Numer Algor (2012) 61:35–55
DOI 10.1007/s11075-011-9529-7

ORIGINAL PAPER

Approximation of singularly perturbed reaction-diffusion problems by quadratic C^1-splines

Torsten Linß · Goran Radojev · Helena Zarin

Received: 22 September 2011 / Accepted: 20 December 2011 /
Published online: 8 January 2012

Abstract Collocation with quadratic C^1-splines for a singularly perturbed reaction-diffusion problem in one dimension is studied. A modified Shishkin mesh is used to resolve the layers. The resulting method is shown to be almost second order accurate in the maximum norm, uniformly in the perturbation parameter. Furthermore, a posteriori error bounds are derived for the collocation method on arbitrary meshes. These bounds are used to drive an adaptive mesh moving algorithm. Numerical results are presented.

Keywords Reaction-diffusion problems · Spline interpolation · Spline collocation · Singular perturbations · A posteriori error estimation

H. Zarin is supported by the Ministry of Education and Science of the Republic of Serbia under grant 174030.

This publication has eminated from research conducted with support by the DAAD (grant no. 50740187) and the Ministry of Education and Science of the Republic of Serbia under grant "Collocation methods for singularly perturbed problems".

T. Linß (✉)
Fakultät für Mathematik und Informatik, FernUniversität in Hagen,
Lützowstr. 125, 58095 Hagen, Germany
e-mail: torsten.linss@fernuni-hagen.de

G. Radojev · H. Zarin
Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad,
Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia

G. Radojev
e-mail: goran.radojev@dmi.uns.ac.rs

H. Zarin
e-mail: helena.zarin@dmi.uns.ac.rs

Mathematics Subject Classifications (2010) 65L10 · 65L11 · 65L60 · 65L70

1 Introduction

Consider the reaction-diffusion problem of finding $u \in C^2(0,1) \cap C[0,1]$ such that

$$\mathcal{L}u := -\varepsilon^2 u'' + ru = f \ \text{ in } \ (0,1), \quad u(0) = \gamma_0, \quad u(1) = \gamma_1, \tag{1}$$

where $\varepsilon \in (0,1]$ and $r \geq \varrho^2$ on $[0,1]$ with some constant $\varrho > 0$. This is a difficult problem because standard numerical methods fail to capture layers – regions in the vicinity of $x = 0$ and $x = 1$ where the solution of (1) changes rapidly when the perturbation parameter ε tends to zero. As a consequence the performance of these methods deteriorate when $\varepsilon \to 0$.

The ultimate goal for numerical methods applied to (1) is **uniform convergence** or **robustness** with regard to the perturbation parameter. Indicating by sub- and superscripts that the solution of (1) and its approximation depend on the perturbation parameter and on the number N of degrees of freedom, the numerical approximation is said to be uniformly convergent of order $p > 0$ in the norm $\|\cdot\|_*$ if there exist a constant C and an integer N_0 which are both independent of ε such that

$$\left\| u_\varepsilon - u_\varepsilon^N \right\|_* \leq CN^{-p} \quad \text{for all } \ N \geq N_0.$$

Our norm of choice is the supremum norm $\|\cdot\|_\infty$. Special procedures have been devised predominantly in the area of finite differences or finite elements. For a survey, we refer the reader to recent monographs [14, 19] and earlier books [6, 16] and references therein. One possible approach—which we shall also pursue in the present paper is the use of layer-adapted meshes. These are designed to resolve the layers present in the solution of (1). Most of the literature is devoted to difference schemes and various types of FEMs, while there are only very few publications on collocation methods. This motivates our interest in the subject.

Collocation methods with polynomial trial functions play a very important role in the context of spectral methods. Section 9.7 of Funaro's monograph [7] is dedicated to the application of these methods to problems with boundary layers. For problem (1), polynomials of degree $p \approx \varepsilon^{-1}$ must be used to resolve the layers satisfactorily.

A general theory for spline-collocation methods applied to classical, not singularly perturbed, boundary-value problems was derived in [5]. An immediate application of those results to (1) yields error bounds with "constants" that tend to infinity when $\varepsilon \to 0$.

The initial goal of our investigation was to extend the general theory from [5] to spline-collocation on arbitrary layer-adapted meshes for the reaction-diffusion problem (1). However, so far we have managed to analyse quadratic C^1-splines on a special modified Shishkin mesh [22] only. Collocation with C^1-splines of arbitrary order as well as transition to convection-diffusion problems present an open task.

In recent years the groups of Kadalbajoo and of Rao have extensively published on collocation methods for a number of singularly perturbed problems, including (1). Here we mention [8] and [18] only. Unfortunately all these papers are flawed by various mistakes in the analysis.

The only reliable paper we are aware of that studies quadratic C^1-spline collocation for (1) on a layer-adapted mesh is [21]. Therein the authors use a nodal basis to represent the collocation spline with the value at the mesh points as degrees of freedom. The method is interpreted as a difference scheme. This in turn is analysed on a Shishkin mesh using truncation-error and stability arguments. The maximum error in the mesh points is shown to be uniformly bounded by $CN^{-2}\ln^2 N$ with a constant independent of ε.

In the present paper a B-spline basis is used instead and results in a completely different convergence analysis compared to [21]. It may have the potential of being extended to C^1-spline collocation with piece-wise polynomial of arbitrary degree. However, some stability issues remain open. A posteriori error bounds for the collocation method will also be given. We shall illustrate how these can be used to design an adaptive algorithm that automatically adapts to the structure of the solution.

For the analog of (1) posed on a rectangle, we expect that the a priori analysis can be extended to biquadratic C^1-splines on tensor-product modified Shishkin meshes. Also, a posteriori estimates can possibly be derived along the lines of [11]. However, details need to be checked. This is ongoing work.

The paper is organised as follows. First, in Section 2 properties of the differential operator $\mathcal{L}$ and of the solution u of (1) are quoted from the literature. Based on this information the construction of the layer-adapted mesh from [22] will be explained. Section 3 is concerned with interpolation-error bounds. The main results of the paper, Theorems 4 and 5, are presented in Section 4. Finally, Section 5 contains results of numerical experiments in order to illustrate our theoretical findings.

Notation Throughout, C will denote a generic positive constant that is independent of the perturbation parameter ε and of the number N of degrees of freedom. For any set $D \subset [0, 1]$ and any function v defined on D we set $\|v\|_{\infty,D} := \sup_{x\in D} |v(x)|$. If $D = [0, 1]$ then we drop D from the notation.

2 Properties of the exact solution and adapted meshes

2.1 The Green's function

Stability of a differential operator is best characterised by its Green's function. Let $\mathcal{G}$ be the Green's function associated with the operator $\mathcal{L}$ in (1). With its help any function $v \in W^{1,\infty}(0,1)$ with $v(0) = v(1) = 0$ can be represented as

$$v(x) = \int_0^1 \mathcal{G}(x,\xi)\,(\mathcal{L}v)\,(\xi)\,\mathrm{d}\xi. \tag{2}$$

For $\mathcal{G}$ and its derivatives we have the following (weighted) L_1-norm estimates [14, Th. 3.31]

$$\|r\mathcal{G}(x,\cdot)\|_1 \le 1, \quad \left\|\mathcal{G}_\xi(x,\cdot)\right\|_1 \le (\varrho\varepsilon)^{-1} \quad \text{and} \quad \left\|\mathcal{G}_{\xi\xi}(x,\cdot)\right\|_1 \le 2\varepsilon^{-2}; \tag{3}$$

with $\|v\|_1 := \int_0^1 |v|$. These bounds will be used in Section 4.3 to derive a posteriori error estimates for the collocation method.

