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On the stability of some second order numerical methods for weak approximation of Itô SDEs

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Abstract In this paper, we first investigate the stability of two weak second order methods introduced by Debrabant and Rößler (Appl Numer Math 59:582–594, 2009) and Platen (Math Comput Simulation 38:69–76, 1995). We then propose a new weak second order predictor-corrector method, with an improved stability properties, based on the Rößler's method as the predictor and the implicit method of Platen as the corrector. The stability functions of these methods, applied to a scalar linear test equation with multiplicative noise, are determined and their regions of stability are then compared with the corresponding stability regions of the test equation. Furthermore, we also investigate mean square stability (MS-stability) of these methods applied to a linear Itô 2-dimensional stochastic differential test equation. Numerical examples will be presented to support the theoretical results.

Keywords Stochastic differential equations · Stochastic stability · Asymptotic stability · MS-stability · Weak convergence · Predictor-corrector methods

1 Introduction

In recent years many efficient numerical methods have been constructed for solving different types of SDEs with different properties (for example see [8, 10, 13]). Stability of numerical schemes for SDEs is essential to avoid

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possible explosion of numerical solution. It is often necessary to use some implicit schemes to overcome this problem in the simulation of solution of stochastic stiff differential equations [3, 14, 15, 21]. There is no unique generally accepted notion of stiffness, and different authors have proposed various definitions, [10, 13]. The stochastic stiffness, similar to the deterministic case, can be characterized by the presence of two or more widely differing time scales in the solutions. It should also be mentioned that for more stiff SDEs appropriate adaptive schemes, e.g., [1, 22], should be used. The implicit schemes usually have a wider range of acceptable step sizes suitable for the solution of stiff systems. Implementation of implicit schemes requires, in general, the solution of an additional algebraic equation in each time step, which may increase the computational effort dramatically. Predictor-corrector methods in the deterministic case are used mainly because of their better numerical stability, inherited from the implicit (corrector) counterparts. These advantages carry over to the stochastic case. Platen [16] proposed a predictor-corrector scheme for weak solution of SDEs which uses weak Taylor or other explicit schemes as predictors and an implicit scheme, which is made explicit by using the predicted value on the right hand side of the corrector implicit scheme. In this paper, we propose a new family of predictor-corrector methods consisting of the weak second order methods of Rößler [4] as predictors and the implicit method used by Platen [16] as corrector. We will simply consider our predictor to be the DIR1 scheme of Rößler, which is weak second order with minimized error coefficients, and our analysis will be focused only on this choice. Obviously, for any other choice of predictor from Rößler's class a similar analysis can be carried out. Hence, in the sequel, we just analyze and investigate the stability properties of the three methods: DIR1, the weak second order Runge-Kutta method of Rößler [4], denoted by method 1, the predictor-corrector (PC) weak second order method of Platen [16], denoted by method 2, and also our new proposed predictor-corrector method, denoted by method 3.

Numerical stability of these methods, following Saito and Mitsui [20], will be carried out for the scalar linear test Itô SDE and also the 2-dimensional linear test Itô SDE. For the 2-dimensional stochastic differential system with one multiplicative noise the mean square stability analysis of these methods will only be carried out. In Section 2, we first give a brief overview of the class of SRK methods proposed by Rößler [4] and also the predictor-corrector method of Platen [16]. Then we introduce our new method and show that it is also a weak second order method. In Section 3, a brief overview of the stability concepts is given. In Section 4, we analyze the stability properties of the above-mentioned methods on linear scalar and also 2-dimensional SDEs. In Section 5, some numerical experiments are carried out to confirm the theoretical results.

Consider the probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t, t \in [t_0, T]\}$ which satisfies the usual conditions. Let $\{X_t, t \in [t_0, T]\}$ denote the process of an Itô SDE

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t,\ X_{t_0} = x_0, \tag{1}$$

for $t \in [t_0, T]$, where $a : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is drift and $b : [t_0, T] \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is the diffusion with d-dimensional column vectors $b^1, b^2, ..., b^m$, W=$\{W_t = (W_t^1, W_t^2, ..., W_t^m),\ t \in [t_0, T]\}$ is the m-dimensional standard Wiener process. Assume that for some $r \in \mathbb{N}$ all of initial moments $E(|X_0|^{2r}) < \infty$. We also assume that SDE (1) satisfies the required conditions of the existence and uniqueness theorem [10].

Definition 1 Consider an equidistant partition of the time interval $[t_0, T]$ with discretization points $t_n = t_0 + nh$ and step size $h = \frac{T-t_0}{N}$, for some $N \in \mathbb{N}$. Then we say a discrete approximation $Y_0, Y_1, ..., Y_n$ (based on step size h) converges weakly with order β to the solution $X = X_t$ as $h \to 0$ at time t_n if for all $F \in \mathcal{C}_P^{2\beta+2}(\mathbb{R}^d, \mathbb{R})$, there exist a constant $C_F > 0$ (independent of h) and $\delta > 0$ such that for each $h \in (0, \delta)$

$$|E(F(X_{t_n})) - E(F(Y_n))| \leqq C_F h^\beta \tag{2}$$

where $\mathcal{C}_P^{2\beta+2}(\mathbb{R}^d, \mathbb{R})$ is the set of all polynomials $F : \mathbb{R}^d \to \mathbb{R}$ for which F and all its partial derivatives up to order β have polynomial growth.

2 The methods 1, 2, and 3

Our proposed predictor-corrector method takes its predictor from weak second order SRK schemes of [4, 17] and its corrector form Platen [16]. Hence, for our analysis and stability comparisons some overview of these methods is needed.

2.1 Method 1, weak second order schemes of Rößler

Consider the class of stochastic Runge-Kutta schemes proposed by Rößler [17, 18] for weak approximation of SDE (1). In this class, the d-dimensional approximation process Y with $Y_n = Y(t_n)$ of an explicit s-stage SRK method is defined by $Y_0 = x_0$ and

$$\begin{aligned} Y_{n+1} = Y_n &+ \sum_{i=1}^{s} \alpha_i a\left(t_n + c_i^{(0)} h_n, H_i^{(0)}\right) h_n \\ &+ \sum_{i=1}^{s}\sum_{k=1}^{m} \left(\beta_i^{(1)} \hat{I}_{(k)} + \beta_i^{(2)} \frac{\hat{I}_{(k,k)}}{\sqrt{h_n}}\right) b^k\left(t_n + c_i^{(1)} h_n, H_i^{(k)}\right) \\ &+ \sum_{i=1}^{s}\sum_{k=1}^{m} \left(\beta_i^{(3)} \hat{I}_{(k)} + \beta_i^{(4)} \sqrt{h_n}\right) b^k\left(t_n + c_i^{(2)} h_n, \hat{H}_i^{(k)}\right) \end{aligned} \tag{3}$$

for $n = 0, 1, 2, ..., N-1$ with stage values

$$H_i^{(0)} = Y_n + \sum_{j=1}^{i-1} A_{ij}^{(0)} a\left(t_n + c_j^{(0)} h_n, H_j^{(0)}\right) h_n + \sum_{j=1}^{i-1}\sum_{l=1}^{m} B_{ij}^{(0)} b^l\left(t_n + c_j^{(1)} h_n, H_j^{(l)}\right) \hat{I}_{(l)},$$

$$H_i^{(k)} = Y_n + \sum_{j=1}^{i-1} A_{ij}^{(1)} a\left(t_n + c_j^{(0)} h_n, H_j^{(0)}\right) h_n + \sum_{j=1}^{i-1} B_{ij}^{(1)} b^k\left(t_n + c_j^{(1)} h_n, H_j^{(k)}\right) \sqrt{h_n},$$

$$\hat{H}_i^{(k)} = Y_n + \sum_{j=1}^{s} A_{ij}^{(2)} a\left(t_n + c_j^{(0)} h_n, H_j^{(0)}\right) h_n$$
$$+ \sum_{j=1}^{s}\sum_{l=1, l\neq k}^{m} B_{ij}^{(2)} b^l\left(t_n + c_j^{(1)} h_n, H_j^{(l)}\right) \frac{\hat{I}_{(k,l)}}{\sqrt{h_n}}$$

for $i = 1, ..., s$ and $k = 1, ..., m$ (see [4, 17] for more details). The random variables of the methods are defined as follows:

$$\hat{I}_{(k,l)} = \begin{cases} \frac{1}{2}\left(\hat{I}_{(k)}\hat{I}_{(l)} - \sqrt{h_n}\tilde{I}_{(k)}\right) & \text{if } k < l, \\ \frac{1}{2}\left(\hat{I}_{(k)}\hat{I}_{(l)} + \sqrt{h_n}\tilde{I}_{(l)}\right) & \text{if } l < k, \\ \frac{1}{2}\left(\hat{I}^2{}_{(k)} - h_n\right) & \text{if } k = l \end{cases} \tag{4}$$

for $1 \leq k, l \leq m$. $\hat{I}_{(k)}$ are three-point distributed random variables with $P(\hat{I}_{(k)} = \pm\sqrt{3h_n}) = \frac{1}{6}$, $P(\hat{I}_{(k)} = 0) = \frac{2}{3}$ and $\tilde{I}_{(k)}$ are two-point distributed random variables satisfying $P(\tilde{I}_{(k)} = \pm\sqrt{h_n}) = \frac{1}{2}$. Frequently, the coefficients of SRK method (3) are presented by the following Butcher tableau:

$c^{(0)}$	$A^{(0)}$	$B^{(0)}$	
$c^{(1)}$	$A^{(1)}$	$B^{(1)}$	
$c^{(2)}$	$A^{(2)}$	$B^{(2)}$	
	α^T	$\beta^{(1)T}$	$\beta^{(2)T}$
		$\beta^{(3)T}$	$\beta^{(4)T}$

Rößler in [4] proposed DIR1 method, as a particular case of this family of schemes. The Butcher tableau of this method, denoted by method 1, with weak order $(p_D, p_S) = (3, 2)$ and minimized leading local error term is presented in the Appendix.

