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Numerical approximations and solution techniques for the space-time Riesz–Caputo fractional advection-diffusion equation

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Abstract In this paper, we consider a space-time Riesz–Caputo fractional advection-diffusion equation. The equation is obtained from the standard advection-diffusion equation by replacing the first-order time derivative by the Caputo fractional derivative of order $\alpha \in (0, 1]$, the first-order and second-order space derivatives by the Riesz fractional derivatives of order $\beta_1 \in (0, 1)$ and $\beta_2 \in (1, 2]$, respectively. We present an explicit difference approximation and an implicit difference approximation for the equation with initial and boundary conditions in a finite domain. Using mathematical induction, we prove that the implicit difference approximation is unconditionally stable and convergent, but the explicit difference approximation is conditionally stable and convergent. We also present two solution techniques: a Richardson extrapolation method is used to obtain higher order accuracy and the short-memory principle is used to investigate the effect of the amount of computations. A numerical example is given; the numerical results are in good agreement with theoretical analysis.

Keywords Numerical approximation · Riesz fractional derivative · Caputo fractional derivative · Stability and convergence · Richardson extrapolation · Short-memory principle

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1 Introduction

In recent years, fractional differential equations have attracted much attention. The fractional advection-diffusion equation provides a useful description of transport dynamics in complex systems which are governed by anomalous diffusion and non-exponential relaxation [22]. There has been significant interest in developing numerical methods for their solution. Meerschaert and Tadjeran [20] presented numerical methods to solve the one-dimensional fractional advection-dispersion equations with Riemann-Liouville fractional derivative on a finite domain. Roop [24] investigated the numerical approximation of the variational solution to the fractional advection-diffusion equation on bounded domains in $\mathbf{R}^2$. Liu et al. [14, 15] transformed the space fractional advection-diffusion equation into a system of ordinary differential equations (method of lines) that was then solved using backward differentiation formulas. Liu et al. [16] also considered a space-time fractional advection-diffusion with Caputo time fractional derivative and Riemann-Liouville space fractional derivatives. The authors proposed an implicit difference method and an explicit difference method, and discussed their stability and convergence. In addition, Del-Castillo-Negrete [4] presented an α -weighted explicit/implicit numerical integration scheme based on the Grünwald-Letnikov representation of the regularized fractional diffusion operator in the flux conserving form. Meerschaert et al. [19] examined some numerical schemes to solve a class of initial-boundary value problems involving fractional partial differential equations with variable coefficients on a finite domain. Recently, some researchers considered solution of space-time fractional diffusion equations from the probabilistic angle of continuous time random walks and Lévy processes [1, 10–12, 17, 18].

Richardson extrapolation is an effective method to improve convergence order. Diethelm et al. [7] presented an extrapolation type algorithm for the numerical solution of fractional order differential equations. An extrapolated Crank-Nicolson method for a one-dimensional fractional diffusion equation was discussed in [28]. Tadjeran and Meerschaert [27] combined the alternating directions implicit approach with a Crank-Nicolson discretization and a Richardson extrapolation to solve the two-dimensional fractional diffusion equation. They obtained an unconditionally stable finite difference method with second-order accuracy. Recently, Chen et al. [2] proposed a multivariate extrapolation of a two-dimensional anomalous sub-diffusion equation.

As is well known, the fractional order differential operator is a non-local operator, which requires more involved computational schemes for its handling. Two approaches, namely the short-memory principle [23] and the logarithmic principle [9], have been suggested to deal with this situation.

In this paper, we consider the following space-time Riesz–Caputo fractional advection-diffusion equation:

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = B_1\frac{\partial^{\beta_1}}{\partial |x|^{\beta_1}}u(x,t) + B_2\frac{\partial^{\beta_2}}{\partial |x|^{\beta_2}}u(x,t) + f(x,t), \tag{1}$$

where α $(0 < \alpha \leq 1)$, β_1 $(0 < \beta_1 < 1)$ and β_2 $(1 < \beta_2 \leq 2)$ are real parameters. The coefficients B_1 and B_2 are both positive constants. The time fractional derivative $\frac{\partial^\alpha}{\partial t^\alpha}u(x,t)$ is the Caputo fractional derivative of order α $(0 < \alpha \leq 1)$ defined by [23]

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,\eta)}{\partial \eta}\frac{d\eta}{(t-\eta)^\alpha}, & 0 < \alpha < 1, \\ \frac{\partial u(x,t)}{\partial t}, & \alpha = 1, \end{cases} \tag{2}$$

while the derivatives $\frac{\partial^{\beta_1}}{\partial |x|^{\beta_1}}$ and $\frac{\partial^{\beta_2}}{\partial |x|^{\beta_2}}$ are Riesz space-fractional derivatives of order β_1 and β_2 respectively, defined by [11, 25, 29]

$$\frac{\partial^\beta}{\partial |x|^\beta}u(x,t) = -c\left[\frac{\partial^\beta}{\partial x^\beta}u(x,t) + \frac{\partial^\beta}{\partial (-x)^\beta}u(x,t)\right], \tag{3}$$

where the coefficient $c = \frac{1}{2\cos(\beta\pi/2)}$, and

$$\frac{\partial^\beta}{\partial x^\beta}u(x,t) = \frac{1}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^{x}\frac{u(\xi,t)d\xi}{(x-\xi)^{\beta+1-n}}, \quad n = \lceil\beta\rceil, \tag{4}$$

$$\frac{\partial^\beta}{\partial (-x)^\beta}u(x,t) = \frac{(-1)^n}{\Gamma(n-\beta)}\frac{\partial^n}{\partial x^n}\int_{x}^{+\infty}\frac{u(\xi,t)d\xi}{(x-\xi)^{\beta+1-n}}, \quad n = \lceil\beta\rceil. \tag{5}$$

For $0 < \beta \leq 2$, the Riesz space-fractional derivative is a symmetric fractional generalization of the second-order derivative and also the first-order derivative (see Appendix).

We derived the fundamental solution for the space-time Riesz–Caputo fractional advection-diffusion equation with an initial condition in [26]. The fundamental solution can be interpreted as a spatial probability density function evolving in time. In [26], we also investigated a discrete random walk model based on an explicit finite difference approximation for the space-time Riesz–Caputo fractional advection-diffusion equation with an initial condition. In this paper, we will discuss numerical approximations and solution techniques for the equation.

The structure of the paper is as follows. In Section 2, we present an explicit difference approximation and an implicit difference approximation for the space-time Riesz–Caputo fractional advection-diffusion equation with initial and boundary conditions in a finite domain. In Sections 3 and 4, we discuss the stability and convergence of the two difference approximations, respectively. In Section 5, we apply the Richardson extrapolation to obtain higher order accuracy and investigated the effect of the short-memory principle. Finally, numerical results are given to evaluate the methods in Section 6.

