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Nontensorial Clenshaw–Curtis cubature

**Alvise Sommariva · Marco Vianello ·
Renato Zanovello**

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Abstract We extend Clenshaw–Curtis quadrature to the square in a nontensorial way, by using Sloan's hyperinterpolation theory and two families of points recently studied in the framework of bivariate (hyper)interpolation, namely the Morrow–Patterson–Xu points and the Padua points. The construction is an application of a general approach to product-type cubature, where we prove also a relevant stability theorem. The resulting cubature formulas turn out to be competitive on nonentire integrands with tensor-product Clenshaw–Curtis and Gauss–Legendre formulas, and even with the few known minimal formulas.

Keywords Orthogonal polynomials · Hyperinterpolation · Quadrature · Cubature · Orthogonal moments · Nontensorial bivariate Clenshaw–Curtis formulas · Morrow–Patterson–Xu points · Padua points

Mathematics Subject Classification (2000) 65D32

1 Introduction

One of the most popular one-dimensional quadrature tools is the Clenshaw–Curtis formula [15]. Recently, it has been object of a renewed interest. Its properties have been revisited and further investigated, in order to explain

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A. Sommariva (✉) · M. Vianello · R. Zanovello
Department of Pure and Applied Mathematics, University of Padova,
via Trieste 63, 35121 Padova, Italy
e-mail: alvise@math.unipd.it

M. Vianello
e-mail: marcov@math.unipd.it

rigorously the fact that its performance is very often comparable with that of Gauss–Legendre quadrature [19], despite the theoretically lower degree of precision; see [43, 46]. As known, one of the strengths of Clenshaw–Curtis formula is that its nodes are known explicitly and that it can be implemented quite efficiently by the FFT algorithm [43], see also the recent paper [45] concerning fast computation of the weights. Once the weights are known, its tensorial extension is straightforward.

In this paper we give an answer to the following question: is it possible to extend the construction of Clenshaw–Curtis formulas to the square, with a general weight function, in a *nontensorial* fashion? Extension of Clenshaw–Curtis quadrature to general one-dimensional weight functions has been studied since the 1970s, for example in the framework of the so-called "product integration rules"; see e.g. [6, 21, 23, 32, 37, 38] and references therein.

The answer is positive, and rests on the concept of *hyperinterpolation*, introduced by Sloan [36]. Indeed, Clenshaw–Curtis quadrature ultimately consists of integrating the truncated Fourier–Chebyshev expansion of the given function, where the Fourier coefficients are computed by the Chebyshev–Gauss–Lobatto formula. Hyperinterpolation gives an extension of discretized Fourier expansion with respect to orthogonal polynomials over general multi-dimensional regions (or lower-dimensional manifolds), and integration of any hyperinterpolant provides a generalized Clenshaw–Curtis cubature formula. Even in the case of tensor-product domains, where tensor-product orthogonal expansions can be used (see, e.g., [24, 42]), hyperinterpolation is intrinsically nontensorial and thus generates *nontensorial* cubature formulas. As known, there are also other ways of constructing useful nontensorial cubature formulas, like the so-called sparse grids introduced by Smolyak in the 1960s (cf., e.g., [7, 28, 30, 31, 40] and references therein). In the present bidimensional context numerical tests and comparisons with available implementations (like, e.g., [8]), have shown that nontensorial formulas generated via (hyper)interpolation seem more effective.

The paper is organized as follows. In the next section, we briefly recall the method of polynomial hyperinterpolation over general regions and a related convergence result concerning generalized product integration. Moreover, we prove a theorem on the stability of the relevant product integration formulas. Then, we apply such results to the construction of nontensorial Clenshaw–Curtis cubature formulas over the square, using two families of nodes recently studied in the literature on polynomial interpolation and hyperinterpolation, namely the "Morrow–Patterson–Xu points" (cf. [2, 9, 11, 27, 44]) and the "Padua points" (cf. [3, 5, 10, 12, 14]). Finally, we compare the performance of the resulting formulas (implemented in Matlab) with tensor-product Clenshaw–Curtis and Gauss–Legendre formulas on several test functions. The numerical results show that such nontensorial Clenshaw–Curtis cubature formulas are more accurate than the tensorial version (at close cardinalities of the cubature point sets). They appear also more accurate than available Matlab implementations of nontensorial formulas based on sparse grids. Moreover, they are competitive with tensor-product Gauss–Legendre cubature and even with the few known minimal cubature formulas, on integrands which are not "too regular" (nonentire).

2 From hyperinterpolation to stable cubature

Polynomial hyperinterpolation of multivariate continuous functions over compact domains or manifolds, originally introduced by Sloan [36], is a discretized orthogonal projection on polynomial subspaces, which provides an approximation method more general than polynomial interpolation. Though the idea is very general and flexible, and the problem in some sense much easier than multivariate polynomial interpolation, till now it has been used effectively in few cases: originally for the sphere [34, 39], and more recently the square [9, 11], the disk [22], and the cube [13].

Indeed, hyperinterpolation requires two basic ingredients, i.e. the explicit knowledge of a family of orthogonal polynomials w.r.t. any measure μ on the domain, and a "good" cubature formula for that measure (positive weights and high algebraic degree of exactness). It is always convergent in the $L^2_{d\mu}$ norm, and it becomes an effective approximation tool in the uniform norm when its norm as a projection operator (the so-called Lebesgue constant) grows slowly (cf. [11, 22, 34, 39]).

