Numer Algor (2011) 56:253–295
DOI 10.1007/s11075-010-9384-y

ORIGINAL PAPER

# Matrix decomposition algorithms for elliptic boundary value problems: a survey

**Bernard Bialecki · Graeme Fairweather · Andreas Karageorghis**

Received: 11 November 2009 / Accepted: 30 March 2010 /
Published online: 27 April 2010


**Abstract** We provide an overview of matrix decomposition algorithms (MDAs) for the solution of systems of linear equations arising when various discretization techniques are applied in the numerical solution of certain separable elliptic boundary value problems in the unit square. An MDA is a direct method which reduces the algebraic problem to one of solving a set of independent one-dimensional problems which are generally banded, block tridiagonal, or almost block diagonal. Often, fast Fourier transforms (FFTs) can be employed in an MDA with a resulting computational cost of $O(N^2 \log N)$ on an $N \times N$ uniform partition of the unit square. To formulate MDAs, we require knowledge of the eigenvalues and eigenvectors of matrices arising in corresponding two–point boundary value problems in one space dimension. In many important cases, these eigensystems are known explicitly, while in others, they must be computed. The first MDAs were formulated almost fifty years ago, for finite difference methods. Herein, we discuss more recent developments in the formulation and application of MDAs in spline collocation, finite element Galerkin and spectral methods, and the method

B. Bialecki
Department of Mathematical and Computer Sciences,
Colorado School of Mines, Golden, CO 80401-1887, USA
e-mail: bbialeck@mines.edu

G. Fairweather (✉)
Mathematical Reviews, American Mathematical Society,
416 Fourth Street, Ann Arbor, MI 48103, USA
e-mail: gxf@ams.org

A. Karageorghis
Department of Mathematics and Statistics,
University of Cyprus, Nicosia 1678, Cyprus
e-mail: andreask@ucy.ac.cy

of fundamental solutions. For ease of exposition, we focus primarily on the Dirichlet problem for Poisson's equation in the unit square, sketch extensions to other boundary conditions and to more involved elliptic problems, including the biharmonic Dirichlet problem, and report extensions to three dimensional problems in a cube. MDAs have also been used extensively as preconditioners in iterative methods for solving linear systems arising from discretizations of non-separable boundary value problems.

**Keywords** Elliptic boundary value problems · Poisson's equation · Biharmonic equation · Matrix decomposition algorithms · Fast Fourier transforms · Finite difference methods · Finite element Galerkin methods · Spline collocation methods · Spectral methods · Method of fundamental solutions

**AMS 2000 Subject Classifications** 65F05 · 65N22 · 65N30 · 65N35

## 1 Introduction

When applied to the numerical solution of certain elliptic boundary value problems (BVPs) in the unit square, various discretization techniques give rise to systems of linear algebraic equations of the form

$$\big(A_1 \otimes B_2 + B_1 \otimes A_2\big)\mathbf{u} = \mathbf{f}, \tag{1.1}$$

where the matrices $A_1$, $B_1$ are $M_1 \times M_1$ and $A_2$, $B_2$ are $M_2 \times M_2$, and $\otimes$ denotes the matrix tensor product.[1] For the efficient solution of such systems, matrix decomposition algorithms (MDAs) have been proposed. Such an algorithm is a direct method which reduces the problem of solving (1.1) to one of solving a set of independent one-dimensional problems. Often, fast Fourier transforms (FFTs) can be employed in MDAs with a resulting computational cost of $O(M_1 M_2 \log M_1)$ on a uniform partition of the unit square. In the literature, an MDA in which FFTs are employed is often referred to as a Fourier algorithm.

To formulate an MDA, we require nonsingular matrices $Y$ and $Z$ such that

$$Y A_1 Z = \Lambda_A, \quad Y B_1 Z = \Lambda_B, \tag{1.2}$$

where $\Lambda_A$ and $\Lambda_B$ are diagonal matrices. System (1.1) is then equivalent to

$$\big(Y \otimes I\big)\big(A_1 \otimes B_2 + B_1 \otimes A_2\big)\big(Z \otimes I\big)\big(Z^{-1} \otimes I\big)\mathbf{u} = \big(Y \otimes I\big)\mathbf{f},$$

[1] If the matrix $A = (a_{i,j})$ is $M_A \times N_A$ and $B$ is $M_B \times N_B$, then the matrix $A \otimes B$ is the $M_A M_B \times N_A N_B$ block matrix whose $(i, j)$ block is $a_{i,j}B$.

which, on using (1.2) and a property of the matrix tensor product,[2] can be written as

$$\left(\Lambda_A \otimes B_2 + \Lambda_B \otimes A_2\right)\tilde{\mathbf{u}} = \tilde{\mathbf{f}}, \tag{1.3}$$

where

$$\tilde{\mathbf{u}} = \left(Z^{-1} \otimes I\right)\mathbf{u}, \quad \tilde{\mathbf{f}} = \left(Y \otimes I\right)\mathbf{f}.$$

If $\Lambda_A = \text{diag}(\lambda_i^A)_{i=1}^{M_1}$ and $\Lambda_B = \text{diag}(\lambda_i^B)_{i=1}^{M_1}$, the solution of (1.3) reduces to the solution of the $M_2 \times M_2$ independent systems

$$\left(\lambda_i^A B_2 + \lambda_i^B A_2\right) \tilde{\mathbf{u}}_i = \tilde{\mathbf{f}}_i, \quad i = 1, 2, \ldots, M_1,$$

where

$$\tilde{\mathbf{u}}_i = \left[\tilde{u}_{i,1}, \ldots, \tilde{u}_{i,M_2}\right]^T, \quad \tilde{\mathbf{f}}_i = \left[\tilde{f}_{i,1}, \ldots, \tilde{f}_{i,M_2}\right]^T. \tag{1.4}$$

We thus have the following algorithm for solving (1.1):

**Algorithm MDA**

**Step 1.** Compute $\tilde{\mathbf{f}} = (Y \otimes I)\mathbf{f}$.
**Step 2.** Solve $\left(\lambda_i^A B_2 + \lambda_i^B A_2\right) \tilde{\mathbf{u}}_i = \tilde{\mathbf{f}}_i, \quad i = 1, 2, \ldots, M_1.$
**Step 3.** Compute $\mathbf{u} = (Z \otimes I)\tilde{\mathbf{u}}$.

Frequently, the elements of the matrix $Z$ are sines and/or cosines and $Y$ can be expressed in terms of $Z$ or $Z^T$. Then, in steps 1 and 3, FFT routines can be employed to perform the matrix multiplications at a cost of $O(M_1 M_2 \log M_1)$. Step 2 comprises $M_1$ independent systems of order $M_2$ which are often banded, block tridiagonal, or almost block diagonal [4, 89], and can be solved at a cost of $O(M_1 M_2)$. Thus the total cost of Algorithm MDA is $O(M_1 M_2 \log M_1)$; that is, $O(N^2 \log N)$ when $M_1, M_2 = O(N)$.

MDAs were first considered for the solution of the standard five–point finite difference approximation to Poisson's equation in the unit square subject to homogeneous Dirichlet boundary conditions (BCs),

$$-\Delta u = f(x, y), \quad (x, y) \in \Omega, \qquad u(x, y) = 0, \quad (x, y) \in \partial\Omega, \tag{1.5}$$

where $\Delta \equiv D_x^2 + D_y^2$ denotes the Laplacian[3] and $\Omega = (0, 1) \times (0, 1)$ with boundary $\partial\Omega$. The early history of MDAs is quite convoluted. The first such method similar to Algorithm MDA was presented in 1960 in [47]; see also [158]. Tensor product methods were introduced in 1964 by Lynch et al. [149, 150], for the solution of finite difference approximations of (1.5) and

[2] $(A \otimes B)(C \otimes D) = AC \otimes BD$.
[3] Differentiation with respect to $x$ and $y$ is denoted by $D_x^2 = \partial^2/\partial x^2, \quad D_y^2 = \partial^2/\partial y^2$, etc.

other separable problems, but without any mention of [47]. In these methods, the diagonalization is performed in both variables so that in Step 2 of the algorithm the systems are diagonal. According to [150], tensor product methods of this type were developed in 1960 in [82] for the five–point finite difference approximation of (1.5) but an inefficient solution procedure was formulated. In 1965, Hockney [112] introduced a method which combines cyclic reduction with an MDA. He claimed that the method of [150] was essentially that proposed by Hyman [116] in 1951. Subsequent developments of MDAs for finite difference methods are described in [61, 79, 121, 162, 188, 189, 197].

The purpose of this paper is to provide an overview of the extensive literature on the development and application of MDAs for the efficient implementation of various discretization techniques with emphasis on works appearing over the last two decades. Most of the discussion centers on the solution of (1.5). Methods for solving this problem easily extend to Neumann, mixed, and periodic BCs on the vertical sides of $\Omega$ and general linear BCs on the horizontal sides, and to elliptic partial differential equations (PDEs) of the form

$$-D_x^2 u + L_y u = f(x, y), \qquad (x, y) \in \Omega, \tag{1.6}$$

with

$$L_y u = -a(y) D_y^2 u + b(y) \frac{\partial u}{\partial y} + c(y) u. \tag{1.7}$$

Note that Eq. 1.6 includes (1.5) in polar coordinates and in axisymmetric cylindrical and spherical coordinate systems.

An outline of the paper is as follows. In Section 2, by way of introduction and for later reference, we consider MDAs for the five-point and nine-point finite difference methods for solving (1.5). This is followed in Section 3 by a description of MDAs in the orthogonal spline collocation approximation of second order elliptic problems, biharmonic problems and related time dependent problems. In Section 4, we describe MDAs for finite element Galerkin methods, including recent work on the determination of eigensystems when the spaces of $C^0$ piecewise biquadratic functions and piecewise Hermite bicubics are employed. $C^2$ cubic and $C^1$ quadratic spline collocation methods are frequently used techniques in the numerical solution of BVPs for ordinary and partial differential equations and for the spatial discretization in time dependent problems [29]. When used in their basic form, these methods are suboptimal and yield approximations which are no more than second order accurate (cf. [57]) although they are often used in practice; see, for example, [58, 59, 130, 195, 196]. In Section 5, we describe MDAs for cubic and quadratic spline collocation methods, in particular, methods developed recently in [3, 31, 32] for the cubic case and in [33–35, 151] for the quadratic case which are globally optimal and possess superconvergence properties. In Section 6, MDAs for various spectral methods for Poisson and biharmonic problems are described. Section 7 is devoted to a discussion of the use of MDAs to solve the linear systems arising when the method of fundamental solutions [90] and related methods are applied to certain axisymmetric problems. Concluding remarks are given in Section 8. We make no attempt to conduct a rigorous comparison of the various methods; the only comparison reported is based on computational cost. The choice of method in a particular situation is invariably dictated by the user's likes and dislikes.

While the emphasis throughout is on two-dimensional problems, we do make mention of methods that have been developed for three-dimensional problems in a cube. These methods give rise to linear systems of the form

$$\left(A_1 \otimes B_2 \otimes B_3 + B_1 \otimes A_2 \otimes B_3 + B_1 \otimes B_2 \otimes A_3\right)\mathbf{u} = \mathbf{f}. \tag{1.8}$$

If $A_1$ and $B_1$ satisfy (1.2), then (1.8) can be transformed in an obvious way to obtain

$$\left(\Lambda_A \otimes B_2 \otimes B_3 + \Lambda_B \otimes A_2 \otimes B_3 + \Lambda_B \otimes B_2 \otimes A_3\right)\tilde{\mathbf{u}} = \tilde{\mathbf{f}},$$

or

$$\left(\lambda_i^B A_2 \otimes B_3 + B_2 \otimes \left\{\lambda_i^A B_3 + \lambda_i^B A_3\right\}\right) \tilde{\mathbf{u}}_i = \tilde{\mathbf{f}}_i, \quad i = 1, \ldots, M_1. \tag{1.9}$$

Each system in (1.9) can then be solved using Algorithm MDA provided $A_2$ and $B_2$ (or $A_3$ and $B_3$) are simultaneously diagonalizable as in (1.2).

Throughout this paper, we use the following. Suppose $\mathcal{I}, \mathcal{J}, \mathcal{M}, \mathcal{N}$ are finite sets of increasing indices. Without loss of generality, we assume that

$$\mathcal{I} = \{1, \ldots, I'\}, \quad \mathcal{J} = \{1, \ldots, J'\}, \quad \mathcal{M} = \{1, \ldots, M'\}, \quad \mathcal{N} = \{1, \ldots, N'\}.$$

Then the matrix form of

$$\phi_{i,j} = \sum_{m \in \mathcal{M}} c_{i,m}^{(1)} \sum_{n \in \mathcal{N}} c_{j,n}^{(2)} \psi_{m,n}, \quad i \in \mathcal{I}, \quad j \in \mathcal{J}, \tag{1.10}$$

is

$$\Phi = (C_1 \otimes C_2)\Psi, \tag{1.11}$$

where

$$C_1 = \left[c_{i,m}^{(1)}\right]_{i \in \mathcal{I},\, m \in \mathcal{M}}, \quad C_2 = \left[c_{j,n}^{(2)}\right]_{j \in \mathcal{J},\, n \in \mathcal{N}},$$

and

$$\boldsymbol{\Phi} = \left[\phi_{1,1}, \ldots, \phi_{1,J'}, \ldots, \phi_{I',1}, \ldots, \phi_{I',J'}\right]^T,$$

$$\boldsymbol{\Psi} = \left[\psi_{1,1}, \ldots, \psi_{1,N'}, \ldots, \psi_{M',1}, \ldots, \psi_{M',N'}\right]^T.$$

## 2 Finite difference methods

First we consider the basic five–point finite difference approximation of (1.5). To describe this method, let

$$\rho = \{t_i\}_{i=0}^{N+1}, \quad t_i = ih, \quad h = 1/(N+1), \tag{2.1}$$

where $N$ is a positive integer, be a uniform partition of [0, 1], and set $x_i = t_i$, $y_j = t_j$, $i, j = 0, \ldots, N+1$. Denote by $U_{i,j}$ an approximation to $u(x_i, y_j)$ defined by the standard second order accurate finite difference equations

$$-\left(\Delta_x^2 + \Delta_y^2\right) U_{i,j} = f(x_i, y_j), \quad i, j = 1, \ldots, N, \tag{2.2}$$

$$U_{i,0} = U_{i,N+1} = 0, \quad i = 0, \ldots, N+1,$$

$$U_{0,j} = U_{N+1,j} = 0, \quad j = 1, \ldots, N, \tag{2.3}$$

where

$$\Delta_x^2 U_{i,j} = \frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{h^2},$$

and $\Delta_y^2 U_{i,j}$ is defined similarly. If $\mathbf{u}$ and $\mathbf{f}$ are given by

$$\mathbf{u} = \left[U_{1,1}, \ldots, U_{1,N}, \ldots, U_{N,1}, \ldots, U_{N,N}\right]^T,$$

$$\mathbf{f} = \left[f(x_1, y_1), \ldots, f(x_1, y_N), \ldots, f(x_N, y_1), \ldots, f(x_N, y_N)\right]^T, \tag{2.4}$$

then, on using (1.10) and (1.11), the difference equations (2.2) can be written in the form

$$\left(J \otimes I + I \otimes J\right)\mathbf{u} = h^2\mathbf{f}, \tag{2.5}$$

where $J$ is the $N \times N$ tridiagonal matrix

$$J = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}. \tag{2.6}$$

Clearly, (2.5) is of the form (1.1) with $A_1 = A_2 = J$, and $B_1 = B_2 = I$. Moreover, it is well known that (1.2) holds with $\Lambda_A = \Lambda_J$, $\Lambda_B = I$ and $Y = Z$, where

$$\Lambda_J = \operatorname{diag}\left(\lambda_i^J\right)_{i=1}^N, \qquad \lambda_i^J = 4\sin^2\frac{i\pi}{2(N+1)}, \tag{2.7}$$

and $Z$ is the symmetric orthogonal matrix given by

$$Z = \left(\frac{2}{N+1}\right)^{1/2} \left[\sin \frac{mn\pi}{N+1}\right]_{m,n=1}^{N}. \tag{2.8}$$

With these matrices, system (2.5) can be solved using Algorithm MDA. In this case, steps 1 and 3 of the algorithm can be carried out using FFTs at a cost of $O(N^2 \log N)$. Step 2 consists of $N$ tridiagonal linear systems, each of which can be solved in $O(N)$ operations. The total cost of the algorithm is thus $O(N^2 \log N)$.

The difference equations arising from the fourth order accurate nine–point difference approximation to (1.5) [46], can be solved in a similar manner, cf. [161]. In this method, the difference approximation $\{U_{i,j}\}_{i,j=0}^{N+1}$ is defined by

$$-\left(\Delta_x^2 + \Delta_y^2 + \frac{h^2}{6}\Delta_x^2\Delta_y^2\right) U_{i,j} = \left(1 + \frac{h^2}{12}\Delta\right) f(x_i, y_j), \quad i, j = 1, \ldots, N,$$

together with (2.3). These equations can be written in the form

$$\left(J \otimes I + \frac{1}{6}(6I - J) \otimes J\right) \mathbf{u} = \mathbf{f}, \tag{2.9}$$

where

$$\mathbf{f} = \left[f_{1,1}, \ldots, f_{1,N}, \ldots, f_{N,1}, \ldots, f_{N,N}\right]^T, \quad f_{i,j} = h^2 \left(1 + \frac{h^2}{12}\Delta\right) f(x_i, y_j).$$

Comparing (2.9) with (1.1), we have

$$A_1 = A_2 = J, \quad B_1 = \frac{1}{6}(6I - J), \quad B_2 = I, \quad Y = Z, \quad \Lambda_B = \frac{1}{6}(6I - \Lambda_J),$$

with $\Lambda_J$ and $Z$ defined by (2.7) and (2.8), respectively. Note that step 2 again involves the solution of tridiagonal linear systems.

