Numer Algor (2012) 60:27–50
DOI 10.1007/s11075-011-9510-5

ORIGINAL PAPER

Multistep Hermite collocation methods for solving Volterra Integral Equations

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Received: 15 March 2011 / Accepted: 28 September 2011 /
Published online: 16 October 2011

Abstract In this paper, we propose a new class of multistep collocation methods for solving nonlinear Volterra Integral Equations, based on Hermite interpolation. These methods furnish an approximation of the solution in each subinterval by using approximated values of the solution, as well as its first derivative, in the r previous steps and m collocation points. Convergence order of the new methods is determined and their linear stability is analyzed. Some numerical examples show efficiency of the methods.

Keywords Volterra Integral Equations · Multistep collocation methods · Hermite interpolation · Linear stability · Convergence

Mathematics Subject Classification (2010) 65R20

1 Introduction

Consider a nonlinear Volterra Integral Equation (VIE) of the form

$$y(t) = g(t) + \int_0^t K(t, \tau, y(\tau))d\tau, \quad t \in I := [0, T], \tag{1}$$

where $g : I \to \mathbb{R}$ is a sufficiently smooth function and $K : D \times \mathbb{R} \to \mathbb{R}$, with $D := \{(t, \tau) : \ 0 \le \tau \le t \le T\}$, is continuous and satisfies in Lipschits condition

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with respect to y. Under these assumptions (1) has a unique continuous solution [5].

One of the most practical methods for solving VIEs of the second kind is approximating the exact solution by piecewise collocation polynomial. In the literature many authors have analyzed one-step collocation methods for VIEs (see [3, 5] and references of them). As it is well known, the collocation methods are based on approximating the solution of VIE by piecewise polynomial which interpolates the exact solution of the equation in certain points which are called collocation points. Approximate solutions by collocation methods are usually discontinuous on mesh points. Note that for $m \geq 2$ the choice of $c_1 = 0$ and $c_m = 1$ yields continuous approximations where m denotes the number of collocation parameters and $c_j, \quad j = 1, 2, \cdots, m$ are collocation parameters. Some smooth piecewise collocation approximations lead to divergent collocation polynomials for VIEs of the second kind [4]. Collocation methods have uniform convergence order m for any choice of collocation parameters and local super convergence order $2m - 2$ in the mesh points (Gauss and Lobatto points) or $2m - 1$ (Radau II points) [5].

Recently Conte et al. proposed two-step collocation methods for VIEs which are obtained by using collocation technique and relaxing some collocation conditions in order to obtain good stability properties. These methods introduce continuous approximations for solution [7].

Also in [8, 9] multistep collocation methods were proposed. In these methods, the solution at each mesh point depends on the numerical solution at a fixed number of previous time steps and m collocation points in the previous subinterval. The approximate solution by multistep collocation method is continuous on the mesh points. The r step m points multistep collocation methods have uniform convergence order $m + r$ and local superconvergence order $2m + r - 1$ at the mesh points, for special choice of collocation parameters. These methods have extensive stability region, whiles there is not any A-stable method in this class.

Differentiating (1) yields the integro-differential equation

$$y'(t) = g'(t) + \int_0^t K_t(t, \tau, y(\tau))d\tau + K(t, t, y(t)), \quad t \in I, \tag{2}$$

with $K_t(t, \tau, y(\tau)) = \frac{\partial K(t,\tau,y(\tau))}{\partial t}$. In the case that the kernel does not depend on t, (2) is an ordinary differential equation with initial condition $y(0) = g(0)$ and it may be integrated by any ordinary differential equation solver. In this paper, we use both of (1) and (2), so that the approximate solution in each subinterval depends on the values of approximated solution and its first derivative in the fixed number r of previous time steps, and also the values of approximate solution and its first derivative in the m collocation points. We want to find more smooth piecewise approximations such that the approximate solution is continuously differentiable on the mesh points and some methods are A-stable. Also by this technique we can find higher order methods with extensive stability region. We show that the new methods, which we call multistep Hermite collocation methods (MHCMs) will have the uniform convergence order $2m + 2r$ for r steps and m collocation points.

This paper is organized as follows: in Section 2, we describe construction of the new methods. In Section 3, convergence order of the new method is determined. In Section 4, we analyze stability properties of the methods and in Section 5, we show performance of the methods by some examples.

2 Construction the method

Let us discretize the interval I by introducing a uniform mesh with stepsize $h := t_{n+1} - t_n$ as

$$I_h := \{t_n := nh,\ n = 0, 1, \cdots, N,\quad Nh := T\}.$$

Corresponding to this mesh, (1) can be rewritten as

$$y(t) = F_n(t) + \Phi_n(t),\quad t \in [t_n, t_{n+1}], \tag{3}$$

where

$$F_n(t) = g(t) + \int_0^{t_n} K(t, \tau, y(\tau))d\tau,$$

$$\Phi_n(t) = \int_{t_n}^{t} K(t, \tau, y(\tau))d\tau, \tag{4}$$

are called lag-term and increment function, respectively. Similarly, for (2), we have

$$y'(t) = H_n(t) + \Psi_n(t) + K(t, t, y(t)), \tag{5}$$

where

$$H_n(t) = g_t(t) + \int_0^{t_n} K_t(t, \tau, y(\tau))d\tau,$$

$$\Psi_n(t) = \int_{t_n}^{t} K_t(t, \tau, y(\tau))d\tau. \tag{6}$$

Let $0 < c_1 < c_2 < \cdots < c_m \leq 1$ be m fixed collocation parameters and $t_{n,j} := t_n + c_j h$ be collocation points. For approximating the solution of (1) restricted to the interval $[t_n, t_{n+1}]$, approximated values of the solution and its first derivative in the r previous steps and m collocation points, $t_{n,j}$, are used. Thus to construct the method, we seek a collocation polynomial of the form

$$u_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) U_{n,j} + h \sum_{k=0}^{r-1} \chi_k(s) y'_{n-k} + h \sum_{j=1}^{m} \rho_j(s) U'_{n,j}, \tag{7}$$

for $s \in [0, 1]$, $n = r, r+1, \cdots, N-1$ in such a way that the conditions

$$\begin{aligned} U_{n,j} &= u_n(t_{n,j}), \\ U'_{n,j} &= u'_n(t_{n,j}) \end{aligned} \tag{8}$$

are satisfied.

Then the approximation for $y'(t)$, in the interval $[t_n, t_{n+1}]$, is obtained from

$$hu'_n(t_n+sh) = \sum_{k=0}^{r-1} \varphi'_k(s) y_{n-k} + \sum_{j=1}^{m} \psi'_j(s) U_{n,j} + h \sum_{k=0}^{r-1} \chi'_k(s) y'_{n-k} + h \sum_{j=1}^{m} \rho'_j(s) U'_{n,j}. \tag{9}$$

The polynomials $\varphi_k(s)$, $\psi_j(s)$, $\chi_k(s)$ and $\rho_j(s)$ for $k = 0, 1, \cdots, r-1$ and $j = 1, 2, \cdots, m$ are of degree $2(m+r)-1$ and their coefficients are determined by equating both sides of (7) and (9) at the points t_{n-k} and satisfying in the conditions (8) which lead to the following interpolation conditions:

$$\begin{array}{llll} \varphi_k(-i) = \delta_{ik}, & \varphi'_k(-i) = 0, & \varphi_k(c_j) = 0 & \varphi'_k(c_j) = 0, \\ \psi_j(-i) = 0, & \psi'_j(-i) = 0, & \psi_j(c_l) = \delta_{jl}, & \psi'_j(c_l) = 0, \\ \chi_k(-i) = 0, & \chi'_k(-i) = \delta_{ik}, & \chi_k(c_j) = 0, & \chi'_k(c_j) = 0, \\ \rho_j(-i) = 0, & \rho'_j(-i) = 0, & \rho_j(c_l) = 0, & \rho'_j(c_l) = \delta_{jl}. \end{array} \tag{10}$$

So by using Hermite interpolation formula (see [15]), we have

$$\begin{aligned} \varphi_k(s) &= L_{k0}(s) - L'_{k0}(-k) L_{k1}(s), \\ \chi_k(s) &= L_{k1}(s), \quad k = 0, 1, \cdots, r-1, \end{aligned} \tag{11}$$

with

$$L_{k0}(s) = \prod_{\substack{i=0 \\ i \neq k}}^{r-1} \left(\frac{s+i}{-k+i} \right)^2 \prod_{j=1}^{m} \left(\frac{s-c_j}{-k-c_j} \right)^2$$

$$L_{k1}(s) = (s+k) L_{k0}(s),$$

and

$$\begin{aligned} \psi_j(s) &= L_{j0}(s) - L'_{j0}(c_j) L_{j1}(s), \\ \rho_j(s) &= L_{j1}(s), \quad j = 1, 2, \cdots, m, \end{aligned} \tag{12}$$

with

$$L_{j0}(s) = \prod_{i=0}^{r-1} \left(\frac{s+i}{c_j+i} \right)^2 \prod_{\substack{i=1 \\ i \neq j}}^{m} \left(\frac{s-c_i}{c_j-c_i} \right)^2,$$

$$L_{j1}(s) = (s-c_j) L_{j0}(s).$$

The exact MHCM is then obtained by imposing the collocation conditions for both (3) and (5), i.e. the collocation polynomials (7) and (9) exactly satisfy