Now we briefly summarize the structure of hyperinterpolation. Let $\Omega \subset \mathbb{R}^d$ be a compact subset (or lower dimensional manifold), and μ a positive and finite measure on Ω. We focus here our attention on *absolutely continuous measures*, i.e. measures with an nonnegative integrable density with respect to the standard Lebesgue (or surface) measure. The results below could be extended, with some restrictions, to other measures, like for example discrete measures.

For every function $f \in C(\Omega)$ the μ-orthogonal projection of f on $\Pi^d_n(\Omega)$ (the subspace of d-variate polynomials of total degree $\leq n$ restricted to Ω) is

$$\mathcal{S}_n f(\boldsymbol{x}) = \sum_{|\alpha| \leq n} c_\alpha \, p_\alpha(\boldsymbol{x}) \ , \quad c_\alpha := \int_\Omega f(\boldsymbol{x}) \, p_\alpha(\boldsymbol{x}) \, d\mu \ , \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_d)$ is a d-dimensional point, α is a d-index of length $|\alpha|$

$$\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \ , \quad |\alpha| := \alpha_1 + \ldots + \alpha_d \ , \tag{2}$$

and the set of polynomials $\{p_\alpha \, , \, 0 \leq |\alpha| \leq n\}$ is any μ-orthonormal basis of $\Pi^d_n(\Omega)$, with p_α of total degree $|\alpha|$ (concerning the theory of multivariate orthogonal polynomials, we refer the reader to the recent monograph by Dunkl and Xu [18]). Clearly, $\mathcal{S}_n p = p$ for every $p \in \Pi^d_n(\Omega)$.

Now, given a cubature formula for μ with $N = N(n)$ nodes $\boldsymbol{\xi} \in X_n \subset \Omega$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_d)$, and positive weights $\{w_{\boldsymbol{\xi}}\}$, which is exact for polynomials of degree $\leq 2n$,

$$\int_\Omega p(\boldsymbol{x}) \, d\mu = \sum_{\boldsymbol{\xi} \in X_n} w_{\boldsymbol{\xi}} \, p(\boldsymbol{\xi}) \quad \forall p \in \Pi^d_{2n}(\Omega) \ , \tag{3}$$

we obtain from (1) the polynomial approximation of degree n by the *discretized Fourier coefficients* $\{\tilde{c}_\alpha\}$

$$f(\boldsymbol{x}) \approx \mathcal{L}_n f(\boldsymbol{x}) := \sum_{|\alpha|\le n} \tilde{c}_\alpha \, p_\alpha(\boldsymbol{x}) \ , \quad \tilde{c}_\alpha := \sum_{\boldsymbol{\xi}\in X_n} w_{\boldsymbol{\xi}} \, f(\boldsymbol{\xi}) \, p_\alpha(\boldsymbol{\xi}) \ , \tag{4}$$

where $\tilde{c}_\alpha = c_\alpha$ and thus $\mathcal{L}_n p = \mathcal{S}_n p = p$ for every $p \in \Pi_n^d(\Omega)$. It is known that necessarily $N \ge \dim\big(\Pi_n^d(\Omega)\big)$, and that (4) is a polynomial interpolation at X_n whenever the equality holds [36]. As for the convergence of hyperinterpolation, we have the following basic estimate, valid for every $f \in C(\Omega)$

$$\| f - \mathcal{L}_n f \|_{L^2_{d\mu}(\Omega)} \le 2\sqrt{\mu(\Omega)}\, E_n(f) \to 0 \ , \quad n \to \infty \ , \tag{5}$$

where $E_n(f) := \inf\big\{ \| f - p \|_\infty \ , \ \ p \in \Pi_n^d(\Omega) \big\}$. The convergence rate can then be estimated by a multivariate version of Jackson theorem [33], which shows that $E_n(f) = \mathcal{O}(n^{-p})$ for $f \in C^p(\Omega)$, $p \in \mathbb{R}^+$.

Moreover, hyperinterpolation allows to generalize product integration rules (cf., e.g., [38]) in the following way, as discussed in [36, pp. 245–246]. Given $h \in L^2_{d\mu}(\Omega)$ and $f \in C(\Omega)$, we can approximate the integral of hf in $d\mu$ as

$$\begin{aligned} \int_\Omega h(\boldsymbol{x}) \ f(\boldsymbol{x}) \, d\mu &\approx \int_\Omega h(\boldsymbol{x}) \, \mathcal{L}_n f(\boldsymbol{x}) \, d\mu \\ &= \sum_{|\alpha|\le n} \tilde{c}_\alpha \, m_\alpha = \sum_{\boldsymbol{\xi}\in X_n} \lambda_{\boldsymbol{\xi}} \ f(\boldsymbol{\xi}) \ , \end{aligned} \tag{6}$$

where the generalized "orthogonal moments" $\{m_\alpha\}$ and the cubature weights $\{\lambda_{\boldsymbol{\xi}}\}$ are defined by

$$m_\alpha := \int_\Omega h(\boldsymbol{x}) \, p_\alpha(\boldsymbol{x}) \, d\mu \ , \quad \lambda_{\boldsymbol{\xi}} := w_{\boldsymbol{\xi}} \sum_{|\alpha|\le n} p_\alpha(\boldsymbol{\xi}) \, m_\alpha \ . \tag{7}$$

Observe that the cubature formula (6) is *exact* for every $f \in \Pi_n^d(\Omega)$, and that the $\{m_\alpha\}$ are just the Fourier coefficients of h with respect to the μ-orthonormal basis $\{p_\alpha\}$.