For the constant coefficient Helmholtz equation on rectangular domains in two and three dimensions subject to Dirichlet, Neumann, and periodic BCs, Boisvert [52–54] derived fourth order finite difference methods, and developed a software package for the solution of the resulting linear systems using MDAs with FFTs [55, 56]. A method claimed to be fourth order for solving the Helmholtz equation in three dimensions which also involves solving tridiagonal systems is described in [152].

In [172, 173], a variant of the cyclic reduction method is developed for the solution of (1.5) using the standard five–point difference approximation, and the finite element Galerkin method with piecewise bilinear functions which is described in Section 4 of the present paper. In this method, the block reduction step is performed in such a way that the resulting system of equations has a sparse right–hand side and this sparsity is then exploited in an MDA–type method known as the partial solution technique; cf. [16, 133–135, 199, 200]. The parallel implementation of this method is discussed in [1, 159, 173]. In

[102], this approach is used in the solution of the Helmholtz equation in two and three dimensions with piecewise bilinear and piecewise trilinear functions, respectively.

In [107], Hendrickx and Van Barel developed a method for solving (1.5) in Cartesian and polar coordinates, called a Kronecker product method, which is equivalent to the MDA. The approach adopted in [107] is the same as that formulated in [63, 64] for a domain decomposition method and is also used in [98].

When Robin BCs are imposed on one or both vertical sides of the unit square, the linear system resulting from the finite difference methods cannot be reduced to a collection of smaller independent linear systems as in step 2 of Algorithm MDA. In this case, iterative methods have been employed to solve the system corresponding to (1.1). Such methods involving MDA–type techniques have been developed in [163–165], and in [111, 115] an MDA is used as a preconditioner. In [108], the approach of [107] is extended to Neumann problems and used in an algorithm for solving the Robin problem for the Helmholtz equation. Preconditioners involving MDAs for the solution of Helmholtz problems in two and three dimensions are also discussed in [85, 86, 101, 118, 141, 167, 173].

MDAs have also been employed in finite difference methods for the solution of various other problems. Their use in the solution of Helmholtz problems with nonlocal boundary conditions arising in electromagnetic scattering from a large cavity is discussed in [18, 202–204]. For their application in the solution of the Navier–Stokes equations in two and three dimensions, see [95, 201]. For the polar form of Poisson's equation on a disk,

$$r^2 D_r^2 u + r D_r u + D_\theta^2 u = f(r, \theta), \qquad (r, \theta) \in (0, 1) \times (0, 2\pi), \tag{2.10}$$

fourth order finite difference–Fourier spectral schemes and a fourth order finite difference scheme and their solutions by MDA-type methods are considered in [138, 140, 156]. Extensions of the fourth order finite difference–Fourier spectral scheme to cylindrical and spherical coordinates are discussed in [139]. The basic second order finite difference method is used in [49] to discretize the biharmonic problem

$$\Delta^2 u = f(x, y), \quad (x, y) \in \Omega, \quad u = D_n u = 0, \quad (x, y) \in \partial\Omega, \tag{2.11}$$

where $D_n$ is the outward normal derivative on $\partial\Omega$, a problem of considerable practical interest particularly in elasticity and fluid dynamics. The resulting linear system is solved using the Sherman-Morrison formula in which the auxiliary problem corresponds to the biharmonic problem with $\Delta u$ rather than $D_n u$ specified on two opposite sides of $\Omega$. The auxiliary problem can be solved using a variant of Algorithm MDA at a cost of $O(N^2 \log N)$. The Sherman-Morrison approach of [49] was used recently in [19] for the fourth order Stephenson type discretization [183] of (2.11). However, in contrast to [49], diagonalization is performed in both directions when solving the auxiliary problem. The cost of this algorithm is also $O(N^2 \log N)$.

## 3 Orthogonal spline collocation methods

With the partition $\rho$ defined in (2.1), let

$$\mathcal{M}_k^r(\rho) = \left\{ v \in C^k[0,1] : v|_{[x_{i-1},x_i]} \in P_r, i = 1, \ldots, N+1 \right\}, \tag{3.1}$$

and

$$\mathcal{M}_k^{r,0}(\rho) = \left\{ v \in \mathcal{M}_k^r(\rho) : v(0) = v(1) = 0 \right\},$$

where $P_r$ denotes the set of all polynomials of degree $\leq r$. Note that

$$\dim \mathcal{M}_k^{r,0}(\rho) = (N+1)(r-k) \equiv M. \tag{3.2}$$

Let $\{\sigma_k\}_{k=1}^{r-1}$ be the nodes of the $(r-1)$-point Gauss-Legendre quadrature rule on [0, 1], and let the Gauss points in [0, 1] be defined by

$$\xi_{(i-1)(r-1)+k} = x_{i-1} + h\sigma_k, \quad k = 1, 2, \ldots, r-1, \quad i = 1, \ldots, N+1. \tag{3.3}$$

Then the orthogonal spline collocation (OSC) method for solving (1.5) consists in finding $U \in \mathcal{M}_1^{r,0}(\rho) \otimes \mathcal{M}_1^{r,0}(\rho)$,[4] $r \geq 3$, such that

$$-\Delta U(\xi_i, \xi_j) = f(\xi_i, \xi_j), \quad i, j = 1, \ldots, M. \tag{3.4}$$

Consider the case in which $r = 3$; that is, the space of piecewise Hermite bicubics. Then $M = 2N + 2$ from (3.2), and from (3.3),

$$\xi_{2i-1} = x_{i-1} + h\sigma_1, \quad \xi_{2i} = x_{i-1} + h\sigma_2, \quad i = 1, \ldots, N+1, \tag{3.5}$$

where $\sigma_1 = (3 - \sqrt{3})/6$ and $\sigma_2 = (3 + \sqrt{3})/6$.

Define $v_i, s_i \in \mathcal{M}_1^{3,0}(\rho)$ as follows: for $i = 1, \ldots, N$,

$$v_i(x_j) = \delta_{i,j}, \quad v_i'(x_j) = 0, \quad j = 0, \ldots, N+1,$$

and, for $i = 0, \ldots, N+1$,

$$s_i(x_j) = 0, \quad s_i'(x_j) = h^{-1}\delta_{i,j}, \quad j = 0, \ldots, N+1,$$

where $\delta_{i,j}$ denotes the Kronecker delta. Explicit expressions for these functions are given in [87]. With

$$\begin{aligned} \{\phi_m\}_{m=1}^M &= \left\{ v_1, \ldots, v_N, s_0, \ldots, s_{N+1} \right\}, \\ \{\psi_n\}_{n=1}^M &= \left\{ s_0, v_1, s_1, \ldots, v_N, s_{N+1} \right\} \end{aligned} \tag{3.6}$$

[4]For two spaces $V$ and $W$ of functions, $V \otimes W$ denotes the space of functions consisting of all finite linear combinations of products $\psi_1(x)\psi_2(y)$ with $\psi_1 \in V$ and $\psi_2 \in W$.

as bases for $\mathcal{M}_1^{3,0}(\rho)$, we write the piecewise Hermite bicubic approximation in the form

$$U(x,y)=\sum_{m=1}^{M}\sum_{n=1}^{M}U_{m,n}\phi_m(x)\psi_n(y). \tag{3.7}$$

Substituting (3.7) in (3.4) and using (1.10) and (1.11), we obtain the system (1.1) with

$$A_1=\left[a_{i,m}^{(1)}\right]_{i,m=1}^{M},\quad a_{i,m}^{(1)}=-\phi_m''(\xi_i),\qquad B_1=\left[b_{j,n}^{(1)}\right]_{j,n=1}^{M},\quad b_{j,n}^{(1)}=\phi_n(\xi_j), \tag{3.8}$$

$$A_2=\left[a_{i,m}^{(2)}\right]_{i,m=1}^{M},\quad a_{i,m}^{(2)}=-\psi_m''(\xi_i),\qquad B_2=\left[b_{j,n}^{(2)}\right]_{j,n=1}^{M},\quad b_{j,n}^{(2)}=\psi_n(\xi_j), \tag{3.9}$$

$$\begin{aligned}\mathbf{u}&=\left[U_{1,1},\ldots,U_{1,M},\ldots,U_{M,1},\ldots,U_{M,M}\right]^T,\\ \mathbf{f}&=\left[f(\xi_1,\xi_1),\ldots,f(\xi_1,\xi_M),\ldots,f(\xi_i,\xi_1),\ldots,f(\xi_M,\xi_M)\right]^T.\end{aligned} \tag{3.10}$$

Explicit expressions for the matrices $A_1$, $A_2$, $B_1$ and $B_2$ are given in [20]. In [30], real nonsingular matrices $\Lambda=\mathrm{diag}(\lambda_j)_{j=1}^{M}$ and $Z$ are determined such that

$$Z^TB_1^TA_1Z=\Lambda,\qquad Z^TB_1^TB_1Z=I, \tag{3.11}$$

where

$$\Lambda=\mathrm{diag}\left(\lambda_1^-,\ldots,\lambda_N^-,\lambda_0,\lambda_1^+,\ldots,\lambda_N^+,\lambda_{N+1}\right)$$

with

$$\lambda_j^{\pm}=12\left(\frac{8+\eta_j\pm\mu_j}{7-\eta_j}\right)h^{-2},\quad j=1,\ldots,N,\quad \lambda_0=36h^{-2},\quad \lambda_{N+1}=12h^{-2},$$

and

$$\eta_j=\cos\left(\frac{j\pi}{N+1}\right),\qquad \mu_j=\sqrt{43+40\eta_j-2\eta_j^2},$$

$$Z=3\sqrt{3}\left[\begin{array}{c|c}S\Lambda_\alpha^- & \mathbf{0}|S\Lambda_\alpha^+|\mathbf{0}\\ \tilde{C}\Lambda_\beta^- & C\Lambda_\beta^+\end{array}\right], \tag{3.12}$$

where $\mathbf{0}$ is the $N-$dimensional zero column vector, $S$ is given by (2.8),

$$C=\left(\frac{2}{N+1}\right)^{1/2}\left[\cos\frac{mn\pi}{N+1}\right]_{m,n=0}^{N+1},$$

$$\tilde{C}=\left(\frac{2}{N+1}\right)^{1/2}\left[\cos\frac{mn\pi}{N+1}\right]_{m=0,n=1}^{N+1,N},$$

and $\Lambda_\alpha^{\pm}$, $\Lambda_\beta^{\pm}$ are diagonal matrices. On comparing (3.11) with (1.2), we see that the system (1.1) with (3.8)–(3.10) can be solved using Algorithm MDA with

$\Lambda_A = \Lambda$ and $\Lambda_B = I$, and $Y = Z^T B_1^T$. Since $Z$ is a matrix of sines and cosines, FFTs can be used in steps 1 and 3. The linear systems in step 2 are almost block diagonal and can be solved at a cost of $O(N^2)$ [4, 89], which is also the cost of the multiplications by $B_1$. Thus the total cost of the algorithm is $O(N^2 \log N)$.

Earlier, Sun and Zamani [187] developed an MDA for solving the OSC equations (3.4). Their algorithm is based on the fact that the eigenvalues of the matrix $B_1^{-1}A_1$ are real and distinct [81] and hence there exists a real nonsingular matrix $Q$ and a diagonal matrix $\Lambda$ such that $B_1^{-1}A_1 = Q\Lambda Q^{-1}$. They determine $Q$ which is essentially the inverse of the matrix $Z$ of (3.12). While the resulting algorithm also requires $O(N^2 \log N)$ operations, it is arguably more complicated than Algorithm MDA which hinges on the existence of a real nonsingular matrix $Z$ satisfying (3.11). In particular, the utilization of the second equation in (3.11) distinguishes Algorithm MDA and makes it not only more straightforward but also more efficient.

Algorithm MDA can be generalized to problems in which, on the sides $y = 0, 1$ of $\partial\Omega$, $u$ satisfies either the Robin BCs,

$$\alpha_0 u(x, 0) + \beta_0 D_y u(x, 0) = g_0(x),$$
$$\alpha_1 u(x, 1) + \beta_1 D_y u(x, 1) = g_1(x), \qquad x \in [0, 1],$$

where $\alpha_i$, $\beta_i$, $i = 0, 1$, are constants, or the periodic BCs,

$$u(x, 0) = u(x, 1), \qquad D_y u(x, 0) = D_y u(x, 1), \qquad x \in [0, 1].$$

On the sides $x = 0, 1$ of $\partial\Omega$, $u$ may be subject to either Dirichlet, Neumann, mixed Dirichlet-Neumann or periodic BCs. Details are given in [20, 37, 88]. Other extensions have been considered, to problems in three dimensions [168] and to OSC with higher degree piecewise polynomials [186]. For the case of Poisson's equation with pure Neumann or pure periodic BCs, Bialecki and Remington [44] formulated a matrix decomposition approach for determining the least squares solution of the singular OSC equations when $r = 3$. Algorithm MDA can also be generalized to elliptic equations of the form (1.6) and (1.7) [27].

Applications of Algorithm MDA to computing OSC approximations of certain separable BVPs with variable coefficients are described in [27]. MDAs similar to those of [28] which require the solution of an eigenvalue problem are described in [166] for the OSC solution of Poisson's equation and Helmholtz equation in three dimensions using both $C^1$ cubics and $C^2$ quintics. The authors claim that these algorithms are competitive with FFT–based methods since the cost of solving one-dimensional collocation eigenvalue problems is low compared to the total cost.

In [174], an eigenvalue analysis is presented for spline collocation differentiation matrices corresponding to periodic BCs. In particular, the circulant structure of piecewise Hermite cubic matrices is used to develop a matrix decomposition FFT algorithm for the OSC solution of a general second order PDE with constant coefficients. The proposed algorithm, whose cost is $O(N^2 \log N)$, requires the use of complex arithmetic.

In [23], Algorithm MDA is extended to the OSC solution of (2.10) with Dirichlet or Neumann BCs. For the homogeneous Dirichlet BC, the starting point in [23] is the problem

$$\left.\begin{array}{l} Lu = f(r,\theta), \quad (r,\theta) \in (0,1)\times(0,2\pi), \\ u(0,\theta) = u_0, \quad u(1,\theta) = 0, \quad \theta \in [0,2\pi], \\ u(r,0) = u(r,2\pi), \quad D_\theta u(r,0) = D_\theta u(r,2\pi), \quad r \in (0,1), \\ \displaystyle\int_0^{2\pi} D_r u(0,\theta) d\theta = 0, \end{array}\right\} \tag{3.13}$$

where

$$Lu = r^2 D_r^2 u + r D_r u + D_\theta^2 u, \tag{3.14}$$

and $u_0$ is the unknown value of $u$ at the center of the disk. A derivation of the integral condition in (3.13) is given in [23, 42]. To describe an OSC method for the solution of this problem, we introduce the following notation. For positive integers $N_r$ and $N_\theta$, let $\rho_r = \{r_i\}_{i=0}^{N_r+1}$ and $\rho_\theta = \{\theta_j\}_{j=0}^{N_\theta+1}$ be uniform partitions of $[0,1]$ and $[0,2\pi]$, respectively. With $V_r = \mathcal{M}_1^3(\rho_r)$ and

$$V_\theta = \left\{v \in \mathcal{M}_1^3(\rho_\theta) : v(0) = v(2\pi),\ v'(0) = v'(2\pi)\right\},$$

let $V$ and $\tilde{V}$ be the spaces of piecewise Hermite bicubics given by

$$V = V_r \otimes V_\theta, \qquad \tilde{V} = \left\{v \in V : v(0,\theta) = c, \quad \theta \in [0,2\pi]\right\},$$

where $c$ is a constant. Also, let $\{\xi_i^{(r)}\}_{i=1}^{M_r}$ and $\{\xi_j^{(\theta)}\}_{j=1}^{M_\theta}$, where $M_r = 2N_r + 2$ and $M_\theta = 2N_\theta + 2$, be the sets of collocation points in the $r$ and $\theta$ coordinates, respectively, corresponding to (3.5). The piecewise Hermite bicubic OSC method for the solution of (3.13) consists in finding $U \in \tilde{V}$ such that

$$\left.\begin{array}{l} LU\left(\xi_i^{(r)}, \xi_j^{(\theta)}\right) = f\left(\xi_i^{(r)}, \xi_j^{(\theta)}\right), \quad i = 1,\ldots,M_r, \quad j = 1,\ldots M_\theta, \\ U\left(1, \xi_j^{(\theta)}\right) = 0, \quad j = 1,\ldots M_\theta, \\ \displaystyle\int_0^{2\pi} D_r U(0,\theta)\, d\theta = 0. \end{array}\right\} \tag{3.15}$$

We seek the solution $U$ in the form (cf. [190])

$$U = U^{(1)} + cU^{(2)}, \tag{3.16}$$

where $c$ is to be determined, and where $U^{(1)},\ U^{(2)} \in V$ are such that

$$\left.\begin{aligned} &LU^{(1)}\left(\xi_i^{(r)}, \xi_j^{(\theta)}\right) = f\left(\xi_i^{(r)}, \xi_j^{(\theta)}\right), \quad i = 1, \ldots, M_r, \quad j = 1, \ldots M_\theta, \\ &U^{(1)}\left(0, \xi_j^{(\theta)}\right) = U^{(1)}\left(1, \xi_j^{(\theta)}\right) = 0, \quad j = 1, \ldots M_\theta, \end{aligned}\right\} \tag{3.17}$$

and

$$\left.\begin{aligned} &LU^{(2)}\left(\xi_i^{(r)}, \xi_j^{(\theta)}\right) = 0, \quad i = 1, \ldots, M_r, \quad j = 1, \ldots M_\theta, \\ &U^{(2)}\left(0, \xi_j^{(\theta)}\right) = 1, \quad U^{(2)}\left(1, \xi_j^{(\theta)}\right) = 0, \quad j = 1, \ldots M_\theta. \end{aligned}\right\} \tag{3.18}$$

Clearly, in order for $U$ of (3.16) to satisfy the last equation in (3.15), $c$ must be determined from

$$c = -\int_0^{2\pi} D_r U^{(1)}(0, \theta)\, d\theta \Big/ \int_0^{2\pi} D_r U^{(2)}(0, \theta)\, d\theta. \tag{3.19}$$