2.2 Method 2, predictor-corrector of Platen

Platen in [16], has proposed an implicit method with weak order 2 for solving stochastic differential equation (1) in autonomous case as follows:

$$Y_{n+1} = Y_n + \frac{1}{2}\{a(Y_{n+1}) + a(Y_n)\}h + \phi_n \tag{5}$$

where

$$\phi_n = \frac{1}{4}\sum_{j=1}^{m}\left[b^j\left(\bar{R}_+^j\right) + b^j\left(\bar{R}_-^j\right) + 2b^j + \sum_{r=1,r\neq j}^{m}\left(b^j\left(\bar{U}_+^r\right) + b^j\left(\bar{U}_-^r\right) - 2b^j\right)h^{-\frac{1}{2}}\right]\hat{I}_{(j)}$$
$$+\frac{1}{4}\sum_{j=1}^{m}\left[\left(b^j\left(\bar{R}_+^j\right) - b^j\left(\bar{R}_-^j\right)\right)\left\{\left(\hat{I}_{(j)}\right)^2 - h\right\} + \sum_{r=1,r\neq j}^{m}\left(b^j\left(\bar{U}_+^r\right) - b^j\left(\bar{U}_-^r\right)\right)\left\{\hat{I}_{(j)}\hat{I}_{(r)} + V_{r,j}\right\}\right]h^{-\frac{1}{2}}$$

with supporting values

$$\bar{R}_\pm^j = Y_n + a(Y_n)h \pm b^j\sqrt{h}, \quad \bar{U}_\pm^j = Y_n \pm b^j\sqrt{h},$$

where $\hat{I}_{(k)}$ are defined as before and V_{j_1,j_2} are independent two-point distributed random variables with $P(V_{j_1,j_2} = \pm\sqrt{h}) = \frac{1}{2}$, for $j_2 = 1, ..., j_1 - 1$ and $V_{j_1,j_1} = -h$ and $V_{j_1,j_2} = -V_{j_2,j_1}$ for $j_2 = j_1 + 1, ..., m$ and $j_1 = 1, ..., m$. Clearly, in the contrast to the previous notations we have here $\hat{I}_{(j_1,j_2)} = \frac{1}{2}\{\hat{I}_{(j_1)}\hat{I}_{(j_2)} + V_{j_1,j_2}\}$. So we can write ϕ_n according to $\hat{I}_{(k)}$ and $\hat{I}_{(j_1,j_2)}$. In this paper, this method will be applied with just one correction in each step.

2.3 Method 3, a new predictor-corrector

We construct a weak second order predictor-corrector method based on DIR1 method as predictor and the implicit weak method of Platen, used in method 2, as its corrector:

(**prediction**) : $\bar{Y}_{n+1}$ is obtained from DIR1 method 1 with initial value Y_n.

$$(\textbf{correction}):\quad Y_{n+1} = Y_n + \frac{1}{2}\{a(\bar{Y}_{n+1}) + a(Y_n)\}h + \phi_n \tag{6}$$

Clearly, in this method only $2m - 1$ independent random variables have to be simulated in each step. We now indicate the weak second order convergency of this method. In this paper, this method will be applied with just one correction in each step.

The convergence property of this method Platen in [16], by applying Itô expansion [10], described a systematic way to derive second weak order implicit and predictor-corrector schemes for autonomous form of SDE (1). Similar to the notations in [16], for sufficiently smooth function f on $\mathbb{R}^d$ consider:

$$f_{(j),n} = L^j f(X_{t_n}), \quad \text{for} \quad j = 0, 1, ..., m$$
$$f_{(j_1,j_2),n} = L^{j_1} f_{(j_2),n}(X_{t_n}) \quad \text{for} \quad j_1, j_2 = 0, 1, ..., m; \quad n = 0, 1, ..., N-1,$$

where L^0, L^j are operators defined for Itô Taylor expansion. With choosing $f(x) \equiv x$ one can find that for all $l_0 \in [0, 1]$ we have

$$X_{t_{n+1}} - X_{t_n} \approx \{l_0 f_{(0),n+1} + (1 - l_0) f_{(0),n}\}h + \sum_{j=1}^{m} f_{(j),n} \hat{I}_{(j),n}$$
$$+ \sum_{j_1=0}^{m} \sum_{j_2=1}^{m} f_{(j_1,j_2),n} \hat{I}_{(j_1,j_2),n} + (1 - 2l_0) \sum_{j=0}^{m} f_{(j,0),n} \hat{I}_{(j,0),n}. \tag{7}$$

The approximation in (7) means, the conditional moments of both sides coincide to show second order weak convergence corresponding scheme. Obviously, by an application of the deterministic Taylor expansion one can derive the scheme (5). Therefore, from second order convergence property of DIR1 method 1 in the weak sense, the new method 3 has also second order convergency in the weak sense.

3 Stability analysis for SDEs

In this section, we review some stability concepts for the SDEs. There are many different ways of defining stability concepts for SDEs. For the paper to be readable we explain some necessary definitions and results that can be found in many related references, such as [2, 3, 7, 10, 14, 19–21]. Consider the scalar SDE (1) with steady solution $X_t \equiv 0$ such that $a(t, 0) = b(t, 0) = 0$ which is called equilibrium position. Suppose that there exists a unique solution $X_t = X(t; t_0, x_0)$ for all $t \geq t_0$ and for nonrandom initial value x_0.

Definition 2 The equilibrium position, $X_t \equiv 0$, is said to be stochastically asymptotically stable in the large if for all $\varepsilon > 0$ and $t_0 \geq 0$

$$\lim_{x_0 \to 0} P(\sup_{t \geq t0} |X(t; t_0, x_0)| \geq \varepsilon) = 0,$$

and also for all $x_0 \in \mathbb{R}$

$$P(\lim_{t \to \infty} |X(t; t_0, x_0)| = 0) = 1.$$

Definition 3 The equilibrium position, $X_t \equiv 0$, is said to be asymptotically mean square stable if for every $\varepsilon > 0$ and $t_0 \geq 0$, there exists $\delta_1 > 0$ such that

$$E(\|X(t; t_0, x_0)\|^2) < \varepsilon \text{ for all } t \geq t_0 \text{ and } \|x_0\| < \delta_1,$$

and there exists a $\delta_2 > 0$ such that

$$\lim_{t \to \infty} E(\|X(t; t_0, x_0)\|^2) = 0 \text{ for all } \|x_0\| < \delta_2,$$

where $\|.\|$ is the usual Euclidean vector norm.

3.1 Stability for scalar SDEs

Consider the scalar linear test equation with multiplicative noise of Itô type [6, 7, 9, 19]

$$dX_t = \lambda X_t dt + \mu X_t dW_t,\ \lambda, \mu \in \mathbb{C}, \tag{8}$$

with nonrandom condition $X_{t_0} = x_0 \in \mathbb{R}\backslash\{0\}$. The exact solution of (8) is given by $X_t = x_0 \exp\{(\lambda - \frac{1}{2}\mu^2)(t - t_0) + \mu(W_t - W_{t_0})\}$ which is stochastically asymptotically stable in the large [19] if

$$\lim_{t\to\infty} |X_t| = 0 \text{ with probability } 1 \Leftrightarrow \Re(\lambda - \tfrac{1}{2}\mu^2) < 0. \tag{9}$$

We can calculate that

$$|X_t|^2 = |x_0|^2 \exp\left\{2\Re\left(\lambda - \frac{1}{2}\mu^2\right)(t - t_0) + 2\Re(\mu)(W_t - W_{t_0})\right\}$$

which yields

$$E(|X_t|^2) = |x_0|^2 \exp\left\{2\Re\left(\lambda - \frac{1}{2}\mu^2\right)(t - t_0) + 2(\Re(\mu))^2(t - t_0)\right\}.$$

Thus, the equilibrium position of SDE (8) is asymptotically MS-stable if

$$\lim_{t\to\infty} E(|X_t|^2) = 0 \Leftrightarrow 2\Re(\lambda) + |\mu|^2 < 0, \tag{10}$$

for $\lambda, \mu \in \mathbb{C}$. Obviously, we have the inequality $\Re(2\lambda - \mu^2) \leq 2\Re(\lambda) + |\mu|^2$. That means, for linear SDE (8) the MS-stability implies asymptotic stability in large.

3.2 Mean square stability for linear 2-dimensional SDEs

Consider the 2-dimensional linear Itô system with multiplicative noise

$$\begin{aligned} dX(t) &= MX(t)dt + SX(t)dW(t) \\ X(0) &= X_0 = 1 \end{aligned} \tag{11}$$

where

$$X(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix},\ M = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},\ S = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix},\ \text{and } \mathbf{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Definition 4 The logarithmic matrix norm of A, denoted as $\mu_p[A]$ is defined by

$$\mu_p[A] = \lim_{h\to 0^+} \frac{\|I + hA\|_p - 1}{h} \tag{12}$$

where $\|.\|_p$ is the p^{th} matrix norm and I is the identity matrix with size equivalent to that of A. The logarithmic matrix norm in l_∞ can be easily established (see e.g., [12, 13]) as

$$\mu_\infty[A] = \max_i \left\{ a_{ii} + \sum_{j \neq i} |a_{ij}| \right\}. \tag{13}$$

Let $X(t)$ be a solution vector of (11) and define $P(t) = E(X(t)X(t)^T)$, where $X(t)^T$ is vector transpose. Then the symmetric matrix $P(t)$ given by

$$P(t) = \begin{bmatrix} E(X_1(t))^2 & E(X_1(t)X_2(t)) \\ E(X_1(t)X_2(t)) & E(X_2(t))^2 \end{bmatrix} \tag{14}$$

that satisfies the IVP [10]

$$\frac{dP}{dt} = MP + PM^T + SPS^T \tag{15}$$

with $P(0) = X_0 X_0^T$. The symmetry property of P leads to the following ODE system [20]

$$\frac{dY(t)}{dt} = \Omega Y(t), \tag{16}$$

where

$$Y(t) = \begin{bmatrix} Y^1(t) \\ Y^2(t) \\ Y^3(t) \end{bmatrix} = \begin{bmatrix} E(X_1(t))^2 \\ E(X_2(t))^2 \\ E(X_1(t)X_2(t)) \end{bmatrix}$$

and

$$\Omega = \begin{bmatrix} 2\lambda_1 + \alpha^2 & \beta^2 & 2\alpha\beta \\ \gamma^2 & 2\lambda_2 + \delta^2 & 2\gamma\delta \\ \alpha\gamma & \beta\delta & \lambda_1 + \lambda_2 + \alpha\delta + \beta\gamma \end{bmatrix}. \tag{17}$$

The following lemma is a natural consequence of the logarithmic matrix norm [20].