(3) and (5) at the collocation points $t_{n,i}$, which leads to the coupled system of $2m$ equations in the unknowns $U_{n,i}$ and $U'_{n,i}$ in the form

$$\begin{cases} U_{n,i} = F_{n,i} + \Phi_{n,i}, \\ U'_{n,i} = H_{n,i} + \Psi_{n,i} + K(t_{n,i}, t_{n,i}, U_{n,i}), \end{cases} \tag{13}$$

where

$$F_{n,i} = g(t_{n,i}) + h\sum_{\nu=0}^{n-1}\int_0^1 K(t_{n,i}, t_\nu + sh, u_\nu(t_\nu + sh))ds,$$

$$\Phi_{n,i} = h\int_0^{c_i} K(t_{n,i}, t_n + sh, u_n(t_n + sh))ds,$$

$$H_{n,i} = g'(t_{n,i}) + h\sum_{\nu=0}^{n-1}\int_0^1 K_t(t_{n,i}, t_\nu + sh, u_\nu(t_\nu + sh))ds,$$

$$\Psi_{n,i} = h\int_0^{c_i} K_t(t_{n,i}, t_n + sh, u_n(t_n + sh))ds,$$

then $y_{n+1} = u_n(t_{n+1})$ and $hy'_{n+1} = hu'_{n+1}(t_{n+1})$ are computed by

$$\begin{cases} y_{n+1} = \sum_{k=0}^{r-1}\varphi_k(1)y_{n-k} + \sum_{j=1}^{m}\psi_j(1)U_{n,j} + h\sum_{k=0}^{r-1}\chi_k(1)y'_{n-k} + h\sum_{j=1}^{m}\rho_j(1)U'_{n,j}, \\ hy'_{n+1} = \sum_{k=0}^{r-1}\varphi'_k(1)y_{n-k} + \sum_{j=1}^{m}\psi'_j(1)U_{n,j} + h\sum_{k=0}^{r-1}\chi'_k(1)y'_{n-k} + h\sum_{j=1}^{m}\rho'_j(1)U'_{n,j}. \end{cases} \tag{14}$$

Also the discretized MHCM is obtained by using suitable quadrature formulas for approximating $F_{n,i}$, $\Phi_{n,i}$, $H_{n,i}$ and $\Psi_{n,i}$. The discretized multistep Hermite collocation polynomials for approximating $y(t_n + sh)$ and $y'(t_n + sh)$ take the forms

$$P_n(t_n + sh) = \sum_{k=0}^{r-1}\varphi_k(s)y_{n-k} + \sum_{j=1}^{m}\psi_j(s)Y_{n,j} + h\sum_{k=0}^{r-1}\chi_k(s)y'_{n-k} + h\sum_{j=1}^{m}\rho_j(s)Y'_{n,j} \tag{15}$$

and

$$hP'_n(t_n + sh) = \sum_{k=0}^{r-1}\varphi'_k(s)y_{n-k} + \sum_{j=1}^{m}\psi'_j(s)Y_{n,j} + h\sum_{k=0}^{r-1}\chi'_k(s)y'_{n-k} + h\sum_{j=1}^{m}\rho'_j(s)Y'_{n,j}, \tag{16}$$

where the unknowns $Y_{n,j} := P_n(t_{n,j})$ and $Y'_{n,j} := P'_n(t_{n,j})$ are determined by solving the coupled nonlinear system

$$\begin{cases} Y_{n,i} = \bar{F}_{n,i} + \bar{\Phi}_{n,i}, \\ Y'_{n,i} = \bar{H}_{n,i} + \bar{\Psi}_{n,i} + K(t_{n,i}, t_{n,i}, Y_{n,i}). \end{cases} \tag{17}$$

The approximations of $F_{n,i}$, $\Phi_{n,i}$, $H_{n,i}$ and $\Psi_{n,i}$ are of the form

$$\bar{F}_{n,i} = g(t_{n,i}) + h\sum_{\nu=0}^{n-1}\sum_{l=1}^{\mu_1} b_l K(t_{n,i}, t_\nu + \xi_l h, P_\nu(t_\nu + \xi_l h)),$$

$$\bar{\Phi}_{n,i} = h\sum_{l=1}^{\mu_0} w_{il} K(t_{n,i}, t_n + d_{il}h, P_n(t_n + d_{il}h)),$$

$$\bar{H}_{n,i} = g'(t_{n,i}) + h\sum_{\nu=0}^{n-1}\sum_{l=1}^{\mu_1} b_l K_t(t_{n,i}, t_\nu + \xi_l h, P_\nu(t_\nu + \xi_l h)),$$

$$\bar{\Psi}_{n,i} = h\sum_{l=1}^{\mu_0} w_{il} K_t(t_{n,i}, t_n + d_{il}h, P_n(t_n + d_{il}h)),$$

where $(b_l, \xi_l)_{l=1}^{\mu_1}$ and $(w_{il}, d_{il})_{l=1}^{\mu_0}$ denote quadrature weights and nodes for the intervals $[0, 1]$ and $[0, c_i]$, respectively and μ_0 and μ_1 are positive integers.

Remark 1 The discretized MHCM (15)–(17) provides a continuously differentiable approximation $P(t)$ for the solution $y(t)$ of (1) in $[0, T]$, which is given by

$$P(t)|_{[t_n,t_{n+1}]} = P_n(t).$$

We note that usually the piecewise polynomials constructed by collocation methods for VIEs do not use the numerical solution in the previous steps and give a discontinuous approximation of the solution, i.e. $u(t) \in S_{m-1}^{(-1)}(I_h)$ where

$$S_\mu^{(d)}(I_h) = \left\{v \in C^d(I) : v|_{(t_n,t_{n+1}]} \in \Pi_\mu \ \ (0 \le n \le N-1)\right\}.$$

Here, Π_μ denotes the space of polynomials of degree not exceeding μ.

The given approximated solution by the multistep collocation method uses r previous steps and does not use the derivative of the approximate solution, so it is a non smooth continuous approximation, i.e. $u(t) \in S_{m+r-1}^{(0)}(I_h)$. In this new extension the Hermite collocation polynomial is an smooth continuous approximation to the solution of (1), i.e. $u(t) \in S_{2m+2r-1}^{(1)}(I_h)$.

The idea of introduced methods which use the derivatives of approximate solution in some points has been widely developed in the context of ordinary differential equations (ODEs). In particular [6, 10, 11] deal with linear multistep methods and second derivative multistep methods. The reason of the interest in constructing new methods lies in the fact that they are high order methods with extensive stability region, as we will show in the next sections.

3 Convergence order

In this section we will analyze convergence order of the MHCMs and discretized MHCMs.

Lemma 1 *Consider the linear VIE*

$$y(t) = g(t) + \int_0^t K(t,\tau)y(\tau)d\tau, \quad t \in I := [0, T], \tag{18}$$

with $y \in \mathbb{R}$, $K \in C^p(D)$, $g \in C^p(I)$ *and* $p = 2m + 2r$. *Then for any choice of distinct collocation parameters* $0 < c_1 < c_2 < \cdots < c_m \le 1$, *the exact solution* $y(t)$ *of* (18) *satisfies*

$$y(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y(t_{n-k}) + \sum_{j=1}^{m} \psi_j(s) y(t_{n,j}) + h \sum_{k=0}^{r-1} \chi_k(s) y'(t_{n-k}) + h \sum_{j=1}^{m} \rho_j(s) y'(t_{n,j}) + h^p R_{m,r,n}(s), \quad s \in [0, 1], \tag{19}$$

where the polynomials φ_k, ψ_j, χ_k *and* ρ_j *are given in* (11) *and* (12), *and*

$$R_{m,r,n}(s) = \int_{-r+1}^{1} K_{m,r}(s,\nu) y^{(p)}(\nu) d\nu,$$

$$K_{m,r}(s,\nu) = \frac{1}{(p-1)!}\Bigg\{(s-\nu)_+^{p-1} - \sum_{k=0}^{r-1} \varphi_k(s)(-k-\nu)_+^{p-1} - \sum_{j=1}^{m} \psi_j(s)(c_j - \nu)_+^{p-1} - h(p-1)\sum_{k=0}^{r-1} \chi_k(s)(-k-\nu)_+^{p-2} - h(p-1)\sum_{j=1}^{m} \rho_j(s)(c_j-\nu)_+^{p-2}\Bigg\}.$$

Proof By the Peano theorem for interpolation [3, 12], it follows that the thesis is true for $s \in [-r+1, 1]$. □

Theorem 1 *Let* $\varepsilon(t) = y(t) - u(t)$ *and* $\varepsilon'(t) = y'(t) - u'(t)$ *be the error of exact MHCM and* $p = 2m + 2r$. *Suppose that*

(i) *the given functions in VIE* (1) *satisfy* $K \in C^p(D \times \mathbb{R})$ *and* $g \in C^p(I)$,
(ii) *the starting errors are* $\|\varepsilon\|_{\infty,[0,t_r]} = O(h^p)$ *and* $\|\varepsilon'\|_{\infty,[0,t_r]} = O(h^{p-1})$,
(iii) $\rho(\boldsymbol{H}) < 1$, *where* ρ *denotes the spectral radius and*

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & \tilde{\boldsymbol{A}} \\ \hat{\boldsymbol{A}} & \bar{\boldsymbol{A}} \end{bmatrix} \tag{20}$$

with

$$\boldsymbol{A}=\left[\begin{array}{c|c}\boldsymbol{0}_{r-1,1} & \boldsymbol{I}_{r-1}\\ \hline \varphi_{r-1}(1) & \varphi_{r-2}(1),\cdots,\varphi_0(1)\end{array}\right],\tilde{\boldsymbol{A}}=\left[\begin{array}{c|c}\boldsymbol{0}_{r-1,1} & \boldsymbol{I}_{r-1}\\ \hline \chi_{r-1}(1) & \chi_{r-2}(1),\cdots,\chi_0(1)\end{array}\right],$$

$$\hat{\boldsymbol{A}}=\left[\begin{array}{c|c}\boldsymbol{0}_{r-1,1} & \boldsymbol{I}_{r-1}\\ \hline \varphi'_{r-1}(1) & \varphi'_{r-2}(1),\cdots,\varphi'_0(1)\end{array}\right],\bar{\boldsymbol{A}}=\left[\begin{array}{c|c}\boldsymbol{0}_{r-1,1} & \boldsymbol{I}_{r-1}\\ \hline \chi'_{r-1}(1) & \chi'_{r-2}(1),\cdots,\chi'_0(1)\end{array}\right]. \tag{21}$$

Then

$$\|\varepsilon\|=O(h^p).$$

Proof We will carry out the proof in the case of linear VIE (18). The proof can be straightforwardly extended to the case of the VIE (1) by the mean value theorem.