Application of the Cauchy–Schwarz inequality and (5) lead immediately to the estimate

$$\left| \int_\Omega h(\boldsymbol{x}) \ f(\boldsymbol{x}) \, d\mu - \sum_{\boldsymbol{\xi}\in X_n} \lambda_{\boldsymbol{\xi}} \ f(\boldsymbol{\xi}) \right| \le \|h\|_{L^2_{d\mu}(\Omega)} \, 2\sqrt{\mu(\Omega)}\, E_n(f) \ , \tag{8}$$

which ensures *convergence* for every $f \in C(\Omega)$. Observe that in general the weights $\{\lambda_{\boldsymbol{\xi}}\}$ are not all positive. Nevertheless, applying the Banach–Steinhaus theorem (cf. [35]) to the sequence of cubature functionals, as in the proof of the classical Polya–Steklov theorem (cf. [20]), we get that their norms remain bounded as $n \to \infty$

$$\exists K > 0 \ : \ \sum_{\boldsymbol{\xi}\in X_n} \left|\lambda_{\boldsymbol{\xi}}\right| \le K \ , \tag{9}$$

ensuring *stability* of the cubature formula (6). The stability formula (9) can be made more precise, in the form of a limit theorem. We state and prove the result, since it is quite general and apparently is not reported in the literature on hyperinterpolation. It is relevant to the construction of nontensorial Clenshaw–Curtis cubature in the next section.

Theorem 1 *Let all the assumptions for the construction of the cubature formula* (6) *be satisfied, and in particular let be* $h \in L^2_{d\mu}(\Omega)$*. Then the sum of the absolute values of the cubature weights has a finite limit*

$$\lim_{n\to\infty} \sum_{\boldsymbol{\xi}\in X_n} \left|\lambda_{\boldsymbol{\xi}}\right| = \int_{\Omega} |h(\boldsymbol{x})|\, d\mu \,. \tag{10}$$

Proof First, observe that in view of (7) and (1)

$$\sum_{\boldsymbol{\xi}\in X_n} \left|\lambda_{\boldsymbol{\xi}}\right| = \sum_{\boldsymbol{\xi}\in X_n} w_{\boldsymbol{\xi}}\ |\mathcal{S}_n h(\boldsymbol{\xi})| \,,$$

since the $\{m_\alpha\}$ are the Fourier coefficients of h and the weights $\{w_{\boldsymbol{\xi}}\}$ are positive by assumption. Fix $\varepsilon > 0$ and let π_ε be a polynomial such that $\|\pi_\varepsilon - h\|_{L^2_{d\mu}(\Omega)} \leq \varepsilon$, which exists since $\Pi^d(\Omega)$ is dense in $L^2_{d\mu}(\Omega)$ (cf. [35]). Consider now the inequality

$$\begin{aligned}
&\left|\sum_{\boldsymbol{\xi}\in X_n} \left|\lambda_{\boldsymbol{\xi}}\right| - \int_{\Omega} |h(\boldsymbol{x})|\, d\mu\right| \\
&\quad\leq \left|\sum_{\boldsymbol{\xi}\in X_n} w_{\boldsymbol{\xi}}\ \left(|\mathcal{S}_n h(\boldsymbol{\xi})| - |\pi_\varepsilon(\boldsymbol{\xi})|\right)\right| \\
&\qquad + \left|\sum_{\boldsymbol{\xi}\in X_n} w_{\boldsymbol{\xi}}\ |\pi_\varepsilon(\boldsymbol{\xi})| - \int_{\Omega} |\pi_\varepsilon(\boldsymbol{x})|\, d\mu\right| \\
&\qquad + \left|\int_{\Omega} |\pi_\varepsilon(\boldsymbol{x})|\, d\mu - \int_{\Omega} |h(\boldsymbol{x})|\, d\mu\right| ,
\end{aligned} \tag{11}$$

and notice that the second summand on the right-hand side is infinitesimal as $n \to \infty$, since the cubature formula (3) is convergent on continuous functions, being algebraic and with positive weights (by a simple generalization of Polya-Steklov theorem, cf. e.g. [20] for the basic one-dimensional version). As for the third summand in (11), by the Cauchy–Schwarz inequality in $L^2_{d\mu}(\Omega)$ we have

$$\begin{aligned}
\left|\int_{\Omega} |\pi_\varepsilon(\boldsymbol{x})|\, d\mu - \int_{\Omega} |h(\boldsymbol{x})|\, d\mu\right| &\leq \int_{\Omega} |\pi_\varepsilon(\boldsymbol{x}) - h(\boldsymbol{x})|\, d\mu \\
&\leq \sqrt{\mu(\Omega)}\, \|\pi_\varepsilon - h\|_{L^2_{d\mu}(\Omega)} \leq \sqrt{\mu(\Omega)}\, \varepsilon \,.
\end{aligned}$$

