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ORIGINAL PAPER

Solving Fredholm integral equations by approximating kernels by spline quasi-interpolants

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Abstract We study two methods for solving a univariate Fredholm integral equation of the second kind, based on (left and right) partial approximations of the kernel K by a discrete quartic spline quasi-interpolant. The principle of each method is to approximate the kernel with respect to one variable, the other remaining free. This leads to an approximation of K by a degenerate kernel. We give error estimates for smooth functions, and we show that the method based on the left (resp. right) approximation of the kernel has an approximation order $O(h^5)$ (resp. $O(h^6)$). We also compare the obtained formulae with projection methods.

Keywords Fredholm integral equation · Quasi-interpolation ·
Left approximation · Right approximation

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1 Introduction. Notations

We consider the Fredholm linear integral equation of the second kind

$$u(x) = f(x) + \int_a^b K(x,s)u(s)ds, \quad x \in I := [a,b], \tag{1}$$

where the kernel $K : I \times I \to \mathbb{R}$ is continuous, so that the associated integral operator is compact (see e.g. [1]). Degenerate kernel methods have been studied by many authors, in particular [1] and [5] describe a construction of appropriate degenerate kernels via interpolation. For constructing such kernel approximation, the author in [10] uses orthonormal expansion and trigonometric interpolation. In [6], the authors approximate the bivariate kernel by tensor product splines. Our method is different, and leads to simpler and less expensive computations. We want to solve (1) by approximating the univariate left and right sections of K:

$$x \to K(x,s) \quad \text{and} \quad s \to K(x,s)$$

by a spline quasi-interpolant. Such an operator Q can be defined as

$$Qf := \sum_{j\in J} f(t_j)L_j,$$

where $\{t_j,\ j \in J\}$ is a sequence of data points in I and $\{L_j,\ j \in J\}$ an associated sequence of *quasi-Lagrange splines* that are (finite) linear combinations of B-splines generating some space $\mathbb{S}$ of splines of degree d. We assume $\mathbb{P}_d \subset \mathbb{S}$ and Q is *exact on* $\mathbb{P}_d$, where $\mathbb{P}_d$ is the space of polynomials of degree d, that means $Qm_r = m_r$ for all monomials $m_r(x) := x^r$ for $0 \le r \le d$. We thus obtain left and right approximations of the kernel:

$$\tilde{K}(x,s) := \sum_{j\in J} K(t_j,s)L_j(x) \quad \text{and} \quad \overline{K}(x,s) := \sum_{j\in J} K(x,t_j)L_j(s).$$

Substituting in the integral equation leads to left and right approximations of the solution:

$$\tilde{u}(x) = f(x) + \sum_{j\in J} \tilde{c}_j L_j(x), \quad \text{and} \quad \bar{u}(x) := f(x) + \sum_{j\in J} \bar{c}_j K_j(x),$$

where $K_j(x) := K(x,t_j)$, $j \in J$. Assuming the sets of functions $\{L_j,\ j \in J\}$ and $\{K_j,\ j \in J\}$ are linearly independent, then the two vectors $\tilde{\mathbf{c}} := (c_j,\ j \in J)$ and $\bar{\mathbf{c}} := (\bar{c}_j,\ j \in J)$ are solutions of the systems of linear algebraic equations

$$\tilde{\mathbf{c}} = \tilde{A}\tilde{\mathbf{c}} + \tilde{b}, \quad \bar{\mathbf{c}} = \bar{A}\bar{\mathbf{c}} + \bar{b},$$

where the coefficients of matrices and vectors are respectively defined by

$$\tilde{A}(i,j) := \int_a^b K(t_i,s)L_j(s)ds \qquad \tilde{b}_i := \int_a^b K(t_i,s)f(s)ds,$$

and

$$\bar{A}(i,j) := \int_a^b L_j(s)K(s,t_i)ds, \qquad \bar{b}_i := \int_a^b L_i(s)f(s)ds.$$

The aim of this paper is to study more specifically these two methods for C^3 quartic spline quasi-interpolants since, as we will see below, it has interesting approximation properties. In particular it is much better than C^1 quadratic and C^2 cubic QIs and slightly better than C^4 quintic ones. Of course, the methods can be extended to other classes of spline functions (see for example [8, 16]).

Here is an outline of the paper. In Section 2, we briefly introduce univariate spline quasi-interpolants on an interval. In Section 3, we give a detailed study of quartic spline operators which are used in the next sections. In Section 4, we study the left and right approximations of the kernel by a quartic spline quasi-interpolant and their associated linear systems. In Section 5, we study error estimates for the corresponding solutions. In Section 6, we compare the two methods studied here to classical projection methods of interpolation or Galerkin types (see [1] and [2]), and we give some numerical examples.