By Lemma 1, we have

$$y(t_n+sh)=\sum_{k=0}^{r-1}\varphi_k(s)y(t_{n-k})+\sum_{j=1}^{m}\psi_j(s)y(t_{n,j})+h\sum_{k=0}^{r-1}\chi_k(s)y'(t_{n-k})$$
$$+h\sum_{j=1}^{m}\rho_j(s)y'(t_{n,j})+h^pR_{m,r,n}(s),\quad s\in[0,1]. \tag{22}$$

By subtracting (7) from (22), the error of exact MHCM, $\varepsilon(t)$, takes the local representation

$$\varepsilon(t_n+sh)=\sum_{k=0}^{r-1}\varphi_k(s)\varepsilon_{n-k}+\sum_{j=1}^{m}\psi_j(s)\varepsilon_{n,j}+h\sum_{k=0}^{r-1}\chi_k(s)\varepsilon'_{n-k}$$
$$+h\sum_{j=1}^{m}\rho_j(s)\varepsilon'_{n,j}+h^pR_{m,r,n}(s), \tag{23}$$

with $n\geq r$, $\varepsilon_{n-k}=\varepsilon(t_{n-k})$, $\varepsilon_{n,j}=\varepsilon(t_{n,j})$, $\varepsilon'_{n-k}=\varepsilon'(t_{n-k})$ and $\varepsilon'_{n,j}=\varepsilon'(t_{n,j})$. Differentiating (23) leads to

$$h\varepsilon'(t_n+sh)=\sum_{k=0}^{r-1}\varphi'_k(s)\varepsilon_{n-k}+\sum_{j=1}^{m}\psi'_j(s)\varepsilon_{n,j}+h\sum_{k=0}^{r-1}\chi'_k(s)\varepsilon'_{n-k}$$
$$+h\sum_{j=1}^{m}\rho'_j(s)\varepsilon'_{n,j}+h^pR'_{m,r,n}(s). \tag{24}$$

Replacing n by $l-1$ in (23) and (24), and $s=1$, lead to

$$\varepsilon_l^{(1)}=\mathbf{A}\varepsilon_{l-1}^{(1)}+\mathbf{S}\varepsilon_{l-1}^{(2)}+h\tilde{\mathbf{A}}\varepsilon'^{(1)}_{l-1}+h\tilde{\mathbf{S}}\varepsilon'^{(2)}_{l-1}+h^p\tilde{\boldsymbol{\rho}}_{m,r,l-1} \tag{25}$$

and

$$h\varepsilon'^{(1)}_l=\hat{\mathbf{A}}\varepsilon_{l-1}^{(1)}+\hat{\mathbf{S}}\varepsilon_{l-1}^{(2)}+h\bar{\mathbf{A}}\varepsilon'^{(1)}_{l-1}+h\bar{\mathbf{S}}\varepsilon'^{(2)}_{l-1}+h^p\bar{\boldsymbol{\rho}}_{m,r,l-1}, \tag{26}$$

where $\mathbf{A}$, $\hat{\mathbf{A}}$, $\bar{\mathbf{A}}$ and $\tilde{\mathbf{A}}$ are given in (21) and

$$\mathbf{S} = \begin{pmatrix} \mathbf{0}_{r-1,m} \\ \boldsymbol{\psi}(1)^T \end{pmatrix}, \tilde{\mathbf{S}} = \begin{pmatrix} \mathbf{0}_{r-1,m} \\ \boldsymbol{\rho}(1)^T \end{pmatrix}, \tilde{\boldsymbol{\rho}}_{m,r,j} = \begin{pmatrix} \mathbf{0}_{r-1,1} \\ R_{m,r,j}(1) \end{pmatrix},$$

$$\hat{\mathbf{S}} = \begin{pmatrix} \mathbf{0}_{r-1,m} \\ \boldsymbol{\psi}'(1)^T \end{pmatrix}, \bar{\mathbf{S}} = \begin{pmatrix} \mathbf{0}_{r-1,m} \\ \boldsymbol{\rho}'(1)^T \end{pmatrix}, \bar{\boldsymbol{\rho}}_{m,r,j} = \begin{pmatrix} \mathbf{0}_{r-1,1} \\ R'_{m,r,j}(1)^T \end{pmatrix},$$

$$\varepsilon_l^{(1)} = [\varepsilon_{l-r+1}, \cdots, \varepsilon_l]^T \in \mathbb{R}^r, \varepsilon_l^{(2)} = [\varepsilon_{l,1}, \cdots, \varepsilon_{l,m}]^T \in \mathbb{R}^m,$$

$$\varepsilon'^{(1)}_l = \left[\varepsilon'_{l-r+1}, \cdots, \varepsilon'_l\right]^T \in \mathbb{R}^r, \varepsilon'^{(2)}_l = \left[\varepsilon'_{l,1}, \cdots, \varepsilon'_{l,m}\right]^T \in \mathbb{R}^m,$$

$$\boldsymbol{\psi}(1) = [\psi_1(1), \cdots, \psi_m(1)]^T,$$

$$\boldsymbol{\psi}'(1) = [\psi'_1(1), \cdots, \psi'_m(1)]^T.$$

Combining (25) and (26) gives the following matrix equation

$$\mathcal{E}_l^{(1)} = \mathbf{H}\mathcal{E}_{l-1}^{(1)} + \mathbf{G}\mathcal{E}_{l-1}^{(2)} + \mathbf{Q}_{m,r,l-1}h^p, \tag{27}$$

where $\mathbf{H}$ is given by (20) and

$$\mathbf{G} = \begin{pmatrix} \mathbf{S} & h\tilde{\mathbf{S}} \\ \hat{\mathbf{S}} & h\bar{\mathbf{S}} \end{pmatrix}, \quad \mathbf{Q}_{m,r,j} = \begin{pmatrix} \tilde{\boldsymbol{\rho}}_{m,r,j} \\ \bar{\boldsymbol{\rho}}_{m,r,j} \end{pmatrix},$$

$$\mathcal{E}_{l-1}^{(1)} = \begin{pmatrix} \varepsilon_l^{(1)} \\ h\varepsilon'^{(1)}_l \end{pmatrix}, \mathcal{E}_{l-1}^{(2)} = \begin{pmatrix} \varepsilon_l^{(2)} \\ \varepsilon'^{(2)}_l \end{pmatrix}.$$

The solution of difference equation (27) is

$$\mathcal{E}_l^{(1)} = \mathbf{H}^{l-r+1}\mathcal{E}_{r-1}^{(1)} + \sum_{j=r-1}^{l-1} \mathbf{H}^{l-j-1}\left(\mathbf{G}\mathcal{E}_j^{(2)} + h^p\mathbf{Q}_{m,r,j}\right) \tag{28}$$

(see [13]). On the other hand, by evaluating (18) for $t = t_{n,i}$, we obtain

$$y(t_{n,i}) = g(t_{n,i}) + h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)y(t_l + sh)ds$$
$$+ h\int_0^{c_i} K(t_{n,i}, t_n + sh)y(t_n + sh)ds. \tag{29}$$

The first equation in (13), for the linear case, is equivalent to

$$u_n(t_{n,i}) = g(t_{n,i}) + h\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)u_l(t_l + sh)ds$$
$$+ h\int_0^{c_i} K(t_{n,i}, t_n + sh)u_n(t_n + sh)ds. \tag{30}$$

By subtracting (30) from (29), we get

$$\varepsilon_{n,i} = h \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh)\varepsilon(t_l + sh)ds + h \int_0^{c_i} K(t_{n,i}, t_n + sh)\varepsilon(t_n + sh)ds. \quad (31)$$

