right|$$
$$+\left|E\Big[y_{n+1}^B - y_{n+1}^A | y_n = y(t_n)\Big]\right|$$
$$\le K(1 + |y_n|^2)^{\frac{1}{2}} h^2.$$

On the other hand, because of the above results and Assumptions 1 and 2 we have

$$\left|y^B_{n+1} - y^A_{n+1}\right| \le \left|\mathcal{O}(h^2)\right| + K\left|Y^B_n - Y^A_n\right| \left[\sum_{j=1}^{m}\left|I_{(j)}\right| + \sum_{j_2=1}^{m}\sum_{j_1=1}^{m}\left|I(j_1, j_2)\right|\right],$$

therefore because $|Y^B_n - Y^A_n| \le \mathcal{O}(h)$ and inequality (3.5), then we have

$$\left|E\Big[\left|y^A_{n+1} - y^B_{n+1}\right|^2 \left|y_n = y(t_n)\Big]\right| \le \mathcal{O}(h^3),$$

that implies $|E[|y(t_{n+1}) - y^B_{n+1}|^2|y_n = y(t_n)]|^{1/2} \le \mathcal{O}(h^{3/2})$, so the proof is finished. □

Theorem 3.4 *Let y^C_k be the numerical approximation of $y(t_k)$ at time T after k steps with step-size $h = T/N$ for $N = 1, 2, ...,$ that is generated by differential equation*

$$y^C_{n+1} = Y^B_n + \sum_{j=1}^{m} g_j\left(Y^B_n\right) I_{(j)} + \frac{1}{2}\sum_{j=1}^{m} L^j g_j\left(Y^B_n\right)\left[\Delta W^j_n\right]^2$$

$$+ \sum_{j_1=1}^{m}\sum_{\substack{j_2=1\\ j_2\ne j_1}}^{m} L^{j_1} g_{j_2}\left(Y^B_n\right) I_{(j_1, j_2)} + C_n\left(Y^B_n - y^C_{n+1}\right),$$

where Y^B_n is defined by (3.7) and C_n in (2.4), then for all $k = 0, 1, ..., N$ we have

$$\left(E\Big[\left(y^C_k - y(t_k)\right)^2 \big| y_0 = y(t_0)\Big]\right)^{\frac{1}{2}} = \mathcal{O}(h).$$

Proof Based on definition of y^B_{n+1} in (3.7) we can write

$$y^B_{n+1} - y^C_{n+1} = \left(I - (I - C_n)^{-1}\right)\left(\sum_{j=1}^{m} g_j\left(Y^B_n\right) I_{(j)} + \frac{1}{2}\sum_{j=1}^{m} L^j g_j\left(Y^B_n\right)\left[\Delta W^j_n\right]^2\right.$$

$$\left. + \sum_{j_1=1}^{m}\sum_{\substack{j_2=1\\ j_2\ne j_1}}^{m} L^{j_1} g_{j_2}\left(Y^B_n\right) I_{(j_1, j_2)}\right)$$

$$= (I - C_n)^{-1} C_n\left(\sum_{j=1}^{m} g_j\left(Y^B_n\right) I_{(j)} + \frac{1}{2}\sum_{j=1}^{m} L^j g_j\left(Y^B_n\right)\left[\Delta W^j_n\right]^2\right.$$

$$\left. + \sum_{j_1=1}^{m}\sum_{\substack{j_2=1\\ j_2\ne j_1}}^{m} L^{j_1} g_{j_2}\left(Y^B_n\right) I_{(j_1, j_2)}\right).$$

Now with the help of the symmetry property of the ΔW_n^j in the above expression we obtain

$$\left|E\Big(y_{n+1}^B - y_{n+1}^C \big| y_n = y(t_n)\Big)\right|$$

$$= \left| E\left((I+C_n)^{-1} C_n \left(\frac{1}{2}\sum_{j=1}^{m} L^j g_j\left(Y_n^B\right)\left[\Delta W_n^j\right]^2 + \sum_{j_1=1}^{m}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{m} L^{j_1} g_{j_2}\left(Y_n^B\right) I_{(j_1,j_2)} \right) \middle| y_n = y(t_n) \right) \right|$$

$$\leq K\left(1+|Y_n^B|^2\right)^{1/2}\left(E\left(\frac{1}{2}\sum_{j=1}^{m}\left|C_n\left[\Delta W_n^j\right]^2\right|\right) + E\left(\sum_{j_1=1}^{m}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{m}\left|C_n I_{(j_1,j_2)}\right|\right)\right) \leq \mathcal{O}(h^2),$$

which K is a constant. The final inequality is established because of the $E(|C_n[\Delta W_n^j]^2||y_n = y(t_n)) \leq \mathcal{O}(h^2)$ and $E(|C_n I_{(j_1,j_2)}||y_n = y(t_n)) \leq \mathcal{O}(h^2)$ for all $j, j_1, j_2 = 1, ..., m$ that comes from definition of C_n in (2.4).