2 Quartic splines on an interval

2.1 B-splines and monomials of degree 4

Let $\mathscr{X}_n := \{x_k, 0 \le k \le n\}$ be a partition of the interval $I = [a, b]$ into n subintervals, i.e. $x_k := a + kh_k$, with $h_k = x_k - x_{k-1}$ and $1 \le k \le n$. We consider the space $\mathscr{S}_4 = \mathscr{S}_4(I, \mathscr{X}_n)$ of quartic splines of class C^3 on this partition. Its canonical basis is formed by the $n+4$ normalized B-splines $\{B_k, k \in \Gamma_n\}$, with $\Gamma_n := \{1, 2, \ldots, n+4\}$. The support of B_k is the interval $[x_{k-5}, x_k]$, if we add multiple knots at the endpoints. We recall (see [16, Theorem 4.21 and Remark 4.1], e.g.) the representation of monomials using symmetric functions $\text{symm}_r(\mathscr{N}_k)$ of interior knots $\mathscr{N}_k = \{x_{k-4}, x_{k-3}, x_{k-2}, x_{k-1}\}$ in $\text{supp}(B_k)$. For $0 \le r \le 4$, one has:

$$m_r(x) = x^r = \sum_{k\in\Gamma_n} (-1)^{4-r}\frac{r!}{4!} D^{4-r}\psi_k(0) B_k(x) = \sum_{k\in\Gamma_n} \theta_k^{(r)} B_k(x)$$

where for $j \ge 0$, D^j is the derivation operator of the order j and

$$\psi_k(t) = (x_{k-4} - t)(x_{k-3} - t)(x_{k-2} - t)(x_{k-1} - t).$$

From that we deduce

$$\theta_k^{(r)} = \binom{4}{r}^{-1} \text{symm}_r(\mathscr{N}_k), \quad 0 \le r \le 4.$$

In particular, for $r = 0$, one has $\theta_k^{(0)} = 1$, for all $k \in \Gamma_n$, since $\sum_{k\in\Gamma_n} B_k(x) = 1$, and for $r = 1$, we obtain the Greville abscissae:

$$\theta_k = \theta_k^{(1)} = \frac{1}{4}\sum_{\ell=1}^{4} x_{k-\ell}$$

which are the coefficients of $m_1(x) = \sum_{k\in\Gamma_n} \theta_k B_k(x)$.

2.2 Discrete quasi-interpolant of degree 4

The discrete quasi-interpolant (abbr. dQI) of degree 4 used in the following is the spline operator

$$Qf = \sum_{k\in\Gamma_n} \mu_k(f) B_k$$

whose coefficients are linear combinations of discrete values of f on the set of data points

$$\mathscr{T}_n = \{t_j,\ j \in J_n\}, \quad J_n = \{1, 2, \ldots, n+2\}$$

defined by

$$t_1 = a, \quad t_{n+2} = b, \quad t_j = \frac{1}{2}(x_{j-2} + x_{j-1}), \quad 2 \le j \le n+1.$$

The dQI is constructed to be exact on $\mathbb{P}_4$, i.e.

$$Qp = p \quad \text{for all} \quad p \in \mathbb{P}_4,$$

that is equivalent to $Qm_r = m_r$, where

$$m_r(x) = \sum_{k\in\Gamma_n} \mu_k(m_r) B_k(x) = \sum_{k\in\Gamma_n} \theta_k^{(r)} B_k(x), \quad 0 \le r \le 4.$$

Therefore we obtain the five conditions

$$\mu_k(m_r) = \theta_k^{(r)} \quad \text{for} \quad k \in \Gamma_n, \quad 0 \le r \le 4.$$

For $5 \le k \le n$, the functionals $\mu_k(f)$ only use values of f in a neighbourhood of the support of B_k, thus it is natural to express $\mu_k(f)$ in the following way

$$\mu_k(f) = \alpha_k f_{k-3} + \beta_k f_{k-2} + \gamma_k f_{k-1} + \nu_k f_k + \lambda_k f_{k+1}$$

where $f_k = f(t_k)$. The above conditions are equivalent to the systems of linear equations:

$$\alpha_k t_{k-3}^r + \beta_k t_{k-2}^r + \gamma_k t_{k-1}^r + \nu_k t_k^r + \lambda_k t_{k+1}^r = \theta_k^{(r)} \quad 0 \le r \le 4.$$

For $1 \le k \le 4$ and $n+1 \le k \le n+4$ we get respectively:

$$\mu_k(f) = \alpha_k f_1 + \beta_k f_2 + \gamma_k f_3 + \nu_k f_4 + \lambda_k f_5,$$

$$\mu_k(f) = \alpha_k f_{n-2} + \beta_k f_{n-1} + \gamma_k f_n + \nu_k f_{n+1} + \lambda_k f_{n+2}.$$

All these systems have Vandermonde determinants $V_5(t_k, t_{k+1}, t_{k+2}, t_{k+3}, t_{k+4}) \neq 0$. As the t_j, $j \in J_n$ are distinct, they have unique solutions, whence the existence and unicity of the QI.