2 Finite difference approximations

Firstly, we introduce a numerical treatment of the space-time Riesz–Caputo fractional advection-diffusion equation.We consider a discrete form of (1) both in time and space. We introduce a spatial grid $-\infty < ... < x_{i-2} < x_{i-1} < x_i < x_{i+1} < x_{i+2} < ... < \infty$ with the step $h = x_k - x_{k-1}$. We denote the value of the function $u(x)$ at the point x_k as $u_k = u(x_k)$, for $k \in \mathbf{Z}$. Using the relationship between the Riemann-Liouville derivative and the Grünwald-Letnikov scheme [23], we discretize the Riesz fractional advection term $\frac{\partial^{\beta_1} u}{\partial |x|^{\beta_1}}$ by the Grünwald-Letnikov scheme in the case $0 < \beta_1 < 1$, and the Riesz fractional diffusion term $\frac{\partial^{\beta_2} u}{\partial |x|^{\beta_2}}$ by the shifted Grünwald-Letnikov scheme in the case $1 < \beta_2 \leq 2$ [26]:

$$\frac{\partial^{\beta_1}}{\partial |x|^{\beta_1}} u_i \approx \frac{1}{h^{\beta_1}} \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_1)} u_{i+k}, \tag{6}$$

where

$$\begin{cases} \omega_0^{(\beta_1)} = -2c_1, \\ \omega_{\pm k}^{(\beta_1)} = (-1)^{k+1} \binom{\beta_1}{k} c_1, \ k = 1, 2, \ldots \end{cases} \tag{7}$$

$$\frac{\partial^{\beta_2}}{\partial |x|^{\beta_2}} u_i \approx \frac{1}{h^{\beta_2}} \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_2)} u_{i+k}, \tag{8}$$

where

$$\begin{cases} \omega_0^{(\beta_2)} = 2 \binom{\beta_2}{1} c_2, \\ \omega_{\pm 1}^{(\beta_2)} = -\left[\binom{\beta_2}{2} + 1 \right] c_2, \\ \omega_{\pm k}^{(\beta_2)} = (-1)^k \binom{\beta_2}{k+1} c_2, \ k = 2, 3, \ldots, \end{cases} \tag{9}$$

$c_1 = \frac{1}{2\cos(\beta_1 \pi/2)}$, $c_2 = \frac{1}{2\cos(\beta_2 \pi/2)}$. It should be noted that the conditions $\beta_1 \neq 1$, $\beta_2 \neq 2$ are needed for the coefficients c_1 and c_2 to be finite.

At a point x_k at the moment of time t_n we denote the function $u(x, t)$ as $u_k^n = u(x_k, t_n)$ for $k \in \mathbf{Z}$ and $n \in \mathbf{N}$. Adopting the discrete scheme in [13, 26], we discretize the Caputo time fractional derivative as

$$\frac{\partial^\alpha}{\partial t^\alpha} u_i^{n+1} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j \left[u_i^{n+1-j} - u_i^{n-j} \right] + O(\tau^{2-\alpha}), \tag{10}$$

where $b_j = (j+1)^{1-\alpha} - j^{1-\alpha}$, $j = 0, 1, 2, \cdots, n$. It should also be noted that the numerical approximation (10) is similar to the predictor of the predictor-corrector method proposed and investigated in [5, 6], but the meanings of the terms are different. The predictor-corrector method can be seen as a generalization of the classical one-step Adams–Bashforth–Moulton scheme for first-order equations. The integration in the predictor is approximated by the product rectangle rule, while the derivative $\frac{\partial u(x,\eta)}{\partial \eta}$ within the interval $[t_j, t_{j+1}]$ in the integration of the Caputo derivative is approximated by the difference quotient $\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t}$.

Now we replace (1) with the following explicit difference approximation:

$$\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{j=0}^{n} b_j[u_i^{n+1-j}-u_i^{n-j}] = \frac{B_1}{h^{\beta_1}}\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_1)}u_{i+k}^n + \frac{B_2}{h^{\beta_2}}\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_2)}u_{i+k}^n + f_i^n. \tag{11}$$

The above equation can be simplified to

$$u_i^{n+1} = b_n u_i^0 + \sum_{j=0}^{n-1}(b_j - b_{j+1})u_i^{n-j} + B_1\mu_1\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_1)}u_{i+k}^n + B_2\mu_2\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_2)}u_{i+k}^n + \mu_0 f_i^n, \tag{12}$$

where $\mu_0 = \tau^\alpha\Gamma(2-\alpha)$, $\mu_1 = \frac{\tau^\alpha\Gamma(2-\alpha)}{h^{\beta_1}}$, $\mu_2 = \frac{\tau^\alpha\Gamma(2-\alpha)}{h^{\beta_2}}$, $f_i^n = f(x_i, t_n)$.

Next, we consider the following space-time Riesz–Caputo fractional advection-diffusion equation in the finite domain $(0 \le x \le l)$ with the given boundary-initial conditions:

$$\begin{aligned} \frac{\partial^\alpha}{\partial t^\alpha}u(x,t) &= B_1\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u(x,t) + B_2\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u(x,t) \\ &\quad + f(x,t), \quad (x,t)\in\Omega=[0,l]\times[0,T], \\ u(x,0) &= g(x), \qquad 0<x<l, \\ u(0,t) &= u(l,t) = 0, \quad 0<t\le T. \end{aligned} \tag{13}$$

We divide the domain $[0, l]$ into N sub-domains with the step $h = \frac{l}{N}$. We propose to follow the numerical treatment [3], which assumes the same values of the function $u(x, t)$ outside the domain limits as the values predicted on boundary nodes x_0 and x_N:

$$u(x_k, t) = \begin{cases} u(x_0, t) = 0, & k < 0, \\ u(x_N, t) = 0, & k > N. \end{cases} \tag{14}$$

Based on previous considerations [26], our explicit finite difference approximation to (13) can be derived from (12). Here, we need to modify the expression (12) for the discretization of the Riesz derivative to get the following explicit difference approximation:

$$
\begin{aligned}
u_i^{n+1} &= \sum_{j=0}^{n-1}(b_j - b_{j+1})u_i^{n-j} + b_n u_i^0 \\
&\quad + B_1\mu_1 \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} u_{i+k}^n + B_2\mu_2 \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} u_{i+k}^n + \mu_0 f_i^n, \\
&\qquad\qquad\qquad\qquad i = 1, ..., N-1, \\
u_0^{n+1} &= u_N^{n+1} = 0, \\
u_i^0 &= g_i = g(x_i), \qquad\qquad i = 0, 1, ..., N.
\end{aligned} \tag{15}
$$

We can also derive the implicit numerical scheme

$$
\begin{aligned}
&\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j \left[u_i^{n+1-j} - u_i^{n-j}\right] \\
&\quad = \frac{B_1}{h^{\beta_1}} \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} u_{i+k}^{n+1} + \frac{B_2}{h^{\beta_2}} \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} u_{i+k}^{n+1} + f_i^{n+1}.
\end{aligned} \tag{16}
$$

Thus we have the following implicit difference approximation:

$$
\begin{aligned}
&u_i^{n+1} - B_1\mu_1 \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} u_{i+k}^{n+1} - B_2\mu_2 \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} u_{i+k}^{n+1} \\
&\quad = \sum_{j=0}^{n-1}(b_j - b_{j+1})u_i^{n-j} + b_n u_i^0 + \mu_0 f_i^{n+1}, \qquad i = 1, ..., N-1, \\
&u_0^{n+1} = u_N^{n+1} = 0, \\
&u_i^0 = g_i = g(x_i), \qquad\qquad i = 0, 1, ..., N.
\end{aligned} \tag{17}
$$

Note that the coefficients possess the following properties:

Lemma 1 *The coefficients b_j, $j = 1, 2, \cdots$, satisfy* [20, 30]*:*

(1) $b_j > 0$, $j = 1, 2, \cdots$;
(2) $b_j > b_{j+1}$, $j = 0, 1, \cdots$.