On the other hand, for a constant K we have

$$\left|y_{n+1}^B - y_{n+1}^C\right| \leq K\left(1+|Y_n^B|^2\right)^{1/2}$$

$$\times\left(\sum_{j=1}^{m}\left|C_n\Delta W_n^j\right| + \frac{1}{2}\sum_{j=1}^{m}\left|C_n\left[\Delta W_n^j\right]^2\right| + \sum_{j_1=1}^{m}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{m}\left|C_n I_{(j_1,j_2)}\right|\right),$$

therefore, based on Assumptions 1 and 2 and inequality (3.5) and the above relation for a constant K' we have

$$\left|y_{n+1}^B - y_{n+1}^C\right|^2 \leq K'\left(1+|Y_n^B|^2\right)$$

$$\times\left(\sum_{j=1}^{m}\left|C_n\Delta W_n^j\right|^2 + \frac{1}{2}\sum_{j=1}^{m}\left|C_n\left[\Delta W_n^j\right]^2\right|^2 + \sum_{j_1=1}^{m}\sum_{\substack{j_2=1\\ j_2\neq j_1}}^{m}\left|C_n I_{(j_1,j_2)}\right|^2\right)$$

So, because of the definition of C_n in (2.4) and inequality (3.5) we have $E(|y_{n+1}^B - y_{n+1}^C|^2|y_n = y(t_n)) \leq \mathcal{O}(h^3)$ that completes the proof. □

4 Stability properties

In this section we want to analyze stability properties of the split-step balanced methods introduced in (2.5). But for the sake of simplicity, we confine our investigation to a particular case in which the drift increment function is taken from the *Rosenbrock* method (6.1) as in the Appendix. In the following, we consider SDE with a steady state solution $X_t \equiv 0$ such that $f(t, 0) = g(t, 0) = 0$ holds, which is called an equilibrium position.

Definition 4.1 ([14, 16]) The zero solution of (1.1) is said to be

1. Mean-square stable, if for each $\varepsilon > 0$ and $t_0 \geq 0$ there exists $\delta = \delta(t_0, \epsilon) \geq 0$ such that
$$\mathrm{E}(|X(t; X(t_0))|^2) < \varepsilon, \quad t \geq t_0$$
whenever $E(|X(t_0)|^2) < \delta$;
2. Asymptotic mean-square stable, if 1. is satisfied and if there exists $\delta_0 = \delta_0(t_0)$ such that for all $E(|X(t_0)|^2) < \delta_0$
$$\mathrm{E}(|X(t; X(t_0))|^2) \to 0. \ \ for \ \ t \to \infty,$$

In this paper, we will focus on asymptotic mean-square for linear test equation with multiplicative noise

$$dX_t = \lambda X_t dt + \mu X_t dW_t, \quad \lambda, \mu \in \mathbb{C}, \tag{4.1}$$

with nonrandom initial condition $X_{t_0} = x_0 \in \mathbb{R}\backslash\{0\}$. The exact solution of (4.1) is given by $X_t = x_0 exp\{(\lambda - \frac{1}{2}\mu^2)(t - t_0) + \mu(W_t - W_{t_0})\}$ which is asymptotically mean-square stable if

$$\lim_{t\to\infty} E(|X_t|^2) = 0 \Leftrightarrow 2\Re(\lambda) + |\mu|^2 < 0, \tag{4.2}$$

for $\lambda, \mu \in \mathbb{C}$, see [9, 14].

Definition 4.2 We say the method is numerically asymptotically mean-square stable if the numerical solution y_n, generated by method satisfies $\lim_{n\to\infty} E(|y_n|^2) = 0$.

We should now find out what conditions must be imposed in order that the split-step balanced method applied to SDE (4.1), produces numerically stable solutions. After applying a one-step stochastic numerical method to linear scalar test equation (4.1), we then obtain with the parametrization $x = \lambda h$ and $y = \mu\sqrt{h}$, [9] a one-step difference equation of the form

$$Y_{n+1} = R_n(x, y)Y_n = \prod_{i=0}^{n} R_i(x, y)Y_0, \tag{4.3}$$

which frequently $R_n(x, y)$ is called stability function of the numerical method. After calculating $E(|y_n|^2)$, it is clear that the domain of MS-stability of a method is subset of $\mathbb{C}^2$ such as $R_{MS} = \{(x, y) \in \mathbb{C}^2 : \hat{R}_n(x, y) < 1\}$, where $\hat{R}_n(x, y) = E(|R_n(x, y)|^2)$. Since it is not easy to visualize the domains of stability for $\lambda, \mu \in \mathbb{C}$, we restrict our attention to $\lambda, \mu \in \mathbb{R}$ for presenting the figures of stability in the $x - y$ plane.

As was mentioned before, we analyze the mean-square stability of the proposed methods with the increment function from the *Rosenbrock* classic ODE solver, see Appendix. We denote the resulting new split-step balanced methods (2.5) by "RSB". It is clearly seen from [4] that with this selection of drift increment function with just two stages, $s = 2$, in the sense of (2.2), the corresponding *Rosenbrock ODE* solver will have order two which is more than needed for split-step balanced methods (2.5). Now, if we apply the *RSB* methods to the linear test equation (4.1) with λ and $\mu \neq 0$ that satisfy (4.2) then setting $K_1(x) = (1-\gamma x)^{-1}x$ and $K_2(x) = K_1(x)\big[1 + (a_{21}+\gamma_{21})K_1(x)\big]$, the stage values k_1, k_2 and the Y_n^C in the method will be given as:

$$
\begin{aligned}
k_1 &= K_1(x)y_n,\\
k_2 &= K_2(x)y_n,\\
Y_n^C &= \frac{1}{1+\frac{1}{2}y^2}\big[1 + b_1K_1(x) + b_2K_2(x)\big]y_n.
\end{aligned}
$$