In the case of a uniform partition, the coefficient functionals are respectively defined by the following formulas (see [11]):

$$\begin{aligned}
\mu_1(f) &= f_1,\\
\mu_2(f) &= \frac{17}{105} f_1 + \frac{35}{32} f_2 - \frac{35}{96} f_3 + \frac{21}{160} f_4 - \frac{5}{224} f_5,\\
\mu_3(f) &= -\frac{19}{45} f_1 + \frac{377}{288} f_2 + \frac{61}{288} f_3 - \frac{59}{480} f_4 + \frac{7}{288} f_5,\\
\mu_4(f) &= \frac{47}{315} f_1 - \frac{77}{144} f_2 + \frac{251}{144} f_3 - \frac{97}{240} f_4 + \frac{47}{1008} f_5,\\
\mu_{n+1}(f) &= \frac{47}{315} f_{n+2} - \frac{77}{144} f_{n+1} + \frac{251}{144} f_n - \frac{97}{240} f_{n-1} + \frac{47}{1008} f_{n-2},\\
\mu_{n+2}(f) &= -\frac{19}{45} f_{n+2} + \frac{377}{288} f_{n+1} + \frac{61}{288} f_n - \frac{59}{480} f_{n-1} + \frac{7}{288} f_{n-2},\\
\mu_{n+3}(f) &= \frac{17}{105} f_{n+2} + \frac{35}{32} f_{n+1} - \frac{35}{96} f_n + \frac{21}{160} f_{n-1} - \frac{5}{224} f_{n-2},\\
\mu_{n+4}(f) &= f_{n+2},
\end{aligned}$$

and, for $5 \leq k \leq n$

$$\mu_k(f) = \frac{47}{1152}(f_{k-4} + f_k) - \frac{107}{288}(f_{k-3} + f_{k-1}) + \frac{319}{192} f_{k-2}.$$

It is easy to verify that $|\mu_2|_\infty = |\mu_{n+3}|_\infty \approx 1.77$, $|\mu_3|_\infty = |\mu_{n+2}|_\infty \approx 2.09$, $|\mu_4|_\infty = |\mu_{n+1}|_\infty \approx 2.88$, $|\mu_k|_\infty \approx 2.49$ for $1 \leq k \leq 5$, from which we deduce $\|Q\|_\infty \leq 2.88$. In addition, for $f \in C^5(I)$, we have the following error estimate (see [4, chap.5]):

$$\|f - Qf\|_{\infty, I_k} \leq 4\, d_{\infty, J_k}(f, \mathbb{P}_4),$$

where

$$I_k = [x_{k-1}, x_k], \quad J_k = [x_{k-3}, x_{k+4}], \quad 1 \leq k \leq n,$$

and

$$d_{\infty, J_k}(f, \mathbb{P}_4) = \max\{\|f - p\|_{\infty, J_k},\ p \in \mathbb{P}_4\}.$$

Then

$$\|f - Qf\|_\infty = O(h^5).$$

We can write the quasi-interpolant Q under the *quasi-Lagrange* form:

$$Qf = \sum_{j \in J_n} f_j L_j,$$

where the *quasi-Lagrange functions* L_j are linear combinations of five B-splines. In general, the support of L_j is the union of seven intervals. For example, the functionals using the value f_1 are $\{\mu_1, \mu_2, \mu_3, , \ \mu_4, \ \mu_5\}$, therefore we will have

$$L_1 := B_1 + \frac{17}{105}B_2 - \frac{19}{45}B_3 + \frac{47}{315}B_4 + \frac{47}{1152}B_5.$$

For $6 \le j \le n-3$, we have

$$L_j := \frac{47}{1152}(B_{j+3} + B_{j-1}) - \frac{107}{288}(B_j + B_{j+2}) + \frac{319}{912}B_{j+1}.$$

This representation is used in Sections 3 and 4 below.

2.3 Superconvergence of the quasi-interpolant at some points

Before giving numerical examples, we study the superconvergence phenomenon of Q at some specific points. More precisely, we determine the points of I where the convergence order is in $O(h^6)$ instead of $O(h^5)$. We first remark that

$$Qf(a) = \mu_1(f) = f(a) \text{ and } Qf(b) = \mu_{n+4}(f) = f(b),$$

i.e. the endpoints a and b are *interpolation points* of Q.

Now, by computing $Qf(x_j)$, $4 \le j \le n-4$, and $Qf(t_j)$, $6 \le j \le n-3$ we respectively obtain

$$\begin{aligned}
Qf(x_j) &= \frac{1}{24}(\mu_{j+1}(f) + \mu_{j+4}(f)) + \frac{11}{24}(\mu_{j+2}(f) + \mu_{j+3}(f)) \\
&= \frac{47}{27648}\{(f_{j-2} + f_{j+5}) + 89(f_{j-1} + f_{j+4}) \\
&\qquad - 2277(f_j + f_{j+3}) + 15965(f_{j+1} + f_{j+2})\} \\
Qf(t_j) &= \frac{1}{384}(\mu_{j-1}(f) + \mu_{j+3}(f)) + \frac{19}{96}(\mu_j(f) + \mu_{j+2}(f)) + \frac{115}{192}\mu_{j+1}(f) \\
&= \frac{47}{442368}\{(f_{j-4} + f_{j+4}) + 3144(f_{j-3} + f_{j+3}) - 19804(f_{j-2} + f_{j+2}) \\
&\qquad + 50168(f_{j-1} + f_{j+1}) + 375258 f_j)\}.
\end{aligned}$$

Table 1 Errors on the QI for the functions $(f_i)_{i=1\ldots3}$

n	E_1	Order	E_2	Order	E_3	Order
16	5.91(−03)		3.30(−02)		1.11(−01)	
32	1.82(−04)	5.02	1.40(−03)	4.56	3.46(−03)	5.00
64	7.76(−06)	4.55	2.72(−05)	5.68	1.09(−04)	4.99
128	2.60(−07)	4.90	1.01(−06)	4.75	3.36(−06)	5.02
256	8.11(−09)	5.00	3.94(−08)	4.68	1.03(−07)	5.03