Thus we restrict the attention to the first summand in (11) and write the chain of inequalities

$$
\begin{aligned}
&\left|\sum_{\boldsymbol{\xi}\in X_n} w_{\boldsymbol{\xi}}\ (|\mathcal{S}_n h(\boldsymbol{\xi})| - |\pi_\varepsilon(\boldsymbol{\xi})|)\right| \\
&\quad\le \sum_{\boldsymbol{\xi}\in X_n} w_{\boldsymbol{\xi}}\ |\mathcal{S}_n h(\boldsymbol{\xi}) - \pi_\varepsilon(\boldsymbol{\xi})| \\
&\quad\le \sqrt{\mu(\Omega)} \left(\sum_{\boldsymbol{\xi}\in X_n} w_{\boldsymbol{\xi}}\ (\mathcal{S}_n h(\boldsymbol{\xi}) - \pi_\varepsilon(\boldsymbol{\xi}))^2\right)^{1/2} \\
&\quad= \sqrt{\mu(\Omega)}\, \|\mathcal{S}_n h - \pi_\varepsilon\|_{L^2_{d\mu}(\Omega)} \\
&\quad\le \sqrt{\mu(\Omega)}\ \left(\|\mathcal{S}_n h - h\|_{L^2_{d\mu}(\Omega)} + \|h - \pi_\varepsilon\|_{L^2_{d\mu}(\Omega)}\right) \\
&\quad\le \sqrt{\mu(\Omega)}\ \left(\|\mathcal{S}_n h - h\|_{L^2_{d\mu}(\Omega)} + \varepsilon\right)\ ,\quad n > \deg(\pi_\varepsilon)\ ,
\end{aligned}
$$

where the second is an application of Cauchy–Schwarz inequality to the discrete inner product defined by $\langle f, g\rangle := \sum_{\boldsymbol{\xi}\in X_n} w_{\boldsymbol{\xi}}\, f(\boldsymbol{\xi})g(\boldsymbol{\xi})$, and the equality stems from exactness of (3) in $\Pi^d_{2n}(\Omega)$. The proof is complete, since the first summand on the right-hand side is infinitesimal as $n \to \infty$ (recall that $\mathcal{S}_n h$ is a partial sum of a Fourier orthogonal series for h). □

3 Nontensorial Clenshaw–Curtis cubature

An interesting and immediate consequence of the quite general cubature formula (6) is that when μ is an absolutely continuous measure, namely

$$d\mu = w(\boldsymbol{x})\, d\boldsymbol{x}\ ,\quad w \in L^1_+(\Omega)\ , \tag{12}$$

for which are known a family of orthogonal polynomials and an algebraic cubature formula with degree of exactness at least $2n$, we can obtain a convergent and stable algebraic cubature formula with degree of exactness at least n for any integrable weight function λ

$$\int_\Omega \lambda(\boldsymbol{x})\ f(\boldsymbol{x})\, d\boldsymbol{x} \approx \sum_{\boldsymbol{\xi}\in X_n} \lambda_{\boldsymbol{\xi}}\ f(\boldsymbol{\xi})\ , \tag{13}$$

provided that

$$h := \frac{\lambda}{w} \in L^2_{d\mu}(\Omega)\ ,\quad \text{or equivalently}\quad \frac{\lambda^2}{w} \in L^1(\Omega)\ . \tag{14}$$

The stability result (10) becomes in this case

$$\lim_{n\to\infty} \sum_{\boldsymbol{\xi}\in X_n} \left|\lambda_{\boldsymbol{\xi}}\right| = \int_\Omega |\lambda(\boldsymbol{x})|\, d\boldsymbol{x}\ . \tag{15}$$

The classical Clenshaw–Curtis quadrature falls just in this frame, where $\Omega = [-1, 1]$, $w(x_1) = 1/\sqrt{1-x_1^2}$, $\lambda \equiv 1$, the orthogonal polynomials are the Chebyshev polynomials of the first kind and the underlying quadrature formula (3) is the Chebyshev–Gauss–Lobatto formula. In this case there is an explicit formula for the orthogonal moments (7), and it is known that the weights $\{\lambda_\xi\}$ are all positive; cf. [24, 43]. Several one-dimensional generalizations of Clenshaw–Curtis quadrature have been studied, where different weight functions λ appear; e.g. [6, 21, 23, 32, 37] and references therein. For these, the limit formula (10) was proved even for $\lambda \in L^p(-1, 1)$, $p > 1$, cf. [37, 38].

In order to extend Clenshaw–Curtis quadrature to cubature over the square in a nontensorial way, we consider hyperinterpolation with respect to the product Chebyshev orthonormal basis (cf. [18])

$$\boldsymbol{x} = (x_1, x_2) \in \Omega = (-1, 1)^2 \ , \quad w(\boldsymbol{x}) = \frac{1}{\pi^2} \frac{dx_1\, dx_2}{\sqrt{1-x_1^2}\sqrt{1-x_2^2}} \ ,$$

$$p_\alpha(\boldsymbol{x}) = \hat{T}_{\alpha_1}(x_1)\, \hat{T}_{\alpha_2}(x_2) \ , \quad \alpha = (\alpha_1, \alpha_2) \ , \quad 0 \le \alpha_1 + \alpha_2 \le n \ , \tag{16}$$

where $\hat{T}_k(t)$, $t \in [-1, 1]$, is the normalized Chebyshev polynomial of degree k, that is $\hat{T}_0(t) = 1$, $\hat{T}_k(t) = \sqrt{2} \cos(k \arccos t)$. The cardinality of the basis is $\dim\big(\Pi_n^2(\Omega)\big) = (n+1)(n+2)/2$.