2.2 Derivative bounds

For any a priori error analysis, bounds on the derivatives of the exact solution are required. These are provided by the following Lemma, see [14, Th. 3.35]

Lemma 1 *Let $r, f \in C^4[0,1]$. Then*

$$\left|u^{(k)}(x)\right| \le C\left\{1 + \varepsilon^{-k}\mathrm{e}^{-\varrho x/\varepsilon} + \varepsilon^{-k}\mathrm{e}^{-\varrho(1-x)/\varepsilon}\right\}, \quad \textit{for}\ x \in (0,1), \quad k = 0,\ldots,4.$$

Furthermore, the solution can be decomposed as $u = v + w_0 + w_1$. For $k = 0,\ldots,4$, the regular solution component v satisfies $\left\|v^{(k)}\right\|_\infty \le C$, while for the layer parts w_0 and w_1 we have

$$\left|w_0^{(k)}(x)\right| \le C\varepsilon^{-k}\mathrm{e}^{-\varrho x/\varepsilon}, \quad \left|w_1^{(k)}(x)\right| \le C\varepsilon^{-k}\mathrm{e}^{-\varrho(1-x)/\varepsilon}, \quad x \in [0,1].$$

2.3 Layer-adapted meshes

The solution changes rapidly near $x = 0$ and $x = 1$. Hence, the mesh has to be refined there. Various meshes have been proposed in the literature. Most frequently analysed are the exponentially graded mesh of Bakhvalov [1] and the piecewise uniform mesh of Shishkin [16, 20].

Here we shall use the smoothed Shishkin mesh proposed by Vulanović [22]. It is constructed as follows. Let $N+1$ be the number of mesh points. Let $q \in (0, 1/2)$ and $\sigma > 0$ be mesh parameters. Define the Shishkin-mesh transition point by

$$\lambda := \min\left\{\frac{\sigma\varepsilon}{\varrho}\ln N, q\right\}.$$

Remark 1 For the mere sake of simplicity in the representation, we assume, that $\lambda = \sigma\varepsilon\varrho^{-1} \ln N$. Otherwise, the method can be analysed in a classical way. We shall also assume that qN is an integer. This is easily achieved, for example, by choosing $q = 1/4$ and N divisible by 4.

The mesh $\Delta : x_0 < x_1 < \cdots < x_N$ is generated by $x_i = \varphi(i/N)$ with the mesh generating function

$$\varphi(t) := \begin{cases} \frac{\lambda}{q} t & t \in [0, q], \\ p(t-q)^3 + \frac{\lambda}{q} t & t \in [q, 1/2], \\ 1 - \varphi(1-t) & t \in [1/2, 1], \end{cases}$$

where p is chosen such that $\varphi(1/2) = 1/2$, i.e., $p = \frac{1}{2}\left(1 - \frac{\lambda}{q}\right)\left(\frac{1}{2} - q\right)^{-3}$. Note, that $\varphi \in C^1[0, 1]$ with $\|\varphi'\|_\infty, \|\varphi''\|_\infty \le C$. Therefore, the mesh sizes $h_i = x_i - x_{i-1}$, $i = 1, \ldots, N$ satisfy

$$h_i = \int_{(i-1)/N}^{i/N} \varphi'(t)\,\mathrm{d}t \le CN^{-1} \quad \text{and}$$
$$|h_{i+1} - h_i| = \left| \int_{(i-1)/N}^{i/N} \int_t^{t+1/N} \varphi''(s)\,\mathrm{d}s\,\mathrm{d}t \right| \le CN^{-2}. \tag{4}$$

Also, for $i = 1, \ldots, N$,

$$h_i \ge N^{-1} \min_{t \in [0,1]} |\varphi'(t)| = \frac{\lambda}{qN} = \frac{\sigma\varepsilon}{q\varrho} \frac{\ln N}{N}.$$

Hence,

$$\frac{\varepsilon}{h_i} \le \frac{q\varrho}{\sigma} \frac{N}{\ln N}. \tag{5}$$

These properties and the explicit control we have over the transition point are essential in our a priori error analysis.

In Section 5 we shall consider the original Shishkin mesh and the Bakhvalov mesh too. The former is generated with the mesh generating function

$$\varphi_S(t) := \begin{cases} \frac{\lambda}{q} t & t \in [0, q], \\ \frac{q - \lambda}{q(1-2q)}(t-q) + \frac{\lambda}{q} t & t \in [q, 1/2], \\ 1 - \varphi_S(1-t) & t \in [1/2, 1], \end{cases}$$

while the Bakhvalov mesh [1] is generated with

$$\varphi_B(t) = \begin{cases} \chi(t) := \dfrac{\sigma\varepsilon}{\varrho} \ln \dfrac{q}{q-t} & t \in [0, \tau], \\ \pi(t) := \chi(\tau) + \chi'(\tau)(t-\tau) & t \in [\tau, 1/2], \\ 1 - \varphi_B(1-t) & t \in [1/2, 1], \end{cases}$$

where the point τ satisfies $(1-2\tau)\chi'(\tau) = (1-2\chi(\tau))$. Geometrically this means that $(\tau, \chi(\tau))$ is the contact point of the tangent π to χ that passes through the point $(1/2, 1/2)$.

Remark 2 Both the standard Shishkin mesh and the Bakhvalov mesh do not satisfy $|h_{i+1} - h_i| \le CN^{-2}$. This inequality is violated where the meshes change from fine to coarse. Because of this our analysis does not extend to these meshes.

3 The interpolation error

In this section we study the interpolation error for piecewise quadratic splines. Let the mesh intervals be denoted by $J_i := [x_{i-1}, x_i]$. Their midpoints are $x_{i-1/2} := (x_{i-1} + x_i)/2 = x_{i-1} + h_i/2$, $i = 1, \ldots, N$. For, $m, \ell \in \mathbb{N}$, $m < \ell$, let

$$\mathcal{S}_\ell^m(\Delta) := \left\{ s \in C^m[0,1] : s|_{J_i} \in \Pi_\ell, \;\; \text{for} \;\; i = 1, \ldots, N \right\}$$

and $\mathcal{S}_{\ell,0}^m(\Delta) := \left\{ s \in \mathcal{S}_\ell^m(\Delta) : s(0) = s(1) = 0 \right\}$, where Π_ℓ is the space of polynomials of highest degree ℓ.

3.1 $\mathcal{S}_2^0$-interpolation

Given an arbitrary function $g \in C^0[0,1]$, consider the interpolation problem of finding $I_2^0 g \in \mathcal{S}_2^0(\Delta)$ with

$$\left(I_2^0 g\right)_i = g_i, \;\; i = 0, \ldots, N, \quad \text{and} \quad \left(I_2^0 g\right)_{i-1/2} = g_{i-1/2}, \;\; i = 1, \ldots, N, \tag{6}$$

where here and throughout we use $d_i = d(x_i)$ and $d_{i-1/2} = d(x_{i-1/2})$ to denote the values of $d \in C^0[0,1]$ in the mesh points and in the midpoints of the mesh intervals.