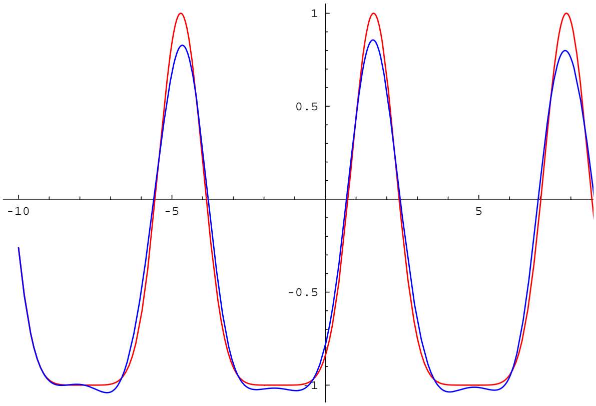

Fig. 1 f_2 in red and Qf_2 in blue

Theorem 1 *Assume $f \in C^5(I)$, then by using the Taylor series expansions of f at the points $\{x_j,\ 4 \leq j \leq n-4\}$ and $\{t_j,\ 6 \leq j \leq n-3\}$, we get the following superconvergence results:*

$$Qf(x_j) - f(x_j) = \frac{229}{27648} h^6 f^{(6)}(\theta_j) \quad and \quad Qf(t_j) - f(t_j) = \frac{55}{6912} h^6 f^{(6)}(\bar{\theta}_j)$$

where $\theta_j \in]t_{j-2}, t_{j+5}[$ and $\bar{\theta}_j \in]t_{j-4}, t_{j+4}[$.

2.4 Numerical examples

Consider the following functions defined on $I = [-5, 5]$ by

$$f_1(x) = \cos\left(\frac{\pi x}{2}\right), \quad f_2(x) = \sin(x - \cos(x)), \quad \text{and} \quad f_3(x) = (1 - x^2)(2 - x^3).$$

The following table gives the max errors $E_i = ||f_i - Qf_i||_\infty$, for $i = 1, 2, 3$, and the observed numerical convergence orders.

Table 1 shows that the convergence order of the quartic QI is in $O(h^5)$, according to the results of Section 2.2. Figure 1 shows the graphs of the function f_2 and of its quasi-interpolant Qf_2 on the interval $[-10, 10]$ with a steplength $h = 1$.

3 Left and right approximations of the kernel

In this section, we approximate the kernel K of (1) with respect to one of the two variables, the other remaining free. This leads to an approximation of K by a degenerate kernel which has two distinct forms according to the choice of that variable.

3.1 Left approximation of the kernel

In this subsection, the univariate function $x \to K(x, s)$ (left section of the kernel) is approximated by a spline quasi-interpolant (expressed in its quasi-Lagrange form):

$$K(x, s) \approx \tilde{K}(x, s) = \sum_{j \in J_n} K(t_j, s) L_j(x).$$

The approximate operator is defined by

$$\tilde{K}_n u(x) := \sum_{j \in J_n} L_j(x) \int_a^b K(t_j, s) u(s) ds,$$

and the approximate solution satisfies

$$\tilde{u}_n(x) - \sum_{j \in J_n} L_j(x) \int_a^b K(t_j, s) \tilde{u}_n(s) ds = f(x),$$

therefore $\tilde{u}_n$ is of the form:

$$\tilde{u}_n(x) = f(x) + \sum_{j \in J_n} c_j L_j(x).$$

Substituting in the integral equation, we get

$$\sum_{i \in J_n} c_i L_i(x) - \sum_{i \in J_n} L_i(x) \int_a^b K(t_i, s)[f(s) + \sum_{j \in J_n} c_j L_j(s) ds] = 0,$$

we deduce, in view of the linear independence of quasi-Lagrange splines:

$$c_i - \sum_{j \in J_n} c_j \int_a^b K(t_i, s) L_j(s) ds = \int_a^b K(t_i, s) f(s) ds.$$

Thus, in practice, we have to approximate the two families of integrals:

$$\int_a^b K(t_i, s) L_j(s) ds \quad \text{and} \quad \int_a^b K(t_i, s) f(s) ds.$$

The first can be computed by using product type or Gauss quadrature formulas (abbr. QF) with B-spline weights [3, 9]. The second can be computed by using classical QF of Newton–Cotes or Gauss types, or product type QF (see e.g. [7, 17]), or new QF obtained by integrating spline quasi-interpolants [11–13].