Now if we set $Y(x,y) = \frac{1}{1+\frac{1}{2}y^2}\big[1 + b_1K_1(x) + b_2K_2(x)\big]$, we can obtain the following difference equation for this class of methods

$$
y_{n+1} = (1+C_n)^{-1}\left[1 + \mu\Delta W_n + \frac{1}{2}\mu^2\Delta W_n^2 + C_n\right]Y(x,y)y_n. \tag{4.4}
$$

Set $R(x,y,\Delta W_n) = (1+C_n)^{-1}\big[1 + \mu\Delta W_n + \frac{1}{2}\mu^2\Delta W_n^2 + C_n\big]Y(x,y)$ then we have

$$
\begin{aligned}
R^2 = (1+C_n)^{-2}\Big[&1 + \mu^2\Delta W_n^2 + \frac{1}{4}\mu^4\Delta W_n^4 + C_n^2 + 2\mu\Delta W_n + \mu^2\Delta W_n^2 + 2C_n\\
&+\mu^3\Delta W_n^3 + 2\mu\Delta W_nC_n + C_n\mu^2\Delta W_n^2\Big]Y^2(x,y),
\end{aligned}
$$

and after taking the expectation and according to the properties of standard normal distribution we have

$$
\begin{aligned}
E[R^2] = E\Big[&(1+C_n)^{-2}\Big[1 + \mu^2\Delta W_n^2 + \frac{1}{4}\mu^4\Delta W_n^4\\
&+ C_n^2 + \mu^2\Delta W_n^2 + C_n\mu^2\Delta W_n^2 + 2C_n\Big]\Big]Y^2(x,y),
\end{aligned}
$$

Then in the terms of x and y this becomes

$$
E[R^2] = Y^2(x,y) + E\Big[(1+C_n)^{-2}\Big[(2+C_n)y^2z_n^2 + \frac{1}{4}y^4z_n^4\Big]\Big]Y^2(x,y), \tag{4.5}
$$

where $z_n = \frac{\Delta W_n}{\sqrt{h}}$ is random variable with standard normal distribution. So the mean square stability region of the method is

$$
R_{MS} = \{(x,y)\in\mathbb{C}^2 : Y^2(x,y) + E\Big[(1+C_n)^{-2}\Big[(2+C_n)y^2z_n^2 + \frac{1}{4}y^4z_n^4\Big]\Big]Y^2(x,y) < 1\}. \tag{4.6}
$$

Now, because of the complexity of structural of the $E(R^2)$ in the general case, motivated by [21], in the following we try to investigate the mean-square stability of RSB methods presented in (2.5) for two special cases.

Remark 4.3 For simplicity in the following two Cases 1 and 2, we choose $b_1 = b_2 = 0.5$ and $\gamma = 1 - \frac{1}{\sqrt{2}}$. We leave a thorough discussion and numerical testing of any other choices to our subsequent work.

Case 1 Suppose $c_0 = -\lambda$ and $c_1 = 0$, in this case $C_n = -x$ and (4.5) reduces to

$$f_1 = Y(x, y)^2 + Y(x, y)^2 \frac{(2 - x)y^2 + \frac{3y^4}{4}}{(1 - x)^2}. \tag{4.7}$$

Now in order to have a better visualization of R_{MS} in this case, Fig. 1 gives the mean-square stability region of RSB method in a special case of (6.2). The light gray area shows the mean-square stability region of RSB method and the dark gray area shows the mean-square stability region of the test equation (4.1). In Fig. 1, it is clear that the mean-square stability region of RSB method includes the mean-square stability region of the test (4.1).

Remark 4.4 One can see from Fig. 1 that as the step-size h decreases the mean-square stability region of RSB method appears to have a tendency to display a similar characteristic as reported in [1], according which the stability region of our method tends towards the mean-square stability region of the test equation (4.1).

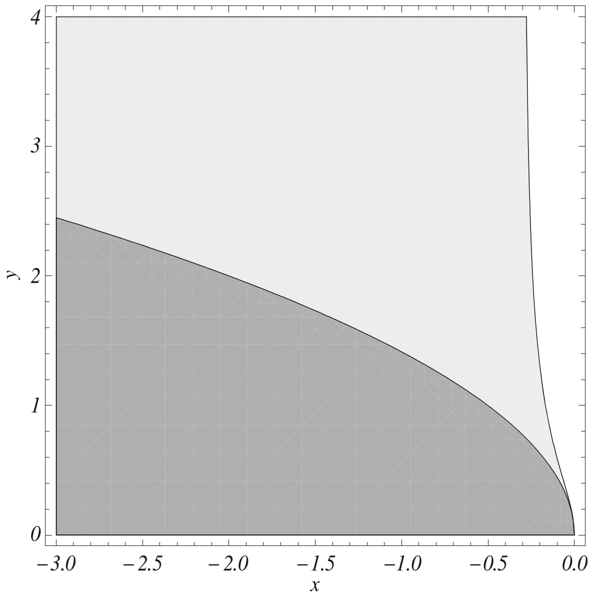

Fig. 1 Mean-square stability regions for the test equation (4.1) (*dark gray area*) and RSB method (*light grey area*) in the Case 1