Lemma 3.1 *System* (11) *is asymptotically mean square stable with respect to logarithmic matrix norm* μ_∞ *if and only if* $\mu_\infty[\Omega] < 0$.

Using direct Lemma 3.1 and direct computation of $\mu_\infty[\Omega]$, the mean square stability criterion for the SDE system (11) can be described in the following theorem whose proof is given in [20].

Theorem 3.2 *The system* (11) *is mean square stable with respect to* μ_∞ *if*

$$\max\{2\lambda_1 + (|\alpha| + |\beta|)^2, 2\lambda_2 + (|\gamma| + |\delta|)^2\} < 0$$

In this paper, for simplicity we assume that $\lambda_1 < \lambda_2 < 0$.

4 Numerical stability of methods 1–3

The larger the intersection of the domain of numerical stability and that of the equation (8) or (11), the larger step size h can be made within the truncation restriction. It should be mentioned that if the domain of stability of the test equation (8) or (11) is subset of the domain of numerical stability, the numerical method is said to be A-stable.

We now ask what condition must be imposed in order that a numerical method applied to SDEs (8), (11) produces numerically stable solutions.

Definition 5 We say a method is numerically asymptotically stable or MS-stable if the numerical solution Y_n, generated by a method, satisfies $\lim_{n\to\infty} |Y_n| = 0$ with probability one or $\lim_{n\to\infty} E(|Y_n|^2) = 0$, respectively.

4.1 Stability analysis of the methods 1–3 applied to linear test equation (8)

Consider the linear test equation (8). If we apply a numerical method to this equation, we will obtain a one-step difference equation of the following form

$$Y_{n+1} = R_n(\hat{h}, k)Y_n = \prod_{i=0}^{n} R_i(\hat{h}, k)Y_0 \tag{18}$$

with parametrization $\hat{h} = \lambda h$ and $k = \mu\sqrt{h}$ (see [7, 11] for more details). We call $R_n(\hat{h}, k)$ the stability function of the scheme. Using the following Lemma mentioned in [7], and in the same line of [5], we can determine the domain of asymptotic stability of a method.

Lemma 4.1 *Suppose that, given a sequence of real-valued, nonnegative, independent and identically distributed random variables* $(R_n(\hat{h}, k))_{n\in\mathbb{N}_0}$*, consider the sequence of random variables* $(|Y_n|)_{n\in\mathbb{N}_0}$ *defined by* (18) *where* $Y_0 \neq 0$ *with probability one. Suppose the random variables* $log(R_n(\hat{h}, k))$ *are square-integrable then we have*

$$\lim_{n\to\infty} |Y_n| = 0, \textit{ with probability } 1 \Leftrightarrow E(log(R_n(\hat{h}, k))) < 0. \tag{19}$$

Clearly, from Lemma 4.1 the domain of asymptotic stability of a method is a subset of $\mathbb{C}^2$ such as

$$R_{AS} = \{(\hat{h}, k) \in \mathbb{C}^2 : E(log(R_n(\hat{h}, k))) < 0\}.$$

The domain of MS-stability of a method is subset of $\mathbb{C}^2$ such as

$$R_{MS} = \{(\hat{h}, k) \in \mathbb{C}^2 : \hat{R}_n(\hat{h}, k) < 1\}, \text{ where } \hat{R}_n(\hat{h}, k) = E(|R_n(\hat{h}, k)|^2).$$

Since it is not easy to visualize the domains of stability for $\lambda, \mu \in \mathbb{C}$, we restrict our attention to $\lambda, \mu \in \mathbb{R}$ for presenting the figures of stability in the $\hat{h} - k^2$

plane. For $\lambda, \mu \in \mathbb{R}$ the region of asymptotic stability of SDE (8), in the $\hat{h} - k^2$ plane, is

$$\left\{\left(\hat{h}, k^2\right) \in \mathbb{R}^2 : \; k^2 > 2\hat{h}, \; k^2 \geq 0\right\}$$

and the region of MS-stability is

$$\left\{\left(\hat{h}, k^2\right) \in \mathbb{R}^2 : \; k^2 < -2\hat{h}, \; k^2 \geq 0\right\}.$$

We now calculate the asymptotic stability functions, $R_n(\hat{h}, k)$, and MS-stability functions, $\hat{R}_n(\hat{h}, k)$, for methods 1–3. From applying each method to the linear test equation (8) we obtain, the recursive formula $Y_{n+1} = R_n(\hat{h}, k)Y_n$ with asymptotic stability function $R_n(\hat{h}, k) = \Gamma - \Lambda + h^{-\frac{1}{2}}\Sigma \hat{I}_{(1),n} + h^{-1}\Lambda \hat{I}^2_{(1),n}$ for asymptotic stability, and the formula $z_{n+1} = \hat{R}_n(\hat{h}, k)z_n$ with MS-stability function $\hat{R}_n(\hat{h}, k) = |\Gamma|^2 + |\Sigma|^2 + 2|\Lambda|^2$ for MS-stability. In the above formulas the values of $\Lambda, \; \Gamma, \; \sum$ corresponding to each of those methods are given as follows.

For method 1:

$$\Lambda = \frac{1}{2}k^2, \sum = \left(1 + \hat{h} + \left(-\frac{1}{30}\sqrt{6} + \frac{1}{5}\right)\hat{h}^2\right)k,$$
$$\Gamma = 1 + \hat{h} + \frac{1}{2}\hat{h}^2 + \frac{1}{6}\hat{h}^3.$$

For method 2:

$$\Lambda = \left(\frac{1}{2} + \frac{1}{4}\hat{h}\right)k^2, \sum = \left(1 + \hat{h} + \frac{1}{2}\hat{h}^2\right)k,$$
$$\Gamma = 1 + \hat{h} + \frac{1}{2}\hat{h}^2 + \frac{1}{4}\hat{h}^3. \tag{20}$$

For method 3:

$$\Lambda = \left(\frac{1}{2} + \frac{1}{4}\hat{h}\right)k^2,$$
$$\sum = \left(1 + \hat{h} + \frac{1}{2}\hat{h}^2 + \frac{1}{2}\left(-\frac{1}{30}\sqrt{6} + \frac{1}{5}\right)\hat{h}^3\right)k,$$
$$\Gamma = 1 + \hat{h} + \frac{1}{2}\hat{h}^2 + \frac{1}{4}\hat{h}^3 + \frac{1}{12}\hat{h}^4.$$

Now, from Lemma 4.1 we can obtain the domains of asymptotic stability and MS-stability, as expressed in the next proposition.

Proposition 1 *For SDE* (8) *with* $\lambda, \mu \in \mathbb{C}$, *the three methods* 1–3, *using their corresponding* $\Gamma, \; \Lambda$, *and* Σ, *are*

(a) *numerically asymptotically stable if* $|(\Gamma + 2\Lambda)^2 - 3(\Sigma)^2||\Gamma - \Lambda|^4 < 1$,
(b) *numerically MS-stable if* $|\Gamma|^2 + |\Sigma|^2 + 2|\Lambda|^2 < 1$.

The regions of stability for these methods in the $\hat{h} - k^2$ plane are illustrated in Fig. 1. This figure has been produced by software Mathematica. The stability regions of the test equation (8) are indicated by two light-grey tones, in which the lighter regions correspond to asymptotic stability and the light ones correspond to of MS-stability. Then, the regions of numerical asymptotic stability and numerical MS-stability are also indicated in the same figure but by two dark-grey tones. The regions of numerical MS-stability, actually the darkest, are darker than the regions of asymptotic stability. It is clearly seen that the regions of MS-stability are subset of the regions of asymptotic stability. Comparison of the illustrated regions confirms that the proposed method

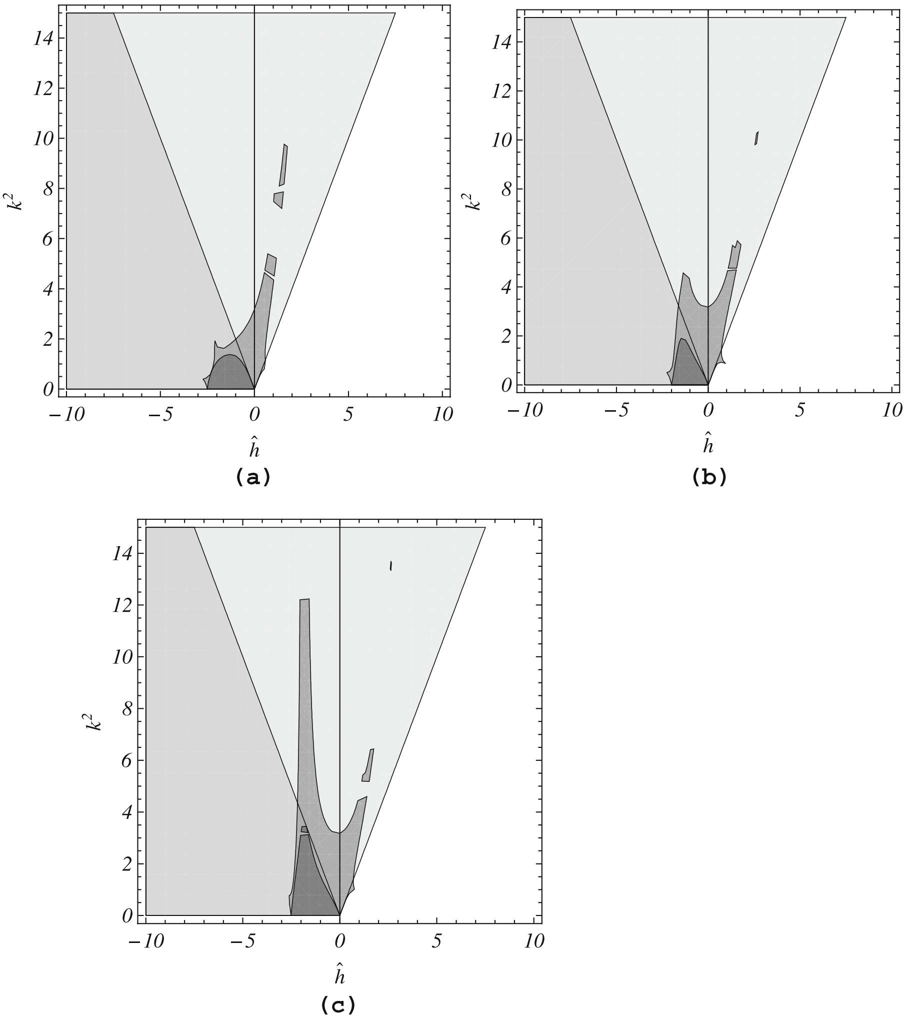

Fig. 1 Stability regions in $\hat{h} - k^2$ plane (**a**) for method 1 ; (**b**) for method 2 ; (**c**) for method 3

of this paper, method 3, has reasonably larger regions of both asymptotic and mean square stabilities, notably in the direction of k^2, which means that the method 1 is stable for sensitively larger diffusion parameter μ through $k = \mu\sqrt{h}$.