By the hypothesis on the starting errors, it follows that

$$\varepsilon(t_l + sh) = h^p q_l(s), \quad l = 0, 1, \cdots, r-1 \quad (32)$$

and

$$h\varepsilon'(t_l + sh) = h^p q_l'(s), \quad l = 0, 1, \cdots, r-1, \quad (33)$$

with $\|q_l\|_\infty \le C_1$ and $\|q_l'\|_\infty \le C_2$ independent of h. By substituting (32), (33) and (23) in (31), we obtain

$$(\mathbf{I} - hC_n^{(n)})\varepsilon_n^{(2)} - h^2 E_n^{(n)} {\varepsilon'}_n^{(2)} = h^{p+1} \sum_{l=0}^{n} \bar{R}_n^{(l)} + h \sum_{l=r}^{n-1} C_n^{(l)} \varepsilon_l^{(2)} + h \sum_{l=r}^{n} B_n^{(l)} \varepsilon_l^{(1)}$$
$$- h^2 \sum_{l=r}^{n-1} E_n^{(l)} \varepsilon_n^{(2)} + h^2 \sum_{l=r}^{n} D_n^{(l)} {\varepsilon'}_l^{(1)}, \quad (34)$$

where $\bar{R}_n^{(l)} \in \mathbb{R}^m$, $C_n^{(l)},\ E_n^{(l)} \in \mathbb{R}^{m\times m}$ and $B_n^{(l)},\ D_n^{(l)} \in \mathbb{R}^{m\times r}$ are defined as

$$\left(\bar{R}_n^{(l)}\right)_i := \begin{cases} \int_0^1 K(t_{n,i}, t_l + sh) q_l(s) ds, & l = 0, 1, \cdots, r-1, \\ \int_0^1 K(t_{n,i}, t_l + sh) R_{m,r,l}(s) ds, & l = r, \cdots, n-1, \\ \int_0^1 K(t_{n,i}, t_n + sh) R_{m,r,n}(s) ds, & l = n, \end{cases}$$

$$\left(B_n^{(l)}\right)_{ik} := \begin{cases} \int_0^1 K(t_{n,i}, t_l + sh)\varphi_k(s) ds, & l = r, \cdots, n-1, \\ \int_0^{c_i} K(t_{n,i}, t_n + sh)\varphi_k(s) ds & l = n, \end{cases}$$

$$\left(C_n^{(l)}\right)_{ij} := \begin{cases} \int_0^1 K(t_{n,i}, t_l + sh)\psi_j(s) ds, & l = r, \cdots, n-1, \\ \int_0^{c_i} K(t_{n,i}, t_n + sh)\psi_j(s) ds, & l = n, \end{cases}$$

$$\left(D_n^{(l)}\right)_{ik} := \begin{cases} \int_0^1 K(t_{n,i}, t_l + sh)\chi_k(s) ds, & l = r, \cdots, n-1, \\ \int_0^{c_i} K(t_{n,i}, t_n + sh)\chi_k(s) ds, & l = n, \end{cases}$$

$$\left(E_n^{(l)}\right)_{ij} := \begin{cases} \int_0^1 K(t_{n,i}, t_l + sh)\rho_j(s) ds, & l = r, \cdots, n-1, \\ \int_0^{c_i} K(t_{n,i}, t_n + sh)\rho_j(s) ds, & l = n. \end{cases}$$

Similarly we have

$$\varepsilon'_{n,i} = h\sum_{l=0}^{n-1}\int_0^1 K_t(t_{n,i}, t_l + sh)\varepsilon(t_l + sh)ds + K(t_{n,i}, t_{n,i})\varepsilon_{n,i}$$
$$+ h\int_0^{c_i} K_t(t_{n,i}, t_n + sh)\varepsilon(t_n + sh)ds \tag{35}$$

for the approximation error of $y'(t)$. By substituting (32), (33) and (23) in (35), we have

$$\left(I - h^2\tilde{E}_n^{(n)}\right)\varepsilon'^{(2)}_n - \left(K(t_{n,i}, t_{n,i}) + h\tilde{C}_n^{(n)}\right)\varepsilon_n^{(2)}$$
$$= h^{p+1}\sum_{l=0}^{n}\tilde{R}_n^{(l)} + h\sum_{l=r}^{n-1}\tilde{C}_n^{(l)}\varepsilon_l^{(2)} + h\sum_{l=r}^{n}\tilde{B}_n^{(l)}\varepsilon_l^{(1)} + h^2\sum_{l=r}^{n-1}\tilde{E}_n^{(l)}\varepsilon'^{(2)}_l$$
$$+ h^2\sum_{l=r}^{n}\tilde{D}_n^{(l)}\varepsilon'^{(1)}_l \tag{36}$$

with

$$\left(\tilde{R}_n^{(l)}\right)_i := \begin{cases} \int_0^1 K_t(t_{n,i}, t_l + sh)q_l(s)ds, & l = 0, 1, \cdots, r-1, \\ \int_0^1 K_t(t_{n,i}, t_l + sh)R_{m,r,l}(s)ds, & l = r, \cdots, n-1, \\ \int_0^1 K_t(t_{n,i}, t_n + sh)R_{m,r,n}(s)ds, & l = n, \end{cases}$$

$$\left(\tilde{B}_n^{(l)}\right)_{ik} := \begin{cases} \int_0^1 K_t(t_{n,i}, t_l + sh)\varphi_k(s)ds, & l = r, \cdots, n-1, \\ \int_0^{c_i} K_t(t_{n,i}, t_n + sh)\varphi_k(s)ds & l = n, \end{cases}$$

$$\left(\tilde{C}_n^{(l)}\right)_{ij} := \begin{cases} \int_0^1 K_t(t_{n,i}, t_l + sh)\psi_j(s)ds, & l = r, \cdots, n-1, \\ \int_0^{c_i} K_t(t_{n,i}, t_n + sh)\psi_j(s)ds, & l = n, \end{cases}$$

$$\left(\tilde{D}_n^{(l)}\right)_{ik} := \begin{cases} \int_0^1 K_t(t_{n,i}, t_l + sh)\chi_k(s)ds, & l = r, \cdots, n-1, \\ \int_0^{c_i} K_t(t_{n,i}, t_n + sh)\chi_k(s)ds, & l = n, \end{cases}$$

$$\left(\tilde{E}_n^{(l)}\right)_{ij} := \begin{cases} \int_0^1 K_t(t_{n,i}, t_l + sh)\rho_j(s)ds, & l = r, \cdots, n-1, \\ \int_0^{c_i} K_t(t_{n,i}, t_n + sh)\rho_j(s)ds, & l = n. \end{cases}$$

Combination of (34) and (36) leads to the following matrix equation

$$W_n^{(n)}\mathcal{E}_n^{(2)} = h\sum_{l=r}^{n}\bar{W}_n^{(l)}\mathcal{E}_l^{(1)} + h\sum_{l=r}^{n-1}\hat{W}_n^{(l)}\mathcal{E}_l^{(2)} + h^{p+1}\sum_{l=0}^{n}\hat{R}_n^{(l)}, \tag{37}$$

where

$$W_n^{(n)} = \begin{bmatrix}\mathbf{I}-hC_n^{(n)} & -h^2E_n^{(n)}\\ -K(t_{n,i},t_{n,i})+h\tilde{C}_n^{(n)} & \mathbf{I}-h^2\tilde{E}_n^{(n)}\end{bmatrix}, \quad \bar{W}_n^{(l)} = \begin{bmatrix}B_n^{(l)} & D_n^{(l)}\\ \tilde{B}_n^{(l)} & \tilde{D}_n^{(l)}\end{bmatrix},$$

$$\hat{W}_n^{(l)} = \begin{bmatrix}C_n^{(l)} & hE_n^{(l)}\\ \tilde{C}_n^{(l)} & h\tilde{E}_n^{(l)}\end{bmatrix}, \quad \hat{R}_n^{(l)} = \begin{bmatrix}\bar{R}_n^{(l)}\\ \tilde{R}_n^{(l)}\end{bmatrix}.$$

Substituting (28) in (37) yields

$$\begin{aligned}W_n^{(n)}\mathcal{E}_n^{(2)} &= h\sum_{l=r}^{n}\bar{W}_n^{l}\mathbf{H}^{l-r+1}\mathcal{E}_{r-1}^{(1)} + h\sum_{j=r}^{n-1}\sum_{l=j+1}^{n}\bar{W}_n^{l}\mathbf{H}^{l-j-1}\mathbf{G}\mathcal{E}_j^{(2)}\\ &\quad + h\sum_{l=r}^{n}\bar{W}_n^{(l)}\mathbf{H}^{l-r}\mathbf{G}\mathcal{E}_{r-1}^{(2)} + h\sum_{l=r}^{n-1}\hat{W}_n^{(l)}\mathcal{E}_l^{(2)}\\ &\quad + h^{p+1}\sum_{j=r-1}^{n-1}\sum_{l=j+1}^{n}\bar{W}_n^{(l)}\mathbf{H}^{l-j-1}\mathbf{Q}_{m,r,j} + h^{p+1}\sum_{l=0}^{n}\hat{R}_n^{(l)}, \quad n\ge r. \end{aligned}\tag{38}$$

Now a bound for $\mathcal{E}_n^{(2)}$ can be found by the same way as described in [9] (Theorem 4.2) that leads to the estimate

$$\|\mathcal{E}_n^{(2)}\| \le M_2h^p,$$

then from (28) a bound for $\|\mathcal{E}_n^{(1)}\|$ obtained in the form

$$\|\mathcal{E}_n^{(1)}\| \le M_1h^p.$$

Note that the coefficients M_1 and M_2 depend on the bounds of the matrices in (38). Using the local error representation (23) and two above inequalities together to the expression (32) for the starting errors, complete the proof. □

Now we state similar discussion for discretized MHCMs by the following theorem.