Let $\{\phi_m\}_{m=1}^{M_r}$ and $\{\psi_n\}_{n=1}^{M_\theta}$ be basis functions for $\mathcal{M}_1^{3,0}(\rho_r)$ and $V_\theta$, respectively. If the solution $U^{(1)}$ of (3.17) has the form

$$U^{(1)}(r, \theta) = \sum_{m=1}^{M_r} \sum_{n=1}^{M_\theta} U_{m,n} \phi_m(r) \psi_n(\theta), \tag{3.20}$$

then, on substituting (3.20) into the first equation of (3.17) and using (3.14), we obtain

$$\left(A_r \otimes B_\theta + B_r \otimes A_\theta\right)\mathbf{u} = \mathbf{f}, \tag{3.21}$$

where

$$A_r = \left[a_{i,m}^{(r)}\right]_{i,m=1}^{M_r}, \quad a_{i,m}^{(r)} = \left(\xi_i^{(r)}\right)^2 \phi_m''\left(\xi_i^{(r)}\right) + \xi_i^{(r)} \phi_m'\left(\xi_i^{(r)}\right),$$

$$B_r = \left[b_{i,m}^{(r)}\right]_{i,m=1}^{M_r}, \quad b_{i,m}^{(r)} = \phi_m\left(\xi_i^{(r)}\right),$$

$$A_\theta = \left[a_{j,n}^{(\theta)}\right]_{j,n=1}^{M_\theta}, \quad a_{j,n}^{(\theta)} = \psi_n''\left(\xi_j^{(\theta)}\right),$$

$$B_\theta = \left[b_{j,n}^{(\theta)}\right]_{j,n=1}^{M_\theta}, \quad b_{j,n}^{(\theta)} = \psi_n\left(\xi_j^{(\theta)}\right),$$

and

$$\mathbf{u} = \left[U_{1,1}, \ldots, U_{1,M_\theta}, \ldots, U_{M_r,1}, \ldots, U_{M_r,M_\theta}\right]^T,$$

$$\mathbf{f} = \left[f_{1,1}, \ldots, f_{1,M_\theta}, \ldots, f_{M_r,1}, \ldots, f_{M_r,M_\theta}\right]^T, \qquad f_{i,j} = f\left(\xi_i^{(r)}, \xi_j^{(\theta)}\right).$$

The system (3.21) is solved by Algorithm MDA, with the diagonalization in the $\theta$ variable, at a cost of $O(M_r M_\theta \log M_\theta)$. This requires the OSC eigensystem for periodic BCs given in [44]. Now define the function $z$ by

$$z(r) = \begin{cases} g_1(1 - r/h) - 6g_2(1 - r/h), & r \in [0, h], \\ 0, & r \in [h, 1], \end{cases}$$

where $g_1(r) = -2r^3 + 3r^2$ and $g_2(r) = r^3 - r^2$. Then it is easy to verify that $z \in V_r$ and

$$\xi_i^{(r)} z''\left(\xi_i^{(r)}\right) + z'\left(\xi_i^{(r)}\right) = 0, \quad i = 1, \ldots, M_r, \qquad z(0) = 1, \qquad z(1) = 0.$$

Therefore, $U^{(2)}(r, \theta) = z(r)$ is a solution of (3.18). (It appears that there is no corresponding analytical formula for $U^{(2)}$ for the finite difference scheme of [190].) Using (3.19), (3.20), the fact that $z'(0) = -6/h$, and properties of $\{\psi_n\}_{n=1}^{M_\theta}$, it can be shown that

$$c = \frac{h}{12\pi} \sum_{j=1}^{N_\theta} U_{1,2j-1}.$$

The total cost of the algorithm is $O(M_r M_\theta \log M_\theta)$. Numerical experiments in [23] indicate that the OSC solution $U$ is a fourth order approximation to the exact solution $u$.

For Neumann BCs, the corresponding OSC problem in [23] is singular. Therefore, the MDA is modified to obtain an OSC approximation corresponding to the particular continuous solution $u$ with a specified value at the center of the disk. The cost of this algorithm is the same as in the Dirichlet case.

Earlier, Sun [185] considered the piecewise Hermite bicubic OSC solution of (2.10) on an annulus and on a disk with Dirichlet BCs based on the MDA approach of [187] with the corresponding periodic eigensystem. For the disk, the approach of [190] is used to derive an additional equation corresponding to the center of the disk. The cost of the resulting MDA is also $O(M_r M_\theta \log M_\theta)$.

Bialecki [21] used a domain decomposition approach to develop an algorithm for the piecewise Hermite bicubic OSC solution of (1.5). The square $\Omega$ is divided into parallel strips and the OSC solution is first obtained on the interfaces by solving a collection of independent tridiagonal linear systems. Algorithm MDA is then used to compute the OSC solution on each strip. Assuming that the strips have the same width and that their number is proportional to $N/\log N$, the cost of the domain decomposition solver is $O(N^2 \log(\log N))$. For the same problem as in [21], Bialecki and Dillery [25] analyzed the convergence rates of two Schwarz alternating methods. In the first method, $\Omega$ is divided into two overlapping subrectangles, while three overlapping subrectangles are used in the second method. Fourier analysis is used to obtain explicit formulas for the convergence factors by which the $H^1$ norm of the error is reduced in one iteration of each of the Schwarz methods. It is shown numerically that, while these factors depend on the size of the overlap, they are independent of $h$.

In [24], an overlapping domain decomposition method is considered for the solution of the piecewise Hermite bicubic OSC problem corresponding to (1.5). The square is divided into overlapping squares and the additive Schwarz, conjugate gradient method involves solving independent OSC problems using Algorithm MDA.

Bialecki and Dryja [26] considered the piecewise Hermite bicubic OSC solution of (1.5) where $\Omega$ is the $L$-shaped region given by

$$\Omega = (0,2) \times (0,1) \cup (0,1) \times (1,2). \tag{3.22}$$

The region is partitioned into three non-overlapping squares with two interfaces. On each square, the approximate solution is a piecewise Hermite bicubic that satisfies Poisson's equation at the collocation points in the subregion. The approximate solution is continuous throughout the region and its normal derivatives are equal at the collocation points on the interfaces, but continuity of the normal derivatives across the interfaces is not guaranteed. The solution of the collocation problem is first reduced to finding the approximate solution on the interfaces. The discrete Steklov-Poincaré operator corresponding to the interfaces is self-adjoint and positive definite with respect to the discrete inner product associated with the collocation points on the interfaces. The approximate solution on the interfaces is computed using the preconditioned conjugate gradient (PCG) method with the preconditioner obtained from two discrete Steklov-Poincaré operators corresponding to two pairs of the adjacent squares. Once the solution of the discrete Steklov-Poincaré equation is obtained, the collocation solutions on the subregions are computed using Algorithm MDA. On a uniform partition, the total cost of the algorithm is $O(N^2 \log N)$, where the number of unknowns is proportional to $N^2$.

A common approach to solving the biharmonic problem (2.11) is to use the mixed approach in which an auxiliary function $v = \Delta u$ is introduced to obtain

$$-\Delta u + v = 0, \qquad -\Delta v = -f(x,y), \qquad (x,y) \in \Omega. \tag{3.23}$$

Using this approach, Lou et al. [146] derived existence, uniqueness and convergence results for piecewise Hermite bicubic OSC methods and developed implementations of these methods for the solution of three biharmonic problems. The first problem comprises (3.23) subject to the BCs

$$u = g_1(x,y), \qquad v = g_2(x,y), \qquad (x,y) \in \partial\Omega, \tag{3.24}$$

and the problem becomes one of solving two non–homogeneous Dirichlet problems for Poisson's equation. The resulting linear systems can be solved with cost $O(N^2 \log N)$ on a uniform partition using Algorithm MDA. In the second problem, the boundary condition in the first problem on the horizontal sides of $\partial\Omega$, $v = g_2$, is replaced by $D_y u = g_3$, so that

$$\begin{aligned} u &= g_1(x,y), \quad (x,y) \in \partial\Omega, \\ v &= g_2(\alpha,y), \quad y \in [0,1], \\ D_y u &= g_3(x,\alpha), \quad x \in (0,1), \quad \alpha = 0,1. \end{aligned} \tag{3.25}$$

A variant of Algorithm MDA is formulated for the solution of the corresponding algebraic problem. This algorithm also has cost $O(N^2 \log N)$. The third problem is the biharmonic Dirichlet problem comprising (3.23) subject to the BCs

$$u = g_1(x, y), \quad D_n u = g_2(x, y), \quad (x, y) \in \partial\Omega. \tag{3.26}$$

In this case, the OSC linear system is solved by a direct method which is based on the capacitance matrix technique with the second problem, comprising (3.23) and (3.25), as the auxiliary problem. The total cost of the capacitance matrix method for computing the OSC solution is $O(N^3)$ since the capacitance system is first formed explicitly and then solved by Gauss elimination. A piecewise Hermite bicubic OSC method for the biharmonic Dirichlet problem was developed by Sun [184] who presented an algorithm which uses a Schur complement approach involving the MDA of [187], the total cost of which is $O(N^3 \log N)$.

In [22], Bialecki developed a Schur complement method for obtaining the piecewise Hermite bicubic OSC solution to the biharmonic Dirichlet problem. In this approach, which is similar to that of Knudson [131] for the finite element Galerkin solution with piecewise Hermite bicubics, the discrete biharmonic Dirichlet problem is reduced to a Schur complement system involving the approximation to $v$ on the vertical sides of $\partial\Omega$ and to an auxiliary discrete problem for the biharmonic problem comprising (3.23) and (3.25). The Schur complement system with a symmetric and positive definite matrix is solved by the PCG method with a preconditioner obtained from the discrete problem for a related biharmonic problem with $v$, instead of $D_n u$, specified on the two horizontal sides of $\partial\Omega$; cf. (3.25). The cost of solving the preconditioned system and the cost of multiplying the Schur complement matrix by a vector are each $O(N^2)$. With the number of PCG iterations proportional to $\log N$, the cost of solving the Schur complement system is $O(N^2 \log N)$. The solution of the auxiliary discrete problem is obtained using a variant of Algorithm MDA at a cost of $O(N^2 \log N)$. Thus the total cost of solving the discrete biharmonic Dirichlet problem using this approach is $O(N^2 \log N)$.

Li et al. [142] considered the OSC solution of the following problem governing the transverse vibrations of a thin square plate clamped at its edges:

$$\begin{aligned} D_t^2 u + \Delta^2 u &= f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \\ u(x, y, 0) &= g_0(x, y), \quad D_t u(x, y, 0) = g_1(x, y), \quad (x, y) \in \Omega, \\ u(x, t) &= 0, \quad D_n u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T]. \end{aligned}$$

With $u_1 = D_t u$, and $u_2 = \Delta u$, and $\mathbf{U} = [u_1, u_2]^T$, $\mathbf{F} = [f, 0]^T$, and $\mathbf{G} = [g_1, \Delta g_0]^T$, this problem can be reformulated as the Schrödinger-type problem

$$\begin{aligned} D_t \mathbf{U} - H\Delta\mathbf{U} &= \mathbf{F}, \quad (x, y, t) \in \Omega \times (0, T], \\ \mathbf{U}(x, y, 0) &= \mathbf{G}(x, y), \quad (x, y) \in \Omega, \\ u_1(x, y, t) &= D_n u_1(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T], \end{aligned}$$

where

$$H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

An approximation to the solution $\mathbf{U}$ is determined using the Crank–Nicolson OSC scheme with $r = 3$. To solve the linear systems arising at each time, a variant of the capacitance matrix method of [146] for biharmonic Dirichlet problems is employed. The cost per time step of this method is then $O(N^2 \log N)$. For the other choices of BCs considered in [146], which correspond to (3.24) and (3.25), alternating direction implicit methods are employed in [142].

## 4 Finite element Galerkin methods

To describe the finite element Galerkin (FEG) approximation of (1.5), let $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $H^1(\Omega)$ is the standard Sobolev space. Then the weak form of (1.5) is

$$(\nabla u, \nabla v)_{L^2(\Omega)} = (f, v)_{L^2(\Omega)}, \quad v \in H_0^1(\Omega), \tag{4.1}$$

where $\nabla$ is the gradient operator and $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the usual $L^2$ inner product on $\Omega$. Suppose $S_h$ is a finite dimensional subspace of $H_0^1(\Omega)$ of dimension $M$. Then the FEG approximation, $U \in S_h \otimes S_h$, is defined by

$$(\nabla U, \nabla v_h)_{L^2(\Omega)} = (f, v_h)_{L^2(\Omega)}, \quad v_h \in S_h \otimes S_h. \tag{4.2}$$

If $\{\phi_m\}_{m=1}^M$ is a basis for $S_h$ and we write

$$U(x, y) = \sum_{m=1}^{M} \sum_{n=1}^{M} U_{m,n} \phi_m(x) \phi_n(y), \tag{4.3}$$

then the Galerkin equations (4.2) with $v_h(x, y) = \phi_i(x)\phi_j(y)$ become

$$\sum_{i=m}^{M} \sum_{n=1}^{M} U_{m,n} \left[ \left(\phi_m', \phi_i'\right)\left(\phi_n, \phi_j\right) + \left(\phi_m, \phi_i\right)\left(\phi_n', \phi_j'\right) \right] = \left(f, \phi_i \phi_j\right)_{L^2(\Omega)}, \tag{4.4}$$

where

$$(\phi, \psi) = \int_0^1 \phi(s)\psi(s)ds.$$

If $\mathbf{u}$ and $\mathbf{f}$ are as in (3.10) with $f_{i,j} = (f, \phi_i\phi_j)_{L^2(\Omega)}$, then, using (1.10) and (1.11), we obtain the linear system (1.1) with

$$A_1 = A_2 = \left[a_{i,m}\right]_{i,m=1}^M, \quad a_{i,m} = \left(\phi_i', \phi_m'\right),$$

$$B_1 = B_2 = \left[b_{j,n}\right]_{j,n=1}^M, \quad b_{j,n} = \left(\phi_j, \phi_n\right). \tag{4.5}$$

First we consider the case in which $S_h = \mathcal{M}_0^{1,0}(\rho)$, the space of piecewise linear functions, and choose the standard basis $\{\phi_m\}_{m=1}^N$ defined by

$$\phi_m(x_j) = \delta_{m,j}, \quad j = 1, \ldots, N.$$

Then it can be shown that $A_1 = h^{-1}J$ and $B_1 = \frac{h}{6}(6I - J)$, where $J$ is given by (2.6). Thus the Galerkin equations can be solved using Algorithm MDA with $\Lambda_J$ and $Z$ defined by (2.7) and (2.8), respectively, and $Y = Z$. In [153], MDAs are developed for the FEG method with piecewise bilinear functions for solving certain types of problems arising in fluid dynamics, linear elasticity and electromagnetics.

In the biquadratic case with $S_h = \mathcal{M}_0^{2,0}(\rho)$, we choose the basis $\{\phi_m\}_{m=1}^{2N-1}$ defined by

$$\phi_m(jh/2) = \delta_{mj}, \quad m, j = 1, ..., 2N-1. \tag{4.6}$$

Then, for this case, it is shown in [80] by considering $\mathcal{S}_{2N-1}A_1\mathcal{S}_{2N-1}$ and $\mathcal{S}_{2N-1}B_1\mathcal{S}_{2N-1}$, where $\mathcal{S}_{2N-1}$ is given by (2.8) with $N$ replaced by $2N-1$, that the problem of determining the matrices $\Lambda$ and $Z$ reduces to $N-1$ generalized eigenvalue problems of order 2 from which analytical expressions for the matrices $\Lambda$ and $Z$ are determined.

Now consider the FEG approximation of (1.5) using piecewise Hermite bicubics. For bases for $\mathcal{M}_1^{3,0}(\rho)$, we take $\{\phi_m\}_{m=1}^{M}$ and $\{\psi_n\}_{n=1}^{M}$ given in (3.6). Then the piecewise Hermite bicubic FEG approximation

$$U(x, y) = \sum_{m=1}^{M}\sum_{n=1}^{M} U_{m,n}\phi_m(x)\psi_n(y) \tag{4.7}$$

is obtained by substituting (4.7) in (4.2) with $S_h = \mathcal{M}_1^{3,0}(\rho)$ and $v_h(x, y) = \phi_i(x)\psi_j(y)$. Then, on using (1.10) and (1.11), we obtain the linear system (1.1) with

$$A_1 = \left[a_{i,m}^{(1)}\right]_{i,m=1}^{M}, \quad a_{i,m}^{(1)} = \left(\phi_i', \phi_m'\right), \quad B_1 = \left[b_{i,m}^{(1)}\right]_{i,m=1}^{M}, \quad b_{i,m}^{(1)} = \left(\phi_i, \phi_m\right),$$

$$A_2 = \left[a_{j,n}^{(2)}\right]_{j,n=1}^{M}, \quad a_{j,n}^{(2)} = \left(\psi_j', \psi_n'\right), \quad B_2 = \left[b_{j,n}^{(2)}\right]_{j,n=1}^{M}, \quad b_{j,n}^{(2)} = \left(\psi_j, \psi_n\right),$$

$\mathbf{u}$ is as in (3.10), and

$$\mathbf{f} = \left[f_{1,1}, \ldots, f_{1,M}, \ldots, f_{M,1}, \ldots, f_{M,M}\right]^T, \qquad f_{i,j} = (f, \phi_i\psi_j)_{L^2(\Omega)},$$

In [131], real nonsingular matrices $\Lambda = \text{diag}(\lambda_j)_{j=1}^{M}$ and $Z$ are determined such that

$$Z^T A_1 Z = \Lambda, \qquad Z^T B_1 Z = I. \tag{4.8}$$

Hence the system (1.1) can be solved using Algorithm MDA with $\Lambda_A = \Lambda$, $\Lambda_B = I$ and $Y = Z^T$. In this case, the linear systems in step 2 of Algorithm MDA are block tridiagonal with $2 \times 2$ blocks. In [36], explicit formulas for the matrices $\Lambda$ and $Z$ satisfying (4.8) for Neumann, mixed and periodic BCs as well as those derived in [131] for the Dirichlet case are presented. In contrast to the OSC approach for piecewise Hermite cubics [20], there does not appear to be a systematic way to determine the FEG piecewise Hermite cubic eigensystems. However, from [131], it was found that the structure of the matrix $Z$ for each choice of BCS when the FEG approximation satisfies the natural as well as essential BCs is similar to that arising in the corresponding OSC problem. This observation enabled the determination of the matrix $\Lambda$ and the exact specification of $Z$.