4.2 MS-stability analysis of the methods 1–3 applied to linear test (11)

Here, because of complexity of general investigation, we only study MS-stability of the methods 1–3 on 2-dimensional linear system (11). We obtain the stability matrix and present the stability domains for each method described above. When a numerical scheme is applied to test equation (11), the one step difference equation

$$\hat{Y}_{n+1} = \hat{\Omega}\hat{Y}_n \tag{21}$$

is obtained, where

$$\hat{Y}_n = \begin{bmatrix} \hat{Y}_n^1 \\ \hat{Y}_n^2 \\ \hat{Y}_n^3 \end{bmatrix} = \begin{bmatrix} E(\hat{X}_n^1)^2 \\ E(\hat{X}_n^2)^2 \\ E(\hat{X}_n^1 \hat{X}_n^2) \end{bmatrix}.$$

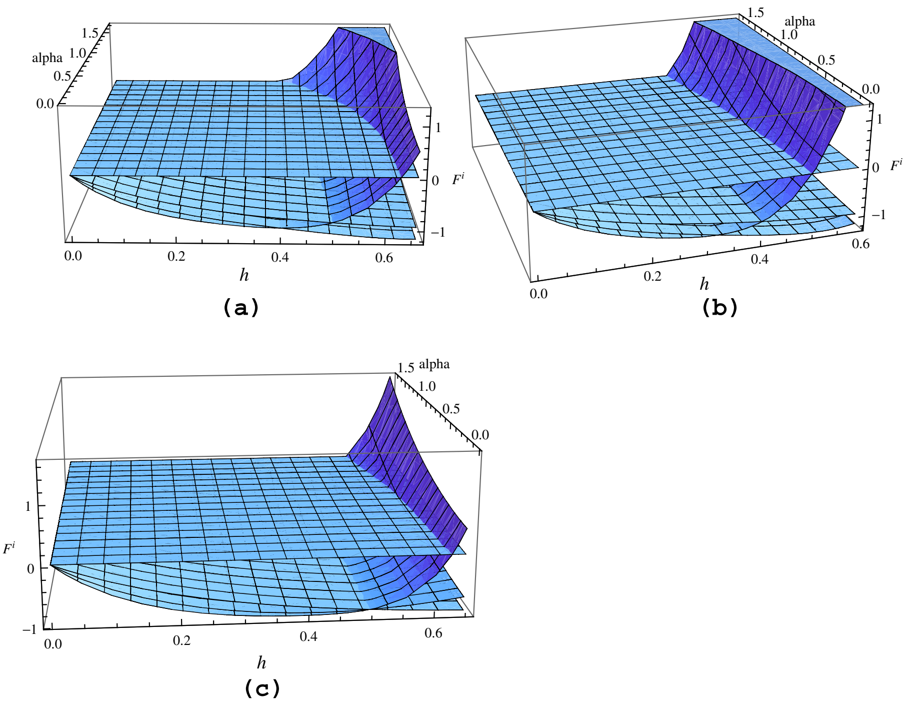

Fig. 2 Stability function of the linear SDE system (24) with varying parameters h and α and fixed $\lambda_1 = -4$, $\lambda_2 = -2$, $\delta = 0.0$ (**a**) method 1; (**b**) method 2; (**c**) method 3

The matrix $\hat{\Omega}$ in (21) is known as the stability matrix. Under the p^{th} matrix norm $\|.\|_p$, it is clear that $\lim_{n\to\infty} \hat{Y}_n = 0$ if $\|\hat{\Omega}\|_p < 1$.

Definition 6 A numerical scheme is said to be MS-stable w.r.t. logarithmic norm $\|.\|_p$ provided that

$$\|\hat{\Omega}\|_p < 1. \tag{22}$$

Hence, by setting

$$\begin{cases} f_1(h) = \hat{\Omega}_{11} + |\hat{\Omega}_{12}| + |\hat{\Omega}_{13}| - 1, \\ f_2(h) = \hat{\Omega}_{22} + |\hat{\Omega}_{21}| + |\hat{\Omega}_{31}| - 1, \\ f_3(h) = \hat{\Omega}_{33} + |\hat{\Omega}_{32}| + |\hat{\Omega}_{31}| - 1, \end{cases} \tag{23}$$

based on the elements of the stability matrix $\hat{\Omega}$, and from (13), (17), the Definition 6 means that for a numerical method to be MS-stable, one should have $f_i(h) < 0$, for $i = 1, 2, 3$.

In the sequel, because of the complexity of the stability matrix in general case, we confine our analysis for the methods 1–3, to some particular forms of SDE (11) under infinity logarithmic matrix norm.

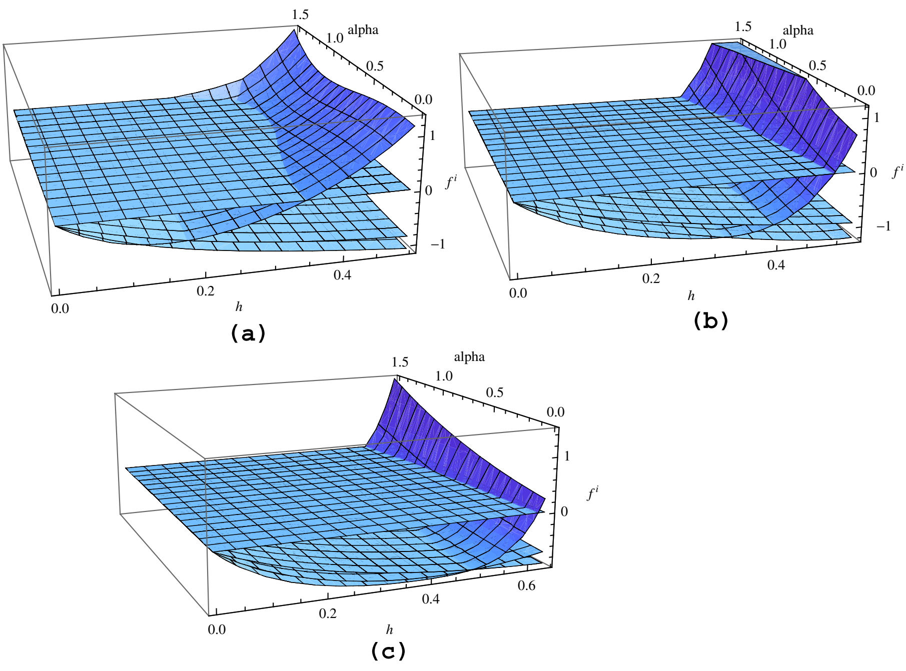

Fig. 3 Stability function of the linear SDE system (24) with varying parameters h and α and fixed $\lambda_1 = -4$, $\lambda_2 = -2$, $\delta = 0.4$ (**a**) method 1; (**b**) method 2; (**c**) method 3

Let us consider the 2-dimensional system given by

$$dX(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} X(t)dt + \begin{bmatrix} \alpha & \delta \\ \delta & \alpha \end{bmatrix} X(t)dW(t)$$
$$X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{24}$$

It should be recalled that according to (17) and the parameters of the test SDE (24), the stability matrix corresponding to each one of the methods 1–3 is the matrix $\hat{\Omega}^{(1)}$, $\hat{\Omega}^{(2)}$ and $\hat{\Omega}^{(3)}$. For the sake of simplicity, the elements of the corresponding stability matrices are listed in the Appendix.