Theorem 2 *Let $e(t) = y(t) - P(t)$ be the error of discretized MHCM and $p = 2m+2r$. Suppose that*

(i) $K \in C^p(D\times\mathbb{R})$ *and* $g \in C^p(I)$,
(ii) *The quadrature formulas for approximating $F_{n,i}$, $\Phi_{n,i}$, $H_{n,i}$ and $\Psi_{n,i}$ are of orders $p+1$, p, $p+1$ and p, respectively,*
(iii) *The starting errors are $\|e\|_{\infty,[0,t_r]} = O(h^p)$ and $\|e'\|_{\infty,[0,t_r]} = O(h^{p-1})$,*
(iv) $\rho(\boldsymbol{H}) < 1$, *where H is given by (20).*

Then

$$\|e\|_\infty = O\left(h^{2m+2r}\right).$$

Proof The result is obtained from Theorem 1, the inequality

$$\|e\|_\infty \le \|\varepsilon\|_\infty + \|u - P\|_\infty$$

and order of quadrature formulas. □

4 Linear stability

In this section, we analyze the stability properties of exact and discretized MHCMs. The stability behavior of a numerical method for (1) is usually analyzed by applying the method with a fixed positive stepsize h to the basic test equation (see [1, 2])

$$y(t) = 1 + \lambda \int_0^t y(\tau)d\tau, \quad t \in [0, T], \quad Re(\lambda) < 0. \tag{39}$$

This test equation is equivalent to the ODE test equation $y' = \lambda y$.

Definition 1 Absolute stability region of the method is the set of all $z := \lambda h \in \mathbb{C}$, such that the numerical solution y_n of test equation (39) with a constant stepsize h, tends to zero as $n \to \infty$. The method is said to be A-stable if its absolute stability region includes the negative complex half plane $\mathbb{C}^-$.

To state the main results of stability properties of the new method, let us define

$$\Omega_{ik} = \int_0^{c_i} \varphi_k(s)ds, \quad \Gamma_{ij} = \int_0^{c_i} \psi_j(s)ds,$$

$$\Delta_{ik} = \int_0^{c_i} \chi_k(s)ds, \quad \Lambda_{ij} = \int_0^{c_i} \rho_j(s)ds,$$

$$\alpha_k = \int_0^1 \varphi_k(s)ds, \quad \beta_j = \int_0^1 \psi_j(s)ds,$$

$$\gamma_k = \int_0^1 \chi_k(s)ds, \quad \eta_j = \int_0^1 \rho_j(s)ds,$$

and introduce the vectors and matrices

$$\mathbf{U}_n = [U_{n,1}, \cdots, U_{n,m}]^T, \quad \mathbf{u} = [1, \cdots, 1]^T \in \mathbb{R}^m,$$

$$\mathbf{y}_n^{(r)} = [y_n, \cdots, y_{n-r+1}]^T, \quad \mathbf{y'}_n^{(r)} = \left[y'_n, \cdots, y'_{n-r+1}\right]^T,$$

$$\boldsymbol{\varphi}(1) = [\varphi_0(1), \cdots, \varphi_{r-1}(1)]^T, \quad \boldsymbol{\psi}(1) = [\psi_1(1), \cdots, \psi_m(1)]^T,$$

$$\boldsymbol{\chi}(1) = [\chi_0(1), \cdots, \chi_{r-1}]^T, \quad \boldsymbol{\rho}(1) = [\rho_1(1), \cdots, \rho_m(1)]^T,$$

$$\boldsymbol{\varphi'}(1) = \left[\varphi'_0(1), \cdots, \varphi'_{r-1}(1)\right]^T, \quad \boldsymbol{\psi'}(1) = \left[\psi'_1(1), \cdots, \psi'_m(1)\right]^T,$$

$$\boldsymbol{\chi'}(1) = \left[\chi'_0(1), \cdots, \chi'_{r-1}\right]^T, \quad \boldsymbol{\rho'}(1) = \left[\rho'_1(1), \cdots, \rho'_m(1)\right]^T,$$

$$\mathbf{E}_1 = \begin{bmatrix} -(\boldsymbol{\psi}(1)^T + z\boldsymbol{\rho}(1)^T) \\ \mathbf{0}_{r,1} \end{bmatrix}, \mathbf{E}_2 = \begin{bmatrix} -(\boldsymbol{\psi'}(1)^T + z\boldsymbol{\rho'}(1)^T) \\ \mathbf{0}_{r,1} \end{bmatrix}, \mathbf{F} = \begin{bmatrix} \mathbf{0}_{1,r} & 0 \\ \mathbf{I}_r & \mathbf{0}_{r,1} \end{bmatrix},$$
$$\mathbf{G}_{2,1} = \begin{bmatrix} 1 & -\boldsymbol{\varphi}(1)^T \\ \mathbf{0}_{r,1} & \mathbf{I}_r \end{bmatrix}, \quad \mathbf{G}_{2,2} = \begin{bmatrix} 0 & -\boldsymbol{\chi}(1)^T \\ \mathbf{0}_{r,1} & \mathbf{0}_{r,r} \end{bmatrix},$$
$$\mathbf{G}_{3,1} = \begin{bmatrix} 0 & -\boldsymbol{\varphi'}(1)^T \\ \mathbf{0}_{r,1} & \mathbf{0}_{r,r} \end{bmatrix}, \quad \mathbf{G}_{3,2} = \begin{bmatrix} 1 & -\boldsymbol{\chi'}(1)^T \\ \mathbf{0}_{r,1} & \mathbf{I}_r \end{bmatrix}.$$

Theorem 3 *The exact MHCM, applied to the test equation (39) leads to the following recurrence relation*

$$\begin{bmatrix} y_{n+1} \\ \boldsymbol{y}_n^{(r)} \\ hy'_{n+1} \\ h\boldsymbol{y'}_n^{(r)} \\ \boldsymbol{U}_n \end{bmatrix} = R(z) \begin{bmatrix} y_n \\ \boldsymbol{y}_{n-1}^{(r)} \\ hy'_n \\ h\boldsymbol{y'}_{n-1}^{(r)} \\ \boldsymbol{U}_{n-1} \end{bmatrix}, \tag{40}$$

where $R(z)$ is the stability matrix and it is given by

$$R(z) = [Q(z)]^{-1} M(z),$$

with

$$Q(z) = \left[\begin{array}{c|c|c|c|c} \boldsymbol{0}_{m,1} & -z\boldsymbol{\Omega} & \boldsymbol{0}_{m,1} & -z\boldsymbol{\Delta} & \boldsymbol{I}_m - z(\boldsymbol{\Gamma} + z\boldsymbol{\Lambda}) \\ \hline \multicolumn{2}{c|}{\boldsymbol{G}_{2,1}} & \multicolumn{2}{c|}{\boldsymbol{G}_{2,2}} & \boldsymbol{E}_1 \\ \hline \multicolumn{2}{c|}{\boldsymbol{G}_{3,1}} & \multicolumn{2}{c|}{\boldsymbol{G}_{3,2}} & \boldsymbol{E}_2 \end{array}\right],$$

$$M(z) = \left[\begin{array}{c|c|c|c|c} \boldsymbol{0}_{m,1} & z(\boldsymbol{u}\boldsymbol{\alpha}^T - \boldsymbol{\Omega}) & \boldsymbol{0}_{m,1} & z(\boldsymbol{u}\boldsymbol{\gamma}^T - \boldsymbol{\Delta}) & M_1 \\ \hline \multicolumn{2}{c|}{\boldsymbol{F}} & \multicolumn{2}{c|}{\boldsymbol{0}_{r+1,r+1}} & \boldsymbol{0}_{r+1,m} \\ \hline \multicolumn{2}{c|}{\boldsymbol{0}_{r+1,r+1}} & \multicolumn{2}{c|}{\boldsymbol{F}} & \boldsymbol{0}_{r+1,m} \end{array}\right],$$

and

$$M_1 = \boldsymbol{I}_m + z\boldsymbol{u}\left(\boldsymbol{\beta}^T + z\boldsymbol{\eta}^T\right) - z(\boldsymbol{\Gamma} + z\boldsymbol{\Lambda}).$$

Proof Second equation in (13), applied to test equation (39), yields

$$U'_{n,i} = \lambda U_{n,i}. \tag{41}$$

Now from (41) and applying (14) to the test equation (39), we obtain

$$y_{n+1} = \boldsymbol{\varphi}(1)^T \mathbf{y}_n^{(r)} + \boldsymbol{\chi}(1)^T h\mathbf{y}'^{(r)}_n + (\boldsymbol{\psi}(1)^T + z\boldsymbol{\rho}(1)^T)\mathbf{U}_n \tag{42}$$

and

$$hy'_{n+1} = \boldsymbol{\varphi}'(1)^T \mathbf{y}_n^{(r)} + \boldsymbol{\chi}'(1)^T h\mathbf{y}'^{(r)}_n + (\boldsymbol{\psi}'(1)^T + z\boldsymbol{\rho}'(1)^T)\mathbf{U}_n \tag{43}$$

which can be written in the following matrices forms

$$\mathbf{G}_{2,1}\begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \end{bmatrix} + \mathbf{G}_{2,2}\begin{bmatrix} hy'_{n+1} \\ h\mathbf{y}'^{(r)}_n \end{bmatrix} + \mathbf{E}_1\mathbf{U}_n = \mathbf{F}\begin{bmatrix} y_n \\ \mathbf{y}_{n-1}^{(r)} \end{bmatrix} \tag{44}$$

and

$$\mathbf{G}_{3,1}\begin{bmatrix} y_{n+1} \\ \mathbf{y}_n^{(r)} \end{bmatrix} + \mathbf{G}_{3,2}\begin{bmatrix} hy'_{n+1} \\ h\mathbf{y}'^{(r)}_n \end{bmatrix} + \mathbf{E}_2\mathbf{U}_n = \mathbf{F}\begin{bmatrix} hy'_n \\ h\mathbf{y}'^{(r)}_{n-1} \end{bmatrix}. \tag{45}$$