Case 2 Suppose $c_0 = -\lambda$ and $c_1 = \mu^2$ in this case $C_n = -x + y^2(z_n^2 - 1)$ and the (4.5) reduces to

$$f_2 = Y(x, y)^2 + Y(x, y)^2 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \times \left[\frac{y^2 z_n^2}{1 - x + y^2(z_n^2 - 1)} + \frac{\frac{y^4 z^4}{4} + y^2 z^2}{\left(1 - x + y^2(z_n^2 - 1)\right)^2} \right] e^{-z_n^2/2} dz_n. \quad (4.8)$$

It seems it is not so simple to find the closed form of f_2. Thus we use the Maple software to compute the integral in (4.8). The integral interval $(-\infty, +\infty)$ is approximated by $[-10, 10]$ because of the magnitude of the integrand in (4.8) becomes sufficiently small when $|x| > 10$.

Now for the Case 2 we also illustrate the mean-square stability regions of RSB method and the test equation (4.1). From Fig. 2 it is clearly seen that, also in this case, the mean-square stability region of RSB method (light gray area) includes the mean-square stability region of the test equation (4.1) (dark gray area).

Remark 4.5 In the next section, we also compare the mean-square stability of the RSB method with a split-step backward balanced *Milstein* $DSSBBM$ that was introduced in [21], see Appendix on systems of $SDEs$. We consider $DSSBBM$ in [21], because of its relatively good stability properties that has been reported.

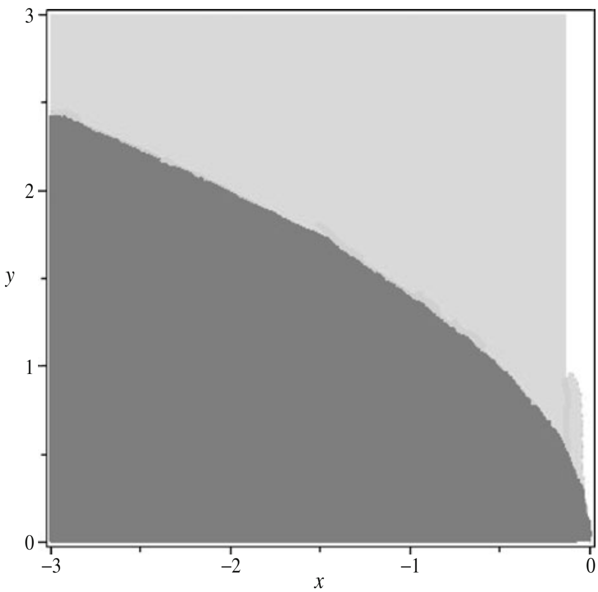

Fig. 2 Mean-square stability regions for the test equation (4.1) (*dark gray area*) and RSB method (*light grey area*) in the Case 2

Table 1 Means of absolute errors, Merr, for problem P1 with $\lambda = 2$, $\mu = 0.5$, $c_0 = 0$, $c_1 = 0$, $X_{t_0} = 1$

h	RSB	$MSSBM$	$DSSBBM$
10^{-2}	4.6×10^{-2}	$\mathbf{1.9 \times 10^{0}}$	$\mathbf{2.9 \times 10^{0}}$
10^{-3}	4.7×10^{-3}	1.8×10^{-1}	4.9×10^{-1}
10^{-4}	4.9×10^{-4}	1.9×10^{-2}	1.1×10^{-1}
10^{-5}	5.0×10^{-5}	1.9×10^{-3}	3.8×10^{-2}

5 Numerical results

In this section numerical results are reported to illustrate the efficiency and superiority of RSB methods which were discussed in previous sections. Denoting $y_N^{(i)}$ and $X^{(i)}(t_N)$ as the numerical solutions and the exact solution at step point t_N in ith simulation, respectively. We use means of absolute errors denoted by "Merr" defined by

$$Merr := \frac{1}{5000} \sum_{i=1}^{5000} |X^{(i)}(t_N) - y_N^{(i)}|,$$

to measure accuracy of the RSB method.

Remark 5.1 One of the benefit of RSB methods is that, we have free parameters that can be chosen for getting better results in accuracy and stability. In these simulations we considered $b_2 = 0.7$ and $\gamma = 1 - \frac{1}{\sqrt{2}}$ for RSB method.

In the tables of numerical results, to highlight some worst performance of the compared methods, the errors greater than one will be shown in bold type.

Problem 1 The first problem is a scalar test equation (4.1) that is considered on $I = [0, 2]$ with initial condition $X_{t_0} = 1$. This equation is stiff in deterministic term, if λ is large and it is stiff in stochastic term, if μ is large. Tables 1, 2, 3, 4 and 5 show the means of absolute errors, Merr, of the RSB and some split-step balanced methods introduced in [21], see Appendix. In Tables 1 and 2 we compare the means of absolute errors, Merr, of the RSB with some split-step balanced methods with $c_0 = c_1 = 0$ for fixed parameter $\lambda = 2$ and $\mu = 0.5$ and more larger $\mu = 1.4$ to show the performance of the method in the case of more stiffness both in the deterministic and stochastic components.