Nontensorial Clenshaw–Curtis cubature is obtained simply by taking $\lambda \equiv 1$ in (13). Observe that $h(\boldsymbol{x}) = \pi^2 \sqrt{1-x_1^2}\sqrt{1-x_2^2}$ is continuous and thus (14) trivially holds. The convergence estimate (8) becomes

$$\left| \int_{[-1,1]^2} f(\boldsymbol{x})\, d\boldsymbol{x} - \sum_{\boldsymbol{\xi} \in X_n} \lambda_{\boldsymbol{\xi}}\, f(\boldsymbol{\xi}) \right| \le 2 \left(\pi \int_{-1}^{1} \sqrt{1-t^2}\, dt \right) E_n(f) = \pi^2\, E_n(f) \ , \tag{17}$$

and the limit relation (10) reads

$$\lim_{n\to\infty} \sum_{\boldsymbol{\xi} \in X_n} |\lambda_{\boldsymbol{\xi}}| = 4 \ . \tag{18}$$

Moreover, since the basis is of product type, there is an explicit formula for the orthogonal moments in (7)

$$m_\alpha = \mu_{\alpha_1}\, \mu_{\alpha_2} \ , \quad \text{where} \quad \mu_k := \begin{cases} 2 & k = 0 \\ \dfrac{2\sqrt{2}}{1-k^2} & k \text{ even}, k \neq 0 \\ 0 & k \text{ odd} \end{cases} \tag{19}$$

Notice that the moments are nonzero only for pairs of even degrees (α_1, α_2).

The key point consists clearly in finding suitable algebraic cubature formulas as in (3) for the product Chebyshev measure, with a low number of nodes. We recall that the number of nodes of a *minimal* formula with degree of exactness

$2n+1$, according to a general result by Möller [26] on centrally symmetric weight functions, is

$$N_{\min} := \dim\left(\Pi^2_{n+1}(\Omega)\right) + \left[\frac{n+1}{2}\right] = \frac{(n+2)(n+3)}{2} + \left[\frac{n+1}{2}\right] . \tag{20}$$

To this respect, two families of Chebyshev-like points in the square are interesting, and we discuss them below. The first family is given by the "Morrow–Patterson–Xu" points, which have been studied in the contexts of minimal cubature [1, 17, 27, 44], interpolation [2, 44] and hyperinterpolation [9, 11, 13]. The second family, termed the "Padua points", has been recently studied in the interpolation context [3, 5, 10, 12, 14]. In what follows we use the following notation for the set of $\nu+1$ one-dimensional Chebyshev–Gauss–Lobatto nodes

$$C_{\nu+1} := \cos\left(\frac{j\pi}{\nu}\right), \quad j = 0, \ldots, \nu , \tag{21}$$

and we denote by $C^{\text{even}}_{\nu+1}$, $C^{\text{odd}}_{\nu+1}$ its restrictions to even and odd indices.

3.1 The Morrow–Patterson–Xu points

Such points give an algebraic cubature formula for the product Chebyshev measure on the square with degree of exactness $2n+1$, which is minimal [cf. (20)] for odd n, and almost minimal (up to 1 node) for even n. We have termed them after Morrow and Patterson [27], who gave originally the explicit formula in the odd case, and Xu [44], who obtained the explicit formula for even instances ("even" and "odd" are interchanged here with respect the usual setting, which refers to degree of exactness $2n-1$). Formulas of this type, even in a more general setting, have been studied by various other authors, for example in [1, 17].

The MPX (Morrow–Patterson–Xu) points are defined as union of the bidimensional Chebyshev-like grids

- Case n odd

$$X_n = X_n^{\text{MPX}} := \left(C^{\text{even}}_{n+2} \times C^{\text{odd}}_{n+2}\right) \cup \left(C^{\text{odd}}_{n+2} \times C^{\text{even}}_{n+2}\right) , \tag{22}$$

  with

$$N := \text{card}(X_n) = \frac{(n+1)(n+3)}{2} .$$

  The weights of the corresponding minimal cubature formula are, for $\boldsymbol{\xi} \in X_n$, $w_{\boldsymbol{\xi}} = (n+1)^{-2}$ for the boundary points, and $w_{\boldsymbol{\xi}} = 2(n+1)^{-2}$ for the interior points.
- Case n even

$$X_n = \left(C^{\text{even}}_{n+2} \times C^{\text{even}}_{n+2}\right) \cup \left(C^{\text{odd}}_{n+2} \times C^{\text{odd}}_{n+2}\right) , \tag{23}$$

with

$$N = \frac{(n+2)^2}{2} .$$

The weights of the corresponding almost minimal cubature formula are, for $\boldsymbol{\xi} \in X_n$, $w_{\boldsymbol{\xi}} = (n+1)^{-2}/2$ for $\boldsymbol{\xi} = (1, 1)$ and $\boldsymbol{\xi} = (-1, -1)$ (two corner points), $w_{\boldsymbol{\xi}} = (n+1)^{-2}$ for the other boundary points and $w_{\boldsymbol{\xi}} = 2(n+1)^{-2}$ for the interior points.

These are good points for both, interpolation of degree $n+1$ (though in a subspace of total degree polynomials, see [44]), and hyperinterpolation of degree n which is the present framework. In both applications it has been proved that the Lebesgue constant (in the uniform norm) has optimal order of growth $\mathcal{O}\left((\log n)^2\right)$; cf. [4, 11]. Notice that in the case of hyperinterpolation the corresponding polynomial of degree n is not interpolant, since $N > \dim\left(\Pi_n^2(\Omega)\right)$.