Theorem 1 *Assume $r, f \in C^4[0,1]$. Then the interpolation error $I_2^0 u - u$ for the solution of* (1) *on a smoothed Shishkin mesh with $\sigma \ge 3$ satisfies*

$$\left\| u - I_2^0 u \right\|_\infty \le CN^{-3} \ln^3 N \qquad \textit{and} \qquad \varepsilon^2 \max_{i=1,\ldots,N} \left| \left(u - I_2^0 u\right)''_{i-1/2} \right| \le CN^{-2} \ln^2 N.$$

Proof First, use the Lagrange representation of the interpolation polynomial and Taylor expansions to verify that, for any $g \in C^4[0,1]$, the interpolation error on each mesh interval satisfies

$$\left\|g - I_2^0 g\right\|_{\infty,J_i} \le \frac{h_i^3}{24}\left\|g'''\right\|_{\infty,J_i}, \qquad \left|\left(g - I_2^0 g\right)''_{i-1/2}\right| \le \frac{h_i^2}{48}\left\|g^{(4)}\right\|_{\infty,J_i}, \tag{7a}$$

and

$$\left\|g - I_2^0 g\right\|_{\infty,J_i} \le \frac{5}{4}\left\|g\right\|_{\infty,J_i}, \qquad \left|\left(g - I_2^0 g\right)''_{i-1/2}\right| \le 2\left\|g''\right\|_{\infty,J_i}. \tag{7b}$$

Recalling the solution decomposition of Lemma 1, we split the error in a similar manner as

$$u - I_2^0 u = \left(v - I_2^0 v\right) + \left(w_0 - I_2^0 w_0\right) + \left(w_1 - I_2^0 w_1\right),$$

because of the linearity of I_2^0. The terms on the right-hand side will be bounded separately.

For the regular solution component v, (7a), (4) and Lemma 1 yield

$$\left\|v - I_2^0 v\right\|_\infty \le CN^{-3} \qquad \text{and} \qquad \max_{i=1,\dots,N}\left|\left(v - I_2^0 v\right)''_{i-1/2}\right| \le CN^{-2}.$$

For the boundary layer w_0 the arguments splits. On $[0,\lambda]$, the local mesh size is $h_i = C\varepsilon N^{-1}\ln N$. Thus,

$$\left\|w_0 - I_2^0 w_0\right\|_{\infty,[0,\lambda]} \le CN^{-3}\ln^3 N \ \text{ and } \ \varepsilon^2 \max_{i=1,\dots,qN}\left|\left(w_0 - I_2^0 w_0\right)''_{i-1/2}\right| \le CN^{-2}\ln^2 N,$$

by (7a) and Lemma 1. On $[\lambda,1]$, use (7b) and the exponential decay of w_0. We obtain

$$\left\|w_0 - I_2^0 w_0\right\|_{\infty,[\lambda,1]} \le CN^{-\sigma} \quad \text{and} \quad \varepsilon^2 \max_{i=qN+1,\dots,N}\left|\left(w_0 - I_2^0 w_0\right)''_{i-1/2}\right| \le CN^{-\sigma}.$$

Similar estimates are obtained for $w_1 - I_2^0 w_1$.

Finally, application of a triangle inequality completes the proof. □

Lemma 2 *Let $s \in \mathcal{S}_2^0(\Delta)$ with $s_{i-1/2} = 0$, $i = 1,\dots,N$. Then*

$$\left\|s\right\|_{\infty,J_i} \le \max\left\{\left|s_{i-1}\right|, \left|s_i\right|\right\} \quad \textit{and} \quad \left\|s''\right\|_{\infty,J_i} \le \frac{8}{h_i^2}\max\left\{\left|s_{i-1}\right|, \left|s_i\right|\right\}, \quad i = 1,\dots,N.$$

Proof Clearly,

$$s(x) = 2\frac{x - x_{i-1/2}}{h_i^2}\left[s_{i-1}(x - x_i) + s_i(x - x_{i-1})\right], \quad x \in J_i.$$

Hence, $|s(x)| \le \max\left\{\left|s_{i-1}\right|, \left|s_i\right|\right\} 2h_i^{-1}\left|x - x_{i-1/2}\right|$ and $s''(x) = 4h_i^{-2}\left(s_{i-1} + s_i\right)$. The proposition of the lemma follows. □

3.2 $\mathcal{S}_2^1$-interpolation

Given an arbitrary function $g \in C^0[0,1]$, consider the interpolation problem of finding $I_2^1 g \in \mathcal{S}_2^1(\Delta)$ with

$$\left(I_2^1 g\right)_0 = g_0, \qquad \left(I_2^1 g\right)_{i-1/2} = g_{i-1/2}, \;\; i = 1, \ldots, N, \qquad \left(I_2^1 g\right)_N = g_N. \tag{8}$$

For any $s \in \mathcal{S}_2^1(\Delta)$, we have [9, 15]

$$[Ms]_i := a_i s_{i-1} + 3s_i + c_i s_{i+1} = 4a_i s_{i-1/2} + 4c_i s_{i+1/2}, \;\; i = 1, \ldots, N-1 \tag{9}$$

with $a_i := h_{i+1}/(h_i + h_{i+1})$ and $c_i := 1 - a_i = h_i/(h_i + h_{i+1})$.

Lemma 3 *For all vectors $s \in \mathbb{R}^{N+1}$ with $s_0 = s_N = 0$, there holds*

$$\max_{i=1,\ldots,N-1} |s_i| \le \frac{1}{2} \max_{i=1,\ldots,N-1} |[Ms]_i| .$$

Proof Let $k \in \operatorname{argmax}_{i=1,\ldots,N-1} |s_i|$. Then $3s_k = [Ms]_k - a_k s_{k-1} - c_k s_{k+1}$. A triangle inequality implies $3|s_k| \le |[Ms]_k| + (a_k + c_k)|s_k| = |[Ms]_k| + |s_k|$. The proposition of the lemma follows. □

Theorem 2 *Assume $r, f \in C^4[0,1]$. Then the interpolation error for the solution u of (1) on a smoothed Shishkin mesh with $\sigma \ge 4$ satisfies*

$$\max_{i=0,\ldots,N} \left|\left(u - I_2^1 u\right)_i\right| \le CN^{-4} \ln^4 N, \tag{10a}$$

$$\left\|u - I_2^1 u\right\|_\infty \le CN^{-3} \ln^3 N, \tag{10b}$$

$$\varepsilon^2 \max_{i=1,\ldots,N} \left|\left(u - I_2^1 u\right)''_{i-1/2}\right| \le CN^{-2} \ln^2 N. \tag{10c}$$

Remark 3 The first bound (10a) constitutes a superconvergence result because the interpolation error in the mesh nodes is one order smaller than the interpolation error on the whole domain, compare (10b). For classical problems without layers this property is well known even for biquadratic interpolation, see [9, Theorem 5.5] and [3]. Similar superconvergence properties are not observed in the collocation method.

Proof of Theorem 2

(i) First, for an arbitrary $g \in C^4[0,1]$, the interpolation error satisfies $\left(g - I_2^1 g\right)_0 = \left(g - I_2^1 g\right)_N = 0$ and

$$\begin{aligned}\left[M\left(g - I_2^1 g\right)\right]_i = a_i g_{i-1} - 4a_i g_{i-1/2} + 3g_i - 4c_i g_{i+1/2} \\ + c_i g_{i+1} =: \tau_{g,i}, \quad i = 1, \ldots, N,\end{aligned} \tag{11}$$

which follows from (8) and (9). Taylor expansions yield

$$\left|\tau_{g,i}\right| \le 8 \|g\|_{\infty,[x_{i-1},x_{i+1}]} \tag{12a}$$

and

$$\left|\tau_{g,i}\right| \le \frac{1}{12} h_i h_{i+1} \left|h_{i+1} - h_i\right| \left|g_i'''\right| + \frac{5}{96} \max\left\{h_i, h_{i+1}\right\}^4 \left\|g^{(4)}\right\|_{\infty,[x_{i-1},x_{i+1}]} . \tag{12b}$$

Again, we decompose the interpolation error:

$$u - I_2^1 u = \left(v - I_2^1 v\right) + \left(w_0 - I_2^1 w_0\right) + \left(w_1 - I_2^1 w_1\right).$$