Table 2 Means of absolute errors, Merr, for problem P1 with $\lambda = 2$, $\mu = 1.4$, $c_0 = 0$, $c_1 = 0$, $X_{t_0} = 1$

h	RSB	$MSSBM$	$DSSBBM$
10^{-2}	8.1×10^{-1}	$\mathbf{2.9 \times 10^{0}}$	$\mathbf{1.01 \times 10^{1}}$
10^{-3}	8.3×10^{-2}	1.5×10^{-1}	$\mathbf{4.5 \times 10^{0}}$
10^{-4}	1.1×10^{-2}	1.4×10^{-2}	9.1×10^{-1}
10^{-5}	8.5×10^{-4}	1.1×10^{-3}	2.8×10^{-1}

Table 3 Means of absolute errors, Merr, for problem P1 with $\lambda = 2, \mu = 0.01, c_0 = 5, c_1 = 4, X_{t_0} = 1$

h	RSB	$MSSBM$	$MSSBBM$	$MSSBDBM$	$DSSBBM$	$DSSBDBM$
10^{-1}	8.3×10^{-1}	$\mathbf{3.27 \times 10^{1}}$	$\mathbf{3.3 \times 10^{1}}$	$\mathbf{4.39 \times 10^{1}}$	$\mathbf{3.21 \times 10^{1}}$	$\mathbf{4.51 \times 10^{1}}$
10^{-2}	7.6×10^{-2}	$\mathbf{2.5 \times 10^{0}}$	$\mathbf{2.5 \times 10^{0}}$	$\mathbf{1.36 \times 10^{1}}$	$\mathbf{2.5 \times 10^{0}}$	$\mathbf{1.36 \times 10^{1}}$
10^{-3}	9.7×10^{-3}	2.1×10^{-1}	2.2×10^{-1}	$\mathbf{1.8 \times 10^{0}}$	2.3×10^{-1}	$\mathbf{1.63 \times 10^{0}}$
10^{-4}	9.8×10^{-4}	1.6×10^{-2}	1.7×10^{-2}	1.7×10^{-1}	2.1×10^{-2}	2.1×10^{-1}
10^{-5}	6.5×10^{-5}	3.2×10^{-3}	3.6×10^{-3}	2.3×10^{-2}	2.4×10^{-3}	2.2×10^{-3}

Table 4 Means of absolute errors, Merr, for problem P1 with $\lambda = 3, \mu = 0.01, c_0 = 5, c_1 = 4, X_{t_0} = 1$

h	RSB	$MSSBM$	$MSSBBM$	$MSSBDBM$	$DSSBBM$	$DSSBDBM$
10^{-2}	5.6×10^{-1}	$\mathbf{3.85 \times 10^{1}}$	$\mathbf{3.85 \times 10^{1}}$	$\mathbf{1.36 \times 10^{2}}$	$\mathbf{3.85 \times 10^{1}}$	$\mathbf{3.87 \times 10^{1}}$
10^{-3}	7.1×10^{-2}	$\mathbf{3.61 \times 10^{0}}$	$\mathbf{3.62 \times 10^{0}}$	$\mathbf{1.73 \times 10^{1}}$	$\mathbf{3.6 \times 10^{0}}$	$\mathbf{3.6 \times 10^{0}}$
10^{-4}	7.2×10^{-3}	3.6×10^{-1}	3.2×10^{-1}	$\mathbf{1.8 \times 10^{0}}$	3.5×10^{-1}	3.6×10^{-1}
10^{-5}	6.5×10^{-4}	3.2×10^{-2}	4.6×10^{-2}	2.1×10^{-1}	3.4×10^{-2}	3.5×10^{-2}

Table 5 Means of absolute errors, Merr, for problem P1 with $\lambda = 1.5, \mu = 1.5, c_0 = 1, c_1 = 0.1, X_{t_0} = 1$

h	RSB	$MSSBM$	$MSSBBM$	$MSSBDBM$	$DSSBBM$	$DSSBDBM$
10^{-2}	7.1×10^{-1}	7.5×10^{-1}	$\mathbf{1.5 \times 10^{0}}$	$\mathbf{1.4 \times 10^{0}}$	$\mathbf{4.5 \times 10^{0}}$	$\mathbf{4.0 \times 10^{0}}$
10^{-3}	8.2×10^{-2}	8.2×10^{-2}	1.8×10^{-1}	1.9×10^{-1}	$\mathbf{2.1 \times 10^{0}}$	$\mathbf{2.2 \times 10^{0}}$
10^{-4}	8.6×10^{-3}	5.6×10^{-3}	1.6×10^{-2}	1.3×10^{-2}	3.7×10^{-1}	3.8×10^{-1}
10^{-5}	9.9×10^{-5}	4.8×10^{-4}	1.1×10^{-3}	1.2×10^{-3}	1.1×10^{-1}	1.4×10^{-3}

Table 6 Means of absolute errors, Merr, for problem P2 with $\alpha = 1, \beta = 0.01, T = 2, c_0 = 0, c_1 = 0$

h	RSB	$MSSBM$	$DSSBBM$
2^{-1}	8.4×10^{-5}	5.7×10^{-2}	5.9×10^{-2}
2^{-2}	4.9×10^{-4}	2.7×10^{-2}	2.9×10^{-2}
2^{-3}	1.6×10^{-4}	1.2×10^{-2}	1.4×10^{-2}
2^{-4}	4.4×10^{-5}	4.1×10^{-3}	6.9×10^{-3}
2^{-5}	1.2×10^{-5}	2.3×10^{-3}	2.9×10^{-3}