3.2 The Padua points

Such points give an algebraic cubature formula for the product Chebyshev measure on the square which is exact on all polynomials of degree $2n+2$ that are orthogonal to $T_{2n+2}(x_1)$ in the Chebyshev measure, and thus in particular on all polynomials of degree $2n+1$. It is not minimal, but we can term it “near minimal” because its cardinality has the same asymptotic growth of the minimal case [cf. (20)], that is $\sim n^2/2$, and the additional points are “only” $\left[\frac{n+3}{2}\right]$.

The Padua points are defined as union of the bidimensional Chebyshev-like grids

$$X_n = X_n^{\text{Pad}} := \left(C_{n+2}^{\text{odd}} \times C_{n+3}^{\text{even}}\right) \cup \left(C_{n+2}^{\text{even}} \times C_{n+3}^{\text{odd}}\right) , \tag{24}$$

with

$$N := \text{card}(X_n) = \dim\left(\Pi_{n+1}^2(\Omega)\right) = \frac{(n+2)(n+3)}{2} .$$

The weights of the corresponding near minimal cubature formula are, for $\boldsymbol{\xi} \in X_n$,

$$w_{\boldsymbol{\xi}} = \frac{1}{(n+2)(n+3)} \cdot \begin{cases} 1/2 & \text{if } \boldsymbol{\xi} \text{ is a vertex point} \\ 1 & \text{if } \boldsymbol{\xi} \text{ is an edge point} \\ 2 & \text{if } \boldsymbol{\xi} \text{ is an interior point} \end{cases} \tag{25}$$

Since $N > \dim\left(\Pi_n^2(\Omega)\right)$, the hyperinterpolation polynomial (4) is not interpolant.

These points are also good points for polynomial interpolation (of degree $n+1$). They have been introduced experimentally in [10], and then studied from the theoretical [3, 5] and the computational [12, 14] point of view. In fact, the Padua points are the first known example of non tensor-product optimal interpolation in two variables, since they are unisolvent and their Lebesgue constant has optimal order of growth $\mathcal{O}\left((\log n)^2\right)$. It is also worth recalling that there are other three families of Padua points, obtained from (4) by successive 90-degree rotations of the square. We refer to [12] for a full description.

One noteworthy feature of the Padua points is that they lie on an algebraic curve, the so-called "generating" curve, namely $T_{n+1}(x_1) + T_{n+2}(x_2) = 0$ for the family (24) or $\gamma_{n+1}(t) := [-\cos((n+2)t), -\cos((n+1)t)]$, $t \in [0, \pi]$ in parametric form, and that they correspond to self-intersections and boundary contacts of this curve. The cubature formula above for the bivariate Chebyshev measure stems just from quadrature along the generating curve, and has been the key to obtain an explicit form of the fundamental Lagrange polynomials [3]. An important fact is that such cubature is exact not only in $\Pi^2_{2n+1}(\Omega)$, but also for all bivariate polynomials of degree $2n+2$ which are μ-orthogonal to $T_{2n+2}(x_1)$, and in particular on all polynomials of the form pq with $p, q \in \Pi^2_{n+1}(\Omega)$ and either p or q are μ-orthogonal to $T_{n+1}(x_1)$.

3.2.1 Improving exactness at the Padua points

A consequence of the extended exactness property just described is that hyperinterpolation at the Padua points can be made exact in a bigger space than $\Pi^2_n(\Omega)$, namely in

$$V_{n+1} := \left\{p \in \Pi^2_{n+1}(\Omega) : \;\; p \perp T_{n+1}(x_1)\right\} = (\mathrm{span}\{T_{n+1}(x_1)\})^{\perp} , \tag{26}$$

or in other words the space of bivariate polynomials of degree $n+1$ whose representation in the Chebyshev orthogonal basis does not contain $T_{n+1}(x_1)$. The new hyperinterpolation polynomial is

$$\mathcal{L}_{V_{n+1}} f(\boldsymbol{x}) := \sum_{|\alpha| \le n+1} \tilde{c}_\alpha \, p_\alpha(\boldsymbol{x}) , \tag{27}$$

where the discretized Fourier–Chebyshev coefficients $\{\tilde{c}_\alpha\}$ are exactly as in (4), the new nontensorial Clenshaw–Curtis cubature formula becomes

$$\int_{[-1,1]^2} f(\boldsymbol{x}) \, d\boldsymbol{x} \approx \sum_{|\alpha| \le n+1} \tilde{c}_\alpha \, m_\alpha = \sum_{\boldsymbol{\xi} \in X_n} \hat{\lambda}_{\boldsymbol{\xi}} \, f(\boldsymbol{\xi}) ,$$

$$\hat{\lambda}_{\boldsymbol{\xi}} := w_{\boldsymbol{\xi}} \sum_{|\alpha| \le n+1} p_\alpha(\boldsymbol{\xi}) \, m_\alpha , \tag{28}$$

cf. (16), (19), (24) and (25), and the convergence estimate (17) becomes

$$\left| \int_{[-1,1]^2} f(\boldsymbol{x}) \, d\boldsymbol{x} - \sum_{\boldsymbol{\xi} \in X_n} \hat{\lambda}_{\boldsymbol{\xi}} \, f(\boldsymbol{\xi}) \right| \le \pi^2 \, E_{V_{n+1}}(f) , \tag{29}$$

where $E_{V_{n+1}}(f) := \inf\{\|f - p\|_\infty , \;\; p \in V_{n+1}\}$. Moreover, it is easy to show that (18) holds with $\hat{\lambda}_{\boldsymbol{\xi}}$ replacing $\lambda_{\boldsymbol{\xi}}$.