Lemma 2 *The coefficients $\omega_k^{(\beta_1)}$ and $\omega_k^{(\beta_2)}$, $(k \in \mathbf{Z})$ satisfy* [26]*:*

(1) $\omega_0^{(\beta_1)} < 0$, $\omega_0^{(\beta_2)} < 0$;
(2) $\omega_{\pm k}^{(\beta_1)} \geq 0$, $\omega_{\pm k}^{(\beta_1)} \geq 0$ *for* $k = 1, 2, \ldots$;
(3) $\sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_1)} = \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_2)} = 0.$

3 Stability of the explicit and implicit difference approximations

Firstly, we consider the stability of the explicit difference approximation (15). We assume that the initial data has error ε_i^0. Let $\widetilde{g}_i^0 = g_i^0 + \varepsilon_i^0$ $(i = 1, ..., N-1)$,

u_i^k and $\widetilde{u}_i^k (i = 1, ..., N-1)$ be the numerical solutions of Eq. 15 corresponding to the initial data g_i^0 and $\widetilde{g}_i^0$ $(i = 1, ..., N-1)$, respectively. Then $\varepsilon_i^k = u_i^k - \widetilde{u}_i^k$ satisfies

$$\varepsilon_i^1 = b_0 \varepsilon_i^0 + B_1 \mu_1 \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} \varepsilon_{i+k}^0 + B_2 \mu_2 \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} \varepsilon_{i+k}^0$$

$$\varepsilon_i^{n+1} = [1 - b_1 + B_1 \mu_1 \omega_0^{(\beta_1)} + B_2 \mu_2 \omega_0^{(\beta_2)}] \varepsilon_i^n + \sum_{j=1}^{n-1} (b_j - b_{j+1}) \varepsilon_i^{n-j}$$

$$+ b_n \varepsilon_i^0 + B_1 \mu_1 \sum_{k=-i, k \neq 0}^{N-i} \omega_k^{(\beta_1)} \varepsilon_{i+k}^n + B_2 \mu_2 \sum_{k=-i, k \neq 0}^{N-i} \omega_k^{(\beta_2)} \varepsilon_{i+k}^n,$$

$$n = 1, 2, \ldots. \tag{18}$$

In the following theorem, we use $E^k = [\varepsilon_1^k, \varepsilon_2^k, ..., \varepsilon_{N-1}^k]^T$.

Theorem 1 (Stability of the explicit difference approximation) *For*

$$\tau^\alpha \leq \frac{2 - 2^{1-\alpha}}{2\Gamma(2-\alpha) \left[\dfrac{B_1 c_1}{h^{\beta_1}} - \dfrac{B_2 c_2 \beta_2}{h^{\beta_2}} \right]}, \tag{19}$$

the explicit difference approximation (15) *for the space-time Riesz–Caputo fractional advection-diffusion equation is stable.*

Proof The stability condition is equivalent to

$$\| E^{n+1} \|_\infty \leq \| E^0 \|_\infty, n = 0, 1, 2, \cdots.$$

We will use mathematical induction to get the above result. For $n = 0$, let $|\varepsilon_l^1| = \max_{1 \leq j \leq N-1} |\varepsilon_j^1|$.

If $\tau^\alpha \leq \frac{2-2^{1-\alpha}}{2\Gamma(2-\alpha)\left[\frac{B_1 c_1}{h^{\beta_1}} - \frac{B_2 c_2 \beta_2}{h^{\beta_2}}\right]}$, we obtain $b_0 + B_1 \mu_1 \omega_0^{\beta_1} + B_2 \mu_2 \omega_0^{\beta_2} > 0$. Thus, applying Lemma 1 and Lemma 2, we have

$$\begin{aligned}
\| E^1 \|_\infty = |\varepsilon_l^1| &\leq \left[b_0 + B_1 \mu_1 \omega_0^{\beta_1} + B_2 \mu_2 \omega_0^{\beta_2} \right] |\varepsilon_l^0| \\
&\quad + B_1 \mu_1 \sum_{k=-l, k \neq 0}^{N-l} \omega_k^{(\beta_1)} |\varepsilon_{l+k}^0| + B_2 \mu_2 \sum_{k=-l, k \neq 0}^{N-l} \omega_k^{(\beta_2)} |\varepsilon_{l+k}^0| \\
&\leq \left[1 + B_1 \mu_1 \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_1)} + B_2 \mu_2 \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_2)} \right] \max_{1 \leq j \leq N-1} |\varepsilon_j^0| \\
&\leq \max_{1 \leq j \leq N-1} |\varepsilon_j^0| \\
&= \| E^0 \|_\infty.
\end{aligned}$$

Suppose that $\| E^j \|_\infty \leq \| E^0 \|_\infty$, $j = 1, 2, \cdots, n$. Let $|\varepsilon_l^{n+1}| = \max_{1 \leq j \leq N-1} |\varepsilon_j^{n+1}|$.

If $\tau^\alpha \leq \frac{2-2^{1-\alpha}}{2\Gamma(2-\alpha)\left[\frac{B_1c_1}{h^{\beta_1}}-\frac{B_2c_2\beta_2}{h^{\beta_2}}\right]}$, then $1-b_1+B_1\mu_1\omega_0^{\beta_1}+B_2\mu_2\omega_0^{\beta_2}\geq 0$. Using Lemma 1 and Lemma 2 again, we have

$$\begin{aligned}\|E^{n+1}\|_\infty &= |\varepsilon_l^{n+1}| \\ &\leq \left[1-b_1+B_1\mu_1\omega_0^{(\beta_1)}+B_2\mu_2\omega_0^{(\beta_2)}\right]|\varepsilon_l^n| + \sum_{j=1}^{n-1}(b_j-b_{j+1})|\varepsilon_l^{n-j}| \\ &\quad + b_n|\varepsilon_l^0| + B_1\mu_1\sum_{k=-l,k\neq 0}^{N-l}\omega_k^{(\beta_1)}|\varepsilon_{l+k}^n| + B_2\mu_2\sum_{k=-l,k\neq 0}^{N-l}\omega_k^{(\beta_2)}|\varepsilon_{l+k}^n| \\ &\leq \Bigg[\left(1-b_1+B_1\mu_1\omega_0^{(\beta_1)}+B_2\mu_2\omega_0^{(\beta_2)}\right)+B_1\mu_1\sum_{k=-l,k\neq 0}^{N-l}\omega_k^{(\beta_1)} \\ &\quad + B_2\mu_2\sum_{k=-l,k\neq 0}^{N-l}\omega_k^{(\beta_2)} + \sum_{j=1}^{n-1}(b_j-b_{j+1})+b_n\Bigg]\|E^0\|_\infty \\ &\leq \left[1+B_1\mu_1\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_1)}+B_2\mu_2\sum_{k=-\infty}^{+\infty}\omega_k^{(\beta_2)}\right]\|E^0\|_\infty \\ &= \|E^0\|_\infty.\end{aligned}$$

Hence, the proof is completed. □

Secondly, we consider the stability of the implicit difference approximation (17).