The three error components are analysed separately.
Lemma 1, (4) and (12b) give $\left|\tau_{v,i}\right| \le CN^{-4}$. Then Lemma 3 implies $\left|\left(v - I_2^1 v\right)_i\right| \le CN^{-4},\ i = 1, \dots, N$.
For the layer component, the argument splits. First, for $i < qN$, $h_i = h_{i+1} = C\varepsilon N^{-1} \ln N$. Therefore, $\left|\tau_{w_0,i}\right| \le CN^{-4} \ln^4 N$, by (12b) and Lemma 1. For $i \ge qN$, we use the exponential decay of w_0 and (12a):

$$\left|\tau_{w_0,i}\right| \le C \exp\left(-\frac{\varrho(\lambda - h_{qN})}{\varepsilon}\right) \le CN^{-\sigma} N^{\sigma/(qN)} \le CN^{-4}.$$

Consequently, $\left|\tau_{w_0,i}\right| \le CN^{-4} \ln^4 N,\ \ i = 1, \dots, N$, and Lemma 3 yields $\left|\left(w_0 - I_2^1 w_0\right)_i\right| \le CN^{-4} \ln^4 N,\ i = 1, \dots, N$. The same bound is obtained for $\left|w_1 - I_2^1 w_1\right|$. Then the use of a triangle inequality establishes (10a).

(ii) A triangle inequality gives

$$\begin{aligned}\left\|u - I_2^1 u\right\|_\infty &\le \left\|u - I_2^0 u\right\|_\infty + \left\|I_2^0 u - I_2^1 u\right\|_\infty \\ &\le \left\|u - I_2^0 u\right\|_\infty + \max_{i=0,\dots,N} \left|\left(u - I_2^1 u\right)_i\right|,\end{aligned}$$

by Lemma 2 and because $\left(I_2^0 u\right)_i = u_i,\ \ i = 0, \dots, N$. Now, Theorem 1 and (10a) imply (10b).

(iii) Again starting from a triangle inequality, we verify (10c):

$$\begin{aligned}\left|\left(u - I_2^1 u\right)_{i-1/2}''\right| &\le \left|\left(u - I_2^0 u\right)_{i-1/2}''\right| + \left|\left(I_2^0 u - I_2^1 u\right)_{i-1/2}''\right| \\ &\le \left|\left(u - I_2^0 u\right)_{i-1/2}''\right| + \frac{8}{h_i^2} \max_{i=0,\dots,N} \left|\left(u - I_2^1 u\right)_i\right|,\end{aligned}$$

by Lemma 2 and because $\left(I_2^0 u\right)_i = u_i,\ \ i = 0, \dots, N$. Then, Theorem 1, (10a) and (5) give (10c). □

4 The collocation method

We shall discretise (1) by seeking a spline in $\mathcal{S}_2^1(\Delta)$ that satisfies the boundary conditions and the differential equation (1) in certain points. For problems that are not singularly perturbed, it is well known that the best choice for collocation with quadratic C^1-splines are the midpoints of the partition, see [5].

Let Δ be an arbitrary partition of [0, 1]. Our discretisation is: Find $u_\Delta \in \mathcal{S}_2^1(\Delta)$ such that

$$u_{\Delta,0} = \gamma_0, \quad (\mathcal{L}u_\Delta)_{i-1/2} = f_{i-1/2}, \;\; i = 1, \ldots, N, \quad u_{\Delta,N} = \gamma_1. \tag{13}$$

Let $\{\varphi_i\}_{i=0}^{N+1}$ be the B-spline basis in $\mathcal{S}_2^1(\Delta)$, see Appendix A. Then we may represent u_Δ as

$$u_\Delta := \sum_{k=0}^{N+1} \alpha_k \varphi_k,$$

where the α_k are determined by collocation. A careful calculation shows that (13) is equivalent to

$$\alpha_0 = \gamma_0, \quad [\boldsymbol{L\alpha}]_{i-1/2} = f_{i-1/2}, \;\; i = 1, \ldots, N, \quad \alpha_{N+1} = \gamma_1 \tag{14}$$

with $\boldsymbol{\alpha} := (\alpha_0, \ldots, \alpha_{N+1})^T \in \mathbb{R}^{N+2}$ and

$$\begin{aligned}[\boldsymbol{L\alpha}]_{i-1/2} := &-\varepsilon^2 \left[\frac{2(\alpha_{i+1} - \alpha_i)}{h_i(h_i + h_{i+1})} - \frac{2(\alpha_i - \alpha_{i-1})}{h_i(h_{i-1} + h_i)}\right] \\ &+ r_{i-1/2}\left[q_i^+ \alpha_{i+1} + \left(1 - q_i^+ - q_i^-\right)\alpha_i + q_i^- \alpha_{i-1}\right], \quad i = 1, \ldots, N, \\ q_i^+ := &\frac{h_i}{4(h_i + h_{i+1})} \quad \text{and} \quad q_i^- := \frac{h_i}{4(h_i + h_{i-1})},\end{aligned}$$

where we have formally set $h_0 = h_{N+1} = 0$.

4.1 Stability

The operator $\boldsymbol{L}$ is not inverse monotone. Nonetheless, we can establish its maximum-norm stability.

Theorem 3 *Assume, there exists a constant $\kappa > 0$ such that*

$$\max\{h_{i+1}, h_{i-1}\} \ge \kappa h_i, \quad i = 2, \ldots, N-1, \quad h_1 \ge \kappa h_2, \quad \textit{and} \quad h_N \ge \kappa h_{N-1}. \tag{15}$$

Then the operator $\boldsymbol{L}$ is maximum-norm stable with

$$\begin{aligned}\|\boldsymbol{\gamma}\|_\infty := \max_{i=1,\ldots,N} |\gamma_i| &\le \frac{2(1+\kappa)}{\kappa} \max_{i=1,\ldots,N} \left|\frac{[\boldsymbol{L\gamma}]_{i-1/2}}{r_{i-1/2}}\right| \\ &\le \frac{2(1+\kappa)}{\kappa\varrho^2} \|\boldsymbol{L\gamma}\|_\infty \quad \textit{for all } \boldsymbol{\gamma} \in \mathbb{R}_0^{N+2},\end{aligned}$$

where $\mathbb{R}_0^{N+2} := \left\{v \in \mathbb{R}^{N+2} : v_0 = v_{N+1} = 0\right\}$.

Proof Set $m_{i-1/2} := r_{i-1/2}\left(1 - q_i^+ - q_i^-\right)$, $i = 1, \ldots, N$. Note that $q_i^+, q_i^- \in (0, 1/4)$, $i = 1, \ldots, N$. Therefore $m_i > 0$, $i = 1, \ldots, N$.

For arbitrary vectors $\boldsymbol{\gamma} \in \mathbb{R}_0^{N+2}$, define the operator $\boldsymbol{\Lambda}$ by

$$\left[\boldsymbol{\Lambda\gamma}\right]_{i-1/2} := -\frac{\varepsilon^2}{m_{i-1/2}}\left(\frac{2(\gamma_{i+1}-\gamma_i)}{h_i(h_i+h_{i+1})} - \frac{2(\gamma_i-\gamma_{i-1})}{h_i(h_{i-1}+h_i)}\right) + \gamma_i, \quad i=1,\ldots,N.$$

Because of the positivity of all m_i, $\boldsymbol{\Lambda}$ is well defined. After eliminating γ_0 and γ_{N+1} which are both zero, $\boldsymbol{\Lambda}$ is a square matrix whose offdiagonal entries are all non-positive and its row sums are at least 1. Therefore, the M-criterion [17] implies $\left\|\boldsymbol{\Lambda}^{-1}\right\|_\infty \le 1$.