Now, with the purpose of finding a recurrence relation, we compute $\mathbf{U}_n - \mathbf{U}_{n-1}$, to get

$$\mathbf{U}_n - \mathbf{U}_{n-1} = \bar{\mathbf{F}}_n - \bar{\mathbf{F}}_{n-1} + \boldsymbol{\Phi}_n - \boldsymbol{\Phi}_{n-1}$$

with

$$\begin{aligned}
&\bar{\mathbf{F}}_n - \bar{\mathbf{F}}_{n-1} \\
&\quad = \lambda h \int_0^1 u_{n-1}(t_{n-1} + sh)ds \\
&\quad = z\left(\sum_{k=0}^{r-1} \alpha_k y_{n-1-k} + \sum_{j=1}^{m} \beta_j U_{n-1,j} + h\sum_{k=0}^{r-1} \gamma_k y'_{n-1-k} + z\sum_{j=1}^{m} \eta_j U_{n-1,j}\right) \\
&\quad = z\mathbf{u}\left(\boldsymbol{\alpha}^T \mathbf{y}_{n-1}^{(r)} + h\boldsymbol{\gamma}^T \mathbf{y}'^{(r)}_{n-1} + (\boldsymbol{\beta}^T + z\boldsymbol{\eta}^T)\mathbf{U}_{n-1}\right).
\end{aligned} \tag{46}$$

On the other hand, we have from (4)

$$\begin{aligned}
\Phi_{n,i} := \Phi_n(t_{n,i}) = \lambda h \int_0^{c_i} u_n(t_n + sh)ds \\
= z\left(\sum_{k=0}^{r-1} \Omega_{ik} y_{n-k} + \sum_{j=1}^{m} \Gamma_{ij} U_{n,j} + h\sum_{k=0}^{r-1} \Delta_{ik} y'_{n-k} + z\sum_{j=1}^{m} \Lambda_{ij} U_{n,j}\right).
\end{aligned}$$

Therefore

$$\boldsymbol{\Phi}_n = z\boldsymbol{\Omega}\mathbf{y}_n^{(r)} + hz\boldsymbol{\Delta}\mathbf{y}'^{(r)}_n + z(\boldsymbol{\Gamma} + z\boldsymbol{\Lambda})\mathbf{U}_n$$

and

$$\mathbf{\Phi}_n-\mathbf{\Phi}_{n-1}=z\mathbf{\Omega}\left(\mathbf{y}_n^{(r)}-\mathbf{y}_{n-1}^{(r)}\right)+z\mathbf{\Delta}\left(h\mathbf{y}'^{(r)}_n-h\mathbf{y}'^{(r)}_{n-1}\right)+z(\mathbf{\Gamma}+z\mathbf{\Lambda})(\mathbf{U}_n-\mathbf{U}_{n-1}). \tag{47}$$

Now computation of difference $\mathbf{U}_n-\mathbf{U}_{n-1}$, by substituting (46) and (47), leads to

$$\begin{aligned}(\mathbf{I}_m &- z(\mathbf{\Gamma}+z\mathbf{\Lambda}))\mathbf{U}_n - z\mathbf{\Omega}\mathbf{y}_n^{(r)} - z\mathbf{\Delta}h\mathbf{y}'^{(r)}_n\\ &= (\mathbf{I}_m + z\mathbf{u}(\boldsymbol{\beta}^T + z\boldsymbol{\eta}^T) - z(\mathbf{\Gamma}+z\mathbf{\Lambda}))\mathbf{U}_{n-1} + z(\mathbf{u}\boldsymbol{\alpha}^T-\mathbf{\Omega})\mathbf{y}_{n-1}^{(r)}\\ &\quad + z(\mathbf{u}\boldsymbol{\gamma}^T-\mathbf{\Delta})h\mathbf{y}'^{(r)}_{n-1}.\end{aligned} \tag{48}$$

Finally from (44), (45) and (48), we obtain

$$Q(z)\begin{bmatrix} y_{n+1}\\ \mathbf{y}_n^{(r)}\\ hy'_{n+1}\\ h\mathbf{y}'^{(r)}_n\\ \mathbf{U}_n\end{bmatrix} = M(z)\begin{bmatrix} y_n\\ \mathbf{y}_{n-1}^{(r)}\\ hy'_n\\ h\mathbf{y}'^{(r)}_{n-1}\\ \mathbf{U}_{n-1}\end{bmatrix}$$

which is equivalent to (52). □

Remark 2 In the case $c_m=1$, from the interpolation conditions (10), we have

$$\varphi'_k(1)=\psi'_j(1)=\chi'_k(1)=0,$$
$$\rho'_j(1)=0,\quad j=1,2,\cdots,m-1,\quad \rho'_m(1)=1.$$

So from (14), we have $y'_{n+1}=U'_{n,m}=\lambda U_{n,m}=\lambda y_{n+1}$. Thus the relation (42) is equivalent to

$$y_{n+1}=\left(\boldsymbol{\varphi}(1)^T+z\boldsymbol{\chi}(1)^T\right)\mathbf{y}_n^{(r)}+\left(\boldsymbol{\psi}(1)^T+z\boldsymbol{\rho}(1)^T\right)\mathbf{U}_n \tag{49}$$

and can be written in the matrix form

$$\mathbf{G}\begin{bmatrix} y_{n+1}\\ \mathbf{y}_n^{(r)}\end{bmatrix}+\mathbf{E}_1\mathbf{U}_n=\mathbf{F}\begin{bmatrix} y_n\\ \mathbf{y}_{n-1}^{(r)}\end{bmatrix}, \tag{50}$$

where $\mathbf{G}=\mathbf{G}_{2,1}+z\mathbf{G}_{2,2}$. Also (48) can be written in the form

$$\begin{aligned}(\mathbf{I}_m &- z(\mathbf{\Gamma}+z\mathbf{\Lambda}))\mathbf{U}_n - z(\mathbf{\Omega}+z\mathbf{\Delta})\mathbf{y}_n^{(r)}\\ &= (\mathbf{I}_m + z(\mathbf{u}\boldsymbol{\beta}^T + z\mathbf{u}\boldsymbol{\eta}^T) - z(\mathbf{\Gamma}+z\mathbf{\Lambda}))\mathbf{U}_{n-1} + (z(\mathbf{u}\boldsymbol{\alpha}^T+z\mathbf{u}\boldsymbol{\gamma}^T)\\ &\quad - z(\mathbf{\Omega}+z\mathbf{\Delta}))\mathbf{y}_{n-1}^{(r)}.\end{aligned} \tag{51}$$

Therefore the exact MHCM applied to the test equation (39) leads to the following recurrence relation

$$\begin{bmatrix} y_{n+1}\\ \mathbf{y}_n^{(r)}\\ \mathbf{U}_n\end{bmatrix}=R(z)\begin{bmatrix} y_n\\ \mathbf{y}_{n-1}^{(r)}\\ \mathbf{U}_{n-1}\end{bmatrix}, \tag{52}$$

where $R(z)$ is the stability matrix and it is given by

$$R(z) = [Q(z)]^{-1} M(z),$$

with

$$Q(z) = \left[\begin{array}{c|c|c} \mathbf{0}_{m,1} & -z(\boldsymbol{\Omega} + z\boldsymbol{\Delta}) & \mathbf{I}_m - z(\boldsymbol{\Gamma} + z\boldsymbol{\Lambda}) \\ \hline \multicolumn{2}{c|}{\mathbf{G}} & \mathbf{E}_1 \end{array}\right],$$

$$M(z) = \left[\begin{array}{c|c|c} \mathbf{0}_{m,1} & M_1 & M_2 \\ \hline \multicolumn{2}{c|}{\mathbf{F}} & \mathbf{0}_{r+1,m} \end{array}\right],$$

where

$$M_1 = z\mathbf{u}\left(\boldsymbol{\alpha}^T + z\boldsymbol{\gamma}^T\right) - z(\boldsymbol{\Omega} + z\boldsymbol{\Delta}),$$
$$M_2 = \mathbf{I}_m + z\mathbf{u}\left(\boldsymbol{\beta}^T + z\boldsymbol{\eta}^T\right) - z(\boldsymbol{\Gamma} + z\boldsymbol{\Lambda}).$$

Thus in the case of methods having $c_m = 1$, the stability matrix $R(z)$ has the smaller dimension $m + r + 1$ instead of $m + 2r + 2$.