Table 7 Means of absolute errors, Merr, for problem P2 with $\alpha = 1, \beta = 0.01, T = 2, c_0 = 5, c_1 = 4, X_{t_0} = 0.5$

h	RSB	$MSSBM$	$MSSBBM$	$MSSBDBM$	$DSSBBM$	$DSSBDBM$
2^{-1}	1.4×10^{-3}	5.7×10^{-2}	5.6×10^{-2}	$\mathbf{1.5 \times 10^{0}}$	5.2×10^{-2}	$\mathbf{1.9 \times 10^{0}}$
2^{-2}	8.3×10^{-3}	2.7×10^{-2}	2.5×10^{-2}	7.9×10^{-1}	2.9×10^{-2}	7.7×10^{-1}
2^{-3}	2.8×10^{-3}	1.2×10^{-2}	1.1×10^{-2}	2.8×10^{-1}	1.3×10^{-2}	2.6×10^{-1}
2^{-4}	9.4×10^{-4}	4.1×10^{-3}	3.7×10^{-3}	2.5×10^{-1}	5.5×10^{-3}	2.3×10^{-1}
2^{-5}	6.5×10^{-4}	2.3×10^{-3}	1.6×10^{-3}	1.5×10^{-1}	2.4×10^{-3}	1.2×10^{-1}
2^{-6}	4.1×10^{-4}	2.1×10^{-3}	2.1×10^{-3}	6.1×10^{-2}	2.1×10^{-3}	6.9×10^{-2}

Table 8 Means of absolute errors, Merr, for problem P2 with $\alpha = 1, \beta = 0.1, T = 1, c_0 = 0, c_1 = 0, X_{t_0} = 0.5$

h	RSB	$MSSBM$	$DSSBBM$
2^{-1}	3.4×10^{-3}	5.7×10^{-2}	5.9×10^{-2}
2^{-2}	4.0×10^{-4}	3.7×10^{-2}	3.9×10^{-2}
2^{-3}	2.6×10^{-4}	3.6×10^{-2}	3.6×10^{-2}
2^{-4}	1.2×10^{-4}	3.8×10^{-2}	3.8×10^{-2}
2^{-5}	5.2×10^{-5}	4.1×10^{-2}	4.9×10^{-2}

In the Tables 3 and 4 we compare the Merr of the split-step balanced methods with $c_0 = 5$ and $c_1 = 4$ for different values of h, for fixed $\mu = 0.01$ and values of $\lambda = 2$ and $\lambda = 3$ to show the comparison with more related schemes.

Finally we consider the parameters $\lambda = \mu = 1.5$ that again make the problem 1 stiff both in the deterministic component and stochastic component and demonstrate the results of our method and some more related schemes in Table 5.

The numerical comparisons reported in the Tables 1–5 are in favor of the new RSB method, which has a much better results in the sense of accuracy and advantage of using larger stepsize.

Problem 2 The second equation is a nonlinear SDE in Itô sense as below

$$dy(t) = -(\alpha + \beta^2 y)(1 - y^2)dt + \beta(1 - y^2)dW(t) \;\; I = [0, T], \;\; y(0) = 0.5.$$

The exact solution of this equation is given by [14]

$$y(t) = \frac{(1 + y_0)exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1 + y_0)exp(-2\alpha t + 2\beta W(t)) - y_0 + 1}.$$

This problem, with parameters $\alpha = 1$ and $\beta = 1$, has been considered in [3] as a stiff test problem for testing some stiff methods.

Tables 6, 7, 8 and 9 give the Means of absolute errors, Merr, of the RSB and some split-step balanced methods introduced in [21], see Appendix. Tables 6 and 7 show the numerical comparison of the methods for fixed parameters $\alpha = 1$ and $\beta = 0.01$, but with two sets of control parameters $c_0 = 0, \; c_1 = 0$ and $c_0 = 5, \; c_1 = 4$, respectively. Tables 8 and 9 compare the methods for fixed parameter $\alpha = 1$ and two values $\beta = 0.1$ and $\beta = 1$, respectively, to show the effect of an increase in the stiffness of the stochastic parameter.

Table 9 Means of absolute errors, Merr, for problem P2 with $\alpha = 1, \beta = 1, T = 1, c_0 = 1, c_1 = 0.5, X_{t_0} = 0.5$

h	RSB	$MSSBM$	$MSSBBM$	$MSSBDBM$	$DSSBBM$	$DSSBDBM$
2^{-1}	2.4×10^{-1}	$\mathbf{4.27 \times 10^{2}}$	2.1×10^{-1}	5.9×10^{-1}	5.6×10^{-1}	7.9×10^{-1}
2^{-2}	3.0×10^{-1}	$\mathbf{2.7 \times 10^{1}}$	1.5×10^{-1}	5.2×10^{-1}	5.1×10^{-1}	6.7×10^{-1}
2^{-3}	9.7×10^{-2}	8.2×10^{-2}	$1. \times 10^{-1}$	5.1×10^{-1}	5.3×10^{-1}	6.4×10^{-1}
2^{-4}	5.4×10^{-2}	2.1×10^{-2}	5.7×10^{-2}	5.6×10^{-1}	5.5×10^{-1}	6.1×10^{-1}
2^{-5}	3.0×10^{-2}	1.3×10^{-2}	3.6×10^{-2}	5.5×10^{-1}	6.4×10^{-1}	6.2×10^{-1}

As we can see in Tables 6–8, and particularly noticing the errors in bold type in Table 9, it is clear that the new RSB method has a much better results in the sense of accuracy and advantage of using larger stepsize.