3.2 Right approximation of the kernel

Now, the univariate function $s \to K(x, s)$ (right section of the kernel) is approximated by a spline quasi-interpolant:

$$K(x, s) \approx \overline{K}(x, s) = \sum_{j \in J_n} K(x, t_j) L_j(s).$$

The approximate operator is then

$$\overline{K}_n u(x) := \sum_{j\in J_n} K(x, t_j) \int_a^b L_j(s)u(s)ds$$

and the approximate solution then satisfies

$$\overline{u}_n(x) - \sum_{j\in J_n} K(x, t_j) \int_a^b L_j(s)\overline{u}_n(s)ds = f(x).$$

Therefore $\overline{u}_n$ is of the form:

$$\overline{u}_n(x) = f(x) + \sum_{i\in J_n} \overline{c}_i \overline{K}_i(x) \quad \text{with} \quad \overline{K}_i(x) := K(x, t_i),$$

and we deduce, *if the functions $\overline{K}_i$ are linearly independent*,

$$\overline{c}_i - \sum_{j\in J_n} \overline{c}_j \int_a^b L_j(s)\overline{K}_i(s)ds = \int_a^b L_i(s) f(s)ds, \quad i \in J_n.$$

Remark

1) In this method, there is a collocation w.r.t. the second variable and a weighted mean w.r.t. the first.
2) All the integrals appearing in the matrix or in the vector of the r.h.s. can be computed by using *quadrature formulas with B-spline weights*.

4 Error estimates on the approximate solutions

4.1 Left approximation of the kernel

Let $\mathscr{K}$ (resp. $\tilde{\mathscr{K}}_n$) be the integral operator with kernel K (resp. $\tilde{K}_n$), then we have [1, 2]:

$$\|\mathscr{K} - \tilde{\mathscr{K}}_n\|_\infty = \max_{x\in I} \int_a^b |K(x, t) - \tilde{K}_n(x, t)|dt.$$

For a smooth kernel K, the following error estimate holds

$$|K(x, t) - \tilde{K}_n(x, t)| \le Ch^5 \|D^{5,0}K\|_{\infty,\Omega}, \quad \Omega := [a, b]^2$$

therefore we deduce

$$\|\mathscr{K} - \tilde{\mathscr{K}}_n\|_\infty = O(h^5).$$

On the other hand, if $(I - \mathscr{K})^{-1}$ also exists, then for all sufficiently large values of n, $(I - \tilde{\mathscr{K}}_n)^{-1}$ exists and the following bound holds:

$$\|(I - \tilde{\mathscr{K}}_n)^{-1}\|_\infty \le \frac{\|(I - \mathscr{K})^{-1}\|_\infty}{1 - \|(I - \mathscr{K})^{-1}\|_\infty \|\tilde{\mathscr{K}}_n - \mathscr{K}\|_\infty}.$$

Using classical results on the approximation of the kernel (see [1, Theorem 2.1.1]), we obtain

$$\|u - \tilde{u}_n\|_\infty \le \|(I - \tilde{\mathscr{K}_n})^{-1}\|_\infty \|\tilde{\mathscr{K}_n} - \mathscr{K}\|_\infty \|u\|_\infty,$$

and therefore

$$\|u - \tilde{u}_n\|_\infty \le C_1 h^5 \|u\|_\infty$$

where C_1 is a constant independent of n.

4.2 Right approximation of the kernel

Let $\overline{\mathscr{K}}_n$ be the integral operator with kernel $\overline{K}_n$, then we have

$$\|\mathscr{K} - \overline{\mathscr{K}}_n\|_\infty = \max_{x\in I} \int_a^b |K(x,t) - \overline{K}_n(x,t)| dt.$$

In order to give the approximation order of $\|\mathscr{K} - \overline{\mathscr{K}}_n\|_\infty$, we use the quadrature formula, based on Q_4, for approximating $\int_a^b |K(x,t) - \overline{K}_n(x,t)| dt$. This quadrature formula, introduced in [13] and studied in [11], has an approximation order $O(h^6)$ and is defined by

$$\mathscr{I}_4(f) := h \sum_{j=6}^{n-3} f_j + h \left[\frac{206}{1575}(f_1 + f_{n+2}) + \frac{107}{128}(f_2 + f_{n+1}) + \frac{6019}{5760}(f_3 + f_n) \right.$$
$$\left. + \frac{9467}{9600}(f_4 + f_{n-1}) + \frac{13469}{13440}(f_5 + f_{n-2}) \right] = \sum_{j=1}^{n+2} w_j f_j.$$

Then we have

$$\int_a^b |K(x,t) - \overline{K}_n(x,t)| dt = h \sum_{j=1}^{n+2} w_j |K(x,t_j) - \overline{K}_n(x,t_j)| + O(h^6).$$

On the other hand, for a smooth kernel K, the following error estimate holds

$$|K(x,t) - \overline{K}_n(x,t)| \le C' h^5 \|D^{0,5} K\|_{\infty,\Omega}, \quad \Omega := [a,b]^2,$$

and consequently, we deduce the following inequality

$$\int_a^b |K(x,t) - \overline{K}_n(x,t)| dt \le h \sum_{j=6}^{n-3} |K(x,t_j) - \overline{K}_n(x,t_j)|$$
$$+ C'' h^6 \|D^{0,5} K\|_{\infty,\Omega} + O(h^6).$$

According to Section 2.3, the $\{t_j,\ 6 \le j \le n-3\}$ are superconvergence points of Q and for $6 \le j \le n-3$, we get

$$|K(x,t_j) - \overline{K}_n(x,t_j)| \le \frac{55}{6912} h^6 \|D^{0,6} K\|_{\infty,\Omega},$$

therefore

$$h\sum_{j=6}^{n-3}|K(x,t_j)-\overline{K}_n(x,t_j)|\le(b-a)h^6\frac{55}{6912}\|D^{0,6}K\|_{\infty,\Omega}.$$

Then, finally we obtain

$$\|\mathscr{K}-\overline{\mathscr{K}}_n\|_\infty=O(h^6),$$

and from classical results on the approximation of the kernel (see [2]) we deduce that

$$\|u-\overline{u}_n\|_\infty\le C_2h^6\|u\|_\infty.$$

where C_2 is a constant independent of n.