For the solution of (1.5), Bank [17] formulated matrix decomposition-like algorithms for solving the FEG linear systems (4.4) arising from tensor product $C^0$ quadratics and piecewise Hermite bicubics. These algorithms are examples of methods that involve matrix block diagonalization rather than standard diagonalization which is used in Algorithm MDA. While the total cost of each of Bank's methods is $O(N^2 \log N)$, they require twice as much work as the corresponding methods of [36] and [80].

In [148], explicit formulas are given for the matrix $\Lambda$ resulting from the piecewise Hermite bicubic FEG discretization and a method for solving (1.1) corresponding to (1.5) is developed. This method, whose cost is claimed to be half of that in [17], is based on the preliminary elimination of half of the unknowns and the use of the FACR($l$) method with $l = O(\log \log N)$ [188], or the marching algorithm. No numerical evidence is provided to demonstrate the efficacy of this approach. Moreover, there is no mention of how this approach could be extended to other BCs.

Kaufman and Warner [128, 129] developed and implemented MDAs based on (4.8) for the FEG method applied to more general elliptic problems in which the eigensystems cannot be determined explicitly. These problems are such that the matrices $A_1$ and $B_1$ are symmetric and positive definite, and hence there exist a real diagonal matrix $\Lambda$ and a real nonsingular matrix $Z$ satisfying (4.8). However, in general, $\Lambda$ and $Z$ are not known explicitly and must be computed. Since FFTs cannot be used, the total cost of the algorithm is $O(N^3)$ on an $N \times N$ partition, which, however, can be nonuniform.

The Schur complement approach and an MDA were employed in [131] to solve the linear system resulting from the Hermite bicubic finite element approximation of the biharmonic Dirichlet problem comprising (3.23) and (3.26), at a cost of $O(N^2 \log N)$.

## 5 Optimal superconvergent spline collocation methods

### 5.1 Nodal cubic spline collocation methods

Let $\rho$ be the partition defined by (2.1). For the space of cubic splines, $\mathcal{M}_2^3(\rho)$, we choose the standard $B$–spline basis which we denote by $\{\mathcal{B}_m\}_{m=-1}^{N+2}$; see, for example, [31]. Then, as a basis for $\mathcal{M}_2^{3,0}(\rho)$, we choose $\{\phi_m\}_{m=0}^{N+1}$, where

$$\left.\begin{aligned} &\phi_0 = \mathcal{B}_0 - 4\mathcal{B}_{-1}, \quad \phi_1 = \mathcal{B}_1 - \mathcal{B}_{-1}, \\ &\phi_m = \mathcal{B}_{m-1}, \quad m = 2, \ldots, N-1, \\ &\phi_N = \mathcal{B}_N - \mathcal{B}_{N+2}, \quad \phi_{N+1} = \mathcal{B}_{N+1} - 4\mathcal{B}_{N+2}. \end{aligned}\right\} \tag{5.1}$$

In the nodal cubic spline collocation (NCSC) method for the solution of (1.5), we seek $U \in \mathcal{M}_2^{3,0}(\rho) \otimes \mathcal{M}_2^{3,0}(\rho)$ such that

$$-\Delta U(x_i, y_j) = f(x_i, y_j), \tag{5.2}$$

for $i = 0, N+1, \quad 1 \le j \le N$, and $1 \le i \le N, \quad 0 \le j \le N+1$, and

$$-D_x^2 D_y^2 U(x_i, y_j) = D_y^2 f(x_i, y_j), \quad i, j = 0, N+1. \tag{5.3}$$

With

$$U(x, y) = \sum_{m=0}^{N+1} \sum_{n=0}^{N+1} U_{m,n} \phi_m(x) \phi_n(y), \tag{5.4}$$

Equation 5.2 together with (5.3) comprise $(N+2)^2$ equations in the $(N+2)^2$ unknown coefficients $\{U_{m,n}\}_{m,n=0}^{N+1}$. Using (5.2) with $i = 0, N+1$ and $1 \le j \le N$, and (5.3), we can determine $\{U_{m,n}\}_{n=0}^{N+1}$, $m = 0, N+1$, by solving two tridiagonal systems with the same coefficient matrix at a cost of $O(N)$. Once this is done, on substituting (5.4) into (5.2) with $1 \le i \le N, 0 \le j \le N+1$, and using (1.10) and (1.11), we obtain (1.1) with

$$\begin{aligned} \mathbf{u} &= \left[U_{1,0}, \ldots, U_{1,N+1}, \ldots, U_{N,0}, \ldots, U_{N,N+1}\right]^T, \\ \mathbf{f} &= \left[f_{1,0}, \ldots, f_{1,N+1}, \ldots, f_{N,0}, \ldots, f_{N,N+1}\right]^T, \end{aligned} \tag{5.5}$$

where

$$f_{i,j} = f(x_i, y_j) + \sum_{m=0,N+1} \sum_{n=0}^{N+1} U_{m,n} \left\{\phi_m''(x_i)\phi_n(y_j) + \phi_m(x_i)\phi_n''(y_j)\right\}, \tag{5.6}$$

$$A_1 = \frac{6}{h^2} J, \quad B_1 = 6I - J, \tag{5.7}$$

with $J$ defined in (2.6), and

$$A_2 = \frac{6}{h^2} \begin{bmatrix} 6 & \mathbf{0}^T & 0 \\ -\mathbf{e}_1 & J & -\mathbf{e}_N \\ 0 & \mathbf{0}^T & 6 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & \mathbf{0}^T & 0 \\ \mathbf{e}_1 & B_1 & \mathbf{e}_N \\ 0 & \mathbf{0}^T & 0 \end{bmatrix}, \tag{5.8}$$

where $\mathbf{e}_1 = [1, 0, \ldots, 0]^T$, $\mathbf{e}_N = [0, \ldots, 0, 1]^T$. The system (1.1) can be solved using Algorithm MDA with $\Lambda_A = 6\Lambda_J/h^2$, $\Lambda_B = 6I - \Lambda_J$, and $Y = Z$, where $\Lambda_J$ and $Z$ are given in (2.7) and (2.8), respectively. Note that step 2 of Algorithm MDA involves solving a set of tridiagonal systems.

It is well known that the NCSC method is suboptimal, in fact, only second order accurate and no better [57]. Optimal methods were first derived by Houstis et al. [114] by extending methods for two–point boundary value problems (TPBVPs). Specifically, they formulated a one–step method (OSM) based on work of Archer [7, 8] and Daniel and Swartz [71] which involves perturbing the differential operator in the governing equation, and a two-step method (TSM) based on a deferred corrections approach of Fyfe [93]. In [31], MDAs are formulated and implemented for the OSM and TSM of [114] applied to the Helmholtz problem. Since a TSM is twice as expensive as the corresponding OSM because it requires twice as many FFTs, we focus on OSMs in the following.

In the optimal (fourth order) OSM for (1.5) (see [114], p. 61), we seek $u_h \in \mathcal{M}_2^{3,0}(\rho) \otimes \mathcal{M}_2^{3,0}(\rho)$ of the form (5.4) by collocating an equation which is obtained by suitably perturbing the Laplacian in (1.5). As in the NCSC method, it is possible to first determine, at a cost of $O(N)$, the coefficients $\{U_{m,n}\}_{n=1}^{N}$, $m = 0, N+1$, in $U(x, y)$ of (5.4) using (5.2) for $i = 0, N+1$ and $1 \le j \le N$, and (5.3). Once this is done, the remaining collocation equations can be written as a system of equations of the form (1.1), where

$$A_1 = \frac{1}{2h^2} J(12I - J), \qquad A_2 = \frac{1}{h^2} \begin{bmatrix} 36 & \mathbf{0} & 0 \\ \mathbf{a} & h^2 A_1 & \mathbf{b} \\ 0 & \mathbf{0} & 36 \end{bmatrix}, \tag{5.9}$$

with

$$\mathbf{a} = [-2, -1/2, 0, \ldots, 0]^T, \quad \mathbf{b} = [0, \ldots, 0, -1/2, -2]^T, \tag{5.10}$$

$B_1$ and $B_2$ are as in (5.7) and (5.8), respectively, and $\mathbf{u}$ and $\mathbf{f}$ are as in (5.5) with the elements of $\mathbf{f}$ given in terms of the $f(x_i, y_j)$ and the previously determined coefficients, $\{U_{m,n}\}_{n=1}^{N}$, $m = 0, N+1$; cf. (5.6). See [31] for details. Thus the system (1.1) can be solved using Algorithm MDA with

$$\Lambda_A = \frac{1}{2h^2} \Lambda_J (12I - \Lambda_J), \qquad \Lambda_B = 6I - \Lambda_J, \tag{5.11}$$

and $Y = Z$, where $\Lambda_J$ and $Z$ are given in (2.7) and (2.8), respectively. Note that the linear systems in step 2 of the algorithm are pentadiagonal. In [45], this MDA is used as a preconditioner in the iterative solution of an optimal NCSC scheme for a very general variable coefficient Dirichlet BVP on a rectangle.

The OSM has two undesirable features. First, unlike the TSM, it does not possess superconvergence properties, and secondly, when the same approach is applied to problems with BCs other than Dirichlet, it is suboptimal, providing approximations that are no better than third order accurate. (One would not expect optimality for the Neumann, mixed and periodic problems because the method of Archer [7, 8] applied to TPBVPs with such BCs has been shown to be only third order accurate.) Moreover, it is possible to formulate an MDA for the OSM only for Dirichlet BCs whereas for the TSM, MDAs have been developed for all four types of BCs. However, for these BCs, Bialecki et al. [32] formulated and implemented optimal superconvergent OSMs for which the collocation equations can be solved using MDAs. These OSMs differ from that of [31, 114] in that they are constructed by judiciously perturbing not only the differential operator but also the right hand side of the differential equation.

In the OSM of [32] for (1.5) with $U$ of the form (5.4), we can again determine the coefficients $\{U_{m,n}\}_{n=0}^{N+1}$, $m = 0, N+1$, in $U(x, y)$ of (5.4) at a cost of $O(N)$. Then we obtain a system of the form (1.1) in which $A_1$, $B_2$, $B_1$ are as in (5.9), (5.8), (5.7), respectively, and

$$A_2 = \frac{1}{h^2}\begin{bmatrix} 40 & \mathbf{c} & 0 \\ \mathbf{a} & h^2 A_1 & \mathbf{b} \\ 0 & \mathbf{d} & 40 \end{bmatrix},$$

with $\mathbf{c} = [-5/2, 2, -1/2, 0, \ldots, 0]$, $\mathbf{d} = [0, \ldots, -1/2, 2, -5/2]$, and $\mathbf{a}$ and $\mathbf{b}$ are given in (5.10). Hence Algorithm MDA is applicable with $\Lambda_A$ and $\Lambda_B$ as in (5.11) and $Y = Z$ of (2.8). When Neumann, Dirichlet–Neumann, Neumann-Dirichlet or periodic BCs are specified on the vertical sides of the unit square, the matrices $A_1$ and $B_1$ are expressed in terms of the corresponding finite difference tridiagonal matrices.

In [3], a different type of OSM is derived for (1.5). In it, we seek $U \in \mathcal{M}_2^{3,0}(\rho) \otimes \mathcal{M}_2^{3,0}(\rho)$ satisfying

$$-\Delta U(x_i, y_j) + \frac{h^2}{6} D_x^2 D_y^2 U(x_i, y_j) = f_h(x_i, y_j), \qquad 0 \le i, j \le N+1, \tag{5.12}$$

where

$$f_h(x_i, y_j) = f(x_i, y_j) + \frac{h^2}{12} \Delta f(x_i, y_j). \tag{5.13}$$

Using (5.12) with $i = 0, N+1$ and $0 \le j \le N+1$, and with $1 \le i \le N$ and $j = 0, N+1$, it is possible to determine $\{U_{m,n}\}_{n=0}^{N+1}$, $m = 0, N+1$, and $\{U_{m,n}\}_{m=1}^{N}$, $n = 0, N+1$, in $U(x, y)$ of (5.4) at a cost of $O(N)$. Then we obtain

$$\left( J \otimes I + \frac{1}{6}(6I - J) \otimes J \right) \mathbf{u} = \frac{h^2}{36} \mathbf{f}, \tag{5.14}$$

where

$$\mathbf{u} = \left[ U_{1,1}, \ldots, U_{1,N}, \ldots, U_{N,1}, \ldots, U_{N,N} \right]^T,$$

$$\mathbf{f} = \left[ f_{1,1}, \ldots, f_{1,N}, \ldots, f_{N,1}, \ldots, f_{N,N} \right]^T,$$

and the elements of $\mathbf{f}$ are given in terms of the $f_h(x_i, y_j)$ of (5.13) and the previously determined coefficients, $\{U_{m,n}\}_{n=0}^{N+1}$, $m = 0, N+1$, and $\{U_{m,n}\}_{m=1}^{N}$, $n = 0, N+1$. The system (5.14) has the same coefficient matrix as that in (2.9) and can be solved by Algorithm MDA in a similar way at a cost of $O(N^2 \log N)$. The method has the advantage that, in contrast to the OSMs of [31, 32], step 2 of the MDA involves the solution of tridiagonal systems instead of penta-diagonal systems. This scheme is used in [2] to approximate the biharmonic Dirichlet problem comprising (3.23) and (3.26), and the resulting linear system is solved using the Schur complement approach and an MDA. This algorithm costs $O(N^2 \log N + mN^2)$ with $m$ iterations of the preconditioned conjugate gradient method.

One practical advantage of optimal NCSC methods over OSC methods is that, for a given partition, there are fewer unknowns with the same degree of piecewise polynomials, thereby reducing the size of the linear systems. However, in contrast to the optimal NCSC methods, OSC methods do not require a uniform partition of the domain and can also be used with higher degree splines [28, 186].

### 5.2 Quadratic spline collocation

With the partition $\rho$ of (2.1), let $\{\tau_i\}_{i=1}^{N+1}$ be the set of midpoints of the subintervals $[t_{i-1}, t_i]$, that is, $\tau_i = (t_{i-1} + t_i)/2$. In the following, the collocation points are $\{(\tau_i, \tau_j)\}_{i,j=1}^{N+1}$.

As a basis for $\mathcal{M}_1^2(\rho)$, we choose the standard $B$-splines, $\{\mathcal{B}_m\}_{m=0}^{N+2}$; cf. [113, Section 2]. Then, as a basis for $\mathcal{M}_1^{2,0}(\rho)$, we choose $\{\phi_m\}_{m=1}^{N+1}$, where

$$\phi_1 = \mathcal{B}_1 - \mathcal{B}_0, \quad \phi_m = \mathcal{B}_m, \quad m = 2, \ldots, N, \quad \phi_{N+1} = \mathcal{B}_{N+1} - \mathcal{B}_{N+2}.$$

Houstis et al. [113] formulated optimal quadratic spline collocation (QSC) TSMs and OSMs for second order TPBVPs. Based on this work, Christara [68] developed biquadratic collocation TSMs and OSMs for elliptic problems with Dirichlet and Neumann BCs, and obtained optimal global accuracy and superconvergence results. Constas [70] implemented the TSM of [68] for Helmholtz problems with Dirichlet, Neumann, and periodic BCs, and solved the linear systems using MDAs; see also [69].

Superconvergent QSC OSMs for the solution of (1.5) are considered in [33–35]. These methods involve perturbations of the right hand side of the differential equation as well as the differential operator, and the resulting collocation equations can be solved using an MDA. If we seek an approximate solution $U \in \mathcal{M}_1^{2,0}(\rho) \otimes \mathcal{M}_1^{2,0}(\rho)$ with

$$U(x, y) = \sum_{m=1}^{N+1} \sum_{n=1}^{N+1} U_{m,n} \phi_m(x) \phi_n(y), \tag{5.15}$$

the collocation equations can be written in the form (1.1) where

$$A_1 = \frac{1}{24h^2}(24I - T)T, \quad B_1 = B_2 = \frac{1}{8}(8I - T), \tag{5.16}$$

with $T = T_{N+1}(3, 3)$, where

$$T_{N+1}(a, b) = \begin{bmatrix} a & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & b \end{bmatrix}, \tag{5.17}$$

and

$$A_2 = \frac{1}{24h^2}\begin{bmatrix} 48 & -24 & 0 & 0 & & \\ -20 & 42 & -20 & -1 & & \\ -1 & -20 & 42 & -20 & -1 & \\ & \ddots & \ddots & \ddots & \ddots & \ddots \\ & -1 & -20 & 42 & -20 & -1 \\ & & -1 & -20 & 42 & -20 \\ & & 0 & 0 & -24 & 48 \end{bmatrix},$$

$$\mathbf{u} = \left[U_{1,1}, \ldots, U_{1,N+1}, \ldots, U_{N+1,1}, \ldots, U_{N+1,N+1}\right]^T,$$

$$\mathbf{f} = \left[f^{\mathcal{D}}_{1,1}, \ldots, f^{\mathcal{D}}_{1,N+1}, \ldots, f^{\mathcal{D}}_{N+1,1}, \ldots, f^{\mathcal{D}}_{N+1,N+1}\right]^T],$$

with

$$f^{\mathcal{D}}_{i,j} = \begin{cases} f(\tau_i, \tau_j) - \dfrac{1}{12} f(0, \tau_j), & i = 1, \\ f(\tau_i, \tau_j), \quad 2 \le i \le N, \\ f(\tau_i, \tau_j) - \dfrac{1}{12} f(1, \tau_j), & i = N+1. \end{cases}$$

If $\Lambda_T = \operatorname{diag}(\lambda_i)_{i=1}^{N+1}$ with

$$\lambda_i = 4\sin^2 \frac{i\pi}{2(N+1)}, \quad i = 1, \ldots, N+1,$$

and $Z = [z_{i,j}]_{i,j=1}^{N+1}$ with

$$z_{i,j} = \begin{cases} \sqrt{\dfrac{2}{N+1}} \sin \dfrac{(2i-1)j\pi}{2(N+1)}, & i = 1, \ldots, N+1, \quad j = 1, \ldots, N, \\ \sqrt{\dfrac{1}{N+1}} (-1)^{i-1}, \quad i = 1, \ldots, N+1, \quad j = N+1, \end{cases}$$

then, from [68, Lemma 4.1], $Z^T T Z = \Lambda_T$. Thus, from (5.16), it follows that the linear system (1.1) can be solved using Algorithm MDA with

$$\Lambda_A = \frac{1}{24h^2}(24I - \Lambda_T)\Lambda_T, \qquad \Lambda_B = \frac{1}{8}(8I - \Lambda_T),$$

and $Y = Z^T$. This approach can be extended to other BCs with one major difference from the cubic case. When a Neumann boundary condition is specified, in the corresponding QSC method, this boundary condition must be perturbed to maintain the optimal global accuracy and superconvergence properties. In [35], a method is formulated for the solution of (1.6), (1.7) subject to non–homogeneous BCs. It should be noted that, while the optimal QSC methods possess third order global accuracy, they yield fourth order accurate approximations at the nodal points, the same nodal accuracy as their cubic spline counterparts.