Let us define

$$\tilde{\Omega}_{ik} = \sum_{l=1}^{\mu_0} w_{il}\varphi_k(d_{il}),\ \tilde{\Gamma}_{ij} = \sum_{l=1}^{\mu_0} w_{il}\psi_j(d_{il}),$$

$$\tilde{\Delta}_{ik} = \sum_{l=1}^{\mu_0} w_{il}\chi_k(d_{il}),\ \tilde{\lambda}_{ij} = \sum_{l=1}^{\mu_0} w_{il}\rho_j(d_{il}),$$

$$\tilde{\alpha}_k = \sum_{l=1}^{\mu_1} b_l\varphi_k(\xi_l), \quad \tilde{\beta}_j = \sum_{l=1}^{\mu_1} b_l\psi_j(\xi_l),$$

$$\tilde{\gamma}_k = \sum_{l=1}^{\mu_1} b_l\chi_k(\xi_l), \quad \tilde{\eta}_j = \sum_{l=1}^{\mu_1} b_l\rho_j(\xi_l)$$

and introduce the vectors and matrices

$$\mathbf{Y} = [Y_{n,1}, \cdots, Y_{n,m}]^T,$$

$$\tilde{\boldsymbol{\alpha}} = [\tilde{\alpha}_0, \cdots, \tilde{\alpha}_{r-1}]^T, \quad \tilde{\boldsymbol{\beta}} = [\tilde{\beta}_1, \cdots, \tilde{\beta}_m]^T,$$

$$\tilde{\boldsymbol{\gamma}} = [\tilde{\gamma}_0, \cdots, \tilde{\gamma}_{r-1}]^T, \quad \tilde{\boldsymbol{\eta}} = [\tilde{\eta}_1, \cdots, \tilde{\eta}_m]^T,$$

$$\tilde{\boldsymbol{\Omega}} = (\tilde{\Omega}_{ik}) \in \mathbb{R}^{m\times r}, \quad \tilde{\boldsymbol{\Gamma}} = (\tilde{\Gamma}_{ij}) \in \mathbb{R}^{m\times m}, \quad \tilde{\boldsymbol{\Delta}} = (\tilde{\Delta}_{ik}) \in \mathbb{R}^{m\times r}, \quad \tilde{\boldsymbol{\Lambda}} = (\tilde{\Lambda}_{ij}) \in \mathbb{R}^{m\times m}.$$

Then:

Theorem 4 *The discretized MHCM, applied to test equation (39), leads to the following recurrence relation*

$$\begin{bmatrix} y_{n+1} \\ \boldsymbol{y}_n^{(r)} \\ hy'_{n+1} \\ h\boldsymbol{y}'^{(r)}_n \\ \boldsymbol{Y}_n \end{bmatrix} = \tilde{R}(z) \begin{bmatrix} y_n \\ \boldsymbol{y}_{n-1}^{(r)} \\ hy'_n \\ h\boldsymbol{y}'^{(r)}_{n-1} \\ \boldsymbol{Y}_{n-1} \end{bmatrix}, \tag{53}$$

where the stability matrix is given by

$$\tilde{R}(z) = \left[\tilde{Q}(z)\right]^{-1} \tilde{M}(z),$$

with

$$\tilde{Q}(z) = \left[\begin{array}{c|c|c|c|c} \boldsymbol{0}_{m,1} & -z\tilde{\boldsymbol{\Omega}} & \boldsymbol{0}_{m,1} & -z\tilde{\boldsymbol{\Delta}} & \boldsymbol{I}_m - z(\tilde{\boldsymbol{\Gamma}} + z\tilde{\boldsymbol{\Lambda}}) \\ \hline \multicolumn{2}{c|}{\boldsymbol{G}_{2,1}} & \multicolumn{2}{c|}{\boldsymbol{G}_{2,2}} & \boldsymbol{E}_1 \\ \hline \multicolumn{2}{c|}{\boldsymbol{G}_{3,1}} & \multicolumn{2}{c|}{\boldsymbol{G}_{3,2}} & \boldsymbol{E}_2 \end{array}\right],$$

$$\tilde{M}(z) = \left[\begin{array}{c|c|c|c|c} \boldsymbol{0}_{m,1} & z(\boldsymbol{u}\tilde{\boldsymbol{\alpha}}^T - \tilde{\boldsymbol{\Omega}}) & \boldsymbol{0}_{m,1} & z(\boldsymbol{u}\tilde{\boldsymbol{\gamma}}^T - \tilde{\boldsymbol{\Delta}}) & \tilde{M}_1 \\ \hline \multicolumn{2}{c|}{\boldsymbol{F}} & \multicolumn{2}{c|}{\boldsymbol{0}_{r+1,r+1}} & \boldsymbol{0}_{r+1,m} \\ \hline \multicolumn{2}{c|}{\boldsymbol{0}_{r+1,r+1}} & \multicolumn{2}{c|}{\boldsymbol{F}} & \boldsymbol{0}_{r+1,m} \end{array}\right],$$

and

$$\tilde{M}_1 = \boldsymbol{I}_m + z\boldsymbol{u}\left(\tilde{\boldsymbol{\beta}}^T + z\tilde{\boldsymbol{\eta}}^T\right) - z\left(\tilde{\boldsymbol{\Gamma}} + z\tilde{\boldsymbol{\Lambda}}\right).$$

Proof The proof is analogous to that of Theorem 3. □

The stability function of the methods with respect to (39) is defined as

$$p(w, z) = \det(w\mathbf{I}_{m+2r+2} - R(z)). \tag{54}$$

To investigate the stability properties of the exact MHCM, it is more convenient to work with the polynomial obtained by multiplying the stability function (54) by its denominator. The resulting polynomial which will be denoted by the same symbol $p(w, z)$, takes the following form

$$p(w, z) = \sum_{i=0}^{m+2r+2} p_i(z) w^i, \tag{55}$$

where $p_i(z)$, $i = 0, 1, \cdots, m + 2r + 2$, are polynomials of degree less than or equal to $2m$. Denoting by $w_1, w_2, \cdots, w_{m+2r+2}$, the roots of the polynomial $p(w, z)$, the region of absolute stability of the methods is defined by

$$\mathcal{S} := \{z \in \mathbb{C} : |w_i(z)| < 1, \ \ i = 1, 2, \cdots, m + 2r + 2\}.$$

To obtain this region we use the boundary locus method [14]. Inserting $w = e^{i\theta}$, the roots of (55) describe the stability region.

4.1 Examples of MHCMs with 2 steps and 1 stage

Consider the MHCMs with 2 steps and 1 collocation parameter c_1. The stability polynomial of this family of methods assumes the form

$$p(w, z) = w^2\big(p_5(z)w^5 + p_4(z)w^4 + \cdots + p_0(z)\big),$$

where $p_i(z)$, $i = 0, 1, \cdots, 5$ are polynomials of degree less than or equal to 2. Performing a computer search based on the boundary locus method, shows that this family of methods are A-stable of order 6 for $c_1 \in [0.22, 0.39] \cup [0.67, 0.72]$.

4.2 Examples of MHCMs with 2 steps and 2 stages

Consider MHCMs with 3 steps and 2 collocation parameters c_1 and c_2. The stability polynomial for this family of methods is of the form

$$p(w, z) = w^2(p_6(z)w^6 + p_5(z)w^5 + \cdots + p_0(z)),$$

where $p_i(z)$, $i = 0, 1, \cdots, 6$ are polynomials of degree less than or equal to 4. Performing an extensive computer search based on the boundary locus method, we obtain A-stable methods of order 8 when both collocation parameters are within the region reported in Fig. 1.

4.3 Examples of MHCMs with 3 steps and 1 stage

Consider the MHCMs with 3 steps and 1 collocation parameter c_1. The stability polynomial of this family of methods assumes the form

$$p(w, z) = w^2\big(p_7(z)w^7 + p_6(z)w^6 + \cdots + p_0(z)\big),$$

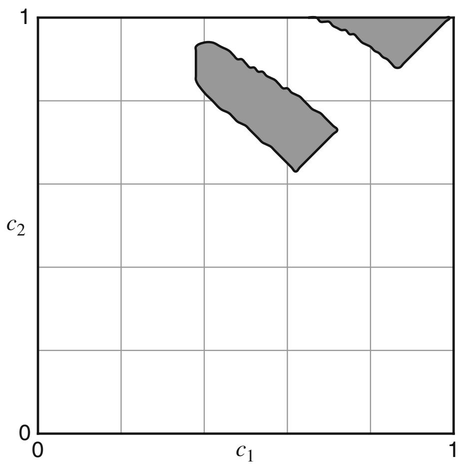

Fig. 1 Acceptable (c_1, c_2) pairs for A-stability of MHCMs with $r = 2,\ m = 2$

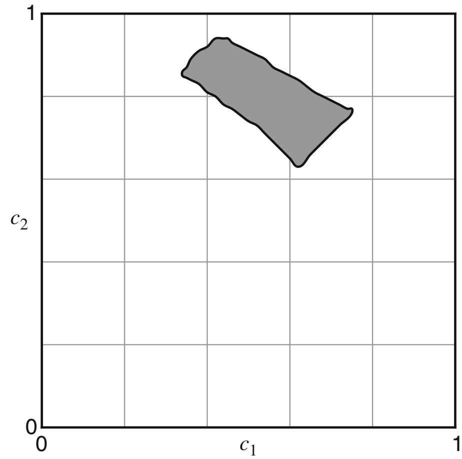

Fig. 2 Acceptable (c_1, c_2) pairs for A-stability of MHCMs with $r = 3,\ m = 2$

where $p_i(z)$, $i = 0, 1, \cdots, 7$ are polynomials of degree less than or equal to 2. Performing a computer search based on the boundary locus method, shows that this family of methods are A-stable of order 8 for $c_1 \in [0.27, 0.32]$.