5 Numerical examples

Now, we give some numerical examples for illustrating the theoretical results. Let us consider the following Fredholm integral equations

$$u_i(x)-\lambda_i\int_{a_i}^{b_i}K_i(x,t)u_i(x)dt=f_i(x),\quad 1\le i\le 2,$$

where the data of each integral equation are given in the following table.

i	λ_i	$[a_i,b_i]$	$K_i(x,t)$	$f_i(x)$	Exact solution
1	$\frac{1}{2}$	$[0,1]$	$(x+t)e^{-xt}$	$e^{-x}-\frac{1}{2}+\frac{1}{2}e^{-(x+1)}$	e^{-x}
2	1	$\left[0,\frac{\pi}{2}\right]$	$\sin(x)\cos(t)$	$\sin(x)$	$2\sin(x)$
3	1	$[0,\pi]$	$\sin(x-t)$	$\cos(x)$	$\frac{\cos(x)+\pi\sin(x)}{4+\pi^2}$

For these examples, we give the infinite norms of the errors between the exact and the approximate solutions given by the left and the right approximations of the kernel, in terms of the number n of subintervals, and we also give their numerical approximation orders noted O_L and O_R respectively.

5.1 Quadrature rules

In order to match with the desired order of the solver for the integral equation, the convergence order of the quadrature formulas used for computing the integrals appearing in the linear system of the left (resp. right) approximation of the kernel method must be at least $O(h^5)$ (resp. $O(h^6)$), for more details (see [1, chap.4]). For the integrals appearing in the r.h.s. of the linear system of the left approximation of the kernel method, we have used the predefined MATLAB function "quadl" which implements a high order method using an adaptive algorithm based on the 4-point Gauss–Lobatto quadrature rule, exact on polynomials of degree 5, given by

$$\int_\alpha^\beta f(t)dt = \frac{\beta-\alpha}{12}\left[f(\alpha)+5f\left(\frac{\alpha+\beta}{2}-\frac{1}{\sqrt{5}}(\beta-\alpha)\right)\right.$$
$$\left.+5f\left(\frac{\alpha+\beta}{2}+\frac{1}{\sqrt{5}}(\beta-\alpha)\right)+f(\beta)\right].$$

All the other integrals appearing in the linear systems of the above two methods are computed by using the 3-point Gauss quadrature formulas with B-spline weights (see [3, 9]). The formula corresponding to the classical quartic B-spline of support $\left[-\frac{5}{2},\frac{5}{2}\right]$ is given by

$$\int_{-\frac{5}{2}}^{\frac{5}{2}} f(t)B(t)dt = \frac{44}{69}f(0)+\frac{25}{138}\left[f\left(-\frac{1}{2}\sqrt{\frac{23}{5}}\right)+f\left(\frac{1}{2}\sqrt{\frac{23}{5}}\right)\right],$$

and those corresponding to the extremal quartic B-splines have specific forms.

5.2 Numerical results

The following numerical experiments confirm that the right approximation of the kernel has an approximation order $O(h^6)$.

n	$\|u_1-u_{1,n}\|_\infty$			
	left	O_L	right	O_R
8	4.5(−8)		1.2(−8)	
16	1.6(−9)	4.9	2.6(−10)	5.5
32	4.8(−11)	5.0	4.6(−12)	5.8
64	1.5(−12)	5.0	7.6(−14)	5.9
128	4.8(−14)	4.9	1.4(−15)	5.8
	$\|u_2-u_{2,n}\|_\infty$			
	left	O_L	right	O_R
8	2.7(−6)		9.5(−7)	
16	8.7(−8)	4.9	1.7(−8)	5.8
32	2.7(−9)	5.0	2.8(−10)	5.9
64	8.3(−11)	5.0	4.4(−12)	6.0
128	2.0(−12)	5.4	6.7(−14)	6.0
	$\|u_3-u_{3,n}\|_\infty$			
	left	O_L	right	O_R
8	6.3(−5)		7.4(−6)	
16	1.7(−6)	5.2	1.5(−7)	5.6
32	4.7(−8)	5.2	2.8(−09)	5.8
64	1.3(−9)	5.1	4.8(−11)	5.9
128	4.0(−11)	5.1	7.8(−13)	5.9

where $u_{i,n}$, $1 \le i \le 2$, is the approximation of u_i.

In the next section, we give a numerical comparison of the above two methods with some methods of the same order existing in the literature.

6 Reminder of projection methods

We first recall some results about projection methods having the same approximation order $O(h^5)$ as the left approximation of the kernel used for solving integral equations. For more details see [1] and [2].