Corresponding to method (5.12), Maack [151] formulated the following OSM QSC method. Define $U \in \mathcal{M}_1^{2,0}(\rho) \otimes \mathcal{M}_1^{2,0}(\rho)$ by

$$-\Delta U(\tau_i, \tau_j) + \frac{h^2}{12} D_x^2 D_y^2 U(\tau_i, \tau_j) = f(\tau_i, \tau_j) + \frac{h^2}{24} \Delta f(\tau_i, \tau_j) \qquad i, j = 1, \ldots, N+1. \tag{5.18}$$

With $U$ as in (5.15), the collocation equations can be written in the form

$$\left( T \otimes I + \frac{1}{6}(6I - T) \otimes T \right) \mathbf{u} = h^2 \mathbf{f},$$

where $T = T_{N+1}(3, 3)$, and

$$\mathbf{u} = \left[ U_{1,1}, \ldots, U_{1,N+1}, \ldots, U_{N+1,1}, \ldots, U_{N+1,N+1} \right]^T,$$

$$\mathbf{f} = \left[ f_{1,1}, \ldots, f_{1,N+1}, \ldots, f_{N+1,1}, \ldots, f_{N+1,N+1} \right]^T,$$

with

$$f_{i,j} = f(\tau_i, \tau_j) + \frac{h^2}{24} \Delta f(\tau_i, \tau_j), \quad i, j = 1, \ldots, N+1.$$

Note that in contrast to the other QSC OSMs discussed in this subsection, the systems of equations arising in Step 2 of Algorithm MDA are tridiagonal. In [91], this method is extended to Helmholtz problems subject to Dirichlet, Neumann, mixed and periodic boundary conditions.

Maack [151] also formulated a superconvergent QSC method for the biharmonic Dirichlet problem (3.23)–(3.26) which involves the Schur complement approach and an MDA.

## 6 Spectral methods

Throughout this section, unless otherwise specified, $\Omega = (-1, 1) \times (-1, 1)$. As the first example of an application of MDA, we consider a special case of the Legendre spectral collocation of [40] for the solution of (1.5). First, for a positive integer $N$, we define $P_N^0$ to be the set of all polynomials of degree $\leq N$ on $[-1, 1]$ which are zero at the endpoints of the interval, and we take $\{\xi_i\}_{i=1}^{N-1}$ and $\{w_i\}_{i=1}^{N-1}$ to be the nodes and weights, respectively, of the $(N-1)$-point Gauss–Legendre quadrature on $[-1, 1]$. We then seek an approximate solution $U \in P_N^0 \otimes P_N^0$ of the form

$$U(x, y) = \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} U_{m,n} \phi_m(x) \phi_n(y) \tag{6.1}$$

such that

$$-\Delta U(\xi_i, \xi_j) = f(\xi_i, \xi_j), \qquad i, j = 1, \ldots, N-1. \tag{6.2}$$

As a basis for $P_N^0$, we choose the functions $\{\phi_m\}_{m=0}^{N-1}$ defined by

$$\phi_m = c_m\big(L_{m-1} - L_{m+1}\big), \qquad m = 1, \ldots, N-1, \tag{6.3}$$

where

$$c_m = (4m+2)^{-1/2}, \quad m = 2, \ldots, N-1, \qquad c_{N-1} = \big[(4N-2)(2-1/N)\big]^{-1/2},$$

and $L_k$ is the Legendre polynomial of degree $k$ on $[-1, 1]$; cf. [176]. Substituting (6.1) into (6.2) and using (1.10) and (1.11), we obtain

$$(A \otimes B + B \otimes A)\,\mathbf{u} = \mathbf{f}, \tag{6.4}$$

where

$$A = \big[a_{i,m}\big]_{i,m=1}^{N-1}, \quad a_{i,m} = -\phi_m''(\xi_i) \qquad B = \big[b_{i,m}\big]_{i,m=1}^{N-1}, \quad b_{i,m} = \phi_m(\xi_i),$$

$$\mathbf{u} = \big[U_{1,1}, \ldots, U_{1,N-1}, \ldots, U_{N-1,1}, \ldots, U_{N-1,N-1}\big]^T,$$

and

$$\mathbf{f} = \big[f_{1,1}, \ldots, f_{1,N}, \ldots, f_{N-1,1}, \ldots, f_{N-1,N-1}\big]^T, \qquad f_{i,j} = f\big(\xi_i, \xi_j\big).$$

With $W = \mathrm{diag}(w_1, \ldots, w_{N-1})$, we premultiply (6.4) by $B^TW \otimes B^TW$ to obtain

$$\Big(\tilde{A} \otimes \tilde{B} + \tilde{B} \otimes \tilde{A}\Big)\,\mathbf{u} = \tilde{\mathbf{f}}, \tag{6.5}$$

where

$$\tilde{A} = B^TWA, \quad \tilde{B} = B^TWB, \quad \tilde{\mathbf{f}} = \big(B^TW \otimes B^TW\big)\,\mathbf{f}.$$

The matrix $\tilde{A} = I$ while $\tilde{B}$ is symmetric, positive definite and pentadiagonal with zeros on the first super- and sub-diagonals. Since $B$ is a dense matrix, the cost of computing $\tilde{\mathbf{f}}$ is $O(N^3)$. Note that (6.5) is of the form (1.1) with $A_1 = A_2 = I$ and $B_1 = B_2 = \tilde{B}$. Since $\tilde{B}$ is symmetric and positive definite, there exists a real diagonal matrix $\Lambda = \mathrm{diag}(\lambda_i)_{i=1}^{N-1}$ with $\lambda_i > 0$ and a real orthogonal matrix $Z$ such that $Z^T\tilde{B}Z = \Lambda$. It follows from the structure of $\tilde{B}$ that the computation of $\Lambda$ and $Z$ reduces to solving two symmetric eigenvalue problems with tridiagonal matrices. Further, using the $QR$ algorithm for determining the eigenvalues and inverse iteration for determining the corresponding eigenvectors, $\Lambda$ and $Z$ can be computed at a cost of $O(N^2)$. The system (6.5) can then be solved by Algorithm MDA with $Y = Z^T$. Note that the costs of steps 1 and 3 in this algorithm are $O(N^3)$ and the cost of step 2 is $O(N^2)$. Hence the total cost of solving (6.4) is $O(N^3)$. In [40], a similar algorithm is formulated for the Helmholtz equation

$$-\Delta u + \kappa u = f(x, y), \quad (x, y) \in \Omega,$$

with constant $\kappa$, subject to Robin BCs. For the case in which the constant $\kappa$ is replaced by the variable coefficient $\kappa(x, y)$, this algorithm is used in conjunction with the PCG method.

In [41], the Legendre spectral collocation solution of (1.5) in an $L$-shaped region and also in a rectangle with a cross point are considered. The computational procedure developed for each case is similar to that of [26] described in Section 3 for OSC. The total cost of each algorithm is $O(N^3)$, where the number of unknowns is proportional to $N^2$.

A similar idea to that developed in [40] is used in [38], where a Legendre spectral collocation method is formulated for the solution of the mixed form of the biharmonic Dirichlet problem (3.23), (3.26) on a square. The solution and its Laplacian are approximated using the basis functions given by (6.3). A Schur complement approach is used to reduce the resulting linear system to one involving the approximation of the Laplacian of the solution on the two vertical sides of the square. The resulting system is again solved by a PCG method leading to an algorithm with total cost $O(N^3)$. The corresponding Galerkin formulation for problem (3.23), (3.26) with $g_1 = g_2 = 0$ is presented in [39].

The work on the biharmonic Dirichlet problem described in [38, 39] is related to MDAs in [50] and [176] for a Legendre spectral Galerkin method applied to the biharmonic equation directly instead of the mixed formulation considered in [39]. Legendre spectral Galerkin MDAs are used in [10] and [11] for the solution of the two- and three-dimensional Helmholtz equations, respectively. In [13], the MDA of [176] is applied to an algorithm for solving the stream function-vorticity version of the Navier-Stokes equations, while in [9], a Legendre spectral Galerkin MDA is applied to Helmholtz Neumann problems. In [137], a Legendre Galerkin MDA is used in the context of the spectral element method for the solution of the two- and three-dimensional Helmholtz problems. In [120], a Legendre Galerkin spectral method is applied to the Poisson equation and the resulting system recast in such a way that the coefficient matrices are symmetric and positive definite, before an MDA is applied. Legendre Galerkin MDAs for Poisson problems in cylindrical-type domains are proposed in [48, 136]. Other applications of Legendre Galerkin MDAs can be found in [51, 67, 97, 180]. An MDA is used in [14] to invert the matrix corresponding to the Laplace operator appearing in the conjugate gradient iteration for the solution of the system resulting from a spectral Galerkin Laguerre-Legendre spectral discretization of the Stokes equations in a semi-infinite channel. Similarly, an MDA is used in [205] in an extension of the method of [14] for the solution of the Navier–Stokes equations in unbounded domains. A Galerkin Legendre-Jacobi MDA is employed in [12] in the solution of the three-dimensional Helmholtz equation in finite cylindrical domains.

One of the first spectral MDAs was proposed in [99], where the numerical solution of Poisson's equation using a Chebyshev polynomial approximation is studied. The discretization is carried out using the recurrence relations for the derivatives of Chebyshev polynomials. In [100], this Chebyshev spectral MDA is extended to three–dimensional Helmholtz problems subject to non–homogeneous linear BCs. The MDA of [99, 100] and variants of it have been applied to systems arising from Fourier-Chebyshev collocation [65, 109, 110, 155], Chebyshev collocation [15, 83, 147, 160] and Chebyshev/wavelet collocation [94] methods to two- and three-dimensional Helmholtz and Poisson problems which arise in fluid dynamics. Chebyshev Galerkin MDAs similar to the Legendre Galerkin MDAs developed in [176] for the solution of second and fourth order problems in two and three dimensions are proposed in [177]. Chebyshev spectral MDAs based on the approach of [99] have been developed for second and fourth order problems by Heinrichs [103, 106]. Various applications of Chebyshev spectral MDAs based on the Petrov-Galerkin method, the approach of [99] and collocation, can be found in [84, 125, 157], respectively. In [119], a Chebyshev spectral Galerkin method based on the so-called quasi-inverse technique and related to an MDA is proposed. Further, a brief description of the Chebyshev MDA of [99] and its development are given in [60, pp. 314–317] and [62, pp. 181–185].

The advantage of using Chebyshev polynomial approximations is that, unlike other polynomial approximations, FFTs may be used. As an example, consider the spectral Chebyshev collocation approach for (1.5) proposed in [43], the development of which is very similar to that of the Legendre spectral collocation method described at the beginning of this section. In this case, $\{\xi_i\}_{i=0}^{N}$ and $\{w_i\}_{i=0}^{N}$ are the nodes and weights, respectively, of the $(N+1)$-point Chebyshev Gauss-Lobatto quadrature on $[-1, 1]$, and we take $\{\phi_m\}_{m=1}^{N-1}$ as a basis for $P_N^0$, where

$$\phi_m(x) = (1 - x^2)T_{m-1}(x), \quad m = 1, \ldots, N-1, \tag{6.6}$$

and $T_{m-1}(x)$ is the Chebyshev polynomial of degree $m-1$. We again obtain a system of equations of the form (6.5); in this case, the matrix $\tilde{A}$ is nonsymmetric and pentadiagonal with zeros on the first super- and sub-diagonals, while $\tilde{B}$ is symmetric and enneadiagonal with zeros on the first and third superdiagonals. Then, from [96],

$$\tilde{A}Z = \tilde{B}Z\Lambda \quad \text{or} \quad Z^{-1}\tilde{A}^{-1}\tilde{B}Z = \Lambda^{-1}, \tag{6.7}$$

where $\Lambda = \text{diag}\,(\lambda_i)_{i=1}^{N-1}$ with distinct, positive $\lambda_i$, and $Z$ is real and nonsingular. Thus, in this case, (6.5) can be solved using Algorithm MDA with $Y = Z^{-1}\tilde{A}^{-1}$, $\Lambda_A = I$ and $\Lambda_B = \Lambda^{-1}$. Because of the structures of $\tilde{A}$ and $\tilde{B}$, the systems arising in Step 2 of the algorithm can be solved at a cost of $O(N^2)$. Since entries of $B$ are given in terms of Chebyshev polynomials, FFTs can be used in the computation of $\tilde{\mathbf{f}}$. The total cost of the algorithm is $O(N^3)$.

Spectral MDAs based on Jacobi polynomial approximations are proposed for the Helmholtz equation in [104, 105]. A spectral Galerkin MDA involving Laguerre functions is developed in [179]. Spectral Galerkin MDAs for Helmholtz problems using ultraspherical polynomials are presented in [73, 76]. This approach is extended to spectral Galerkin MDAs for Helmholtz problems using Jacobi polynomials in [72, 75]. Spectral-Galerkin approaches are also applied to Helmholtz Neumann problems in [78] and to fourth order problems in [72, 77] using Jacobi polynomials, and to problems of order $2n$ in [74] using ultraspherical polynomials.

Other spectral Galerkin MDA applications similar to those described in the preceding can be found in [143–145, 178]. An MDA-related spectral approach for Jacobi polynomials is given in [117].

We next describe a spectral collocation MDA for the solution of Poisson's equation in a disk. Specifically, the solution of (2.10) with homogeneous Dirichlet BCs is obtained in [42] as follows. For a positive integer $M_\theta$, we define $V_\theta = \text{span}\{\psi_0(\theta), \ldots, \psi_{2M_\theta}\}$, where

$$\psi_0(\theta) = 1, \qquad \psi_{2l-1}(\theta) = \cos(l\theta), \qquad \psi_{2l}(\theta) = \sin(l\theta), \qquad l = 1, \ldots, M_\theta.$$

Also, for a positive integer $N$, we define $P_N(0, 1)$ to be the set of all polynomials of degree $\leq N$ on [0, 1] and

$$V = P_N(0, 1) \otimes V_\theta, \qquad \tilde{V} = \{v \in V : v(0, \theta) = c, \theta \in [0, 2\pi]\},$$

where $c$ is a constant. If $\{\xi_i\}_{i=0}^N$ and $\{w_i\}_{i=0}^N$ are again the nodes and weights, respectively, of the $(N+1)$-point Chebyshev Gauss-Lobatto quadrature on $[-1, 1]$, let $\{\xi_i^{(r)}\}_{i=1}^{N-1}$ and $\{\xi_j^{(\theta)}\}_{j=0}^{2M_\theta}$ be the sets of collocation points given by

$$\xi_i^{(r)} = l^{-1}(\xi_i), \qquad \xi_j^{(\theta)} = \frac{2j\pi}{2M_\theta + 1},$$

where $l(r) = 2r - 1$ is the linear function mapping [0, 1] onto $[-1, 1]$. The spectral collocation solution of (3.13) is $U \in \tilde{V}$ satisfying (3.15). We seek $U$ in the form (3.16), where $U^{(1)}, U^{(2)} \in V$ satisfy (3.17) and (3.18), respectively, and $c$ is given by (3.19).