Next, note that

$$\left[\boldsymbol{\Lambda\gamma}\right]_{i-1/2} = \frac{\left[\boldsymbol{L\gamma}\right]_{i-1/2} - r_{i-1/2}\left(q_i^+\gamma_{i+1} + q_i^-\gamma_{i-1}\right)}{m_{i-1/2}}, \quad i=1,\ldots,N.$$

Thus,

$$\|\boldsymbol{\gamma}\|_\infty \le \max_{i=1,\ldots,N}\left|\frac{\left[\boldsymbol{L\gamma}\right]_{i-1/2}}{m_{i-1/2}}\right| + \max_{i=1,\ldots,N}\frac{r_{i-1/2}\left(q_i^+ + q_i^-\right)}{m_{i-1/2}}\|\boldsymbol{\gamma}\|_\infty, \tag{16}$$

because $\left\|\boldsymbol{\Lambda}^{-1}\right\|_\infty \le 1$.

Then, by (15),

$$\max_{i=1,\ldots,N}\frac{r_{i-1/2}\left(q_i^+ + q_i^-\right)}{m_{i-1/2}} \le \frac{2+\kappa}{2+3\kappa} \quad \text{for } i=1,\ldots,N, \tag{17a}$$

and

$$\frac{1}{m_{i-1/2}} \le \frac{4(1+\kappa)}{r_{i-1/2}(2+3\kappa)} \quad \text{for } i=1,\ldots,N. \tag{17b}$$

Inequalities (16) and (17) yield

$$\|\boldsymbol{\gamma}\|_\infty \le \frac{4(1+\kappa)}{2+3\kappa}\max_{i=1,\ldots,N}\left|\frac{\left[\boldsymbol{L\gamma}\right]_{i-1/2}}{r_{i-1/2}}\right| + \frac{2+\kappa}{2+3\kappa}\|\boldsymbol{\gamma}\|_\infty.$$

The proposition of the theorem follows. □

Remark 4 The smoothed Shishkin mesh satisfies (15) with $\kappa = 1$.

Remark 5 Numerical experiments on randomly generated meshes indicate that

$$\|\boldsymbol{\gamma}\|_\infty \le 3\max_{i=1,\ldots,N}\left|\frac{\left[\boldsymbol{L\gamma}\right]_{i-1/2}}{r_{i-1/2}}\right| \quad \text{for all } \boldsymbol{\gamma} \in \mathbb{R}_0^{N+2},$$

without any restrictions on the mesh. However, we have not been able to prove this sharper result.

4.2 Maximum-norm a priori error bound

Theorem 4 *Let u be the solution of (1) and u_Δ its approximation by the collocation method on a smoothed Shishkin mesh with $\sigma \geq 4$. Then*

$$\|u - u_\Delta\|_\infty \leq CN^{-2} \ln^2 N.$$

Proof The interpolant $I_2^1 u$ of u can be represented by means of the B-spline basis as $I_2^1 u = \sum_{k=0}^{N+1} \beta_k \varphi_k$. Clearly, $\boldsymbol{\alpha} - \boldsymbol{\beta} \in \mathbb{R}_0^{N+2}$ and

$$[\boldsymbol{L}(\boldsymbol{\alpha} - \boldsymbol{\beta})]_{i-1/2} = \mathcal{L}\left(u_\Delta - I_2^1 u\right)_{i-1/2} = \varepsilon^2 \left(I_2^1 u - u\right)''_{i-1/2}, \quad i = 1, \ldots, N.$$

Theorems 2 and 3 yield $\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_\infty \leq CN^{-2} \ln^2 N$. Next, note that $\varphi_k \geq 0$ and $\sum_{k=0}^{N+1} \varphi_k = 1$. Therefore, $\left\| I_2^1 u - u_\Delta \right\|_\infty \leq \|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_\infty$. Theorem 2 and a triangle inequality complete the proof. □

Remark 6 Theorem 4 should be compared with similar results for central differencing and piecewise linear Galerkin-FEM, see [14]. All three methods are uniformly convergent of (almost) second-order on layer-adapted meshes. All three approaches generate tridiagonal systems. Therefore they give (approximately) the same accuracy with identical computational costs. Amongst these methods collocation is the least well understood.

4.3 Maximum-norm a posteriori error bounds

Theorem 5 *Let u be the solution of (1) and u_Δ its approximation by the collocation method on an arbitrary mesh Δ. Then*

$$\|u - u_\Delta\|_\infty \leq \eta(ru_\Delta - f, \Delta)$$

with $\eta(q, \Delta) = \eta^I(q, \Delta) + \eta^3(q, \Delta) + \eta^4(q, \Delta)$,

$$\eta^I(q, \Delta) := \left\| \frac{I_2^0 q - q}{r} \right\|_\infty,$$

$$\eta^3(q, \Delta) := \frac{2}{\varrho^2} \max_{i=1,\ldots,N} \left[\max\left\{ |q_i - q_{i-1/2}|, |q_{i-1/2} - q_{i-1}| \right\} \min\left\{1, \frac{h_i \varrho}{4\varepsilon}\right\} \right]$$

and

$$\eta^4(q, \Delta) := \max_{i=1,\ldots,N} \frac{|q_{i-1} - 2q_{i-1/2} + q_i|}{4\varrho^2}.$$

Remark 7 The term η^I captures the data oscillations and inevitably requires sampling of r and f. In view of the collocation condition (13), we have $ru_\Delta - f \approx \varepsilon^2 u''_\Delta$. Therefore, η^3 and η^4 involve discrete third and fourth order derivatives of u_Δ.

Remark 8 A posteriori error bounds for central differencing and P_1-FEM were derived in [10] and [13], resp. In contrast to Theorem 5, these error bounds involve discrete derivatives of order three (central differencing) and of order two (FEM) only.

Proof of Theorem 5 Let $q := ru_\Delta - f$. For a fixed $x \in (0,1)$, the error of the method can be written as

$$(u - u_\Delta)(x) = \int_0^1 \mathcal{G}(x,\xi)\,(\mathcal{L}(u - u_\Delta))(\xi)\,\mathrm{d}\xi = \int_0^1 \mathcal{G}(x,\xi)\,(f - \mathcal{L}u_\Delta)(\xi)\,\mathrm{d}\xi, \tag{18}$$

by (2). In view of (13), we have

$$\sum_{i=1}^N (f - \mathcal{L}u_\Delta)_{i-1/2} \int_{J_i} \mathcal{G}(x,\xi)\,\mathrm{d}\xi = 0.$$

Subtracting the last equation from (18) and employing that $u''_\Delta \equiv u''_{\Delta,i-1/2}$ on J_i, we obtain

$$(u - u_\Delta)(x) = \sum_{i=1}^N \int_{J_i} \mathcal{G}(x,\xi)\left[q_{i-1/2} - q(\xi)\right]\mathrm{d}\xi.$$

Clearly,

$$(u - u_\Delta)(x) = \int_0^1 \left(I_2^0 q - q\right)(\xi)\mathcal{G}(x,\xi)\,\mathrm{d}\xi + \sum_{i=1}^N \int_{J_i} \mathcal{G}(x,\xi)\left[q_{i-1/2} - \left(I_2^0 q\right)(\xi)\right]\mathrm{d}\xi.$$

A direct calculation gives $\left(I_2^0 q\right)(\xi) - q_{i-1/2} = (\xi - x_{i-1/2})R_i(\xi),\ \xi \in J_i,$ with

$$R_i(\xi) = \frac{q_i - q_{i-1}}{h_i} + 2\left(\xi - x_{i-1/2}\right)\frac{q_{i-1} - 2q_{i-1/2} + q_i}{h_i^2}.$$