A further improvement can be obtained by integrating the interpolation (instead of an hyperinterpolation) polynomial at the Padua points (24). This procedure is well located in the present framework, since it has been shown in [12] that the interpolation polynomial can be written as

$$L_{n+1} f(\boldsymbol{x}) := \sum_{|\alpha| \le n+1} c^\star_\alpha \, p_\alpha(\boldsymbol{x}) \; , \tag{30}$$

where only a little correction is needed on one of the hyperinterpolation coefficients (4)

$$c^\star_\alpha = \tilde{c}_\alpha \quad \alpha \neq (n+1, 0) \; , \quad c^\star_{(n+1,0)} = \frac{1}{2}\, \tilde{c}_{(n+1,0)} \; . \tag{31}$$

By unisolvence of the Padua points, $L_{n+1} p = p$ for every $p \in \Pi^2_{n+1}(\Omega)$. Moreover, in [5] it has been proved that there exists a constant $c_p > 0$ such that for every $f \in C(\Omega)$

$$\| f - L_{n+1} f \|_{L^p_{d\mu}(\Omega)} \le c_p \, E_{n+1}(f) \; , \quad 1 \le p < \infty \; . \tag{32}$$

We can then mimic all the procedure in Section 2 concerning construction of generalized product integration rules, and in particular for $h = 1/w$ and w the bivariate Chebyshev density in (16), we get another nontensorial Clenshaw–Curtis cubature formula

$$\int_{[-1,1]^2} f(\boldsymbol{x})\, d\boldsymbol{x} \approx \sum_{|\alpha| \le n+1} c^\star_\alpha \, m_\alpha = \sum_{\boldsymbol{\xi} \in X_n} \lambda^\star_{\boldsymbol{\xi}} \, f(\boldsymbol{\xi}) \; ,$$

$$\lambda^\star_{\boldsymbol{\xi}} := w_{\boldsymbol{\xi}} \sum_{|\alpha| \le n+1} p_\alpha(\boldsymbol{\xi})\, m^\star_\alpha \; , \quad m^\star_\alpha = m_\alpha \quad \alpha \neq (n+1, 0) \; , \quad m^\star_{(n+1,0)} = \frac{1}{2}\, m_{(n+1,0)} \; , \tag{33}$$

cf. (19), (24) and (25), along with the convergence estimate

$$\left| \int_{[-1,1]^2} f(\boldsymbol{x})\, d\boldsymbol{x} - \sum_{\boldsymbol{\xi} \in X_n} \lambda^\star_{\boldsymbol{\xi}} \, f(\boldsymbol{\xi}) \right| \le c_2 \, \frac{\pi^2}{2} \, E_{n+1}(f) \; . \tag{34}$$

Again, (18) holds with $\lambda^\star_{\boldsymbol{\xi}}$ replacing $\lambda_{\boldsymbol{\xi}}$. Notice that at degree of exactness n (replace n by $n-1$ everywhere in the construction above), this formula uses $\mathrm{card}(X_{n-1}) = (n+1)(n+2)/2$ Padua points, which are less than those used by nontensorial Clenshaw–Curtis cubature at the Morrow–Patterson–Xu points [cf. (22) and (23)].

3.3 Implementation and numerical results

In order to summarize, we can say that we have constructed two main nontensorial bivariate versions of Clenshaw–Curtis quadrature with degree of exactness n, by integrating the hyperinterpolation polynomial of degree n at $N = (n+1)(n+3)/2$ (n odd) or $N = (n+2)^2/2$ (n even) Morrow–Patterson–Xu points

$$\int_{[-1,1]^2} \mathcal{L}_n f(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{|\alpha| \le n} \tilde{c}_\alpha \, m_\alpha = \sum_{\boldsymbol{\xi} \in X^{\mathrm{MPX}}_n} \lambda_{\boldsymbol{\xi}} \, f(\boldsymbol{\xi}) \; , \tag{35}$$

or by integrating the interpolation polynomial of degree n at $N = (n+1)(n+2)/2$ Padua points

$$\int_{[-1,1]^2} L_n f(\boldsymbol{x})\, d\boldsymbol{x} = \sum_{|\alpha|\le n} c_\alpha^\star\, m_\alpha = \sum_{\boldsymbol{\xi}\in X_{n-1}^{\mathrm{Pad}}} \lambda_{\boldsymbol{\xi}}^\star\, f(\boldsymbol{\xi})\ , \tag{36}$$

(replace n by $n-1$ everywhere in the construction of Section 3.2.1).

There are (at least) two ways to implement these formulas, the one based on the *coefficients* of the underlying (hyper)interpolation polynomial, and that based on the *weights*. Both rely on knowledge of the orthogonal moments, for which the explicit formula (19) is available. In what follows, we assume that some ordering of the cubature points and of the bi-indices α (i.e. of the polynomial basis) have been fixed.

Concerning computation of the *coefficients*, we can resort to the *matrix formulation* adopted in [9, 12, 14], which takes advantage of the optimized linear algebra subroutines used for example by Matlab [25]. The main difference with respect to that formulation is that here only pairs of even indices are needed, in view of (19).