To solve (3.17) in this case, we introduce

$$\phi_k(r) = \left[1 - l^2(r)\right] T_{k-1}(l(r)), \qquad k = 1, \ldots, N-1,$$

and take

$$U^{(1)}(r, \theta) = \sum_{m=1}^{N-1} \sum_{n=0}^{2M_\theta} U_{m,n} \phi_m(r) \psi_n(\theta). \tag{6.8}$$

Substituting (6.8) into the first equation of (3.17) and using (3.14), we obtain

$$\left(A_r \otimes B_\theta + B_r \otimes A_\theta\right)\mathbf{u} = \mathbf{f}, \tag{6.9}$$

where

$$A_r = \left[a_{i,m}^{(r)}\right]_{i,m=1}^{N-1}, \quad a_{i,m}^{(r)} = \left(\xi_i^{(r)}\right)^2 \phi_m''\left(\xi_i^{(r)}\right) + \xi_i^{(r)} \phi_m'\left(\xi_i^{(r)}\right),$$

$$B_r = \left[b_{i,m}^{(r)}\right]_{i,m=1}^{N-1}, \quad b_{i,m}^{(r)} = \phi_m\left(\xi_i^{(r)}\right),$$

$$A_\theta = \left[a_{n,l}^{(\theta)}\right]_{j,n=0}^{2M_\theta}, \quad a_{j,n}^{(\theta)} = \psi_n''\left(\xi_j^{(\theta)}\right),$$

$$B_\theta = \left[b_{j,n}^{(\theta)}\right]_{j,n=0}^{2M_\theta}, \quad b_{j,n}^{(\theta)} = \psi_n\left(\xi_j^{(\theta)}\right), \tag{6.10}$$

and

$$\mathbf{u} = \left[U_{1,0}, \ldots, U_{1,2M_\theta}, \ldots, U_{N-1,0}, \ldots, U_{N-1,2M_\theta}\right]^T,$$

$$\mathbf{f} = \left[f_{1,0}, \ldots, f_{1,2M_\theta}, \ldots, f_{N-1,0}, \ldots, f_{N-1,2M_\theta}\right]^T, \qquad f_{i,j} = f\left(\xi_i^{(r)}, \xi_j^{(\theta)}\right).$$

with

$$W = \mathrm{diag}\left(w_1, \ldots, w_{N-1}\right), \quad D = \mathrm{diag}\left(1 + \xi_1, \ldots, 1 + \xi_{N-1}\right),$$
$$B = \left[b_{i,n}\right]_{i,n=1}^{N-1}, \quad b_{i,n} = \chi_n(\xi_i),$$

where

$$\chi_n(x) = (1 - x^2) T_{n-1}(x), \quad n = 1, \ldots, N-1,$$

(cf. (6.6)), we premultiply (6.14) by $B^T W D^{-1} \otimes I$ to obtain

$$\left(\tilde{A}_r \otimes B_\theta + \tilde{B}_r \otimes A_\theta\right)\mathbf{u} = \tilde{\mathbf{f}}, \tag{6.11}$$

where $\tilde{A}_r = B^T W D^{-1} A_r$, $\tilde{B}_r = B^T W D^{-1} B_r$ and $\tilde{\mathbf{f}} = (B^T W D^{-1} \otimes I)\mathbf{f}$. System (6.11) can be solved as follows, using a variant of Algorithm MDA in which the diagonalization is performed in the $\theta$ variable. It is shown in [42] that

$$B_\theta^{-1} A_\theta = \Lambda = \mathrm{diag}\left(\lambda_0, \ldots, \lambda_{2M_\theta}\right), \tag{6.12}$$

where

$$\lambda_0 = 0, \qquad \lambda_{2l-1} = \lambda_{2l} = -l^2, \qquad l = 1, \ldots, M_\theta.$$

It follows from (6.12) that (6.11) is equivalent to

$$\left(\tilde{A}_r \otimes I + \tilde{B}_r \otimes \Lambda\right)\mathbf{u} = \tilde{\mathbf{u}}, \tag{6.13}$$

where $\tilde{\mathbf{u}} = (I \otimes B_\theta^{-1})\mathbf{f}$. Introducing

$$\mathbf{u}_j = \left[U_{1,j}, \ldots, U_{N-1,j}\right]^T, \qquad \tilde{\mathbf{u}}_j = \left[\tilde{u}_{1,j}, \ldots, \tilde{u}_{N-1,j}\right]^T, \qquad j = 0, \ldots, 2M_\theta,$$

and using (6.12), we see that (6.13) reduces to

$$\left(\tilde{A}_r + \lambda_j \tilde{B}_r\right)\mathbf{u}_j = \tilde{\mathbf{u}}_j, \qquad j = 0, \ldots, 2M_\theta, \tag{6.14}$$

$2M_\theta + 1$ independent heptadiagonal systems the solution of which yields $U^{(1)}$.

Further, let $z \in P_N(0, 1)$ be such that

$$\xi_i^{(r)} z''\left(\xi_i^{(r)}\right) + z'\left(\xi_i^{(r)}\right) = 1, \quad i = 1, \ldots, N-1, \quad z(0) = 0, \quad z(1) = 0. \tag{6.15}$$

Then

$$U^{(2)}(r, \theta) = z(r) + 1 - r$$

is a solution of (3.18). With

$$z(r) = \sum_{m=1}^{N-1} \beta_m \phi_m(r),$$

the linear system corresponding to (6.15) is a special case of (6.14) with $j = 0$.

Finally, it can be shown that

$$c = -\frac{4\sum_{m=1}^{N-1}(-1)^m U_{m,0}}{1 + 4\sum_{m=1}^{N-1}(-1)^m \beta_m}.$$

Using FFTs, the cost of solving a system with the matrix $B_\theta$ of (6.10) is $O(M_\theta \log M_\theta)$. Also, in the premultiplication of (6.14) by $B^T W D^{-1}$, FFTs can be used to multiply a vector by $B^T$ at a cost of $O(N \log N)$. Moreover, the cost of solving each heptadiagonal linear system is proportional to the number of unknowns. Hence, for $M_\theta = N$, the cost of the algorithm is $O(N^2 \log N)$.

The MDA of [99] was used in [66] for the solution of Poisson's equation in a disk using a spectral collocation scheme.

## 7 The method of fundamental solutions and related techniques

MDAs have also been used in applications of the method of fundamental solutions (MFS) for the solution of certain harmonic and biharmonic axisymmetric problems [92, 181, 182, 191] and axisymmetric problems in linear elasticity and thermoelasticity [126, 127]. As an example, we describe the MFS MDA proposed in [181] for the three-dimensional axisymmetric potential problem,

$$\Delta u = 0 \text{ in } \Omega, \quad u = g(x, y, z), \quad (x, y, z) \in \partial\Omega,$$

where $\Delta$ denotes the Laplacian in three dimensions, $\Omega$ is an axisymmetric region in $\mathcal{R}^3$, and $\partial\Omega$ denotes the boundary of $\Omega$. Axisymmetric means that $\Omega$ is formed by rotating a region $\Omega' \in \mathcal{R}^2$ about the $z$-axis.

In the MFS, the solution $u$ is approximated by

$$U_{MN}(\mathcal{P}) = \sum_{m=1}^{M}\sum_{n=1}^{N} U_{m,n} K(\mathcal{P}, \mathcal{Q}_{m,n}), \qquad \mathcal{P} \in \overline{\Omega}, \tag{7.1}$$

where the specified singularities $\mathcal{Q}_{m,n}$ are located outside $\overline{\Omega}$. The function $K(\mathcal{P}, \mathcal{Q})$ is the fundamental solution of Laplace's equation in $\mathbf{R}^3$ given by

$$K(\mathcal{P}, \mathcal{Q}) = \frac{1}{4\pi |\mathcal{P} - \mathcal{Q}|},$$

where $|\mathcal{P} - \mathcal{Q}|$ denotes the distance between the points $\mathcal{P}$ and $\mathcal{Q}$. Let $\{\mathcal{P}_{i,j}\}_{i=1,j=1}^{M,N}$ be a set of $MN$ collocation points placed on $\partial\Omega$. Then the coefficients $\{U_{m,n}\}_{m=1,n=1}^{M,N}$ are determined by collocating the boundary condition at these points; that is,

$$U_{MN}(\mathcal{P}_{i,j}) = g(\mathcal{P}_{i,j}), \qquad i = 1, \ldots, M, \ \ j = 1, \ldots, N. \tag{7.2}$$

Substituting (7.1) into (7.2), we obtain an $MN \times MN$ linear system

$$G\mathbf{u} = \mathbf{g}, \tag{7.3}$$

where $\mathbf{u}$ and $\mathbf{g}$ are the vectors containing $\{U_{m,n}\}_{m=1,n=1}^{M,N}$ and $\{g(\mathcal{P}_{i,j})\}_{i=1,j=1}^{M,N}$, respectively. If the collocation points and the singularities are positioned appropriately (see [181] for details), the matrix $G$ has the block circulant structure

$$G = \begin{pmatrix} A_1 & A_2 & \cdots & A_M \\ A_M & A_1 & \cdots & A_{M-1} \\ \vdots & \vdots & & \vdots \\ A_2 & A_3 & \cdots & A_1 \end{pmatrix} \equiv \mathrm{circ}[A_1, A_2, \ldots, A_M], \tag{7.4}$$

where the matrices $A_\ell$, $\ell = 1, \cdots, M$, are $N \times N$ with

$$(A_\ell)_{j,n} = \frac{1}{4\pi |\mathcal{P}_{1,j} - \mathcal{Q}_{\ell,n}|}, \qquad j, n = 1, \ldots, N.$$

If $P$ is the $M \times M$ permutation matrix $P = \mathrm{circ}(0, 1, 0, \cdots, 0)$, then

$$G = \sum_{k=1}^{M} P^{k-1} \otimes A_k.$$

If $Z$ is the $M \times M$ unitary matrix

$$Z = \frac{1}{M^{1/2}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{M-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(M-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{M-1} & \omega^{2(M-1)} & \cdots & \omega^{(M-1)(M-1)} \end{pmatrix},$$

where $\omega = e^{2\pi\iota/M}$, $\iota^2 = -1$, then

$$Z^* Z = I_M, \qquad Z^* P Z = D, \tag{7.5}$$

where $Z^*$ is the conjugate transpose of $Z$ and

$$D = \mathrm{diag}\big(d_1, \ldots, d_M\big), \qquad d_m = \omega^{m-1}. \tag{7.6}$$

It follows from (7.5) that

$$(Z^* \otimes I_N)\left(\sum_{k=1}^{M} P^{k-1} \otimes A_k\right)(Z \otimes I_N) = \sum_{k=1}^{M} D^{k-1} \otimes A_k.$$

Thus the system (7.3), (7.4) is equivalent to

$$\left(\sum_{k=1}^{M} D^{k-1} \otimes A_k\right)\tilde{\mathbf{u}} = \tilde{\mathbf{g}}, \tag{7.7}$$

where

$$\tilde{\mathbf{u}} = (Z^{-1} \otimes I_N)\mathbf{u}, \quad \tilde{\mathbf{g}} = (Z^* \otimes I_N)\mathbf{g}.$$

It follows from (7.6) that (7.7) decomposes into the $M$ independent $N \times N$ systems,

$$B_m \tilde{\mathbf{u}}_m = \tilde{\mathbf{g}}_m, \qquad m = 1, 2, \ldots M,$$

where $B_m = \sum_{k=1}^{M} d_m^{k-1} A_k$, and $\tilde{\mathbf{u}}_m$ and $\tilde{\mathbf{g}}_m$ are the corresponding subvectors of $\tilde{\mathbf{u}}$ and $\mathbf{w}$, respectively. The $(r, s)$ entry of the matrix $B_m$, $r, s = 1, \ldots, N$, $m = 1, \ldots, M$, is determined from

$$[(B_1)_{rs}, \ldots, (B_M)_{rs}]^T = M^{1/2} Z \left[(A_1)_{rs}, \ldots, (A_M)_{rs}\right]^T.$$

We thus have the following MDA for solving (7.3):

**MFS MDA**

**Step 1.** Compute $\tilde{\mathbf{g}} = (Z^* \otimes I_N)\mathbf{g}$.

**Step 2.** Construct $B_m = \sum_{k=1}^{M} d_m^{k-1} A_k$, $m = 1, \ldots, M$.

**Step 3.** Solve $B_m \tilde{\mathbf{u}}_m = \tilde{\mathbf{g}}_m$, $\quad m = 1, \ldots, M$.

**Step 4.** Compute $\mathbf{u} = (Z \otimes I_N)\tilde{\mathbf{u}}$.

The cost of performing steps 1 and 4 using FFTs is $O(NM \log M)$. Similarly, step 2 can be carried out at a cost of $O(N^2 M \log M)$. In step 3, we need to solve

$M$ complex linear systems of order $N$. This is done using an $LU$-factorization with partial pivoting at a cost of $O(MN^3)$. Hence the total cost of the algorithm is $O(MN^3)$.

Systems possessing block circulant structures also arise in the application of the MFS to harmonic and biharmonic problems in regular polygonal domains [122], and in the application of the so–called MFS-K to problems in circular domains [123]. MFS MDAs have also been developed for wave scattering by circular cylinders [5, 192, 193]. The MDA approach used, which is essentially that proposed in [181], is also described in [194]. Similar ideas are employed in [124] in the development of an MDA for solving elliptic problems in three dimensions using radial basis functions. In the context of the boundary element method, MDAs are developed for potential problems in [154, 169], linear elasticity problems in [170, 171] and electromagnetic problems in [132]. A MDA for an integral equation method for scattering from cylindrically periodic structures is proposed in [6].

## 8 Concluding remarks

This paper provides a comprehensive account of the formulation and application of MDAs in commonly used discretizations of Poisson problems. While emphasis has been placed on the treatment of problems with homogeneous Dirichlet boundary conditions, methods can be extended to handle non–homogeneous Dirichlet and Neumann conditions and mixed combinations of these, as well as periodic boundary conditions, and to equations of the form (1.6) and (1.7). Methods for Poisson problems have been employed in the treatment of the biharmonic Dirichlet problem written in the form (3.23), (3.26). Furthermore, such methods have been used frequently as preconditioners in iterative techniques for the solution of linear systems arising from the discretization of quite general elliptic equations and of problems in more general regions. In addition, MDAs have been used for the solution of the linear systems resulting when the Laplace or biharmonic operator is discretized in the numerical solution of linear and nonlinear time-dependent problems including the Navier–Stokes equations.

In many papers, results of numerical experiments are provided to demonstrate some measure of the accuracy of the method under consideration. However, the rigorous convergence analysis of several of the techniques presented herein remains an open problem. In particular, there is very little analysis to support the convergence rates of the spline collocation methods of Section 5.

While we have made reference to a number of methods that have extensions to problems in three dimensions, there is still work to be done on the implementation and application of these extensions to non-trivial problems. In general, little information is given on the implementation of MDAs. A challenging project would be the development of a software library comprising standardized implementations of MDAs for the various discretization methods in two and three dimensions. Such a library would be a valuable asset for the community.

The formulation and implementation of extensions of MDAs to higher order PDEs and to systems of PDEs in two and three dimensions, such as the Cauchy-Navier equations of elasticity, are topics of future research. Moreover, there has been considerable research activity recently on the approximation of functions and particular solutions of elliptic PDEs using radial basis functions. This is of much practical importance since once a particular solution of the PDE in question is known, the associated BVP can be reformulated as one governed by a homogeneous PDE, which, in turn, can be solved by a boundary method at considerably less expense and implementational effort. The approximation of functions/particular solutions in certain domains possessing radial symmetry using MDAs similar to that described in Section 7 is also an area worthy of investigation.

**Acknowledgements** The authors wish to thank the referees for their insightful comments and helpful suggestions which resulted in a much improved manuscript. This research was supported in part by the University of Cyprus.

## References

1. Abakumov, A.A., Yeremin, Yu.A., Kuznetsov, Yu.A.: Efficient fast direct method of solving Poisson's equation on a parallelepiped and its implementation in an array processor. Sov. J. Numer. Anal. Math. Model. **3**, 1–20 (1988)
2. Abushama, A.A., Bialecki, B.: Modified nodal cubic spline collocation for biharmonic equations. Numer. Algorithms **43**, 331–353 (2006)
3. Abushama, A.A., Bialecki, B.: Modified nodal cubic spline collocation for Poisson's equation. SIAM J. Numer. Anal. **46**, 397–418 (2008)
4. Amodio, P., Cash, J.R., Roussos, G., Wright, R.W., Fairweather, G., Gladwell, I., Kraut, G.L., Paprzycki, M.: Almost block diagonal linear systems: sequential and parallel solution techniques, and applications. Numer. Linear Algebra Appl. **7**, 275–317 (2000)
5. Anastassiu, H.T., Lymperopoulos, D.G., Kaklamani, D.I.: Accuracy analysis and optimization of the method of auxiliary sources (MAS) for scattering by a circular cylinder. IEEE Trans. Antennas Propag. **52**, 1541–1547 (2004)
6. Anastassiu, H.T., Volakis, J.L., Filipovic, D.S.: Integral equation modeling of cylindrically periodic scatterers in the interior of a cylindrical waveguide. IEEE Trans. Microwave Theor. Tech. **46**, 1713–1720 (1998)
7. Archer, D.: Some collocation methods for differential equations. Ph.D. thesis, Rice University, Houston, Texas (1973)
8. Archer, D.: An $O(h^4)$ cubic spline collocation method for quasilinear parabolic equations. SIAM J. Numer. Anal. **14**, 620–637 (1977)
9. Auteri, F., Parolini, N., Quartapelle, L.: Essential imposition of Neumann condition in Galerkin-Legendre elliptic solvers. J. Comput. Phys. **185**, 427–444 (2003)
10. Auteri, F., Quartapelle, L.: Galerkin spectral method for the vorticity and stream function equations. J. Comput. Phys. **149**, 306–332 (1999)
11. Auteri, F., Quartapelle, L.: Galerkin–Legendre spectral method for the 3D Helmholtz equation. J. Comput. Phys. **161**, 454–483 (2000)
12. Auteri, F., Quartapelle, L.: Spectral elliptic solvers in a finite cylinder. Commun. Comput. Phys. **5**, 426–441 (2009)
13. Auteri, F., Quartapelle, L., Vigevano, L.: Accurate $\omega - \psi$ spectral solution of the singular driven cavity problem. J. Comput. Phys. **180**, 597–615 (2002)