Theorem 2 (Stability of the implicit difference approximation) *The implicit difference approximation defined by* (17) *for the space-time Riesz–Caputo fractional advection-diffusion equation is unconditionally stable.*

Proof According to (17), the error $\varepsilon_i^k = u_i^k - \widetilde{u}_i^k$ satisfies

$$\begin{aligned}&\varepsilon_i^{n+1} - B_1\mu_1\sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}\varepsilon_{i+k}^{n+1} - B_2\mu_2\sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}\varepsilon_{i+k}^{n+1} \\ &\quad = \sum_{j=0}^{n-1}(b_j-b_{j+1})\varepsilon_i^{n-j}+b_n\varepsilon_i^0, \qquad i=1,\ldots,N-1.\end{aligned} \tag{20}$$

Similar to Theorem 1, we will prove

$$\|E^{n+1}\|_\infty \leq \|E^0\|_\infty, n=0,1,2,\cdots$$

by mathematical induction. For $n=0$, let $|\varepsilon_l^1| = \max_{1\leq j\leq N-1}|\varepsilon_j^1|$. Then applying Lemma 1 and Lemma 2, we have

$$\begin{aligned}\|E^1\|_\infty &= |\varepsilon_l^1| \\ &\leq |\varepsilon_l^1| + \left(-B_1\mu_1\sum_{k=-l}^{N-l}\omega_k^{(\beta_1)} - B_2\mu_2\sum_{k=-l}^{N-l}\omega_k^{(\beta_2)}\right)|\varepsilon_{l+k}^1| \\ &= |\varepsilon_l^0| \\ &\leq \|E^0\|_\infty.\end{aligned}$$

Suppose that $\| E^j \|_\infty \le \| E^0 \|_\infty$, $j = 1, 2, \cdots, n$. Let $|\varepsilon_l^{n+1}| = \max_{1 \le j \le N-1} |\varepsilon_j^{n+1}|$. Using Lemma 1 and Lemma 2 again, we have

$$\begin{aligned}
\| E^{n+1} \|_\infty &= |\varepsilon_l^{n+1}| \\
&\le |\varepsilon_l^{n+1}| + \left(-B_1 \mu_1 \sum_{k=-l}^{N-l} \omega_k^{(\beta_1)} - B_2 \mu_2 \sum_{k=-l}^{N-l} \omega_k^{(\beta_2)} \right) |\varepsilon_{l+k}^{n+1}| \\
&\le |\varepsilon_l^{n+1} - B_1 \mu_1 \sum_{k=-l}^{N-l} \omega_k^{(\beta_1)} \varepsilon_{l+k}^{n+1} - B_2 \mu_2 \sum_{k=-l}^{N-l} \omega_k^{(\beta_2)} \varepsilon_{l+k}^{n+1}| \\
&= | \sum_{j=0}^{n-1} (b_j - b_{j+1}) \varepsilon_l^{n-j} + b_n \varepsilon_l^0 | \\
&\le \sum_{j=0}^{n-1} (b_j - b_{j+1}) \| E^{n-j} \|_\infty + b_n \| E^0 \|_\infty \\
&\le \left(\sum_{j=0}^{n-1} (b_j - b_{j+1}) + b_n \right) \| E^0 \|_\infty \\
&\le \| E^0 \|_\infty .
\end{aligned}$$

Hence, the proof is completed. □

4 Convergence of the explicit and implicit difference approximations

Suppose that the continuous problem (13) has a smooth solution $u(x,t) \in C_{x,t}^{3,2}(\Omega)$, where $C_{x,t}^{3,2}(\Omega) = \left\{ u(x,t) \,\middle|\, \frac{\partial^3 u(x,t)}{\partial x^3}, \frac{\partial^2 u(x,t)}{\partial t^2} \in C(\Omega) \right\}$. To analyze its convergence, we find it worthwhile to recall here the following useful lemmas associated with error estimates.

Lemma 3 *Podlubny* [23]

$$ {}_h\delta_+^{\beta_1} u(x_i, t_n) = \frac{\partial^{\beta_1}}{\partial x^{\beta_1}} u(x_i, t_n) + O(h), $$

$$ {}_h\delta_-^{\beta_1} u(x_i, t_n) = \frac{\partial^{\beta_1}}{\partial (-x)^{\beta_1}} u(x_i, t_n) + O(h), $$

where ${}_h\delta_\pm^{\beta_1} u(x_i, t_n) = \frac{1}{h^{\beta_1}} \sum_{k=0}^{\infty} (-1)^k \binom{\beta_1}{k} u(x_{i \mp k}, t_n), \quad 0 < \beta_1 < 1.$

Lemma 4 *Meerschaert and Tadjeran* [21]

$$ {}_h\delta_+^{\beta_2} u(x_i, t_n) = \frac{\partial^{\beta_2}}{\partial x^{\beta_2}} u(x_i, t_n) + O(h), $$

$$ {}_h\delta_-^{\beta_2} u(x_i, t_n) = \frac{\partial^{\beta_2}}{\partial (-x)^{\beta_2}} u(x_i, t_n) + O(h), $$

where ${}_h\delta_\pm^{\beta_2} u(x_i, t_n) = \frac{1}{h^{\beta_2}} \sum_{k=0}^{\infty} (-1)^k \binom{\beta_2}{k} u(x_{i \mp k \pm 1}, t_n), \quad 1 < \beta_2 \le 2.$

According to the above two lemmas, we have the following result:

Lemma 5

$$\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u_i = \frac{1}{h^{\beta_1}}\sum_{k=-\infty}^{\infty}\omega_k^{(\beta_1)}u_{i+k} + O(h),\ \ 0 < \beta_1 < 1,$$

$$\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u_i = \frac{1}{h^{\beta_2}}\sum_{k=-\infty}^{\infty}\omega_k^{(\beta_2)}u_{i+k} + O(h),\ \ 1 < \beta_2 \le 2.$$

In a finite domain, by using the numerical treatment (14), we can get

Lemma 6

$$\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u_i = \frac{1}{h^{\beta_1}}\sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}u_{i+k} + O(h),\ \ 0 < \beta_1 < 1,$$

$$\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u_i = \frac{1}{h^{\beta_2}}\sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}u_{i+k} + O(h),\ \ 1 < \beta_2 \le 2.$$

We now consider the convergence of the explicit difference approximation. Let U be the exact solution of Eq. 1, and u be the numerical solution of the explicit difference approximation (15). Let the error $e = U - u$, and at the mesh points (x_i, t_n), $u_i^n = U_i^n - e_i^n$ $(i = 1, 2, \ldots, N-1;\ n = 0, 1, \ldots, K = \frac{T}{\tau})$. We denote $R^n = [e_1^n, e_2^n, \cdots, e_{N-1}^n]^T$; then $R^0 = [e_1^0, e_2^0, \cdots, e_0^n]^T = 0$.

Substituting $u_i^n = U_i^n - e_i^n$ into (15) leads to the following two cases. When $n = 0$,

$$\begin{aligned} U_i^1 - e_i^1 = U_i^0 - e_i^0 &+ B_1\mu_1\sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}(U_{i+k}^0 - e_{i+k}^0) \\ &+ B_2\mu_2\sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}(U_{i+k}^0 - e_{i+k}^0) + \mu_0 f_i^0, \end{aligned}$$

i.e.,

$$e_i^1 = U_i^1 - U_i^0 - \left[B_1\mu_1\sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}U_{i+k}^0 + B_2\mu_2\sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}U_{i+k}^0\right] - \mu_0 f_i^0. \quad (21)$$

Based on (6), (8), (10) and Lemma 6, we have

$$\begin{aligned} e_i^1 &= \mu_0\left\{\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[U_i^1 - U_i^0\right] - \left[B_1\tfrac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}U_i^0 + B_2\tfrac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}U_i^0 + f_i^0 + O(h)\right]\right\} \\ &= \mu_0\left\{\tfrac{\partial^\alpha}{\partial t^\alpha}U_i^1 + O(\tau^{2-\alpha}) - B_1\tfrac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}U_i^0 - B_2\tfrac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}U_i^0 - f_i^0 + O(h)\right\} \\ &= C\tau^\alpha(\tau + h). \end{aligned} \quad (22)$$