6.1 An interpolating projector

Let $n = 4m$, $h = (b - a)/n$ and $x_j = a + jh$, $\quad j = 0, 1, \ldots, n$, be a uniform partition of $[a, b]$. Denoting V_n the set of all continuous functions whose restrictions to each subinterval $[x_{4i}, x_{4i+4}]$, $0 \le i \le m - 1$, is a polynomial of degree 4, it is clear that $\dim V_n = n + 1$. The Lagrange basis of V_n is formed by the elements $\ell_j(x)_{j=0,\ldots,n}$ satisfying $\ell_j(x_i) = \delta_{ij}$ and their expressions depend on the value of the remainder of the Euclidean division of j by 4 denoted by $r(j) \in \{0,\ 1,\ 2\ 3\}$. Indeed,

if $r(j) = 0$, we have

$$\ell_j(x) = \begin{cases} \frac{1}{24h^4}(x - x_{j-4})(x - x_{j-3})(x - x_{j-2})(x - x_{j-1}), & x_{j-4} \le x \le x_j\,, \\ \frac{1}{24h^4}(x - x_{j+1})(x - x_{j+2})(x - x_{j+3})(x - x_{j+4}), & x_j \le x \le x_{j+4}, \\ 0, & \text{otherwise,} \end{cases}$$

if $r(j) = 1$, we have

$$\ell_j(x) = \begin{cases} -\frac{1}{6h^4}(x - x_{j-1})(x - x_{j+1})(x - x_{j+2})(x - x_{j+3}), & x_{j-1} \le x \le x_{j+3}\,, \\ 0, & \text{otherwise,} \end{cases}$$

if $r(j) = 2$, we have

$$\ell_j(x) = \begin{cases} \frac{1}{4h^4}(x - x_{j-2})(x - x_{j-1})(x - x_{j+1})(x - x_{j+2}), & x_{j-2} \le x \le x_{j+2}\,, \\ 0, & \text{otherwise,} \end{cases}$$

if $r(j) = 3$, we have

$$\ell_j(x) = \begin{cases} -\frac{1}{6h^4}(x - x_{j-3})(x - x_{j-2})(x - x_{j-1})(x - x_{j+1}), & x_{j-3} \le x \le x_{j+1}\,, \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $\mathscr{P}_n$ the continuous quartic interpolating projector which maps $V = C[a, b]$ onto V_n, defined by the $n+1$ following conditions:

$$\mathscr{P}_n u(x_i) = u(x_i), \quad 0 \le i \le n.$$

It can be written in the Lagrange form as follows:

$$\mathscr{P}_n u(x) = \sum_{j=0}^{n} u(x_j)\ell_j(x).$$

Setting $u_n = \sum_{j=1}^{n} X_j \ell_j$, we have $X_i = u_n(x_i)$ for all i, and the approximating equation can be written in the form

$$X_i - \int_a^b K(x_i, s)\left(\sum_{j=0}^{n} X_j \ell_j(s)\right) ds = f(x_i), \quad 0 \le i \le n,$$

or in matrix form as follows

$$(I - M)X = f,$$

where X (resp. f) is the vector of $\mathbb{R}^{n+1}$ with components X_i (resp. $f_i = f(x_i)$), and M is the matrix with entries $M(i, j) = \int_a^b K(x_i, t)\ell_j(t)dt$.

On the other hand, if $u \in C^5([a, b])$ we have the following error estimate (see [2])

$$\|u - \mathscr{P}_n u\|_\infty \le Ch^5 \|u^{(5)}\|_\infty.$$

Moreover, as $\|\mathscr{K} - \mathscr{P}_n \mathscr{K}\|_\infty$ converges to zero, then for n large enough, the operator $(I - \mathscr{P}_n \mathscr{K})^{-1}$ exists and it is bounded. Therefore

$$\|u - u_n\|_\infty \le \|(I - \mathscr{P}_n \mathscr{K})^{-1}\|_\infty \|u - \mathscr{P}_n u\|_\infty.$$

Consequently, for u smooth enough, we obtain

$$\|u - u_n\|_\infty = O(h^5).$$

6.2 Galerkin projector

Let V and V_n be the spaces introduced above. Denote the inner product of V by $\langle ., . \rangle$. Then, the orthogonal projection $\mathscr{P}_n u$ of $u \in V$ onto V_n is defined by the $n+1$ following equations:

$$\langle u - \mathscr{P}_n u, \ell_i \rangle = 0, \quad 0 \le i \le n.$$

Let $u_n = \sum_{j=0}^{n} X_j \ell_j \in V_n$ be the approximating solution of u, and let r_n be the residual function defined by

$$\begin{aligned} r_n(x) &= u_n(x) - \int_a^b K(x, t)u_n(t)dt - f(x) \\ &= \sum_{j=0}^{n} X_j \left(\ell_j(x) - \int_a^b K(x, t)\ell_j(t)dt\right) - f(x), \quad x \in [a, b]. \end{aligned} \tag{2}$$

The quantity (2) can also written in operator notation as

$$r_n = u_n - \mathscr{K} u_n - f.$$

Therefore, if $\mathscr{P}_n r_n = 0$, we obtain $\langle r_n, \ell_i \rangle = 0$ for all $0 \le i \le n$, i.e.

$$\langle u_n - \mathscr{K} u_n, \ell_i \rangle = \langle f, \ell_i \rangle, \quad 0 \le i \le n.$$