14. Azaiez, M., Shen, J., Xu, C., Zhuang, Q.: A Laguerre-Legendre spectral method for the Stokes problem in a semi-infinite channel. SIAM J. Numer. Anal. **47**, 271–292 (2009)
15. Bade, F., Haldenwang, P.: High order scheme for thermally driven flows in an open channel. Comput. Fluids **27**, 273–290 (1999)
16. Banegas, A.: Fast Poisson solvers for problems with sparsity. Math. Comput. **32**, 441–446 (1978)
17. Bank, R.E.: Efficient algorithms for solving tensor product finite element equations. Numer. Math. **31**, 49–61 (1978)
18. Bao, G., Sun, W.: A fast algorithm for the electromagnetic scattering from a large cavity. SIAM J. Sci. Comput. **27**, 553–574 (2005)
19. Ben–Artzi, M., Croisille, J.-P., Fishelov, D.: A fast direct solver for the biharmonic problem in a rectangular grid. SIAM J. Sci. Comput. **31**, 303–333 (2008)
20. Bennett, K.R.: Parallel collocation methods for boundary value problems. Ph.D. thesis, University of Kentucky, Lexington, Kentucky (1991)
21. Bialecki, B.: A fast domain decomposition Poisson solver on a rectangle for Hermite bicubic orthogonal spline collocation. SIAM J. Numer. Anal. **30**, 425–434 (1993)
22. Bialecki, B.: A fast solver for the orthogonal spline collocation solution of the biharmonic Dirichlet problem on rectangles. J. Comput. Phys. **191**, 601–621 (2003)
23. Bialecki, B.: Piecewise Hermite bicubic orthogonal spline collocation for Poisson's equation on a disk (preprint)
24. Bialecki, B., Cai, X.-C., Dryja, M., Fairweather, G.: An additive Schwarz algorithm for piecewise Hermite bicubic orthogonal spline collocation. In: Domain Decomposition Methods in Science and Engineering (Como 1992). Contemp. Math., vol. 157, pp. 237–244. Amer. Math. Soc., Providence, Rhode Island (1994)
25. Bialecki, B., Dillery, D.S.: Fourier analysis of Schwarz alternating methods for piecewise Hermite bicubic orthogonal spline collocation. BIT **33**, 634–646 (1993)
26. Bialecki, B., Dryja, M.: A nonoverlapping domain decomposition method for orthogonal spline collocation problems. SIAM J. Numer. Anal. **41**, 1709–1728 (2003)
27. Bialecki, B., Fairweather, G.: Matrix decomposition algorithms for separable elliptic boundary value problems in two dimensions. J. Comput. Appl. Math. **46**, 369–386 (1993)
28. Bialecki, B., Fairweather, G.: Matrix decomposition algorithms in orthogonal spline collocation for separable elliptic boundary value problems. SIAM J. Sci. Comput. **16**, 330–347 (1995)
29. Bialecki, B., Fairweather, G.: Orthogonal spline collocation methods for partial differential equations. J. Comput. Appl. Math. **128**, 55–82 (2001)
30. Bialecki, B., Fairweather, G., Bennett, K.R.: Fast direct solvers for piecewise Hermite bicubic orthogonal spline collocation equations. SIAM J. Numer. Anal. **29**, 156–173 (1992)
31. Bialecki, B., Fairweather, G., Karageorghis, A.: Matrix decomposition algorithms for modified spline collocation for Helmholtz problems. SIAM J. Sci. Comput. **24**, 1733–1753 (2003)
32. Bialecki, B., Fairweather, G., Karageorghis, A.: Optimal superconvergent one step nodal cubic spline collocation methods. SIAM J. Sci. Comput. **27**, 575–598 (2005)
33. Bialecki, B., Fairweather, G., Karageorghis, A., Nguyen, Q.N.: On the formulation and implementation of optimal superconvergent one step quadratic spline collocation methods for elliptic problems. Technical Report TR/18/2007, Department of Mathematics and Statistics, University of Cyprus (2007)
34. Bialecki, B., Fairweather, G., Karageorghis, A., Nguyen, Q.N.: Optimal superconvergent one step quadratic spline collocation methods for Helmholtz equations. In: Jorgensen, P., Shen, X., Shu, C.-W., Yan, N. (eds.) Recent Advances in Computational Science, pp. 156–174. World Scientific, Singapore (2008)
35. Bialecki, B., Fairweather, G., Karageorghis, A., Nguyen, Q.N.: Optimal superconvergent one step quadratic spline collocation methods. BIT **48**, 449–472 (2008)
36. Bialecki, B., Fairweather, G., Knudson, D.B., Lipman, D.A., Nguyen, Q.N., Sun, W., Weinberg, G.M.: Matrix decomposition algorithms for the finite element Galerkin method with piecewise Hermite cubics. Numer. Algorithms **52**, 1–23 (2009)
37. Bialecki, B., Fairweather, G., Remington, K.A.: Fourier methods for piecewise Hermite bicubic orthogonal spline collocation. East-West J. Numer. Math. **2**, 1–20 (1994)

38. Bialecki, B., Karageorghis, A.: A Legendre spectral collocation method for the biharmonic Dirichlet problem. M2AN Math. Model. Numer. Anal. **34**, 637–662 (2000)
39. Bialecki, B., Karageorghis, A.: A Legendre spectral Galerkin method for the biharmonic Dirichlet problem. SIAM J. Sci. Comput. **22**, 1549–1569 (2000)
40. Bialecki, B., Karageorghis, A.: Legendre Gauss spectral collocation for the Helmholtz equation on a rectangle. Numer. Algorithms **36**, 203–227 (2004)
41. Bialecki, B., Karageorghis, A.: A nonoverlapping domain decomposition method for Legendre spectral collocation problems. J. Sci. Comput. **32**, 373–409 (2007)
42. Bialecki, B., Karageorghis, A.: Spectral Chebyshev-Fourier collocation for the Helmholtz and variable coefficient equations in a disk. J. Comput. Phys. **227**, 8588–8603 (2008)
43. Bialecki, B., Karageorghis, A.: Spectral Chebyshev collocation for Poisson and biharmonic equations (submitted)
44. Bialecki, B., Remington, K.A.: Fourier matrix decomposition methods for the least squares solution of singular Neumann and periodic Hermite bicubic collocation problems. SIAM J. Sci. Comput. **16**, 431–451 (1995)
45. Bialecki, B., Wang, Z.: Modified nodal cubic spline collocation for elliptic equations (submitted)
46. Bickley, W.S.: Finite difference formulae for the square lattice. Q. J. Mech. Appl. Math. **1**, 35–42 (1948)
47. Bickley, W.S., McNamee, J.: Matrix and other direct methods for the solution of systems of linear difference equations. Philos. Trans. R. Soc. Lond., Ser. A. **252**, 69–131 (1960)
48. Bjøntegaard, T., Maday, Y., Rønquist, E.M.: Fast tensor-product solvers: partially deformed three-dimensional domains. J. Sci. Comput. **39**, 28–48 (2009)
49. Bjørstad, P.E.: Fast numerical solution of the biharmonic Dirichlet problem on rectangles. SIAM J. Numer. Anal. **20**, 59–71 (1983)
50. Bjørstad, P.E., Tjøstheim, B.P.: Efficient algorithms for solving a fourth-order equation with the spectral-Galerkin method. SIAM J. Sci. Comput. **18**, 621–632 (1997)
51. Bjørstad, P.E., Tjøstheim, B.P.: High precision solutions of two fourth order eigenvalue problems. Computing **63**, 97–107 (1999)
52. Boisvert, R.F.: Families of high order accurate discretizations of some elliptic problems. SIAM J. Sci. Statist. Comput. **2**, 268–284 (1981)
53. Boisvert, R.F.: High order compact difference formulas for elliptic problems with mixed boundary conditions. In: Vichnevetsky, R., Stepleman, R.S. (eds.) Advances in Computer Methods for Partial Differential Equations IV, pp. 193–199. IMACS, Rutgers University, New Brunswick, New Jersey (1981)
54. Boisvert, R.F.: A fourth-order accurate fast direct method for the Helmholtz equation. In: Birkhoff, G., Schoenstadt, A. (eds.) Elliptic Problem Solvers II, pp. 35–44. Academic, Orlando, Florida (1984)
55. Boisvert, R.F.: A fourth-order-accurate Fourier method for the Helmholtz equation in three dimensions. ACM Trans. Math. Softw. **13**, 221–234 (1987)
56. Boisvert, R.F.: Algorithm 651: HFFT—high-order fast-direct solution of the Helmholtz equation. ACM Trans. Math. Softw. **13**, 235–249 (1987)
57. de Boor, C.: The method of projections as applied to the numerical solution of two point boundary value problems using cubic splines. Ph.D. thesis, University of Michigan, Ann Arbor, Michigan (1966)
58. Bottcher, C., Strayer, M.R.: The basis spline method and associated techniques. In: Bottcher, C., Strayer, M.R., McGrory, J.B. (eds.) Computational Atomic and Nuclear Physics, pp. 217–240. World Scientific, Singapore (1990)
59. Bottcher, C., Strayer, M.R.: Spline methods for conservation equations. In: Lee, D., Robinson, A.R., Vichnevetsky, R. (eds.) Computational Acoustics, vol. 2, pp. 317–338. Elsevier, Amsterdam (1993)
60. Boyd, J.P.: Chebyshev and Fourier Spectral Methods. 2nd edn. Dover, New York (2001)
61. Buzbee, B.L., Golub, G.H., Nielson, C.W.: On direct methods for solving Poisson's equation. SIAM J. Numer. Anal. **7**, 627–656 (1970)
62. Canuto, C., Hussaini, M.Y., Quarteroni, A., Zang, T.A.: Spectral Methods. Fundamentals in Single Domains. Springer, New York (2006)

63. Chan, T.F., Resasco, D.C.: A domain–decomposed fast Poisson solver on a rectangle. SIAM J. Sci. Statist. Comput. **8**, 14–26 (1987)
64. Chan, T.F., Resasco, D.C.: Errata: a domain–decomposed fast Poisson solver on a rectangle. SIAM J. Sci. Statist. Comput. **8**, 457 (1987)
65. Chen, H.B., Nandakumar, K., Finlay, W.H., Ku, H.C.: Three-dimensional viscous flow through a rotating channel: a pseudospectral matrix method approach. Int. J. Numer. Methods Fluids **23**, 379–396 (1996)
66. Chen, H., Su, Y., Shizgal, B.D.: A direct spectral collocation Poisson solver in polar and cylindrical coordinates. J. Comput. Phys. **160**, 453–469 (2000)
67. Chen, Y., Yi, N., Liu, W.: A Legendre-Galerkin spectral method for optimal control problems governed by elliptic equations. SIAM J. Numer. Anal. **46**, 2254–2275 (2008)
68. Christara, C.C.: Quadratic spline collocation methods for elliptic partial differential equations. BIT **34**, 33–61 (1994)
69. Christara, C.C., Ng, K.S.: Fast Fourier transform solvers and preconditioners for quadratic spline collocation. BIT **42**, 702–739 (2002)
70. Constas, A.: Fast Fourier transform solvers for quadratic spline collocation. M.Sc. thesis, Department of Computer Science, University of Toronto (1996)
71. Daniel, J.W., Swartz, B.K.: Extrapolated collocation for two-point boundary value problems using cubic splines. J. Inst. Math. Appl. **16**, 161–174 (1975)
72. Doha, E.H.: Efficient Jacobi Galerkin methods for second- and fourth-order elliptic problems. J. Egypt. Math. Soc. **16**, 161–213 (2008)
73. Doha, E.H., Abd-Elhameed, W.M.: Efficient spectral-Galerkin algorithms for direct solution of second-order equations using ultraspherical polynomials. SIAM J. Sci. Comput. **24**, 548–571 (2002)
74. Doha, E.H., Abd-Elhameed, W.M., Bhrawy, A.H.: Efficient spectral ultraspherical-Galerkin algorithms for the direct solution of 2*n*th-order linear differential equations. Appl. Math. Model. **33**, 1982–1996 (2009)
75. Doha, E.H., Bhrawy, A.H.: Efficient spectral-Galerkin algorithms for direct solution of the integrated forms of second-order equations using ultraspherical polynomials. Numer. Algorithms **42**, 137–164 (2006)
76. Doha, E.H., Bhrawy, A.H.: Efficient spectral-Galerkin algorithms for direct solution of second-order differential equations using Jacobi polynomials. ANZIAM J. **48**, 361–386 (2007)
77. Doha, E.H., Bhrawy, A.H.: Efficient spectral-Galerkin algorithms for direct solution of fourth-order differential equations using Jacobi polynomials. Appl. Numer. Math. **58**, 1224–1244 (2008)
78. Doha, E.H., Bhrawy, A.H., Abd-Elhameed, W.M.: Jacobi spectral Galerkin method for elliptic Neumann problems. Numer. Algorithms **50**, 67–91 (2009)
79. Dorr, F.W. (1970). The direct solution of the discrete Poisson equation on a rectangle. SIAM Rev. **12**, 248–263 (1970)
80. Dui, K., Fairweather, G., Nguyen, Q.N., Sun, W.: Matrix decomposition algorithms for the $C^0$–quadratic finite element Galerkin method. BIT **49**, 509–526 (2009)
81. Dyksen, W.R.: Tensor product generalized ADI methods for separable elliptic problems. SIAM J. Numer. Anal. **24**, 59–76 (1987)
82. Egerváry, E.: Űber eine Methode zur numerischen Lősung der Poissonschen Differenzengleichung für beliebige Gebeite. Acta. Math. Acad. Sci. Hungar. **11**, 341–361 (1960)
83. Ehrenstein, U., Peyret, R.: A Chebyshev collocation method for the Navier-Stokes equations with application to double-diffusive convection. Int. J. Numer. Methods Fluids **9**, 427–452 (1989)
84. Elbardary, E.M.E.: Efficient Chebyshev-Petrov-Galerkin method for solving second order equations. J. Sci. Comput. **34**, 113–126 (2008)
85. Elman, H.C., O'Leary, D.P.: Efficient iterative solution of the three–dimensional Helmholtz equation. J. Comput. Phys. **142**, 163–181 (1998)
86. Ernst, O., Golub, G.H.: A domain decomposition approach to solving the Helmholtz equation with a radiation boundary condition. In: Domain Decomposition Methods in Science

and Engineering (Como, 1992). Contemp. Math., vol. 157, pp. 177–192. Amer. Math. Soc., Providence, Rhode Island (1994)
87. Fairweather, G.: Finite Element Galerkin Methods for Differential Equations. Lecture Notes in Pure and Applied Mathematics, vol. 34. Marcel Dekker, New York (1978)
88. Fairweather, G., Bennett, K.R., Bialecki, B.: Parallel matrix decomposition algorithms for separable elliptic boundary value problems. In: Noye, B.J., Benjamin, B.R., Colgan, L.H. (eds.) Computational Techniques and Applications: CTAC-91, Proceedings of the 1991 International Conference on Computational Techniques and Applications, Adelaide, South Australia, July 1991, pp. 63–74. Computational Mathematics Group, Australian Mathematical Society (1992)
89. Fairweather, G., Gladwell, I.: Algorithms for almost block diagonal linear systems. SIAM Rev. **46**, 49–58 (2004)
90. Fairweather, G., Karageorghis, A.: The method of fundamental solutions for elliptic boundary value problems. Adv. Comput. Math. **9**, 69–95 (1998)
91. Fairweather, G., Karageorghis, A., Maack, J.: Compact optimal quadratic spline collocation methods for Poisson and Helmholtz problems: formulation and numerical verification. Technical Report TR/03/2010, Department of Mathematics and Statistics, University of Cyprus (2010)
92. Fairweather, G., Karageorghis, A., Smyrlis, Y.-S.: A matrix decomposition MFS algorithm for axisymmetric biharmonic problems. Adv. Comput. Math. **23**, 55–71 (2005)
93. Fyfe, D.J.: The use of cubic splines in the solution of two-point boundary value problems. Comput. J. **13**, 188–192 (1969)
94. Garba, A.: A mixed spectral/wavelet method for the solution of the Stokes problem. J. Comput. Phys. **145**, 297–315 (1998)
95. Golub, G.H., Huang, L.C., Simon, H., Tang, W.-P.: A fast Poisson solver for the finite difference solution of the incompressible Navier Stokes equations. SIAM J. Sci. Comput. **19**, 1606–1624 (1998)
96. Gottlieb, D., Lustman, L.: The spectrum of the Chebyshev collocation operator for the heat equation. SIAM J. Numer. Anal. **20**, 909–921 (1983)
97. Guessous, L.: A pseudo-spectral numerical scheme for the simulation of steady and oscillating wall-bounded flows. Numer. Heat Transf., B **45**, 135–157 (2004)
98. Gustafsson, B., Hemmingsson–Frändén, L.: A fast domain decomposition high order Poisson solver. J. Sci. Comput. **14**, 223–243 (1999)
99. Haidvogel, D.B., Zang, T.: The accurate solution of Poisson's equation by expansion in Chebyshev polynomials. J. Comput. Phys. **30**, 167–180 (1979)
100. Haldenwang, P., Labrosse, G., Abboudi, S., Deville, M.: Chebyshev 3–D spectral and 2–D pseudospectral solvers for the Helmholtz equation. J. Comput. Phys. **55**, 115–128 (1984)
101. Heikkola, E., Kuznetsov, Y.A., Lipnikov, K.N.: Fictitious domain methods for the numerical solution of three–dimensional acoustic scattering problems. J. Comput. Acoust. **7**, 161–183 (1999)
102. Heikkola, E., Rossi, T., Toivanen, J.: Fast direct solution of the Helmholtz equation with a perfectly matched or an absorbing boundary condition. Int. J. Numer. Methods Eng. **57**, 2007–2025 (2003)
103. Heinrichs, W.: Improved condition number for spectral methods. Math. Comput. **53**, 103–119 (1989)
104. Heinrichs, W.: Spectral methods with sparse matrices. Numer. Math. **56**, 25–41 (1989)
105. Heinrichs, W.: Algebraic spectral multigrid methods. Comput. Methods Appl. Mech. Eng. **80**, 281–286 (1990)
106. Heinrichs, W.: A stabilized treatment of the biharmonic operator with spectral methods. SIAM J. Sci. Comput. **12**, 1162–1172 (1991)
107. Hendrickx, J., Van Barel, M.: A Kronecker product variant of the FACR method for solving the generalized Poisson equation. J. Comput. Appl. Math. **140**, 369–380 (2002)
108. Hendrickx, J., Vandebril, R., Van Barel, M.: A fast direct method for solving the two–dimensional Helmholtz equation, with Robbins boundary conditions. In: Fast Algorithms for Structured Matrices: Theory and Applications. Contemp. Math., vol. 323, pp. 187–204. Amer. Math. Soc., Providence, Rhode Island (2003)