Then, using (3) and a triangle inequality, we get

$$|(u - u_\Delta)(x)| \le \left\|\frac{I_2^0 q - q}{r}\right\|_\infty + \sum_{i=1}^N \left|\int_{J_i} \mathcal{G}(x,\xi)(\xi - x_{i-1/2})R_i(\xi)\,\mathrm{d}\xi\right|. \tag{19}$$

Next we derive two bounds for the summands in (19). First, a Hölder inequality gives

$$\left|\int_{J_i} \mathcal{G}(x,\xi)(\xi - x_{i-1/2})R_i(\xi)\,\mathrm{d}\xi\right| \le \frac{h_i}{2}\,\|R_i\|_{\infty,J_i} \int_{J_i} \mathcal{G}(x,\xi)\,\mathrm{d}\xi. \tag{20}$$

Second, note that

$$\xi - x_{i-1/2} = \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\frac{(\xi - x_{i-1/2})^2}{2} - \frac{h_i^2}{8}\right].$$

Integration by parts yields

$$\int_{J_i} \mathcal{G}(x,\xi)(\xi - x_{i-1/2}) R_i(\xi)\, \mathrm{d}\xi$$
$$= \int_{J_i} \left[\frac{h_i^2}{8} - \frac{(\xi - x_{i-1/2})^2}{2} \right] \left(\mathcal{G}_\xi(x,\xi) R_i(\xi) + \mathcal{G}(x,\xi) R_i'(\xi) \right) \mathrm{d}\xi.$$

Again, using a Hölder inequality, we obtain the alternative bound

$$\left| \int_{J_i} \mathcal{G}(x,\xi)(\xi - x_{i-1/2}) R_i(\xi)\, \mathrm{d}\xi \right|$$
$$\leq \frac{h_i}{8} \left\{ \| R_i \|_{\infty,J_i} \int_{J_i} \left| \mathcal{G}_\xi(x,\xi) \right| \mathrm{d}\xi + \| R_i' \|_{\infty,J_i} \int_{J_i} \mathcal{G}(x,\xi)\, \mathrm{d}\xi \right\}. \qquad (21)$$

Combine (20) and (21):

$$\left| \int_{J_i} \mathcal{G}(x,\xi)(\xi - x_{i-1/2}) R_i(\xi)\, \mathrm{d}\xi \right|$$
$$\leq \frac{h_i}{2} \| R_i \|_{\infty,J_i} \min\left\{1, \frac{h_i \varrho}{4\varepsilon}\right\} \left\{ \int_{J_i} \mathcal{G}(x,\xi)\, \mathrm{d}\xi + \frac{\varepsilon}{\varrho} \int_{J_i} \left| \mathcal{G}_\xi(x,\xi) \right| \mathrm{d}\xi \right\}$$
$$+ \| R_i' \|_{\infty,J_i} \int_{J_i} \mathcal{G}(x,\xi)\, \mathrm{d}\xi.$$

Summing for $i = 1, \ldots, N$, we have bounded the sum in (19). Then, a discrete Hölder inequality and (3) yield

$$|(u - u_\Delta)(x)| \leq \left\| \frac{I_2^0 q - q}{r} \right\|_\infty + \frac{2}{\varrho^2} \max_{i=1,\ldots,N} \frac{h_i}{2} \| R_i \|_{\infty,J_i} \min\left\{1, \frac{h_i \varrho}{4\varepsilon}\right\}$$
$$+ \frac{1}{\varrho^2} \max_{i=1,\ldots,N} \frac{h_i^2}{8} \| R_i' \|_{\infty,J_i}.$$

Finally, note that

$$R_i' \equiv 2 \frac{q_{i-1} - 2q_{i-1/2} + q_i}{h_i^2} \quad \text{and} \quad \| R_i \|_{\infty,J_i} = \max\left\{ |q_i - q_{i-1/2}|, |q_{i-1/2} - q_{i-1}| \right\}.$$

This completes the proof. □

4.4 An adaptive algorithm

Using the a posteriori estimates of the preceding section an adaptive algorithm can be devised. It is based on an idea by de Boor [4] and uses an equidistribution principle. Its convergence in connection with an error estimator for a central difference scheme was recently studied by Kopteva and Chadha [2].

The idea is to adaptively design a mesh for which the local contributions to the a posteriori error estimator

$$\mu_i(u_\Delta, \Delta) := \left\| \frac{I_2^0 q - q}{r} \right\|_{\infty, J_i} + \frac{|q_{i-1} - 2q_{i-1/2} + q_i|}{4\varrho^2}$$
$$+ \frac{2}{\varrho^2}\left[\max\left\{|q_i - q_{i-1/2}|, |q_{i-1/2} - q_{i-1}|\right\} \min\left\{1, \frac{h_i\varrho}{4\varepsilon}\right\}\right],$$
$$q = ru_\Delta - f,$$

are the same on each mesh interval, i.e., $\mu_{i-1}(u_\Delta, \Delta) = \mu_i(u_\Delta, \Delta)$, for $i = 1, \ldots, N$. This is equivalent to

$$Q_i(u_\Delta, \Delta) = \frac{1}{N}\sum_{j=1}^{N} Q_j(u_\Delta, \Delta), \quad Q_i(u_\Delta, \Delta) := \mu_i(u_\Delta, \Delta)^{1/2}. \tag{22}$$

However, de Boor's algorithm, which we are going to describe now, becomes numerically unstable when the equidistribution principle (22) is enforced strongly. Instead, we shall stop the algorithm as proposed in [2, 12] when

$$\tilde{Q}_i(u_\Delta, \Delta) \le \frac{\gamma}{N}\sum_{j=1}^{N} \tilde{Q}_j(u_\Delta, \Delta),$$

for some user chosen constant $\gamma > 1$. Here we have also modified Q_i by choosing

$$\tilde{Q}_i(u_\Delta, \Delta) := \left(h_i^2 + \mu_i(u_\Delta, \Delta)\right)^{1/2}.$$

Adding this constant floor to μ_i avoids mesh starvation and smoothes the convergence of the adaptive mesh algorithm.

Algorithm (de Boor [4])

1. Fix N, r and a constant $\gamma > 1$. The initial mesh $\Delta^{[0]}$ is uniform with mesh size $1/N$.
2. For $k = 0, 1, \ldots$, given the mesh $\Delta^{[k]}$, compute the discrete solution $u_{\Delta^{[k]}}^{[k]}$ on this mesh using the $\mathcal{S}_2^1$-collocation method. Set $h_i^{[k]} = x_i^{[k]} - x_{i-1}^{[k]}$ for each i. Compute the piecewise-constant monitor function $M^{[k]}$ defined by

$$M^{[k]}(x) := \tilde{Q}_i^{[k]} := \tilde{Q}_i\left(u_{\Delta^{[k]}}^{[k]}, \Delta^{[k]}\right) \quad \text{for} \quad x \in \left(x_{i-1}^k, x_i^k\right).$$

   The total integral of the monitor function is

$$I^{[k]} := \int_0^1 M^{[k]}(t)\,\mathrm{d}t = \sum_{j=1}^{N} h_j^{[k]} \tilde{Q}_j^{[k]}.$$

3. Test mesh: If

$$h_j^{[k]} \tilde{Q}_j^{[k]} \leq \gamma I^{[k]} N^{-1} \quad \text{for all} \quad j = 1, \ldots, N$$

then go to Step 5. Otherwise, continue to Step 4.
4. Generate a new mesh by equidistributing the monitor function $M^{[k]}$, i.e., choose the new mesh $\Delta^{[k+1]}$ such that

$$\int_{x_{i-1}^{[k+1]}}^{x_i^{[k+1]}} M^{[k]}(t)\,\mathrm{d}t = \frac{I^{[k]}}{N}, \quad i = 1, \ldots, N.$$

Return to Step 2.
5. Set $\Delta^* = \Delta^{[k]}$ and $u^*_{\Delta^*} = u^{[k]}_{\Delta^{[k]}}$ then stop.