Clearly, to have MS-stability with respect to infinity logarithmic matrix norm for each of the above-mentioned methods, from (13), (22) we must have $f_i(h) < 0$, $\forall i = 1, 2, 3$. For the visualization of the stability domains of these methods, using the related stability functions, the parameters λ_1, λ_2 are taken fixed. As the Theorem 3.2 shows, the test SDE (24) for $\lambda_1 = -4,\ \lambda_2 = -2$ is MS-stable for $|\alpha| + |\delta| < 2$. Therefore, we have provided 3D Figs. 2, 3, and 4, for $\delta = 0,\ 0.4,\ 0.9$, respectively, with a zero-hyperplane for more visualization. From these figures it is obvious that the method 3 provides stronger MS-stability features. By increasing δ the MS-stability regions of methods 1

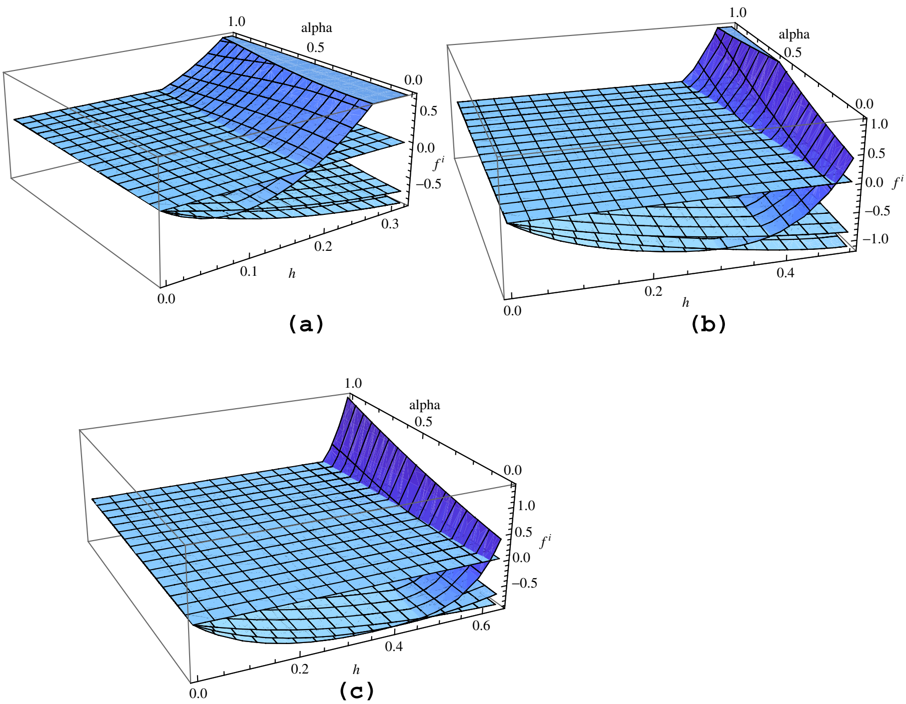

Fig. 4 Stability function of the linear SDE system (24) with varying parameters h and α and fixed $\lambda_1 = -4$, $\lambda_2 = -2$, $\delta = 0.9$ (**a**) method 1; (**b**) method 2; (**c**) method 3

and 2 decrease, while that of method 3 remains approximately unchanged and larger.

5 Numerical simulations

Here, we examine the mean square stability of the second order weak schemes described before by numerical examples. Two linear 2-dimensional systems with the initial condition $X(0) = [1, 1]^T$ are considered. For the three methods 1–3, from (23), we compute and plot $f_i(h)$ with respect to h for each row $i = 1, 2, 3$, of the stability matrix. It should be mentioned for SDEs more general than (24) the off-diagonal elements of the stability matrix $\hat{\Omega}$ are obtained accordingly but are listed in the Appendix.

Example 1 Consider the SDE system

$$dX(t) = \begin{bmatrix} -4 & 0 \\ 0 & -2 \end{bmatrix} X(t)dt + \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix} X(t)dW(t)$$
$$X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (25)$$

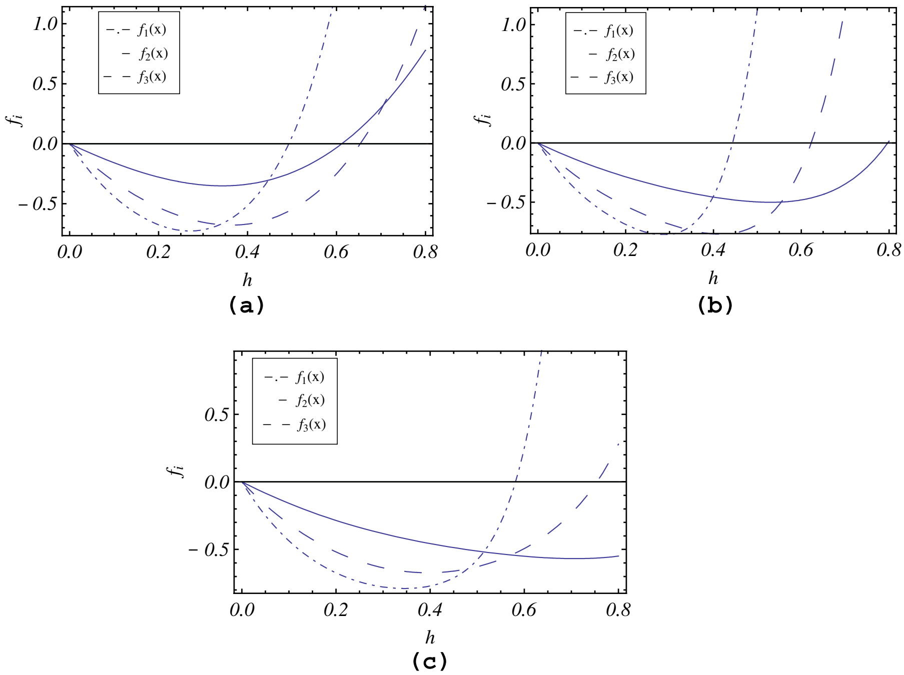

Fig. 5 $f^i(h)$ vs h in $\hat{h} - k^2$ plane for the example 1: (**a**) method 1, (**b**) method 2, (**c**) method 3

The plots of the $f^i(h)$ against step size h in $\hat{h} - k^2$ plane for the three methods 1–3 are illustrated in Fig. 5. From this figure it is obvious that the methods 1, 2 are unstable for $h > 0.45$. The largest h that method 3 can take for stable behavior, in this example, is 0.5813.

Example 2 Consider the SDE system

$$dX(t) = \begin{bmatrix} -4 & 0 \\ 0 & -1.5 \end{bmatrix} X(t)dt + \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix} X(t)dW(t)$$
$$X(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{26}$$

The plots of the $f^i(h)$ against step size h in $\hat{h} - k^2$ plane for the three methods 1–3 are illustrated in Fig. 6. This figure illustrates that the methods 1, 2 are

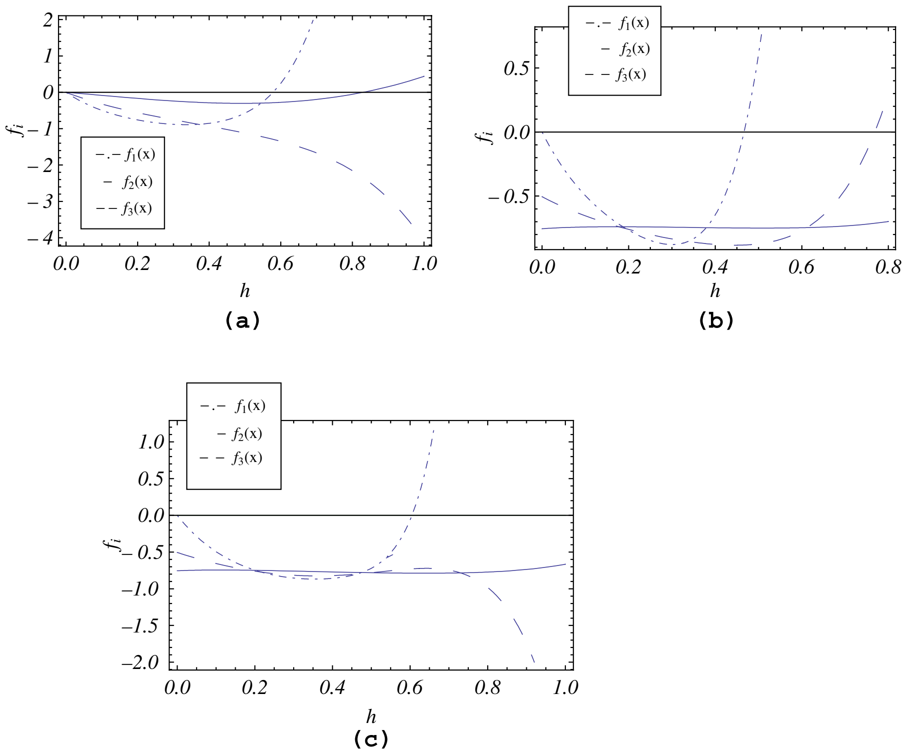

Fig. 6 $f_i(h)$ vs h in $\hat{h} - k^2$ plane for the example 2: (**a**) method 1, (**b**) method 2, (**c**) method 3

unstable for $h > 0.55$. The largest h that method 3 can take for stable behavior, in this example, is 0.6047.

6 Conclusions

In this paper, we introduced a class of weak second order methods of predictor-corrector type in which each second order method of the class of methods introduced by Rößler can be considered as the corrector and the implicit method used by Platen is considered as the corrector. For the purpose of stability comparison, we have discussed the stability of the methods of Rößler and Platen, methods 1, 2, and also that of the new proposed method, method 3. Based on numerical results, it was shown that the new method has reasonably larger regions of asymptotic and mean square stability, for both scalar and 2-dimensional linear test SDEs. Analysis of more complex form of stability for these methods can be continued.

Acknowledgement We appreciate the referees' suggestions and comments which have improved the paper.

Appendix

The Butcher tableau for DRI1 method

Case 1 $m = 1$

$$
\begin{array}{c|ccc|ccc|ccc}
0 & & & & & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \frac{6-\sqrt{6}}{10} & & & & & \\
1 & -1 & 2 & & \frac{3+2\sqrt{6}}{5} & 0 & & & & \\
\hline
0 & & & & & & & & & \\
\frac{342}{491} & \frac{342}{491} & & & 3\sqrt{\frac{38}{491}} & & & & & \\
\frac{342}{491} & \frac{342}{491} & 0 & & -3\sqrt{\frac{38}{491}} & 0 & & & & \\
\hline
0 & & & & 0 & 0 & 0 & & & \\
0 & 0 & & & -\frac{214}{513}\sqrt{\frac{1105}{991}} & -\frac{491}{513}\sqrt{\frac{221}{4955}} & -\frac{491}{513}\sqrt{\frac{221}{4955}} & & & \\
0 & 0 & 0 & & \frac{214}{513}\sqrt{\frac{1105}{991}} & \frac{491}{513}\sqrt{\frac{221}{4955}} & \frac{491}{513}\sqrt{\frac{221}{4955}} & & & \\
\hline
 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \frac{193}{684} & \frac{491}{1368} & \frac{491}{1368} & 0 & \frac{1}{6}\sqrt{\frac{491}{38}} & -\frac{1}{6}\sqrt{\frac{491}{38}} \\
\cline{5-10}
 & & & & -\frac{4955}{7072} & \frac{4955}{14144} & \frac{1}{2} & 0 & -\frac{1}{8}\sqrt{\frac{4955}{221}} & \frac{1}{8}\sqrt{\frac{4955}{221}}
\end{array}
$$