Problem 3 Next, we consider a $d = 4$ dimensional non-linear SDE with non-commutative noise of $m = 2$ dimensional driving Wiener process.

$$d\begin{bmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \\ X_t^4 \end{bmatrix} = \begin{bmatrix} \frac{243}{154}X_t^1 & -\frac{27}{77}X_t^2 & \frac{23}{154}X_t^3 & -\frac{65}{154}X_t^4 \\ \frac{27}{77}X_t^1 & -\frac{243}{154}X_t^2 & \frac{65}{154}X_t^3 & -\frac{23}{154}X_t^4 \\ \frac{5}{154}X_t^1 & -\frac{61}{154}X_t^2 & \frac{162}{77}X_t^3 & -\frac{36}{77}X_t^4 \\ \frac{61}{154}X_t^1 & -\frac{5}{154}X_t^2 & \frac{36}{77}X_t^3 & -\frac{162}{77}X_t^4 \end{bmatrix} dt$$

$$+\frac{1}{9}\sqrt{(X_t^2)^2 + (X_t^3)^2 + \frac{2}{23}}\begin{bmatrix} \frac{1}{13} \\ \frac{1}{14} \\ \frac{1}{13} \\ \frac{1}{15} \end{bmatrix} dW_t^1$$

$$+\frac{1}{8}\sqrt{(X_t^4)^2 + (X_t^1)^2 + \frac{1}{11}}\begin{bmatrix} \frac{1}{14} \\ \frac{1}{16} \\ \frac{1}{16} \\ \frac{1}{13} \end{bmatrix} dW_t^2, \tag{5.1}$$

with initial value $X_0 = [\frac{1}{8}, \frac{1}{8}, 1, \frac{1}{8}]^T$ on $[0, T]$, see [5]. The moments of the solution can be calculated as $E(X_T^i) = \frac{1}{8}exp(2T)$ for $i = 1, 2$. We calculated $E(X_T^1) = \frac{1}{8}exp(2T)$ which is approximated at time $T = 1$ with step-sizes $2^{-1}, ..., 2^{-5}$ and 10^3 simulated trajectories.

Remark 5.2 The random variables $I_{(i,j)}$ for $1 \leq i, j \leq m$ with $i \neq j$, are approximated by a series expansion of the so-called Lévy stochastic series, see [13] and also the Appendix.

As we know, the split-step balanced methods introduced in [21] can only be applied to systems of $SDEs$ with just one Wiener process. But here we have shown the application of RSB to systems with a more general setting. See Table 9 for numerical results in which we have given the absolute errors,

Table 10 Errors for approximation of $E(X_T^1)$ for problem P3 with $T = 1$, $c_0 = 100\mathcal{I}$, $c_1 = 5\mathcal{I}$, $c_2 = 5\mathcal{I}$

h	RSB	*Milstein*
2^{-1}	7.6×10^{-2}	4.7×10^{-1}
2^{-2}	1.6×10^{-2}	3.1×10^{-1}
2^{-3}	4.1×10^{-3}	1.8×10^{-1}
2^{-4}	8.5×10^{-4}	1.1×10^{-1}
2^{-5}	1.6×10^{-5}	5.7×10^{-2}

$|\frac{1}{1000}\sum_{i=1}^{1000} X_{i,T}^{(1)} - \frac{1}{8}exp(2T)|$, of the RSB and *Milstein* methods. It is obviously seen that the new RSB method has a much better results in the sense of accuracy (Table 10).

Problem 4 Here we consider a system of $SDEs$ with noise intensity parameter δ and apply the RSB method and also the $DSSBBM$ method, as a sample of split-step backward balanced *Milstein* methods, to compare them in the sense of mean-square stability. Consider the mean-square stable SDE system

$$dX(t) = \begin{bmatrix} -1 & 0 \\ 0 & -1.5 \end{bmatrix} X(t)dt + \begin{bmatrix} 0.01 & \delta \\ \delta & 0.01 \end{bmatrix} X(t)dW(t)$$

$$X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{5.2}$$

see [7]. We simulated the mean-square of the first component, $E(X_t^{(1)})^2 \approx \frac{1}{1000}\sum_{i=1}^{1000}(X_{i,t}^{(1)})^2$, of the exact solution of (5.1) with $\delta = 0.9, 1$, for step-sizes $h > 0.7$, respectively, by RSB and $DSSBBM$ methods (with $c_0 = 0.4\mathcal{I}$ and $c_1 = 0.01\mathcal{I}$). In this experiment, we have seen the stability of RSB method and instability of $DSSBBM$ method.