When $n \geq 1$,

$$U_i^{n+1} - e_i^{n+1} = \sum_{j=0}^{n-1}(b_j - b_{j+1})\left[U_i^{n-j} - e_i^{n-j}\right] + b_n\left[U_i^0 - e_i^0\right] + \mu_0 f_i^n + B_1\mu_1 \sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}\left[U_{i+k}^n - e_{i+k}^n\right] + B_2\mu_2 \sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}\left[U_{i+k}^n - e_{i+k}^n\right],$$

i.e.,

$$\begin{aligned} e_i^{n+1} &= \sum_{j=0}^{n-1}(b_j - b_{j+1})e_i^{n-j} + B_1\mu_1 \sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}e_{i+k}^n + B_2\mu_2 \sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}e_{i+k}^n \\ &\quad + U_i^{n+1} - \sum_{j=0}^{n-1}(b_j - b_{j+1})U_i^{n-j} - b_nU_i^0 - B_1\mu_1 \sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}U_{i+k}^n \\ &\quad - B_2\mu_2 \sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}U_{i+k}^n - \mu_0 f_i^n, \end{aligned} \tag{23}$$

while

$$\begin{aligned} &U_i^{n+1} - \sum_{j=0}^{n-1}(b_j - b_{j+1})U_i^{n-j} - b_nU_i^0 \\ &\quad = \sum_{j=0}^{n} b_jU_i^{n+1-j} - \sum_{j=0}^{n} b_jU_i^{n-j} \\ &\quad = \sum_{j=0}^{n} b_j\left[U_i^{n+1-j} - U_i^{n-j}\right]. \end{aligned} \tag{24}$$

Based on (6), (8), (10) and Lemma 6,

$$\begin{aligned} &U_i^{n+1} - \sum_{j=0}^{n-1}(b_j - b_{j+1})U_i^{n-j} - b_nU_i^0 - B_1\mu_1 \sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}U_{i+k}^n \\ &\quad - B_2\mu_2 \sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}U_{i+k}^n - \mu_0 f_i^n \\ &\quad = \sum_{j=0}^{n} b_j[U_i^{n+1-j} - U_i^{n-j}] - B_1\mu_1 \sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}U_{i+k}^n \\ &\qquad - B_2\mu_2 \sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}U_{i+k}^n - \mu_0 f_i^n \\ &\quad = \mu_0\left\{\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j[U_i^{n+1-j} - U_i^{n-j}] \right. \\ &\qquad \left. - \left[B_1\frac{1}{h^{\beta_1}} \sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}U_{i+k}^n + B_2\frac{1}{h^{\beta_2}} \sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}U_{i+k}^n + f_i^n\right]\right\} \\ &\quad = \mu_0\left\{\frac{\partial^\alpha}{\partial t^\alpha}U_i^{n+1} - B_1\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}U_i^n - B_2\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}U_i^n - f_i^n + O(\tau^{2-\alpha} + h)\right\} \\ &\quad = C\tau^\alpha(\tau + h). \end{aligned} \tag{25}$$

Thus,

$$
\begin{aligned}
e_i^{n+1} &= \sum_{j=0}^{n-1}(b_j - b_{j+1})e_i^{n-j} + B_1\mu_1 \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} e_{i+k}^n + B_2\mu_2 \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} e_{i+k}^n \\
&\quad + C\tau^\alpha(\tau + h) \\
&= \left[1 - b_1 + B_1\mu_1\omega_0^{(\beta_1)} + B_2\mu_2\omega_0^{(\beta_2)}\right] e_i^n + \sum_{j=1}^{n-1}(b_j - b_{j+1})e_i^{n-j} \\
&\quad + B_1\mu_1 \sum_{k=-i,k\neq 0}^{N-i} \omega_k^{(\beta_1)} e_{i+k}^n + B_2\mu_2 \sum_{k=-i,k\neq 0}^{N-i} \omega_k^{(\beta_2)} e_{i+k}^n + C\tau^\alpha(\tau + h).
\end{aligned} \tag{26}
$$

Now we can get the following result by mathematical induction.

Theorem 3 *If the condition* (19) *is satisfied, then*

$$\|R^n\|_\infty \le C b_{n-1}^{-1}(\tau^{1+\alpha} + \tau^\alpha h), n = 1, 2, \cdots, K.$$

Proof For $n = 1$, let $|e_l^1| = \max_{1\le j\le N-1} |e_j^1|$. From (22), we have

$$\|R^1\|_\infty = |e_l^1| = C\tau^\alpha(\tau + h) = C b_0^{-1}(\tau^{1+\alpha} + \tau^\alpha h).$$

Suppose that $\|R^j\|_\infty \le C b_{j-1}^{-1}(\tau^{1+\alpha} + \tau^\alpha h)$, $j = 1, 2, \cdots, n$, and let $|e_l^{n+1}| = \max_{1\le j\le N-1} |e_j^{n+1}|$. Subject to the condition (19), based on (26), Lemma 1 and Lemma 2, we obtain

$$
\begin{aligned}
\|R^{n+1}\|_\infty &= |e_l^{n+1}| \\
&\le [1 - b_1 + B_1\mu_1\omega_0^{(\beta_1)} + B_2\mu_2\omega_0^{(\beta_2)}]|e_l^n| + \sum_{j=1}^{n-1}(b_j - b_{j+1})|e_l^{n-j}| \\
&\quad + B_1\mu_1 \sum_{k=-l,k\neq 0}^{N-l} \omega_k^{(\beta_1)} |e_{l+k}^n| + B_2\mu_2 \sum_{k=-l,k\neq 0}^{N-l} \omega_k^{(\beta_2)} |e_{l+k}^n| + C\tau^\alpha(\tau + h) \\
&\le |1 - b_1 + B_1\mu_1 \sum_{k=-l}^{N-l} \omega_k^{(\beta_1)} + B_2\mu_2 \sum_{k=-l}^{N-l} \omega_k^{(\beta_2)}|\|R^n\|_\infty \\
&\quad + \sum_{j=1}^{n-1}(b_j - b_{j+1})\|R^{n-j}\|_\infty + C\tau^\alpha(\tau + h) \\
&\le |1 - b_1 + B_1\mu_1 \sum_{k=-l}^{N-l} \omega_k^{(\beta_1)} + B_2\mu_2 \sum_{k=-l}^{N-l} \omega_k^{(\beta_2)} + \sum_{j=1}^{n-1}(b_j - b_{j+1}) + b_n| \\
&\quad \cdot b_n^{-1} \cdot C\tau^\alpha(\tau + h) \\
&= |1 + B_1\mu_1 \sum_{k=-l}^{N-l} \omega_k^{(\beta_1)} + B_2\mu_2 \sum_{k=-l}^{N-l} \omega_k^{(\beta_2)}| \cdot b_n^{-1} \cdot C\tau^\alpha(\tau + h) \\
&\le |1 + B_1\mu_1 \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_1)} + B_2\mu_2 \sum_{k=-\infty}^{+\infty} \omega_k^{(\beta_2)}| \cdot b_n^{-1} \cdot C\tau^\alpha(\tau + h) \\
&= C b_n^{-1} \cdot \tau^\alpha(\tau + h).
\end{aligned}
$$

Thus, the proof is completed. □

Since

$$\lim_{n\to\infty}\frac{b_n^{-1}}{n^\alpha}=\lim_{n\to\infty}\frac{n^{-\alpha}}{(n+1)^{1-\alpha}-n^{1-\alpha}}=\frac{1}{1-\alpha}, \tag{27}$$

there is a constant $\tilde{C}$ for which

$$\|R^n\|_\infty \le \tilde{C}n^\alpha\tau^\alpha(\tau+h).$$

Because $n\tau \le T$ is finite, we obtain the following result.