5 Numerical experiments

We verify the theoretical results of the preceding section by applying the collocation method to the test problem

$$-\varepsilon^2 u''(x) + 4u(x) = \cos 12x, \quad x \in (0, 1), \; u(0) = u(1) = 0. \tag{23}$$

Its exact solution is easily determined.

Although the solution of (23) is available, the maximum norm of the error cannot be determined exactly. Therefore, we evaluate the error in the mesh points and in additional points evenly distributed in each mesh interval. Indicating by a superscript ε that the solution and the collocation spline depend on the perturbation parameter, we approximate the maximum-norm errors by

$$\left\| u^\varepsilon - u^\varepsilon_\Delta \right\|_\infty \approx \chi_N^\varepsilon := \max_{\substack{i=1,\ldots,N \\ m=0,\ldots,M}} \left| \left(u^\varepsilon - u^\varepsilon_\Delta\right)\left(x_{i-1} + mM^{-1}h_i\right) \right|$$

and the uniform errors by

$$\chi_N := \max_{k=0,\ldots,20} \chi_N^{10^{-k}}.$$

In our experiments we have chosen $M = 7$. Larger values will give more accurate approximations of the actual errors. However, the difference is negligible.

Table 1 contains the results of our test computations. Apart from smoothed Shishkin meshes, we also considered standard Shishkin meshes, Bakhvalov meshes and uniform meshes. The table also gives the rates of convergence. These are computed using the following formulae:

$$\tilde{p}_N := \frac{\ln \chi_N - \ln \chi_{2N}}{\ln 2} \quad \text{for the Bakhvalov mesh and the uniform mesh,}$$

$$p_N := \frac{\ln \chi_N - \ln \chi_{2N}}{\ln 2 + \ln \ln N - \ln \ln 2N} \quad \text{for the two meshes of Shishkin type.}$$

Table 1 Maximum-norm errors of the collocation method on layer-adapted meshes

N	Smoothed Shishkin mesh		Standard Shishkin mesh		Bakhvalov mesh		Uniform mesh	
	χ_N	p_N	χ_N	p_N	χ_N	$\tilde{p}_N$	χ_N	$\tilde{p}_N$
2^6	3.198e-03	2.49	2.879e-03	2.29	1.024e-04	2.07	1.574e-01	0.00
2^7	8.375e-04	2.10	8.375e-04	2.10	2.439e-05	2.03	1.575e-01	0.00
2^8	2.588e-04	2.08	2.588e-04	2.08	5.987e-06	2.01	1.575e-01	0.00
2^9	7.800e-05	2.05	7.800e-05	2.05	1.485e-06	2.01	1.574e-01	0.00
2^{10}	2.335e-05	2.03	2.335e-05	2.03	3.698e-07	2.00	1.574e-01	0.00
2^{11}	6.940e-06	2.02	6.940e-06	2.02	9.229e-08	2.00	1.574e-01	0.00
2^{12}	2.046e-06	2.01	2.046e-06	2.01	2.305e-08	2.00	1.574e-01	0.00
2^{13}	5.971e-07	2.00	5.971e-07	2.00	5.761e-09	2.00	1.574e-01	0.00
2^{14}	1.726e-07	2.00	1.726e-07	2.00	1.440e-09	2.00	1.575e-01	0.00
2^{15}	4.947e-08	2.00	4.947e-08	2.00	3.599e-10	2.00	1.575e-01	0.00
2^{16}	1.406e-08	2.00	1.406e-08	2.00	8.998e-11	2.00	1.574e-01	0.00
2^{17}	3.966e-09	2.00	3.966e-09	2.00	2.249e-11	2.00	1.574e-01	0.00
2^{18}	1.111e-09	–	1.111e-09	–	5.623e-12	–	1.574e-01	–

The first formula is standard. The second one estimating the "Shishkin rate" of convergence is motivated by our theoretical estimate $\chi_N \sim (N^{-1} \ln N)^p$ with $p = 2$. The results of our test computations for the smoothed Shishkin mesh (with $\varrho = 2$ and $\sigma = 4$) are in full agreement with Theorem 4. For comparison reasons, Table 1 also contains results for other meshes. Both the standard Shishkin mesh and the Bakhvalov mesh are constructed with the same parameter ϱ and σ, see Section 2.3. Most notably, the original and the smoothed Shishkin mesh give almost precisely the same accuracy, only for rather small N differences can be observed.

As expected, the Bakhvalov mesh outperforms the Shishkin meshes because its convergence is not spoiled by a logarithmic factor. The errors behave like $\mathcal{O}(N^{-2})$. On uniform meshes the method is not uniformly convergent.

Table 2 A posteriori-error estimates for smoothed Shishkin meshes; $\varepsilon = 10^{-12}$

N	χ_N	η	η^I	η^3	η^4	χ_N/η
2^6	3.198e-03	6.468e-02	1.662e-03	5.302e-02	1.000e-02	4.945e-02
2^7	8.375e-04	2.437e-02	2.190e-04	1.991e-02	4.238e-03	3.437e-02
2^8	2.588e-04	8.510e-03	2.730e-05	6.905e-03	1.578e-03	3.042e-02
2^9	7.800e-05	2.807e-03	3.401e-06	2.265e-03	5.386e-04	2.779e-02
2^{10}	2.335e-05	8.878e-04	4.246e-07	7.138e-04	1.736e-04	2.630e-02
2^{11}	6.940e-06	2.724e-04	5.302e-08	2.185e-04	5.381e-05	2.548e-02
2^{12}	2.046e-06	8.168e-05	6.624e-09	6.545e-05	1.623e-05	2.504e-02
2^{13}	5.971e-07	2.407e-05	8.278e-10	1.927e-05	4.797e-06	2.481e-02
2^{14}	1.726e-07	6.996e-06	1.035e-10	5.600e-06	1.397e-06	2.468e-02
2^{15}	4.947e-08	2.011e-06	1.293e-11	1.609e-06	4.017e-07	2.461e-02
2^{16}	1.406e-08	5.723e-07	1.616e-12	4.579e-07	1.144e-07	2.457e-02
2^{17}	3.966e-09	1.616e-07	2.021e-13	1.293e-07	3.231e-08	2.455e-02
2^{18}	1.111e-09	4.529e-08	2.526e-14	3.624e-08	9.058e-09	2.454e-02

Table 3 A posteriori-error estimates for smoothed Shishkin meshes, robustness of the estimator; $N = 2^{14}$

ε	χ_N	η	η^I	η^3	η^4	χ_N/η
1	1.715e-09	8.335e-08	7.159e-13	6.846e-08	1.489e-08	2.058e-02
10^{-2}	1.720e-07	6.971e-06	3.587e-12	5.580e-06	1.392e-06	2.468e-02
10^{-3}	1.726e-07	6.996e-06	8.304e-11	5.599e-06	1.397e-06	2.468e-02
10^{-4}	1.726e-07	6.996e-06	1.013e-10	5.600e-06	1.397e-06	2.468e-02
10^{-6}	1.726e-07	6.996e-06	1.034e-10	5.600e-06	1.397e-06	2.468e-02
10^{-8}	1.726e-07	6.996e-06	1.035e-10	5.600e-06	1.397e-06	2.468e-02
10^{-12}	1.726e-07	6.996e-06	1.035e-10	5.600e-06	1.397e-06	2.468e-02
10^{-16}	1.726e-07	6.996e-06	1.035e-10	5.600e-06	1.397e-06	2.468e-02
10^{-20}	1.726e-07	6.996e-06	1.035e-10	5.600e-06	1.397e-06	2.468e-02

A posteriori-error estimates according to Theorem 5 for the smoothed Shishkin mesh are displayed in Tables 2 and 3. The first column of these tables gives the value of N or ε. The actual error and the error estimates can be found in columns 2 and 3. The following three columns give the three different parts of the estimator, while the last column contains the effectivity index.