6 Conclusions

In this paper we introduce a class of general split-step balanced methods for solving Itô stochastic differential systems with m-dimensional multiplicative noise. The methods of this class are obtained by changing the drift increment function Φ which can be taken from any one-step ODE solver of order at least one. We analyze the strong convergence of the methods in this class for general drift increment functions. For mean-square stability, to be able to get more insight into the properties of the resulting split-step methods, we confine ourselves to the case in which we use the increment function of the well known one-step method of *Rosenbrock*. It is shown that this new split-step balanced method denoted by "RSB" has a large region of stability. Hence as the numerical results also confirm, this method could be appropriate for many drift and diffusion stiff SDE systems with m-dimensional multiplicative noise. We will consider constructing methods with higher strong global convergence orders and better stability properties in future work.

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Appendix

Rosenbrock methods

As we know the general s stage *Rosenbrock* methods for solving ODE systems is as follows:

$$y_{n+1} = y_n + \sum_{j=1}^{s} b_j k_j,$$

$$k_i = hf\left(y_n + \sum_{j=1}^{i-1} \alpha_{ij} k_j\right) + hJ \sum_{j=1}^{i} \gamma_{ij} k_j,\ i = 1, .., s, \tag{6.1}$$

where α_{ij}, γ_{ij}, b_j are the determining coefficients and $J = f'(y_n)$ [8]. Of special interest are the methods for which $\gamma_{ii} = \gamma$ for $i = 1, .., s$ so that we need only one L-U-decomposition per step. The coefficients must be chosen so that the method becomes convergent of a desired order, for example, for two stage *Rosenbrock* method of second order of convergence we can choose

$$b_1 = 1 - b_2,\ \gamma_{21} = \frac{\gamma}{b_2},\ \alpha_{21} = \frac{1}{2b_2} \tag{6.2}$$

with γ and $b_2 \neq 0$ still free. Furthermore, the method is A-stable if $\gamma > \frac{1}{4}$, and is L-stable if $\gamma = 1 \pm \frac{1}{\sqrt{2}}$, see [4].

Some split-step backward balanced *Milstein* methods for stiff stochastic systems

1. Modified split-step backward *Milstein* ($MSSBM$) method

$$Y_n = y_n + hf(h, Y_n) - \frac{1}{2} hg(Y_n)g'(Y_n),$$

$$y_{n+1} = Y_n + \Delta W_n g(Y_n) + \frac{1}{2}\Delta W_n^2 g(Y_n)g'(Y_n); \tag{6.3}$$

2. Drifting split-step backward balanced *Milstein* ($DSSBBM$) method

$$Y_n = y_n + hf(h, Y_n),$$

$$y_{n+1} = Y_n + \Delta W_n g(Y_n) + \frac{1}{2}(\Delta W_n^2 - h)g(Y_n)g'(Y_n) + C_n(Y_n - y_{n+1}). \tag{6.4}$$

3. Modified split-step backward balanced *Milstein* (*MSSBBM*) method

$$Y_n = y_n + hf(h, Y_n) - \frac{1}{2}hg(Y_n)g'(Y_n),$$

$$y_{n+1} = Y_n + \Delta W_n g(Y_n) + \frac{1}{2}\Delta W_n^2 g(Y_n)g'(Y_n) + C_n(Y_n - y_{n+1}). \quad (6.5)$$

4. Drifting split-step backward double balanced *Milstein* (*DSSBDBM*) method

$$Y_n = y_n + hf(h, Y_n) + C_n(y_n - Y_n),$$

$$y_{n+1} = Y_n + \Delta W_n g(Y_n) + \frac{1}{2}(\Delta W_n^2 - h)g(Y_n)g'(Y_n) + C_n(Y_n - y_{n+1}). \quad (6.6)$$

5. Modified split-step backward double balanced *Milstein* (*MSSBDBM*) method

$$Y_n = y_n + hf(h, Y_n) - \frac{1}{2}hg(Y_n)g'(Y_n) + C_n(y_n - Y_n),$$

$$y_{n+1} = Y_n + \Delta W_n g(Y_n) + \frac{1}{2}\Delta W_n^2 g(Y_n)g'(Y_n) + C_n(Y_n - y_{n+1}), \quad (6.7)$$

with control function C_n as defined in (2.3), see [21].

Approximation of the $I_{(i,j)}$ by Lévy stochastic series

The authors in [13] have proposed the following simultaneous representation of the random variables $I_{(i,j)}$: first $I_{(i)} \sim \mathcal{N}(0, h)$, which can be rescaled to $\sqrt{h}\xi_{(i)}$, where $\xi_{(i)} \sim \mathcal{N}(0, 1)$ then

$$I_{(i,j)} = \frac{1}{2}(I_{(i)}I_{(j)} - h\delta_{i,j}) + A_{i,j}(h) = \frac{h}{2}(\xi_{(i)}\xi_{(j)} - \delta_{i,j}) + A_{i,j}(h),$$

$$A_{i,j}(h) = \frac{h}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k}\left\{\chi_{i,k}(\zeta_{j,k} + \sqrt{2}\xi_j) - \chi_{j,k}(\zeta_{i,k} + \sqrt{2}\xi_i)\right\}.$$

Here $\chi_{i,k}$ and $\zeta_{j,k}$ are $\mathcal{N}(0, 1)$-distributed independent random variables for all $r = 1, ..., m$ and $k = 1, 2, ...,$ and δ_{ij} is the *Kronecker* delta. For a practical implementation the infinite sum representing $A_{i,j}(h)$ has to be truncated after, say, p terms and denoted the resulting finite sum by $A_{i,j}^p(h)$, see [13].

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