Theorem 4 (Convergence of the explicit difference approximation) *Suppose that the continuous problem (13) has a smooth solution $u(x,t)\in C_{x,t}^{3,2}(\Omega)$. If the condition (19) is satisfied, then u converges to U as h and τ tend to zero. Furthermore there is a positive constant C such that*

$$|U_i^n-u_i^n|\le C(\tau+h), i=1,2,\cdots,N-1; n=1,2,\cdots,K.$$

The convergence analysis of the implicit difference approximation is similar to that of the explicit finite difference scheme. Substituting $u_i^n=U_i^n-e_i^n$ into (17) leads to the following two cases.

When $n=0$,

$$\begin{aligned}U_i^1-e_i^1&+\left(-B_1\mu_1\sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}-B_2\mu_2\sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}\right)(U_{i+k}^1-e_{i+k}^1)\\&=(U_i^0-e_i^0)+\mu_0 f_i^1,\end{aligned}$$

i.e.,

$$\begin{aligned}e_i^1&+\left(-B_1\mu_1\sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}-B_2\mu_2\sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}\right)e_{i+k}^1\\&=U_i^1-U_i^0+\left(-B_1\mu_1\sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}-B_2\mu_2\sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}\right)U_{i+k}^1-\mu_0 f_i^1.\end{aligned} \tag{28}$$

According to (6), (8), (10) and Lemma 6, we have

$$\begin{aligned}e_i^1&+\left(-B_1\mu_1\sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}-B_2\mu_2\sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}\right)e_{i+k}^1\\&=\mu_0\left\{\tfrac{\tau^{-\alpha}}{\Gamma(2-\alpha)}(U_i^1-U_i^0)-\left[B_1\tfrac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}U_i^1+B_2\tfrac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}U_i^1+f_i^1+O(h)\right]\right\}\\&=\mu_0\left\{\tfrac{\partial^\alpha}{\partial t^\alpha}U_i^1+O(\tau^{2-\alpha})-B_1\tfrac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}U_i^1-B_2\tfrac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}U_i^1-f_i^1+O(h)\right\}\\&=C\tau^\alpha(\tau^{2-\alpha}+h).\end{aligned} \tag{29}$$

When $n\ge 1$,

$$\begin{aligned}(U_i^{n+1}-e_i^{n+1})&-B_1\mu_1\sum_{k=-i}^{N-i}\omega_k^{(\beta_1)}(U_{i+k}^{n+1}-e_{i+k}^{n+1})-B_2\mu_2\sum_{k=-i}^{N-i}\omega_k^{(\beta_2)}(U_{i+k}^{n+1}-e_{i+k}^{n+1})\\&=\sum_{j=0}^{n-1}(b_j-b_{j+1})(U_i^{n-j}-e_i^{n-j})+b_nU_i^0+\mu_0 f_i^{n+1},\end{aligned}$$

i.e.,

$$\begin{aligned} & e_i^{n+1} - B_1\mu_1 \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} e_{i+k}^{n+1} - B_2\mu_2 \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} e_{i+k}^{n+1} \\ & \quad = \sum_{j=0}^{n-1} (b_j - b_{j+1}) e_i^{n-j} \\ & \qquad + \left\{ U_i^{n+1} - B_1\mu_1 \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} U_{i+k}^{n+1} - B_2\mu_2 \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} U_{i+k}^{n+1} \right. \\ & \qquad \left. - \sum_{j=0}^{n-1} (b_j - b_{j+1}) U_i^{n-j} - b_n U_i^0 - \mu_0 f_i^{n+1} \right\}, \end{aligned} \tag{30}$$

and according to (24), (6), (8), (10) and Lemma 6,

$$\begin{aligned} & U_i^{n+1} - B_1\mu_1 \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} U_{i+k}^{n+1} - B_2\mu_2 \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} U_{i+k}^{n+1} \\ & \quad - \sum_{j=0}^{n-1} (b_j - b_{j+1}) U_i^{n-j} - b_n U_i^0 - \mu_0 f_i^{n+1} \\ & \quad = \mu_0 \left\{ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} b_j [U_i^{n+1-j} - U_i^{n-j}] \right. \\ & \qquad \left. - \left[B_1 \frac{1}{h^{\beta_1}} \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} U_{i+k}^{n+1} + B_2 \frac{1}{h^{\beta_2}} \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} U_{i+k}^{n+1} + f_i^{n+1} \right] \right\} \\ & \quad = \mu_0 \left\{ \frac{\partial^\alpha}{\partial t^\alpha} U_i^{n+1} - B_1 \frac{\partial^{\beta_1}}{\partial |x|^{\beta_1}} U_i^{n+1} - B_2 \frac{\partial^{\beta_2}}{\partial |x|^{\beta_2}} U_i^{n+1} - f_i^{n+1} + O(\tau^{2-\alpha} + h) \right\} \\ & \quad = C\tau^\alpha (\tau^{2-\alpha} + h). \end{aligned} \tag{31}$$

Substituting the above equation into (30), we obtain

$$\begin{aligned} & e_i^{n+1} - B_1\mu_1 \sum_{k=-i}^{N-i} \omega_k^{(\beta_1)} e_{i+k}^{n+1} - B_2\mu_2 \sum_{k=-i}^{N-i} \omega_k^{(\beta_2)} e_{i+k}^{n+1} \\ & \quad = \sum_{j=0}^{n-1} (b_j - b_{j+1}) e_i^{n-j} + C\tau^\alpha (\tau^{2-\alpha} + h). \end{aligned} \tag{32}$$

Similar to Theorem 3, we can get the following result by mathematical induction.

Theorem 5 *There is a positive constant C such that*

$$\|R^n\|_\infty \le C b_{n-1}^{-1} (\tau^2 + \tau^\alpha h), n = 1, 2, \cdots, K.$$

Proof For $n = 1$, let $|e_l^1| = \max\limits_{1\leq j\leq N-1} |e_j^1|$. According to Lemma 2 and (22), we have

$$\begin{aligned}\|R^1\|_\infty &= |e_l^1| \\ &\leq |e_l^1| + \left(-B_1\mu_1 \sum_{k=-l}^{N-l} \omega_k^{(\beta_1)} - B_2\mu_2 \sum_{k=-l}^{N-l} \omega_k^{(\beta_2)}\right) |e_{l+k}^1| \\ &= C(\tau^2 + \tau^\alpha h) \\ &= Cb_0^{-1}(\tau^2 + \tau^\alpha h).\end{aligned}$$

Suppose that $\|R^j\|_\infty \leq Cb_{j-1}^{-1}(\tau^2 + \tau^\alpha h)$, $j = 1, 2, \cdots, n$, and let $|e_l^{n+1}| = \max\limits_{1\leq j\leq N-1} |e_j^{n+1}|$. Based on (32), Lemma 1 and Lemma 2, we obtain

$$\begin{aligned}\|R^{n+1}\|_\infty &= |e_l^{n+1}| \\ &\leq \sum_{j=0}^{n-1}(b_j - b_{j+1})\|R^{n-j}\|_\infty + C\tau^\alpha(\tau^{2-\alpha} + h) \\ &\leq \left|e_l^{n+1} + \left(-B_1\mu_1 \sum_{k=-l}^{N-l} \omega_k^{(\beta_1)} - B_2\mu_2 \sum_{k=-l}^{N-l} \omega_k^{(\beta_2)}\right) e_{l+k}^{n+1}\right| \\ &\leq \sum_{j=0}^{n-1}(b_j - b_{j+1}) \cdot b_{n-j-1}^{-1} \cdot C(\tau^2 + \tau^\alpha h) + C\tau^\alpha(\tau^{2-\alpha} + h) \\ &\leq \sum_{j=0}^{n-1}(b_j - b_{j+1}) \cdot b_n^{-1} \cdot C(\tau^2 + \tau^\alpha h) + C\tau^\alpha(\tau^{2-\alpha} + h) \\ &= Cb_n^{-1}(\tau^2 + \tau^\alpha h).\end{aligned}$$

Thus, the proof is completed. □

Because of (27), there is a constant $\tilde{C}$ for which

$$\|R^n\|_\infty \leq \tilde{C}n^\alpha\tau^\alpha(\tau^{2-\alpha} + h).$$

Since $n\tau \leq T$ is finite, we obtain the following result.