Case 2 $m > 1$

In this case, the method differ only in $A^{(2)}$, which is given by

$$A^{(2)} = \begin{bmatrix} \frac{2(-3442595658 + 1259007085\sqrt{6})}{1554073317(-6+\sqrt{6})} & -\frac{8(212963260 + 73915807\sqrt{6})}{1554073317(-6\sqrt{6})} & \frac{4(-1111473969 + 371403611\sqrt{6})}{23311099755} \\ \frac{2}{27}(7-2\sqrt{6}) & \frac{8}{81}(3+\sqrt{6}) & \frac{4}{81}(-3+\sqrt{6}) \\ \frac{2}{27}(7-2\sqrt{6}) & \frac{8}{81}(3+\sqrt{6}) & \frac{4}{81}(-3+\sqrt{6}) \end{bmatrix}.$$

The elements of the stability matrix for the methods 1–3 are, respectively,as follows:

Method 1

$$\begin{aligned} \hat{\Omega}_{11}^{(1)} &= 1 + (\alpha^2 + 2\lambda_1)h + \left(\frac{1}{2}(\alpha^2+\delta^2)^2 + 2\lambda_1^2 + 2\lambda_1\alpha^2\right)h^2 \\ &\quad + \frac{1}{15}(20\lambda_1 + \alpha^2(21-\sqrt{6}))\lambda_1^2 h^3 + \frac{1}{60}(35\lambda_1 + \alpha^2(24-4\sqrt{6}))\lambda_1^3 h^4 \\ &\quad + \frac{1}{150}(25\lambda_1 + \alpha(7-2\sqrt{6})^2)\lambda_1^4 h^5 + \frac{1}{36}\lambda_1^6 h^6, \end{aligned}$$

$$\begin{aligned} \hat{\Omega}_{12}^{(1)} &= \delta^2 h + 2\delta\left(\alpha^2 + \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2\right)^2 h^2 + \left(2\delta^2\lambda_1^2\left(\frac{1}{5} - \frac{1}{30}\sqrt{6}\right) + \frac{\delta^2}{4}(\lambda_1+\lambda_2)^2\right)h^3 \\ &\quad + \delta^2\lambda_1^2(\lambda_1+\lambda_2)\left(\frac{1}{5} - \frac{1}{30}\sqrt{6}\right)h^4 + \lambda_1^4\delta^2\left(\frac{1}{5} - \frac{1}{30}\sqrt{6}\right)^2 h^5, \end{aligned}$$

$$\begin{aligned} \hat{\Omega}_{13}^{(1)} &= 2h\alpha\delta + \alpha\delta(2\alpha^2 + 2\delta^2 + 3\lambda_1 + \lambda_2)h^2 \\ &\quad + \frac{27-2\sqrt{6}}{705}(47\lambda_1 + (27+2\sqrt{6})\lambda_2))\delta\lambda_1\alpha h^3 \\ &\quad + \frac{1}{30}(6-\sqrt{6})(3\lambda_1+\lambda_2)\delta\alpha\lambda_1^2 h^4 + \frac{7-2\sqrt{6}}{75}\alpha\delta\lambda_1^4 h^5, \end{aligned}$$

$$\begin{aligned} \hat{\Omega}_{21}^{(1)} &= \delta^2 h + 2\delta^2\left(\alpha^2 + \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2\right)h^2 + \left(2\delta^2\lambda_2^2\left(\frac{1}{5} - \frac{1}{30}\sqrt{6}\right) + \frac{\delta^2}{4}(\lambda_1+\lambda_2)^2\right)h^3 \\ &\quad + \delta^2\lambda_2^2(\lambda_1+\lambda_2)\left(\frac{1}{5} - \frac{1}{30}\sqrt{6}\right)h^4 + \lambda_2^4\delta^2\left(\frac{1}{5} - \frac{1}{30}\sqrt{6}\right)^2 h^5, \end{aligned}$$

$$\begin{aligned}\hat{\Omega}_{22}^{(1)} &= 1+(\alpha^2+2\lambda_2)h+\left(\frac{1}{2}(\alpha^2+\delta^2)^2+2\lambda_2^2+2\lambda_2\alpha^2\right)h^2\\ &+\frac{1}{15}(20\lambda_2+\alpha^2(21-\sqrt{6}))\lambda_2^2h^3+\ \frac{1}{60}(35\lambda_2+\alpha^2(24-4\sqrt{6}))\lambda_2^3h^4\\ &+\left(\alpha^2\left(\frac{6-\sqrt{6}}{30}\right)^2+\frac{\lambda_2}{6}\right)\lambda_2^4h^5+\frac{1}{36}\lambda_2^6h^6,\end{aligned}$$

$$\begin{aligned}\hat{\Omega}_{23}^{(1)} &= 2h\alpha\delta+\delta\alpha(2\alpha^2+2\delta^2+3\lambda_2+\lambda_1)h^2+\frac{\delta}{15}((27-2\sqrt{6})\lambda_2+15\lambda_1)\lambda_2\alpha h^3\\ &+\frac{6-\sqrt{6}}{30}(\lambda_1+3\lambda_2)\alpha\delta\lambda_2^2h^4+\frac{7-2\sqrt{6}}{75}\alpha\delta\lambda_2^4h^5,\end{aligned}$$

$$\begin{aligned}\hat{\Omega}_{31}^{(1)} &= h\alpha\delta+\frac{\alpha\delta}{2}(2\alpha^2+2\delta^2+3\lambda_1+\lambda_2)h^2+\frac{21-\sqrt{6}}{870}\Big((8-\sqrt{6})\lambda_2^2+29\lambda_1^2\\ &+\lambda_1\lambda_2(21+\sqrt{6})\Big)\delta\alpha h^3+\frac{6-\sqrt{6}}{60}(\lambda_1^2+2\lambda_2^2+\lambda_1\lambda_2)\delta\alpha\lambda_1h^4\\ &+\frac{7-2\sqrt{6}}{150}\alpha\delta\lambda_1^2\lambda_2^2h^5,\end{aligned}$$

$$\begin{aligned}\hat{\Omega}_{32}^{(1)} &= h\alpha\delta+\frac{1}{2}\alpha\delta(2\alpha^2+2\delta^2+3\lambda_2+\lambda_1)h^2\\ &+\frac{6-\sqrt{6}}{60}(\lambda_2^2(8+\sqrt{6})+(6+\sqrt{6})\lambda_1\lambda_2+2\lambda_1^2)\delta\alpha h^3\\ &+\frac{6-\sqrt{6}}{60}(\lambda_1\lambda_2+\lambda_2^2+2\lambda_1^2)\lambda_2\alpha\delta rac7-2\sqrt{6}150\alpha\delta\lambda_1^2\lambda_2^2h^5,\end{aligned}$$

$$\begin{aligned}\hat{\Omega}_{33}^{(1)} &= 1+(\delta^2+\alpha^2+\lambda_2+\lambda_1)h+\left(\frac{1}{2}\alpha^4+\lambda_1\lambda_2+\lambda_2\alpha^2+\frac{1}{2}\lambda_1^2+3\alpha^2\delta^2+\lambda_1\alpha^2\right.\\ &\left.+\frac{1}{2}\lambda_2+\frac{1}{2}\delta^4+\delta^2\lambda_2\right)h^2+\frac{27-2\sqrt{6}}{8460}(24\lambda_1\alpha^2\lambda_2\sqrt{6}+12\lambda_2^2\lambda_1\sqrt{6}+54\lambda_2^3\\ &+162\lambda_1^2\lambda_2-6\lambda_1^2\alpha^2\sqrt{6}+12\lambda_1^2\lambda_2\sqrt{6}-6\lambda_2^2\alpha^2\sqrt{6}+60\lambda_1^2\alpha^2+4\lambda_2^3\sqrt{6}\\ &+162\lambda_2^2\lambda_1+4\lambda_1^3\sqrt{6}+141\delta^2\lambda_1^2+12\lambda_1\delta^2\lambda_2\sqrt{6}+60\lambda_2^2\alpha^2+54\lambda_1^3+141\delta^2\lambda_2^2\\ &+162\lambda_1\lambda_2\delta^2+324\lambda_1\alpha^2\lambda_2)h^3+\frac{6-\sqrt{6}}{360}(2\lambda_1^3\lambda_2\sqrt{6}+2\lambda_1\lambda_2^3\sqrt{6}+6\lambda_1\delta^2\lambda_2^2\\ &+12\lambda_2\delta^2\lambda_1^2+6\delta^2\lambda_1^2\lambda_2+3\lambda_1^2\lambda_2\sqrt{6}+12\lambda_1\lambda_2^3+18\lambda_1^2\lambda_2^2+12\lambda_1^3\lambda_2+6\lambda_1^3\delta^2\\ &+12\lambda_1\alpha^2\lambda_2^2)h^4\frac{7-2\sqrt{6}}{300}(7\lambda_1+2\lambda_1\sqrt{6}+7\lambda_2+2\lambda_2\sqrt{6}+2\alpha^2+2\delta^2)\lambda_1^2\lambda_2^2h^5\\ &+\frac{1}{36}\lambda_1^3\lambda_2^3h^6;\end{aligned}$$