Replacing u_n by its above expression, we obtain:

$$\sum_{j=0}^{n} \langle \ell_j, \ell_i \rangle X_j - \sum_{j=0}^{n} \langle \mathscr{K} \ell_j, \ell_i \rangle X_j = \langle f, \ell_i \rangle, \quad 0 \le i \le n. \tag{3}$$

Let us denote by $G \in \mathbb{R}^{(n+1)\times(n+1)}$ the Gram matrix associated with the basis $\{\ell_j,\ 0 \le j \le n\}$, with entries

$$G(i, j) = \langle \ell_j, \ell_i \rangle = \int_a^b \ell_j \ell_i,$$

by $\overline{M} \in \mathbb{R}^{(n+1)\times(n+1)}$ the matrix with entries

$$\overline{M}(i, j) = \langle \mathscr{K} \ell_j, \ell_i \rangle = \int_a^b \int_a^b K(x, t) \ell_j(t) \ell_i(x) dt dx,$$

by $X \in \mathbb{R}^{n+1}$ the vector with components X_j, and by $c \in \mathbb{R}^{n+1}$ the vector with components

$$c_i = \langle f, \ell_i \rangle = \int_a^b f(x) \ell_i(x).$$

Then, the linear system (3) becomes:

$$(G - \overline{M}) X = c.$$

In general, the entries of the above matrices can be computed by using classical Newton–Cotes or Gauss quadrature formulae.

On the other hand, according to [2], if $u \in C^5([a, b])$ we have the following error estimate

$$\|u - \mathscr{P}_n u\| \le C' h^5 \|u^{(5)}\|_\infty.$$

Then for n large enough, the operator $(I - \mathscr{P}_n \mathscr{K})^{-1}$ exists and is bounded. Therefore

$$\|u - u_n\| \le \|(I - \mathscr{P}_n \mathscr{K})^{-1}\| \, \|u - \mathscr{P}_n u\|.$$

Consequently, for u smooth enough, we obtain

$$\|u - u_n\| = O(h^5).$$

6.3 Comparison with projection methods

In this subsection, we reconsider the examples of integral equations introduced in Section 5 We compute their approximating solutions by using the two projection methods presented in Sections 6.1 and 6.2. Then we compare the obtained numerical results with those obtained by using the left and right approximations of the kernel. All the integrals appearing in the matrices or in the vectors of the r.h.s. are computed by using Gauss–Lobatto quadrature formulae.

Example 1 Exact solution $u(x) = e^{-x}$

$$u(x) - \frac{1}{2}\int_0^1 (x+t)e^{-xt}u(t)dt = e^{-x} - \frac{1}{2} + \frac{1}{2}e^{-(x+1)}$$

n	left	right	intproj	galproj
8	4.5(−8)	1.2(−8)	7.6(−7)	9.5(−7)
16	1.6(−9)	2.6(−10)	2.6(−8)	3.4(−8)
32	4.8(−11)	4.6(−12)	8.5(−10)	1.1(−9)
64	1.5(−12)	7.6(−14)	2.5(−11)	3.7(−11)
128	4.8(−14)	1.4(−15)	8.3(−13)	1.2(−12)

Example 2 Exact solution $u(x) = 2\sin(x)$.

$$u(x) - \int_0^{\frac{\pi}{2}} \sin(x)\cos(t)u(t)dt = \sin(x)$$

n	left	right	intproj	galproj
8	2.7(−6)	9.5(−7)	1.7(−5)	2.0(−5)
16	8.7(−8)	1.7(−8)	5.4(−7)	7.3(−7)
32	2.7(−9)	2.8(−10)	1.7(−8)	2.3(−8)
64	8.3(−11)	4.4(−12)	5.4(−10)	7.4(−10)
128	2.0(−12)	6.7(−14)	1.6(−11)	2.3(−11)

Example 3 Exact solution $u(x) = 2\,\frac{\cos(x)+\pi\sin(x)}{4+\pi^2}$

$$u(x) - \int_0^{\pi} \sin(x-t)u(t)dt = \cos(x)$$

n	left	right	intproj	galproj
8	6.3(−5)	7.4(−6)	1.4(−4)	1.7(−4)
16	1.7(−6)	1.5(−7)	4.8(−6)	5.8(−6)
32	4.7(−8)	2.8(−9)	1.5(−7)	1.7(−7)
64	1.3(−9)	4.8(−11)	4.7(−9)	4.9(−9)
128	4.0(−11)	7.8(−13)	1.4(−10)	1.4(−10)

7 Conclusion

According to the above numerical examples, we remark that the method based on the right approximation of the kernel is better than the others since its approximation order is $O(h^6)$. We also observe that the results given by the left approximation of the kernel are slightly better than those provided by projection methods.

Comparing the computational aspect of these methods, we see that the Galerkin method is more expensive since we have $(n+1)(n+2)$ simple integrals and $(n+1)^2$ double integrals to evaluate instead of $(n+2)(n+3)$ simple integrals in the left and right approximation methods and $(n+1)^2$ simple integrals in the collocation method.

On the other hand, the required solution regularity in the left and the right approximation methods is only $C[a,b]$ instead of the class $C^5[a,b]$ in the projection methods.

We are extending this study to multivariate Fredholm integral equations using multivariate QIs on bounded domains (see [14] and [15]). More details will be given in forthcoming papers.

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