Theorem 6 (Convergence of the implicit difference approximation) *Suppose that the continuous problem* (13) *has a smooth solution $u(x, t) \in C_{x,t}^{3,2}(\Omega)$. The solution u unconditionally converges to U as h and τ tend to zero. Furthermore there is a positive constant C such that*

$$|U_i^n - u_i^n| \leq C(\tau^{2-\alpha} + h), \quad i = 1, 2, \cdots, N-1; \quad n = 1, 2, \cdots, K.$$

5 Richardson extrapolation and short-memory technique

As discussed above, a numerical solution of the implicit difference approximation that is $O(\tau^{2-\alpha}) + O(h)$ accurate can be obtained. Furthermore, the Richardson extrapolation method can be employed to improve convergence order, as discussed in [27, 28]. In this way, the extrapolated solution is computed from

$$U_{\text{extrap}} = 2U_{h,\tau^{2-\alpha}} - U_{h/2,\tau^{2-\alpha}/2}, \tag{33}$$

where $U_{h,\tau^{2-\alpha}}$ and $U_{h/2,\tau^{2-\alpha}/2}$ denote the numerical solutions at the grid point (x, t) on the coarse grid ($h \approx \tau^{2-\alpha}$) and the fine grid $h/2 \approx \tau^{2-\alpha}/2$, respectively. This extrapolation technique is illustrated by a numerical example in the following.

As is well known, the difficulty of solving fractional differential equations is essentially because fractional derivatives are non-local operators, particularly where the application requires a solution to be given over a long time interval. This non-local property means that the next state of a system not only depends on its current state but also on its historical states starting from the initial time. This property is closer to reality and is the main reason why fractional calculus has become more and more useful. To overcome this difficulty, some authors explore techniques for reducing computational cost that keeps the error under control [8, 9, 23]. The simplest approach is to disregard the tail of the integral and to integrate only over a fixed period of recent history. This is commonly referred to as the short-memory principle, which is described in [23] where the error introduced for the Riemann-Liouville fractional derivative was analysed. In this paper, we consider the error introduced for the Caputo fractional derivative. For a fixed memory of length T_0 and for $\alpha \in (0, 1)$, we substitute

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(s)}{(t-s)^\alpha} ds$$

by

$$\frac{1}{\Gamma(1-\alpha)} \int_{t-T_0}^t \frac{y'(s)}{(t-s)^\alpha} ds$$

and use the result of [9]: for any given error bound $\varepsilon > 0$ it is sufficient to choose T_0 to satisfy

$$T_0^{1-\alpha} \geq t^{1-\alpha} - \left(\frac{\varepsilon\Gamma(2-\alpha)}{M}\right), \tag{34}$$

where

$$M = \sup_{s\in[0,t]} |y'(s)|.$$

Therefore if we require a numerical scheme over some interval I to preserve the original accuracy we would need to use a T_0 that satisfies the inequality (34) for all $t \in I$. We found that, in many cases, the short-memory principle leads to suppression of the influence of accumulating rounding error during long-time simulations. We will show the effect of application of the short-memory principle in our numerical example. The numerical results indicate that the computational cost would not be reduced significantly for preserving the convergence order, supporting the assertion that the short-memory principle would not work for the Caputo derivative.

6 Numerical results

In this section, the following space-time Riesz–Caputo fractional advection-diffusion equation in a bounded domain is considered:

$$\begin{cases} \frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = B_1\frac{\partial^{\beta_1}}{\partial|x|^{\beta_1}}u(x,t) + B_2\frac{\partial^{\beta_2}}{\partial|x|^{\beta_2}}u(x,t) + f(x,t), & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 < x < 1, 0 \le t \le T, & \\ u(x,0) = 0, & 0 < x < 1, \\ u(0,t) = u(1,t) = 0, & 0 < t \le T, \end{cases} \tag{35}$$

where

$$\begin{aligned} f(x,t) = & \frac{B_1(t^{\alpha+\beta_1}+t^{\beta_2})}{2\cos(\beta_1\pi/2)}\left\{\frac{2}{\Gamma(3-\beta_1)}\left[x^{2-\beta_1}+(1-x)^{2-\beta_1}\right] - \frac{12}{\Gamma(4-\beta_1)}\right. \\ & \left.\left[x^{3-\beta_1}+(1-x)^{3-\beta_1}\right] + \frac{24}{\Gamma(5-\beta_1)}\left[x^{4-\beta_1}+(1-x)^{4-\beta_1}\right]\right\} \\ & + \frac{B_2(t^{\alpha+\beta_1}+t^{\beta_2})}{2\cos(\beta_2\pi/2)}\left\{\frac{2}{\Gamma(3-\beta_2)}\left[x^{2-\beta_2}+(1-x)^{2-\beta_2}\right] - \frac{12}{\Gamma(4-\beta_2)}\right. \\ & \left.\left[x^{3-\beta_2}+(1-x)^{3-\beta_2}\right] + \frac{24}{\Gamma(5-\beta_2)}\left[x^{4-\beta_2}+(1-x)^{4-\beta_2}\right]\right\} \\ & + \left[\frac{\Gamma(\alpha+\beta_1+1)}{\Gamma(\beta_1+1)}t^{\beta_1} + \frac{\Gamma(\beta_2+1)}{\Gamma(\beta_2+1-\alpha)}t^{\beta_2-\alpha}\right]x^2(1-x)^2. \end{aligned}$$

Table 1 Maximum error behavior versus gridsize reduction at $t = 1.5$ for $\alpha = 0.7$, $\beta_1 = 0.3$, $\beta_2 = 1.5$

$h(\approx \tau^{2-\alpha})$	τ	Maximum error	
		No extrapolation	Extrapolation
0.0200	0.0500	0.00288113	0.00009832
0.0100	0.0300	0.00147675	0.00003299
0.0050	0.0150	0.00075063	0.00001126
0.0025	0.0010	0.00037560	0.00000316

Table 2 Error behavior versus the original numerical solution (explicit) and application of short-memory principle at $t = 5.0$ for $\alpha = 0.7$, $\beta_1 = 0.3$, $\beta_2 = 1.5$, with $h = 0.02$ and $\tau = 0.0001$

x_i (t = 5.0)	Original numerical solution (explicit)	Exact solution	Error	Solution of short-memory ($T_0 = 4.5$)	Error
0.1000	0.130125	0.131061	0.000936	0.130123	0.000937
0.3000	0.696136	0.713553	0.017417	0.696133	0.017420
0.5000	0.986445	1.011271	0.024826	0.986441	0.024830
0.7000	0.696136	0.713553	0.017417	0.696133	0.017420
0.9000	0.130125	0.131061	0.000936	0.130123	0.000937

Table 3 Error behavior versus the original numerical solution (implicit) and application of short-memory principle at $t = 5.0$ for $\alpha = 0.7$, $\beta_1 = 0.3$, $\beta_2 = 1.5$, with $h = 0.01$ and $\tau = 0.03$

x_i (t = 5.0)	Original numerical solution (implicit)	Exact solution	Error	Solution of short-memory ($T_0 = 4.5$)	Error
0.1000	0.13298928	0.13106075	0.00192853	0.13293379	0.00187304
0.3000	0.70323274	0.71355299	0.01032025	0.70310291	0.01045008
0.5000	0.99493396	1.01127124	0.01633728	0.99477536	0.01649588
0.7000	0.70323274	0.71355299	0.01032025	0.70310291	0.01045008
0.9000	0.13298928	0.13106075	0.00192853	0.13293379	0.00187304

The exact solution is

$$u(x, t) = (t^{\alpha+\beta_1} + t^{\beta_2})x^2(1 - x)^2. \quad (36)$$

Here, we take $B_1 = B_2 = 0.5$, $\alpha = 0.7$, $\beta_1 = 0.3$, $\beta_2 = 1.5$.