Table 2 shows results for fixed ε and various numbers of mesh points. It is observed that η^I, which captures the data oscillations behaves like $\mathcal{O}(N^{-3})$. It is the least important part of the estimator. In contrast η^3 and η^4 strongly correlate with the actual error and behave like $\mathcal{O}(N^{-2}\ln^2 N)$. In this experiment the errors are overestimated by a factor of approximately 40.

Table 3 verifies the robustness of the estimator with respect to the perturbation parameter. We have fixed N and varied ε. Similar results are observed for other layer-adapted meshes.

Table 4 gives results for uniform meshes. Again, η^I decreases like $\mathcal{O}(N^{-3})$, while η^3 and η^4 are the dominating terms. The effectivity index is greater than for the Shishkin meshes, the errors are overestimated by a factor of 3.5 only.

Table 4 The a posteriori-error estimator for a uniform mesh, $\varepsilon = 10^{-12}$

N	χ_N	η	η^I	η^3	η^4	χ_N/η
2^6	1.574e-01	5.518e-01	1.200e-05	5.000e-01	5.178e-02	2.853e-01
2^7	1.574e-01	5.518e-01	1.501e-06	5.000e-01	5.178e-02	2.853e-01
2^8	1.574e-01	5.518e-01	1.877e-07	5.000e-01	5.178e-02	2.853e-01
2^9	1.574e-01	5.518e-01	2.346e-08	5.000e-01	5.178e-02	2.853e-01
2^{10}	1.574e-01	5.518e-01	2.932e-09	5.000e-01	5.178e-02	2.853e-01
2^{11}	1.574e-01	5.518e-01	3.666e-10	5.000e-01	5.178e-02	2.853e-01
2^{12}	1.574e-01	5.518e-01	4.582e-11	5.000e-01	5.178e-02	2.853e-01
2^{13}	1.574e-01	5.518e-01	5.727e-12	5.000e-01	5.178e-02	2.853e-01
2^{14}	1.574e-01	5.518e-01	7.159e-13	5.000e-01	5.178e-02	2.853e-01
2^{15}	1.574e-01	5.518e-01	8.949e-14	5.000e-01	5.178e-02	2.853e-01
2^{16}	1.574e-01	5.518e-01	1.119e-14	5.000e-01	5.178e-02	2.853e-01
2^{17}	1.574e-01	5.518e-01	1.398e-15	5.000e-01	5.178e-02	2.853e-01
2^{18}	1.574e-01	5.518e-01	1.748e-16	5.000e-01	5.178e-02	2.853e-01

Table 5 The adaptive algorithm, $\varepsilon = 10^{-12}$

N	χ_N	$\tilde{p}_N$	η	η^I	η^3	η^4	χ_N/η	#iter
2^6	2.237e-04	0.63	2.071e-03	1.066e-04	1.667e-03	2.972e-04	1.080e-01	8
2^7	1.445e-04	2.09	1.115e-03	1.432e-05	9.878e-04	1.132e-04	1.296e-01	14
2^8	3.393e-05	2.17	1.255e-04	1.648e-06	1.093e-04	1.456e-05	2.703e-01	6
2^9	7.528e-06	2.37	3.961e-05	2.149e-07	3.563e-05	3.765e-06	1.901e-01	5
2^{10}	1.457e-06	0.80	8.961e-06	2.599e-08	8.068e-06	8.666e-07	1.626e-01	5
2^{11}	8.373e-07	3.86	2.864e-06	3.504e-09	2.602e-06	2.581e-07	2.924e-01	4
2^{12}	5.750e-08	0.27	4.942e-07	4.111e-10	4.410e-07	5.273e-08	1.164e-01	4
2^{13}	4.760e-08	3.65	1.632e-07	5.099e-11	1.481e-07	1.502e-08	2.917e-01	4
2^{14}	3.786e-09	−0.11	6.380e-08	1.227e-11	5.104e-08	1.276e-08	5.934e-02	3
2^{15}	4.095e-09	3.35	1.408e-08	8.956e-13	1.273e-08	1.348e-09	2.909e-01	3
2^{16}	4.028e-10	1.17	2.327e-09	1.030e-13	2.106e-09	2.205e-10	1.731e-01	3
2^{17}	1.796e-10	1.72	6.174e-10	1.256e-14	5.591e-10	5.833e-11	2.909e-01	3
2^{18}	5.440e-11	–	1.834e-10	1.559e-15	1.686e-10	1.479e-11	2.966e-01	3
Av. rate	1.83		1.95	3.00	1.94	2.02		

Finally, Table 5 contains the results of our test computations for the adaptive algorithm of Section 4.4. The last column gives the number of iterations required until the stopping criterion with $\gamma = 2$ is met. The bottom row displays averaged rates of convergence of the quantities in the respective column. The adaptive strategy is successful, although the actual errors are reduced in a less continuous way than observed for a priori chosen meshes. The algorithm is based on a posteriori error estimates. These upper error bounds are continuously reduced in each level of refinement. The term η^3 is the leading contribution to the error estimator.

Appendix A: B-spline basis functions

The B-spline basis for $\mathcal{S}^1_{2,0}(\Delta)$ is

$$\varphi_0(x) = \begin{cases} \dfrac{(x_1 - x)^2}{h_1^2} & \text{if } x \in [x_0, x_1], \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_1(x) = \begin{cases} \dfrac{h_1^2 - (x_1 - x)^2}{h_1^2} - \dfrac{(x - x_0)^2}{(h_1 + h_2)h_1} & \text{if } x \in [x_0, x_1], \\ \dfrac{(x_2 - x)^2}{(h_1 + h_2)h_1} & \text{if } x \in [x_1, x_2], \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 2, \ldots, N-1$:

$$\varphi_i(x) = \begin{cases} \dfrac{(x - x_{i-2})^2}{(h_{i-1} + h_i)h_{i-1}} & \text{if } x \in [x_{i-2}, x_{i-1}], \\ \dfrac{(x - x_{i-2})(x_i - x)}{(h_{i-1} + h_i)h_i} + \dfrac{(x_{i+1} - x)(x - x_{i-1})}{(h_i + h_{i+1})h_i} & \text{if } x \in [x_{i-1}, x_i], \\ \dfrac{(x_{i+1} - x)^2}{(h_i + h_{i+1})h_{i+1}} & \text{if } x \in [x_i, x_{i+1}], \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi_N(x) = \begin{cases} \dfrac{(x_{N-2} - x)^2}{(h_{N-1} + h_N)h_{N-1}} & \text{if } x \in [x_{N-2}, x_{N-1}], \\ \dfrac{h_N^2 - (x_{N-1} - x)^2}{h_N^2} - \dfrac{(x - x_N)^2}{(h_N + h_{N-1})h_N} & \text{if } x \in [x_{N-1}, x_N], \\ 0 & \text{otherwise} \end{cases}$$

and

$$\varphi_{N+1}(x) = \begin{cases} \dfrac{(x_{N-1} - x)^2}{h_N^2} & \text{if } x \in [x_{N-1}, x_N], \\ 0 & \text{otherwise.} \end{cases}$$

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