109. Hill, R.W., Ball, K.S.: Direct numerical simulations of turbulent forced convection between counter-rotating disks. Int. J. Heat Fluid Flow **20**, 208–221 (1999)
110. Hill, R.W., Ball, K.S.: Parallel implementation of a Fourier-Chebyshev collocation method for incompressible fluid flow and heat transfer. Numer. Heat Transf., B **36**, 309–329 (1999)
111. Ho, A.C., Ng, M.K.: Iterative methods for Robbins problem. Appl. Math. Comput. **165**, 103–125 (2005)
112. Hockney, R.W.: A fast direct solution of Poisson's equation using Fourier analysis. J. Assoc. Comput. Mach. **12**, 95–113 (1965)
113. Houstis, E.N., Christara, C.C., Rice, J.R.: Quadratic-spline collocation methods for two-point boundary value problems. Int. J. Numer. Methods Eng. **26**, 935–952 (1988)
114. Houstis, E.N., Vavalis, E.A., Rice, J.R.: Convergence of $O(h^4)$ cubic spline collocation methods for elliptic partial differential equations. SIAM J. Numer. Anal. **25**, 54–74 (1988)
115. Hu, Y., Ling, X.: Preconditioners for elliptic problems via non–uniform meshes. Appl. Math. Comput. **181**, 1182–1198 (2006)
116. Hyman, M.A.: Non iterative numerical solution of boundary value problems. Appl. Sci. Res., B **2**, 325–351 (1951)
117. Ierley, G.R.: A class of sparse spectral operators for inversion of powers of the Laplacian in $N$ dimensions. J. Sci. Comput. **12**, 57–73 (1997)
118. Ito, K., Qiao, Z., Toivanen, J.: A domain decomposition solver for acoustic scattering by elastic objects in layered media. J. Comput. Phys. **227**, 8685–8698 (2008)
119. Julien, K., Watson, M.: Efficient multi-dimensional solution of PDEs using Chebyshev spectral methods. J. Comput. Phys. **228**, 1480–1503 (2009)
120. Jun, S., Kang, S., Kwon, Y.: A direct solver for the Legendre tau approximation for the two-dimensional Poisson problem. J. Appl. Math. Comput. **23**, 25–42 (2007)
121. Kadalbajoo, M.K., Bharadwaj, K.K.: Fast elliptic solvers—an overview. Appl. Math. Comput. **14**, 331–355 (1984)
122. Karageorghis, A.: Efficient MFS algorithms in regular polygonal domains. Numer. Algorithms **50**, 215–240 (2009)
123. Karageorghis, A.: Efficient Kansa-type MFS algorithm for elliptic problems. Numer. Algorithms (to appear). doi:10.1007/s11075-009-9334-8
124. Karageorghis, A., Chen, C.S., Smyrlis, Y.-S.: Matrix decomposition RBF algorithm for solving 3d elliptic problems. Eng. Anal. Bound. Elem. **33**, 1368–1373 (2009)
125. Karageorghis, A., Kyza, I.: Efficient algorithms for approximating particular solutions of elliptic equations using Chebyshev polynomials. Commun. Comput. Phys. **2**, 501–521 (2007)
126. Karageorghis, A., Smyrlis, Y.-S.: Matrix decomposition MFS algorithms for elasticity and thermo-elasticity problems in axisymmetric domains. J. Comput. Appl. Math. **206**, 774–795 (2007)
127. Karageorghis, A., Smyrlis, Y.-S.: Matrix decomposition algorithms related to the MFS for axisymmetric problems. In: Manolis, G.D., Polyzos, D. (eds.) Recent Advances in Boundary Element Methods, pp. 223–237. Springer, New York (2009)
128. Kaufman, L., Warner, D.: High–order, fast–direct methods for separable elliptic equations. SIAM J. Numer. Anal. **21**, 674–694 (1984)
129. Kaufman, L., Warner, D.: Algorithm 685: a program for solving separable elliptic equations. ACM Trans. Math. Softw. **16**, 325–351 (1990)
130. Kegley, D.R., Jr., Oberacker, V.E., Strayer, M.R., Umar, A.S., Wells, J.C.: Basis spline collocation method for solving the Schrödinger equation in axillary symmetric systems. J. Comput. Phys. **128**, 197–208 (1996)
131. Knudson, D.B.: A piecewise Hermite bicubic finite element Galerkin method for the biharmonic Dirichlet problem. Ph.D. thesis, Colorado School of Mines, Golden, Colorado (1997)
132. Kurz, S., Rain, O., Rjasanow, S.: Application of the adaptive cross approximation technique for the coupled BE–FE solution of symmetric electromagnetic problems. Comput. Mech. **32**, 423–429 (2003)
133. Kuznetsov, Yu.A.: Numerical methods in subspaces. In: Marchuk, G.I. (ed.) Vychislitel'nye Processy i Sistemy II, pp. 265–350. Naukam Moscow (1985) (in Russian)
134. Kuznetsov, Yu.A., Matsokin, A.M.: On partial solution of systems of linear algebraic equations. Sov. J. Numer. Anal. Math. Model. **4**, 453–468 (1989)

135. Kuznetsov, Yu.A., Rossi, T.: Fast direct method for solving algebraic systems with separable symmetric band matrices. East-West J. Numer. Math. **4**, 53–68 (1996)
136. Kwan, Y.-Y.: Efficient spectral-Galerkin methods for polar and cylindrical geometries. Appl. Numer. Math. **59**, 170–186 (2009)
137. Kwan, Y.-Y., Shen, J.: An efficient direct parallel spectral-element solver for separable elliptic problems. J. Comput. Phys. **225**, 1721–1735 (2007)
138. Lai, M.-C.: A simple compact fourth-order Poisson solver on polar geometry. J. Comput. Phys. **182**, 337–345 (2002)
139. Lai, M.-C., Tseng, J.-M.: A formally fourth-order accurate compact scheme for 3D Poisson equation in cylindrical and spherical coordinates. J. Comput. Appl. Math. **201**, 175–181 (2007)
140. Lai, M.-C., Wang, W.-C.: Fast direct solvers for Poisson equation on 2D polar and spherical geometries. Numer. Methods Partial Differ. Equ. **18**, 56–68 (2002)
141. Larsson, E.: A domain decomposition method for the Helmholtz equation in a multilayer domain. SIAM J. Sci. Comput. **120**, 1713–1731 (1999)
142. Li, B., Fairweather, G., Bialecki, B.: Discrete-time orthogonal spline collocation methods for vibration problems. SIAM J. Numer. Anal. **39**, 2045–2065 (2002)
143. Liu, W.B., Shen, J.: A new efficient spectral Galerkin for singular perturbation problems. J. Sci. Comput. **11**, 411–437 (1996)
144. Lopez, J.M., Shen, J. : An efficient spectral-projection method for the Navier–Stokes equations in cylindrical geometries. I. Axisymmetric cases. J. Comput. Phys. **139**, 308–326 (1998)
145. Lopez, J.M., Marques, F., Shen, J.: An efficient spectral-projection method for the Navier–Stokes equations in cylindrical geometries, II. Three-dimensional cases. J. Comput. Phys. **176**, 384–401 (2002)
146. Lou, Z.-M., Bialecki, B., Fairweather, G.: Orthogonal spline collocation methods for biharmonic problems. Numer. Math. **80**, 267–303 (1998)
147. Louchart, O., Randriamampianina, A., Leonardi, E.: Spectral domain decomposition technique for the incompressible Navier-Stokes equations. Numer. Heat Transf., A **34**, 495–518 (1998)
148. Lyashko, A.D., Solov'yev, S.I.: Fourier method of solution of FE systems with Hermite elements for Poisson equation. Sov. J. Numer. Anal. Math. Model. **6**, 121–129 (1991)
149. Lynch, R.E., Rice, J.R., Thomas, D.H.: Direct solution of partial difference equations by tensor product methods. Numer. Math. **6**, 185–199 (1964)
150. Lynch, R.E., Rice, J.R., Thomas, D.H.: Tensor product analysis of partial difference equations. Bull. Am. Math. Soc. **70**, 378–384 (1964)
151. Maack, J.: Quadratic spline collocation for Poisson's and biharmonic equations in the unit square. M.S. thesis, Colorado School of Mines, Golden, Colorado (2009)
152. Marinos, A.Th.: On a direct method for solving Helmholtz's type equations in 3-D rectangular regions. J. Comput. Phys. **88**, 62–85 (1990)
153. Martikainen, J., Rossi, T., Toivanen, J.: A fast direct solver for elliptic problems with a divergence constraint. Numer. Linear Algebra Appl. **9**, 629–652 (2002)
154. Meyer, A., Rjasanow, S.: An effective direct solution method for certain boundary element equations in 3D. Math. Methods Appl. Sci. **13**, 45–53 (1990)
155. Mittal, R.: A Fourier–Chebyshev spectral collocation method for simulating flow past spheres and spheroids. Int. J. Numer. Methods Fluids **30**, 921–937 (1999)
156. Mittal, R.C., Gahlaut, S.: High–order finite–differences schemes to solve Poisson's equation in polar coordinates. IMA J. Numer. Anal. **11**, 261–270 (1991)
157. Nguyen, S., Delcarte, C.: A spectral collocation method to solve Helmholtz problems with boundary conditions involving mixed tangential and normal derivatives. J. Comput. Phys. **200**, 34–49 (2004)
158. Osborne, M.R.: Direct methods for the solution of finite–difference approximations to partial differential equations. Comput. J. **8**, 150–156 (1965/1966)
159. Petrova, S.: Parallel implementation of fast elliptic solver. Parallel Comput. **12**, 1113–1128 (1997)
160. Peyret, R.: Spectral Methods for Incompressible Viscous Flow. Springer, New York (2002)

161. Pickering, W.M.: Some comments on the solution of Poisson's equation using Bickley's formula and fast Fourier transforms. J. Inst. Math. Appl. **19**, 337–338 (1977)
162. Pickering, W.M.: An Introduction to Fast Fourier Transform Methods for Partial Differential Equations, with Applications. Research Studies Press, Wiley, New York (1986)
163. Pickering, W.M., Harley, P.J.: Iterative solution of the Robbins problem using FFT methods. Int. J. Comput. Math. **45**, 243–257 (1992)
164. Pickering, W.M., Harley, P.J.: FFT solution of the Robbins problem. IMA J. Numer. Anal. **13**, 215–233 (1993)
165. Pickering, W.M., Harley, P.J.: On Robbins boundary conditions, elliptic equations and FFT methods. J. Comput. Phys. **122**, 380–383 (1995)
166. Plagne, L., Berthou, J.-Y.: Tensorial basis spline collocation method for Poisson's equation. J. Comput. Phys. **157**, 419–440 (2000)
167. Plessix, R.E., Mulder, W.A.: Separation–of–variables as a preconditioner for an iterative Helmholtz solver. Appl. Numer. Math. **44**, 385–400 (2003)
168. Pozo, R., Remington, K.: Fast three–dimensional elliptic solvers on distributed network clusters. In: Joubert, G.R., et al. (eds.) Parallel Computing: Trends and Applications, pp. 201–208. Elsevier, Amsterdam (1994)
169. Rjasanow, S.: Effective algorithms with circulant–block matrices. Linear Algebra Appl. **202**, 55–69 (1994)
170. Rjasanow, S.: Optimal preconditioner for boundary element formulation of the Dirichlet problem in elasticity. Math. Methods Appl. Sci. **18**, 603–613 (1995)
171. Rjasanow, S.: The structure of the boundary element matrix for the three-dimensional Dirichlet problem in elasticity. Numer. Linear Algebra Appl. **5**, 203–217 (1998)
172. Rossi, T., Toivanen, J.: A nonstandard cyclic reduction method, its variants and stability. SIAM J. Matrix Anal. Appl. **20**, 628–645 (1999)
173. Rossi, T., Toivanen, J.: A parallel fast direct solver for block tridiagonal systems with separable matrices of arbitrary dimension. SIAM J. Sci. Comput. **20**, 1778–1796 (1999)
174. Russell, R.D., Sun, W.: Spline collocation differentiation matrices. SIAM J. Numer. Anal. **34**, 2274–2287 (1997)
175. Samarskii, A.A., Nikolaev, E.S.: Numerical Methods for Grid Equations. Vol. I, Direct Methods. Birkhäuser Verlag, Boston (1989)
176. Shen, J.: Efficient spectral-Galerkin method I. Direct solvers of second- and fourth-order equations using Legendre polynomials. SIAM J. Sci. Comput. **15**, 1489–1505 (1994)
177. Shen, J.: Efficient spectral-Galerkin method II. Direct solvers of second- and fourth-order equations using Chebyshev polynomials. SIAM J. Sci. Comput. **16**, 74–87 (1995)
178. Shen, J.: Efficient spectral-Galerkin methods III. Polar and cylindrical geometries. SIAM J. Sci. Comput. **18**, 1583–1604 (1997)
179. Shen, J.: Stable and efficient spectral methods in unbounded domains using Laguerre functions. SIAM J. Numer. Anal. **38**, 1113–1133 (2000)
180. Shen, J., Wang, L.-L.: Some recent advances on spectral methods for unbounded domains. Commun. Comput. Phys. **5**, 195–241 (2009)
181. Smyrlis, Y.-S., Karageorghis, A.: A matrix decomposition MFS algorithm for axisymmetric potential problems. Eng. Anal. Bound. Elem. **28**, 463–474 (2004)
182. Smyrlis, Y.-S., Karageorghis, A.: The method of fundamental solutions for stationary heat conduction problems in rotationally symmetric domains. SIAM J. Sci. Comput. **27**, 1193–1512 (2006)
183. Stephenson, J.W.: Single cell discretizations of order two and four for biharmonic problems. J. Comput. Phys. **55**, 65–80 (1984)
184. Sun, W.: Orthogonal collocation solution of biharmonic equations. Int. J. Comput. Math. **49**, 221–232 (1993)
185. Sun, W.: A higher order direct method for solving Poisson's equation on a disc. Numer. Math. **70**, 501–506 (1995)
186. Sun, W.: Fast algorithms for high-order spline collocation systems. Numer. Math. **81**, 143–160 (1998)
187. Sun, W., Zamani, N.G.: A fast algorithm for solving the tensor product collocation equations. J. Franklin Inst. **326**, 295–307 (1989)

188. Swarztrauber, P.N.: The methods of cyclic reduction, Fourier analysis and the FACR algorithm for the direct solution of Poisson's equation on a rectangle. SIAM Rev. **19**, 490–501 (1977)
189. Swarztrauber, P.N.: Fast Poisson solvers. In: Studies in Numerical Analysis, MAA Stud. Math., vol. 24, pp. 319–370. Mathematical Association of America, Washington, DC (1984)
190. Swarztrauber, P.N., Sweet, R.A.: The direct solution of the discrete Poisson equation on a disk. SIAM J. Numer. Anal. **10**, 900–907 (1973)
191. Tsangaris, T., Smyrlis, Y.-S., Karageorghis, A.: A matrix decomposition MFS algorithm for problems in hollow axisymmetric domains. J. Sci. Comput. **28**, 31–50 (2006)
192. Tsitsas, N.L., Alivizatos, E.G., Anastassiu, H.T., Kaklamani, D.I.: Optimization of the method of auxiliary sources (MAS) for scattering by an infinite cylinder under oblique incidence. Electromagnetics **25**, 39–54 (2005)
193. Tsitsas, N.L., Alivizatos, E.G., Anastassiu, H.T., Kaklamani, D.I.: Optimization of the method of auxiliary sources (MAS) for oblique incidence scattering by an infinite dielectric cylinder. Electr. Eng. **89**, 353–361 (2007)
194. Tsitsas, N.L., Alivizatos, E.G., Kalogeropoulos, G.H.: A recursive algorithm for the inversion of matrices with circulant blocks. Appl. Math. Comput. **188**, 877–894 (2007)
195. Umar, A.S.: Three-dimensional HF and TDHF calculations with the basis-spline collocation technique. In: Bottcher, C., Strayer, M.R., McGrory, J.B. (eds.) Computational Atomic and Nuclear Physics, pp. 377–390. World Scientific, Singapore (1990)
196. Umar, A.S., Wu, J., Strayer, M.R., Bottcher, C.: Basis-spline collocation method for the lattice solution of boundary value problems. J. Comput. Phys. **93**, 426–448 (1991)
197. Vajteršic, M.: Algorithms for Elliptic Problems: Efficient Sequential and Parallel Solvers. Kluwer, Dordrecht (1993)
198. Van Loan, C.: Computational Frameworks for the Fast Fourier Transform. SIAM, Philadelphia (1992)
199. Vassilevski, P.: Fast algorithm for solving a linear algebraic problem with separation of variables. C. R. Acad. Bulgare Sci. **37**, 305–308 (1984)
200. Vassilevski, P.: Fast algorithm for solving discrete Poisson equation in a rectangle. C. R. Acad. Bulgare Sci. **38**, 1311–1314 (1985)
201. Vedy, E., Viazzo, S., Schiestel, R.: A high–order finite difference method for incompressible fluid turbulence simulations. Int. J. Numer. Methods Fluids **42**, 1155–1188 (2003)
202. Wang, Y., Du, K., Sun, W.: Fast algorithms for the electromagnetic scattering from rectangular cavities. In: Deng, D., Jin, X.-Q., Sun, H.-W. (eds.) Recent Advances in Computational Mathematics, pp. 13–38. International Press, Somerville, Massachusetts, Higher Education Press, Beijing (2008)
203. Wang, Y., Du, K., Sun, W.: A second–order method for the electromagnetic scattering from a large cavity. Numer. Math. Theor. Methods Appl. **1**, 357–382 (2008)
204. Wang, Y., Du, K., Sun, W.: Preconditioning iterative algorithm for the electromagnetic scattering from a large cavity. Numer. Linear Algebra Appl. **16**, 345–363 (2009)
205. Zhang, Q., Shen, J., Wu, C.: A coupled Legendre–Laguerre spectral–element method for the Navier–Stokes equations in unbounded domains. J. Sci. Comput. **42**, 1–22 (2010)