Method 2

$$\hat{\Omega}_{11}^{(2)} = 1 + (2\lambda_1 + \alpha^2)h + \left(2\lambda_1^2 + 2\lambda_1\alpha^2 + \frac{1}{2}(\alpha^2+\delta^2)^2\right)h^2$$
$$+\frac{1}{2}\lambda_1((\alpha^2+\delta^2)^2 + 3\lambda_1^2 + 4\lambda_1\alpha^2)h^3\frac{1}{8}\lambda_1^2((\alpha^2+\delta^2)^2 + 8\lambda_1\alpha^2 + 6\lambda_1^2)h^4$$
$$+\frac{1}{4}\lambda_1^4(\alpha^2+\lambda_1)h^5 + \frac{1}{16}\lambda_1^6h^6,$$

$$\hat{\Omega}_{12}^{(2)} = \delta^2 h + \delta^2(2\alpha^2+\lambda_1+\lambda_2)h^2 + \frac{1}{4}\delta^2(8\lambda_1\alpha^2 + 3\lambda_1^2 + 4\lambda_1\lambda_2 + \lambda_2^2)h^3$$
$$+\frac{1}{4}\delta^2\lambda_1(2\lambda_1\alpha^2 + (\lambda_1+\lambda_2)^2)h^4 + \frac{1}{16}\lambda_1^2\delta^2(\lambda_1+\lambda_2)h^5,$$

$$\hat{\Omega}_{13}^{(2)} = 2\alpha\delta h + \alpha\delta(2\alpha^2 + 3\lambda_1 + 2\delta^2 + \lambda_2)h^2 + \frac{1}{2}\lambda_1\alpha\delta(4\alpha^2 + 4\delta^2 + 5\lambda_1 + 3\lambda_2)h^3$$
$$\times\frac{1}{2}\delta\lambda_1^2\alpha(\alpha^2+\delta^2+2\lambda_1+2\lambda_2)h^4 + \frac{1}{4}\alpha\lambda_1^3\delta(\lambda_1+\lambda_2)h^5,$$

$$\hat{\Omega}_{21}^{(2)} = \delta^2 h + \delta^2(2\alpha^2+\lambda_1+\lambda_2)h^2 + \frac{1}{4}\delta^2(8\lambda_2\alpha^2 + 3\lambda_2^2 + 4\lambda_1\lambda_2 + \lambda_1^2)h^3$$
$$+\frac{1}{4}\delta^2\lambda_2(2\lambda_2\alpha^2 + (\lambda_1+\lambda_2)^2)h^4 + \frac{1}{16}\lambda_2^2\delta^2(\lambda_1+\lambda_2)^2h^6,$$

$$\hat{\Omega}_{22}^{(2)} = 1 + (2\lambda_2 + \alpha^2)h + \left(2\lambda_2^2 + 2\lambda_2\alpha^2 + \frac{1}{2}(\alpha^2+\delta^2)^2\right)h^2$$
$$+\frac{1}{2}\lambda_2((\alpha^2+\delta^2)^2 + 3\lambda_2^2 + 4\lambda_2\alpha^2)h^3 + \frac{1}{8}\lambda_2^2((\alpha^2+\delta^2)^2 + 8\lambda_2\alpha^2 + 6\lambda_2^2)h^4$$
$$+\frac{1}{4}\lambda_2^4(\alpha^2+\lambda_2)h^5 + \frac{1}{16}\lambda_2^6h^6,$$

$$\hat{\Omega}_{23}^{(2)} = 2h\alpha\delta + \delta\alpha(2\alpha^2 + 3\lambda_2 + 2\delta^2 + \lambda_1)h^2 + \frac{1}{2}\delta\lambda_2\alpha(5\lambda_2 + 4\alpha^2 + 4\delta^2 + 3\lambda_1)h^3$$
$$+\frac{1}{2}\delta\alpha\lambda_2(\alpha^2+\delta^2+2\lambda_1+2\lambda_2)h^4 + \frac{1}{4}\lambda_2^3\delta\alpha(\lambda_1+\lambda_2)h^5,$$

$$\hat{\Omega}_{31}^{(2)} = \alpha\delta h + \frac{1}{2}\alpha\delta(2\alpha^2 + 3\lambda_1 + 2\delta^2 + \lambda_2)h^2 + \frac{1}{4}\alpha\delta(4\alpha^2\lambda_2 + 2\delta^2\lambda_2 + \lambda_1\alpha_2$$
$$+ 2\delta^2\lambda_1 + 3\lambda_1\lambda_2 + \lambda_2^2 + 4\lambda_1^2)h^3 + \frac{1}{4}\delta\lambda_1\alpha(\alpha^2\lambda_2 + \delta^2\lambda_2 + (\lambda_1+\lambda_2)^2)h^4$$
$$+ \frac{1}{8}\alpha\delta\lambda_1^2\lambda_2(\lambda_1+\lambda_2)h^5;$$

$$\hat{\Omega}_{32}^{(2)} = h\alpha\delta + \frac{1}{2}\delta\alpha(2\alpha^2 + 3\lambda_2 + 2\delta^2 + \lambda_1)h^2 + \frac{1}{4}\delta\alpha(2\delta^2(\lambda_1+\lambda_2) + 2\alpha^2\lambda_2 + 2\lambda_1\alpha^2 + 4\lambda_2^2$$
$$+ 3\lambda_2\lambda_1 + \lambda_1^2)h^3\frac{1}{4}\delta\alpha\lambda_2(\delta^2\lambda_1 + \lambda_1\alpha^2 + (\lambda_1+\lambda_2)^2)h^4 + \frac{1}{8}\lambda_1\delta\alpha\lambda_2^2(\lambda_1+\lambda_2)h^5,$$

$$\hat{\Omega}_{33}^{(2)} = 1+(\lambda_2+\alpha^2+\delta^2+\lambda_1)h+\left(\frac{1}{2}\delta^4+\delta^2(\lambda_1+\lambda_2)+\frac{1}{2}\alpha^4+3\delta^2\alpha^2+\frac{1}{2}\lambda_1^2+\alpha^2\lambda_2\right.$$
$$\left.+\lambda_1\alpha^2+\frac{1}{2}\lambda_2^2+\lambda_1\lambda_2\right)h^2+\frac{1}{4}(\lambda_1+\lambda_2)(\lambda_1^2+2\delta^2\lambda_1+\lambda_1\lambda_2+2\lambda_1\alpha^2+\lambda_2^2+2\delta^2\lambda_2$$
$$+\alpha^4+2\alpha^2\lambda_2+\delta^4+6\delta^2\alpha^2)h^3+\left(\frac{3}{4}\lambda_2\alpha^2\delta^2\lambda_1+\frac{1}{8}\delta^2(\lambda_2^3+\lambda_1^3)+\frac{1}{4}\lambda_1^2\lambda_2^2+\frac{1}{4}\lambda_1^3\lambda_2\right.$$
$$\left.+\frac{3}{8}\lambda_1\lambda_2^2\delta^2+\frac{1}{4}\lambda_2^3\lambda_1+\frac{1}{2}\lambda_1\alpha^2\lambda_2(\lambda_1+\lambda_2)+\frac{1}{8}\lambda_1\lambda_2(\delta^4+\alpha^4)+\frac{3}{8}\lambda_1^2\lambda_2\delta^2\right)h^4$$
$$+\frac{1}{16}\lambda_1\lambda_2(\delta^2(\lambda_1+\lambda_2)^2+2\lambda_1\lambda_2(2\alpha^2+\lambda_1+\lambda_2))h^5+\frac{1}{16}(\lambda_1\lambda_2)^3h^6$$

and

Method 3

$$\hat{\Omega}_{11}^{(3)} = 1+(\alpha^2+2\lambda_1)h+\left(\frac{1}{2}(\alpha^2+\delta^2)^2+2\lambda_1\alpha^2+2\lambda_1^2\right)h^2+\frac{1}{2}\lambda_1(3\lambda_1^2+4\lambda_1\alpha^2$$
$$+(\alpha^2+\delta^2)^2)h^3+\frac{1}{120}(110\lambda_1^2+(144-4\sqrt{6})\lambda_1\alpha^2+15(\alpha^2+\delta^2)^2)\lambda_1^2h^4$$
$$+\frac{1}{60}(25\lambda_1+(27-2\sqrt{6})\alpha^2)\lambda_1^4h^5+\frac{1}{240}(35\lambda_1+(24-4\sqrt{6})\alpha^2)\lambda_1^5h^6$$
$$+\frac{1}{600}(25\lambda_1+(7-2\sqrt{6})\alpha^2)\lambda_1^6h^7+\frac{1}{144}\lambda_1^8h^8,$$

$$\hat{\Omega}_{12}^{(3)} = \delta^2h+(2\alpha^2+\lambda_2+\lambda_1)\delta^2h^2+\frac{1}{4}(8\lambda_1\alpha^2+4\lambda_1\lambda_2+3\lambda_1^2+\lambda_2^2)\delta^2h^3$$
$$+\frac{1}{2820}(27-2\sqrt{6})(47\lambda_1^2+(54+4\sqrt{6})(\lambda_1\alpha^2+\lambda_1\lambda_2)+(27+2\sqrt{6})\lambda_2^2)\lambda_1\delta^2h^4$$
$$+\frac{1}{22800}(39-4\sqrt{6})(95\lambda_1+(39+4\sqrt{6})\lambda_2)(\lambda_1+\lambda_2)\delta^2\lambda_1^2h^5$$
$$+\frac{1}{120}(6-\sqrt{6})(\lambda_1+\lambda_2)\delta^2\lambda_1^4h^6+\frac{1}{3600}(-6+\sqrt{6})^2\lambda_1^6\delta^2h^7,$$

$$\hat{\Omega}_{13}^{(3)} = 2h\alpha\delta+\delta\alpha(2\alpha^2+2\delta^2+3\lambda_1+\lambda_2)h^2+\frac{1}{2}(4\delta^2+4\alpha^2+3\lambda_2+5\lambda_1)\delta\lambda_1\alpha h^3$$
$$+\frac{1}{30}(15(\alpha^2+\delta^2)+30\lambda_2+(42+2\sqrt{6})\lambda_1)\alpha\delta\lambda_1^2h^4\frac{1}{1380}(11-\sqrt{6})(69\lambda_1$$
$$+(45+2\sqrt{6})\lambda_2)\delta\alpha\lambda_1^3h^5+\frac{1}{120}(6-\sqrt{6})\delta(3\lambda_1+\lambda_2)\alpha\lambda_1^4h^6,$$