To examine the performance of the implicit difference approximation (17) with $h = 0.02$ and $\tau = 0.05$ for this example, the maximum numerical error at time $t = 1.5$ between the exact analytical solution and the numerical solution of implicit difference approximation for the non-extrapolated solution and the extrapolated solution is computed. Table 1 shows that, as the number of spatial subintervals/time steps is decreased (i.e., $\tau^{2-\alpha} \approx h$), an (almost) linear reduction in the maximum error is observed, as expected from the convergence order $O(\tau^{2-\alpha} + h)$ of the method. It is also seen that the higher order accuracy is achieved by applying the Richardson extrapolation.

When $t = 5.0$ and the error bound $\varepsilon = 0.05$, we can get $T_0 > 3.620644$ from inequality (34). Table 2 compares the results of application of the short-memory principle with the original results of explicit difference approximation as we take $T_0 = 4.5$. Here, we take $h = 0.02$, $\tau = 0.0001$. τ satisfies the condition of stability (19), namely $\tau^\alpha = 1.5848931925E - 003 < 2.2735094415E -$

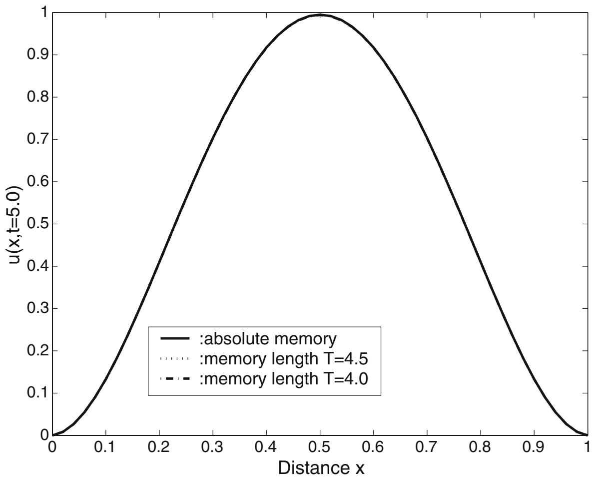

Fig. 1 Absolute memory (numerical solution (implicit)) and short-memory solution ($T_0 = 4.5$ and $T_0 = 4.0$) for $\alpha = 0.7$, $\beta_1 = 0.3$, $\beta_2 = 1.5$, with $h = 0.01$ and $\tau = 0.03$ at $t = 5.0$

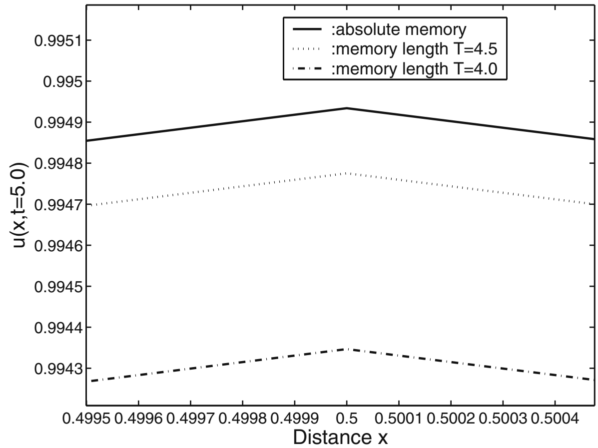

Fig. 2 Absolute memory (numerical solution(implicit)) and short-memory solution ($T_0 = 4.5$ and $T_0 = 4.0$) for $\alpha = 0.7$, $\beta_1 = 0.3$, $\beta_2 = 1.5$, with $h = 0.01$ and $\tau = 0.03$ at $t = 5.0$

003. Table 3 compares the results of application of the short-memory principle with the original results of the implicit difference approximation as we take $T_0 = 4.5$. Here, $h = 0.01$, $\tau = 0.03$(i.e., $\tau^{2-\alpha} \approx h$). It can be seen that the short-memory principle is a good approach to keep the error under control. Figures 1 and 2 show the usefulness of the short-memory principle for the numerical solution of the given example. One can see that even taking the memory length $T_0 = 4.5$ and $T_0 = 4.0$ gives satisfactory accuracy.

We also present Tables 4, 5 and 6 to summarize the consequences of applying the theoretical approach described to control the errors in our numerical example. We can see that the error is kept under control, but at the expense of much greater computational cost. We remark that, unless the interval over which we are finding the solution is sufficiently large, the short-memory principle with order preservation is unlikely to reduce significantly the computational effort compared with the full integral.

Table 4 T_0 calculated from (34) to guarantee error bound for $I = [0, 5]$

Error bound ε	0.1	0.05	0.01	0.005	0.001	0.0005	0.0001
T_0 for $I = [0, 5]$	2.53	3.62	4.70	4.85	4.94	4.98	5.0

Table 5 T_0 calculated from (34) to guarantee error bound for $I = [0, 100]$

Error bound ε	0.1	0.05	0.01	0.005	0.001	0.0005	0.0001
T_0 for $I = [0, 100]$	92.7	96.3	99.3	99.6	99.9	100	100

Table 6 T_0 calculated from (34) to guarantee error bound for $I = [0, 10000]$

Error bound ε	0.1	0.05	0.01	0.005	0.001	0.0005	0.0001
T_0 for $I = [0, 10000]$	9,980	9,990	9,998	9,999	10,000	10,000	10,000

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Appendix

As defined by Eq. 3, the regularized form of the Riesz space-fractional derivative is given by [29]

$$\frac{\partial^\beta}{\partial |x|^\beta} f(x,t) = -\frac{1}{K(\beta)} \int_{0+}^{\infty} \frac{dy}{y^{\beta+1}} \left[f(x-y,t) - 2f(x,t) + f(x+y,t) \right], 0 < \beta \leq 2,$$

$$K(\beta) = \begin{cases} 2\Gamma(-\beta)\cos(\pi\beta/2), & \beta \neq 1, \\ -\pi, & \beta = 1. \end{cases}$$

We note that

$$\begin{aligned} \lim_{\beta\to 1} 2\Gamma(-\beta)\ \cos(\pi\beta/2) &= \lim_{\beta\to 1} \frac{2\pi}{\sin(\pi(-\beta))\,\Gamma(1+\beta)} \cos(\pi\beta/2) \\ &= -\lim_{\beta\to 1} \frac{\pi}{\sin(\pi\beta/2)\Gamma(1+\beta)} \\ &= -\pi. \end{aligned}$$

Hence the Riesz space-fractional derivative is also a continuous extension of the first-order derivative.

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