4.4 Examples of MHCMs with 3 steps and 2 stages

Consider MHCMs with 3 steps and 2 collocation parameters c_1 and c_2. The stability polynomial for this family of methods is of the form

$$p(w, z) = w^2(p_8(z)w^8 + p_7(z)w^7 + \cdots + p_0(z)),$$

where $p_i(z)$, $i = 0, 1, \cdots, 8$ are polynomials of degree less than or equal to 4. Performing an extensive computer search based on the boundary locus method,

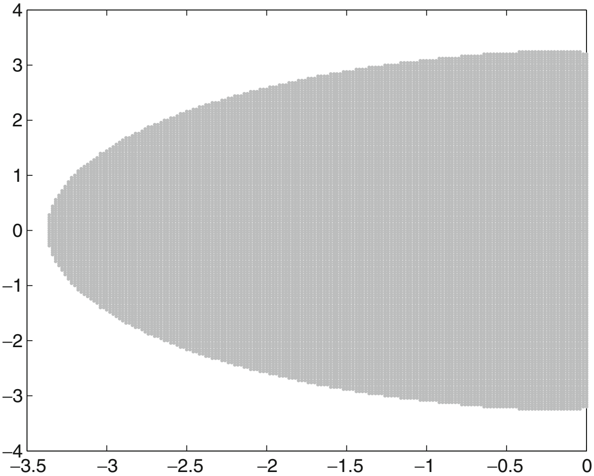

Fig. 3 Stability region for MHCM with $r = 3$, $m = 1$, $c_1 = 1$

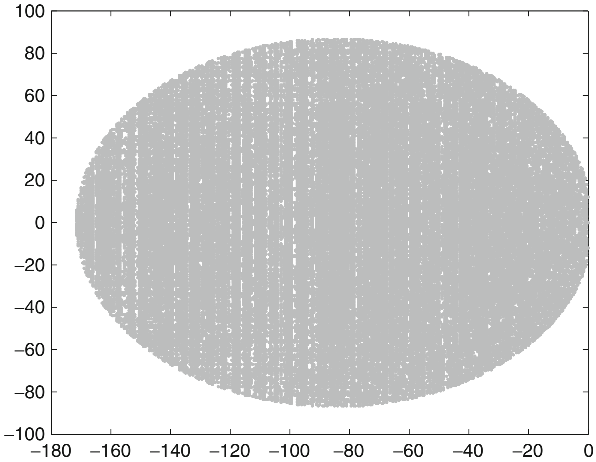

Fig. 4 Stability region for MHCM with $r = 3$, $m = 2$, $c_1 = 0.7$, $c_2 = 1$

we obtain A-stable methods of order 10 when both collocation parameters are within the region reported in Fig. 2.

Remark 3 Figure 3 shows the stability region for MHCM with $r = 3$, $m = 1$ and Fig. 4 shows the stability region for MHCM with $r = 3$, $m = 2$ and $c_1 = \frac{7}{10}$, $c_2 = 1$ as collocation parameters.

Remark 4 In the discretized MHCMs, the order of applied quadrature rules is at least the same proved order for MHCMs in Section 3. These rules are exact for $\varphi_k(s)$, $\chi_k(s)$, $k = 0, 1, \cdots, r-1$, $\psi_j(s)$, $\rho_j(s)$ and $j = 1, 2, \cdots, m$, since these polynomials are of degree $2m + 2r - 1$. Thus we have

$$\tilde{R}(z) = R(z)$$

and so the stability regions plotted in Figs. 1 and 2 do not change for the discretized cases.

5 Numerical experiments

In this section illustrative examples are given to show efficiency of proposed methods. We solve the given problems by MHCMs and compare the results with multistep collocation method [9]. Here the starting values $y_1, \cdots, y_{r-1}$ and $y'_1, \cdots, y'_{r-1}$ are obtained by a one step MHCM of the same order of the present method, i.e. the number of collocation abscissas must be $m + r - 1$. It must be mentioned that in [9] the starting values $y_1, \cdots, y_{r-1}$ have been taken from the known exact solutions.

In practice, we need quadrature rules to obtain numerical solutions. For this purpose, we have to apply the rules that preserve order of the main method. A suitable choice is Gauss quadrature formulas with $m + r - 1$ points.

In what follows, we describe details of the implemented methods:

- **Method 1:** MHCM of convergence order 10 with $r = 3,\ m = 2$, and collocation parameters $c_1 = 0.8,\ c_2 = 1$.
- **Method 2:** MHCM of convergence order 8 with $r = 2,\ m = 2$, and collocation parameters $c_1 = 0.7,\ c_2 = 1$, which is an A-stable method.
- **Method 3:** MHCM of convergence order 8 with $r = 2,\ m = 2$, and collocation parameters $c_1 = 0.55,\ c_2 = 1$, which has bounded stability region.
- **Method 4:** Multistep collocation method of local superconvergence order 8 [9] with $r = 3,\ m = 3$, and collocation parameters $c_1 = \frac{103}{194} - \frac{\sqrt{89355}}{1358},\ c_2 = \frac{103}{194} + \frac{\sqrt{89355}}{1358},\ c_3 = 1$.

Computational experiments are doing by applying the methods 1–4 on the following problems:

I The linear VIE

$$y(t) = e^t + \int_0^t 2\cos(t-\tau)y(\tau)d\tau, \quad t \in [0, 10],$$

with exact solution $y(t) = e^t(1+t)^2$.

II The nonlinear VIE

$$y(t) = 2 - \cos(t) - \int_0^t \sin(ty(\tau) - \tau)d\tau, \quad t \in [0, 5],$$

with exact solution $y(t) \equiv 1$.

III The nonlinear VIE

$$y(t) = 1 + \int_0^t e^{-t}y^2(\tau)d\tau, \quad t \in [0, 5],$$

with the exact solution $y(t) = e^t$.

IV The linear stiff VIE [16]

$$y(t) = \sin t + \lambda(1 - \cos t) - \lambda\int_0^t y(\tau)d\tau, \quad t \in [0, 20],$$

with the exact solution $y(t) = \sin t$.

In the Tables 1, 2 and 3, maximal end point errors are 10^{-cd}, where cd denotes the number of correct digits. Convergence order of the method is defined by $p(h) = Log_2(\frac{e(2h)}{e(h)})$, where $e(h)$ is the maximal absolute end point error. In Table 4, we show the effect of linear stability of the methods 2 and 3 in solving the stiff problem **IV**. The absolute stability region for the method 3, is bounded. For $z = \lambda h$ out of this region, increasing of absolute error is evidently seen while this does not happen for the method 2, which is an A-stable method and the obtained results are acceptable.

Table 1 The results of problem I

	N	8	16	32	64	128	256
Method 1	cd	5.04	7.87	10.77	13.74	16.72	19.72
	$p(h)$		9.40	9.67	9.83	9.91	9.95
Method 2	cd	3.89	6.24	8.61	11.01	13.40	15.80
	$p(h)$		7.82	7.89	7.94	7.96	7.98

Table 2 The results of problem II

	N	8	16	32	64	128	256
Method 1	cd	14.63	17.56	20.53	23.51	26.52	29.52
	$p(h)$		9.74	9.84	9.92	9.96	9.98
Method 2	cd	11.22	13.54	15.90	18.28	20.68	23.09
	$p(h)$		7.71	7.84	7.92	7.96	7.98

Table 3 The results of problem III

	N	8	16	32	64	128	256
Method 1	cd	9.78	12.86	15.91	18.93	21.92	24.96
	$p(h)$		10.23	10.10	10.05	9.96	10.06
Method 2	cd	8.30	10.71	13.12	15.52	17.93	20.34
	$p(h)$		8.01	8.00	8.00	8.00	8.00

Table 4 Comparison absolute errors of the methods 2 and 3 for problem **IV** with $\lambda = 400$

	Method 2		Method 3	
t	$N = 128$	$N = 256$	$N = 128$	$N = 256$
2.5	9.440E−14	3.270E−16	1.481E−12	4.052E−16
5	1.263E−14	5.372E−16	7.485E−11	4.994E−16
10	4.669E−14	8.603E−16	1.888E−07	1.334E−16
15	9.981E−14	3.553E−16	4.766E−04	4.237E−16
20	1.033E−13	4.797E−16	1.203E+00	3.738E−16

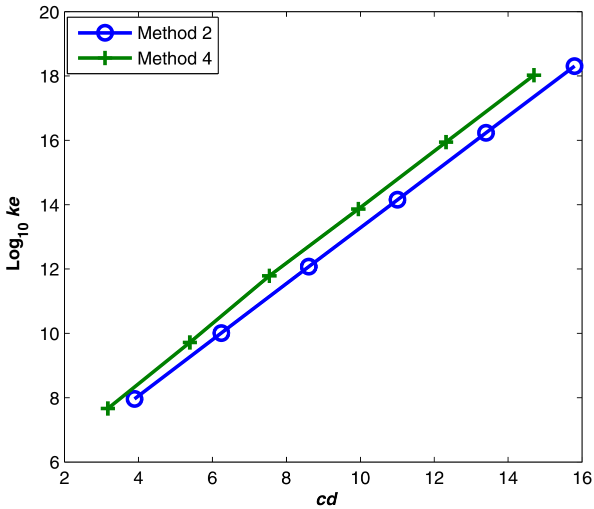

Fig. 5 The number of kernel evaluations for problem **I**

We compare the performance of MHCMs with respect to the multistep collocation methods in terms of computational cost. Figure 5 shows the number of kernel evaluations, *ke*, including kernel derivatives, with respect to the correct digits of methods 2 and 4 for solving the problem **I**.

6 Conclusion

In the introduced methods which are a new class of multistep collocation methods, the first derivative of approximate solution in r previous mesh points and m collocation points, as well as the values of approximate solution in these points, are used. The applied technique not only gets methods of higher orders, but also causes to A-stable methods in some cases and the approximate solution is more smooth than the approximate solution in multistep collocation methods and collocation methods.

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