$$
\begin{aligned}
\hat{\Omega}^{(3)}_{21} &= \delta^2 h + (2\alpha^2 + \lambda_2 + \lambda_1)\delta^2 h^2 + \frac{1}{4}(8\lambda_2\alpha^2 + 4\lambda_1\lambda_2 + 3\lambda_2^2 + \lambda_1^2)\delta^2 h^3 \\
&\quad + \frac{1}{60}((27 - 2\sqrt{6})\lambda_2^2 + 30\lambda_2\alpha^2 + 30\lambda_1\lambda_2 + 15\lambda_1^2)\lambda_2\delta^2 h^4 \\
&\quad + \frac{1}{240}(15\lambda_1 + (39 + 4\sqrt{6})\lambda_2)(\lambda_1 + \lambda_2)\delta^2\lambda_2^2 h^5 \\
&\quad + \frac{1}{120}(6 - \sqrt{6})(\lambda_1 + \lambda_2)\delta^2\lambda_2^4 h^6 + \frac{1}{600}(7 - \sqrt{6})\lambda_2^6\delta^2 h^7, \\
\hat{\Omega}^{(3)}_{22} &= 1 + (\alpha^2 + 2\lambda_2)h + \left(\frac{1}{2}(\alpha^2 + \delta^2)^2 + 2\lambda_2\alpha^2 + 2\lambda_2^2\right)h^2 + \frac{1}{2}(3\lambda_2^2 + 4\lambda_2\alpha^2 \\
&\quad + (\alpha^2 + \delta^2)^2)\lambda_2 h^3 + \frac{1}{120}(110\lambda_2^2 + (144 - 4\sqrt{6})\lambda_2\alpha^2 + 15(\alpha^2 + \delta^2)^2)\lambda_2^2 h^4 \\
&\quad + \frac{1}{60}(25\lambda_2 + (27 - 2\sqrt{6})\alpha)\lambda_2^4 h^5 + \frac{1}{240}(35\lambda_2 + (24 - 4\sqrt{6})\alpha^2)\lambda_2^5 h^6 \\
&\quad + \frac{1}{600}(25\lambda_2 + (7 - 2\sqrt{6})\alpha^2)\lambda_2^6 h^7 + \frac{1}{144}\lambda_2^8 h^8, \\
\hat{\Omega}^{(3)}_{23} &= 2h\alpha\delta + \delta\alpha(2\alpha^2 + 2\delta + 3\lambda_2 + \lambda_1)h^2 + \frac{1}{2}(4\delta^2 + 4\alpha^2 + 5\lambda_2 + 3\lambda_1)\delta\lambda_2\alpha h^3 \\
&\quad + \frac{1}{30}(15\delta^2 + (42 - 2\sqrt{6})\lambda_2 + 15\alpha^2 + 30\lambda_1)\delta\alpha\lambda_2^2 h^4 \\
&\quad + \frac{1}{1740}(21 - \sqrt{6})(29\lambda_1 + (45 - 2\sqrt{6})\lambda_2)\delta\alpha\lambda_2^3 h^5 \\
&\quad + \frac{1}{120}(6 - \sqrt{6})(\lambda_1 + 3\lambda_2)\delta\alpha\lambda_2^4 h^6 + \frac{1}{300}(7 - 2\sqrt{6})\delta\alpha\lambda_2^6 h^7, \\
\hat{\Omega}^{(3)}_{31} &= h\alpha\delta + \frac{1}{2}\delta\alpha(2\alpha^2 + 2\delta^2 + 3\lambda_1 + \lambda_2)h^2 + \frac{1}{4}(2(\delta^2 + \alpha^2)(\lambda_2 + \lambda_1) \\
&\quad + \lambda_2^2 + 3\lambda_1\lambda_2 + 4\lambda_1^2)\delta\alpha h^3 + \frac{1}{60}((6 - \sqrt{6})\lambda_2^3 + 15\lambda_1\lambda_2(\alpha^2 + \delta^2) \\
&\quad + 15\lambda_1\lambda_2(\lambda_2 + 2\lambda_1) + (21 - \sqrt{6})\lambda_1^3)\delta\alpha h^4 \\
&\quad + \frac{1}{240}(6 - \sqrt{6})(2\lambda_1^3 + 4\lambda_2^3 + \lambda_1^2\lambda_2(8 + \sqrt{6}) + \lambda_2^2\lambda_1(6 + \sqrt{6}))\delta\alpha\lambda_1 h^5 \\
&\quad + \frac{1}{240}(6 - \sqrt{6})(2\lambda_2^2 + \lambda_1 + \lambda_1\lambda_2)\delta\lambda_2\alpha\lambda_1^2 h^6 + \frac{1}{600}(7 - 2\sqrt{6})\alpha\delta\lambda_1^3\lambda_2^3 h^7, \\
\hat{\Omega}^{(3)}_{32} &= h\alpha\delta + \frac{1}{2}\delta\alpha(2\alpha^2 + 2\delta^2 + 3\lambda_2 + \lambda_1)h^2 \\
&\quad + \frac{1}{4}(2(\alpha^2 + \delta^2)(\lambda_1 + \lambda_2) + 3\lambda_1\lambda_2 + 4\lambda_2^2 + \lambda_1^2)\delta\alpha h^3 \\
&\quad + \frac{1}{60}((21 - \sqrt{6})\lambda_2^3 + 15\lambda_1\lambda_2(2\lambda_2 + \lambda_1) + (6 - \sqrt{6})\lambda_1^3)\delta\alpha h^4 \\
&\quad + \frac{1}{240}(6 - 2\sqrt{6})(2\lambda_2^3 + \lambda_1\lambda_2^2(8 + \sqrt{6}) + \lambda_1^2\lambda_2(6 + \sqrt{6}) + 4\lambda_1^3)\lambda_2\alpha h^5 \\
&\quad + \frac{1}{240}(6 - \sqrt{6})(\lambda_2^2 + 2\lambda_1^2 + \lambda_1\lambda_2)\delta\lambda_1\alpha\lambda_2^2 h^6 + \frac{1}{600}(7 - 2\sqrt{6})\alpha\delta\lambda_1^3\lambda_2^3 h^7,
\end{aligned}
$$

$$\hat{\Omega}_{33}^{(3)} = 1+(\alpha^2+\delta^2+\lambda_2+\lambda_1)h+\left(\frac{1}{2}\alpha^4+\lambda_1\delta^2+\lambda_1\lambda_2+\lambda_2\alpha^2+\frac{1}{2}\lambda_1^2+3\alpha^2\delta^2+\lambda_1\alpha^2\right.$$
$$\left.+\frac{1}{2}\lambda_2^2+\frac{1}{2}\delta^4+\delta^2\lambda_2\right)h^2+\frac{1}{4}(\lambda_2^2+\lambda_1\lambda_2+\lambda_1^2+2(\lambda_1+\lambda_2)(\alpha^2+\delta^2)+\alpha^4+\delta^4$$
$$+6\alpha^2\delta^2)(\lambda_1+\lambda_2)h^3+\left(-\frac{1}{60}\lambda_1^3\delta^2\sqrt{6}+\frac{1}{8}\lambda_1\delta^4\lambda_2+\frac{1}{10}\lambda 1^3\alpha^2+\frac{3}{4}\delta^2\lambda_2\lambda_1\alpha^2\right.$$
$$+\frac{3}{8}\lambda_1\delta^2\lambda_2^2+\frac{1}{2}\lambda_2\alpha^2\lambda_1^2+\frac{3}{8}\delta^2\lambda_1^2\lambda_2-\frac{1}{60}\lambda_2^3\alpha^2\sqrt{6}+\frac{1}{12}\lambda_2^4+\frac{1}{12}\lambda_1^4+\frac{1}{4}\lambda_1\lambda_2^3$$
$$+\frac{1}{4}\lambda_1^2\lambda_2^2+\frac{1}{4}\lambda_1^3\lambda_2+\frac{9}{40}\lambda_2^3\delta^2+\frac{9}{40}\lambda_1^3\delta^2-\frac{1}{60}\lambda_2^3\delta^2\sqrt{6}+\frac{1}{2}\lambda_1\alpha^2\lambda_2^2$$
$$\left.-\frac{1}{60}\lambda_1^3\alpha^2\sqrt{6}+\frac{1}{10}\lambda_2^3\alpha^2+\frac{1}{8}\lambda_2\alpha^4\lambda_1\right)h^4$$
$$+\frac{1}{1440}(6-\sqrt{6})(3\lambda_2^3\delta^2\lambda_1\sqrt{6}+30\lambda_2^3\delta^2\lambda_1+6\lambda_1^3\lambda_2^2\sqrt{6}+36\lambda_1^2\delta^2\lambda_2^2+24\lambda_1^4\lambda_2$$
$$+30\delta^2\lambda_1^3\lambda_2+24\lambda_2^4\lambda_1+72\lambda_1^2\alpha^2\lambda_2^2+4\lambda_1^4\lambda_2\sqrt{6}+6\lambda_1^2\lambda_2^3\sqrt{6}+4\lambda_2^4\lambda_1\sqrt{6}$$
$$+24\lambda_2^3\alpha^2\lambda_1+36\lambda_1^2\lambda_2^3+36\lambda_1^3\lambda_2^2+12\lambda_1^2\alpha^2\lambda_2^2\sqrt{6}+6\lambda_1^2\delta^2\lambda_2^2\sqrt{6}+12\lambda_1^4\delta^2$$
$$+3\delta^2\lambda_1^3\lambda_2\sqrt{6}+12\lambda_2^4\delta^2+24\lambda_1^3\alpha^2\lambda_2)h^5+\frac{1}{1440}(6-\sqrt{6})(2\lambda_1^3\lambda_2\sqrt{6}$$
$$+2\lambda_1\lambda_2^3\sqrt{6}+6\lambda_1\delta^2\lambda_2^2+12\lambda_2\alpha^2\lambda_1^2+6\delta^2\lambda_1^2\lambda_2+3\lambda_1^2\lambda_2^2\sqrt{6}+12\lambda_1\lambda_2^3$$
$$+18\lambda_1^2\lambda_2^2+12\lambda_1^3\lambda_2+6\lambda_2^3\delta^2+6\lambda_1^3\delta^2+12\lambda_1\alpha^2\lambda_2^2)\lambda_1\lambda_2h^6$$
$$+\frac{1}{1200}(7-2\sqrt{6})((7+2\sqrt{6})(\lambda_1+2\lambda_2)+2\alpha^2+2\delta^2)\lambda_1^3\lambda_2^3h^7$$
$$+\frac{1}{144}\lambda_1^4\lambda_2^4h^8.$$

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