Given a vector $S = (s_1, \ldots, s_N) \in [-1,1]^N$, define the rectangular Chebyshev matrix

$$\Theta_n^{\mathrm{even}}(S) := \begin{bmatrix} \hat{T}_0(s_1) & \cdots & \hat{T}_0(s_N) \\ \hat{T}_2(s_1) & \cdots & \hat{T}_2(s_N) \\ \vdots & \cdots & \vdots \\ \hat{T}_{p_n}(s_1) & \cdots & \hat{T}_{p_n}(s_N) \end{bmatrix} \in \mathbb{R}^{([\frac{n}{2}]+1)\times N}\ , \quad p_n := 2\left[\frac{n}{2}\right]\ , \tag{37}$$

and consider the vectors of the one-dimensional projections of the cubature points

$$Z_{n,i} := (\xi_i)_{\boldsymbol{\xi}\in Z_n} \in \mathbb{R}^N\ , \quad i = 1, 2\ , \tag{38}$$

where

$$Z_n = X_n^{\mathrm{MPX}} \quad \text{or} \quad Z_n = X_{n-1}^{\mathrm{Pad}}\ , \tag{39}$$

and the diagonal matrix

$$\mathcal{D}(f) := \mathrm{diag}\left([w_{\boldsymbol{\xi}}\, f(\boldsymbol{\xi}),\ \boldsymbol{\xi} \in Z_n]\right) \in \mathbb{R}^{N\times N}\ . \tag{40}$$

It is easy to check that the required coefficients correspond to the matrix

$$B_0(f) := \begin{bmatrix} b_{(0,0)} & b_{(0,2)} & \cdots & \cdots & b_{(0,p_n)} \\ b_{(2,0)} & b_{(2,2)} & \cdots & b_{(2,p_n-2)} & 0 \\ \vdots & \vdots & ⋰ & ⋰ & \vdots \\ b_{(p_n-2,0)} & b_{(p_n-2,2)} & 0 & \cdots & 0 \\ b_{(p_n,0)} & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{([\frac{n}{2}]+1)\times([\frac{n}{2}]+1)}\ , \tag{41}$$

which is the upper-left triangular part of the matrix product

$$B(f) := \Theta_1\, \mathcal{D}(f)\, \Theta_2^t\ , \quad \Theta_i := \Theta_n^{\mathrm{even}}(Z_{n,i})\ , \tag{42}$$

cf. (37), where in the case of the Padua points $b_{(p_n,0)}$ has to be substituted by $b_{(p_n,0)}/2$ for n even. It is worth stressing that the matrix product by the diagonal matrix, which corresponds to a scaling of rows or columns, can be conveniently accelerated by using the `sparse` format in Matlab [25].

Observe that also the *moments* (19) can be computed by a *matrix formulation*, as the matrix

$$M_0 := \begin{bmatrix} m_{(0,0)} & m_{(0,2)} & \cdots & \cdots & m_{(0,p_n)} \\ m_{(2,0)} & m_{(2,2)} & \cdots & m_{(2,p_n-2)} & 0 \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ m_{(p_n-2,0)} & m_{(p_n-2,2)} & 0 & \cdots & 0 \\ m_{(p_n,0)} & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{([\frac{n}{2}]+1)\times([\frac{n}{2}]+1)} , \qquad (43)$$

which is the upper-left triangular part of

$$M := (\mu^{\text{even}})^t \, \mu^{\text{even}} \, , \quad \mu^{\text{even}} := (\mu_0, \mu_2, \ldots, \mu_{p_n}) \in \mathbb{R}^{[\frac{n}{2}]+1} \, . \qquad (44)$$

Concerning computation of the weights, a matrix formulation is still possible. Indeed, it is not difficult to show that the vector

$$\left(\text{diagonal of } \; \Theta_1^t \, M_0 \, \Theta_2\right) \in \mathbb{R}^N \, , \qquad (45)$$

cf. (42)-(44), contains exactly the values that multiplied by the w_ξ give the relevant weights λ_ξ and $\lambda_\xi^\star$. Again, in the case of the Padua points $m_{(p_n,0)}$ has to be substituted by $m_{(p_n,0)}/2$ for n even.

The theoretical complexity of the bulks in computing the coefficients (42) and the weights (44) is of the order of $n^4/4$ flops. However, the use of optimized linear algebra subroutines makes the matrix formulation very effective. A first Matlab implementation of nontensorial Clenshaw–Curtis cubature by such a formulation can be found in [41]. A faster implementation based on the bivariate FCT (fast cosine transform) seems also possible, by exploiting the fact that both the families of cubature points are union of subgrids of a Chebyshev grid, and will be object of future work.

An important observation is that both the nontensorial Clenshaw–Curtis formulas above are not positive, i.e. some of the weights can be negative. On the other hand, we have the limit relation (18) which ensures, at least asymptotically, an almost optimal stability. In order to appreciate the initial behavior of the formula, in Table 1 we have reported the distance from the limit (the area of the square) for the MPX and the Padua points at a sequence of degrees. It is clear that the negative weights are few and of small size, and that the formulas are quite stable already at low degrees (in particular, the MPX weights at odd degrees turn out to be all positive).

Table 1 Distance of the sum of the weights absolute values from the area of the square for nontensorial Clenshaw–Curtis cubature at the Morrow–Patterson–Xu and Padua points

n	5	10	15	20	25	30	35	40
MPX pts	4.4E-16	2.3E-3	0.0	2.3E-4	8.9E-16	5.5E-5	5.3E-15	2.0E-5
Padua pts	4.1E-2	2.5E-3	8.9E-4	2.4E-4	1.4E-4	5.7E-5	4.0E-5	2.0E-5

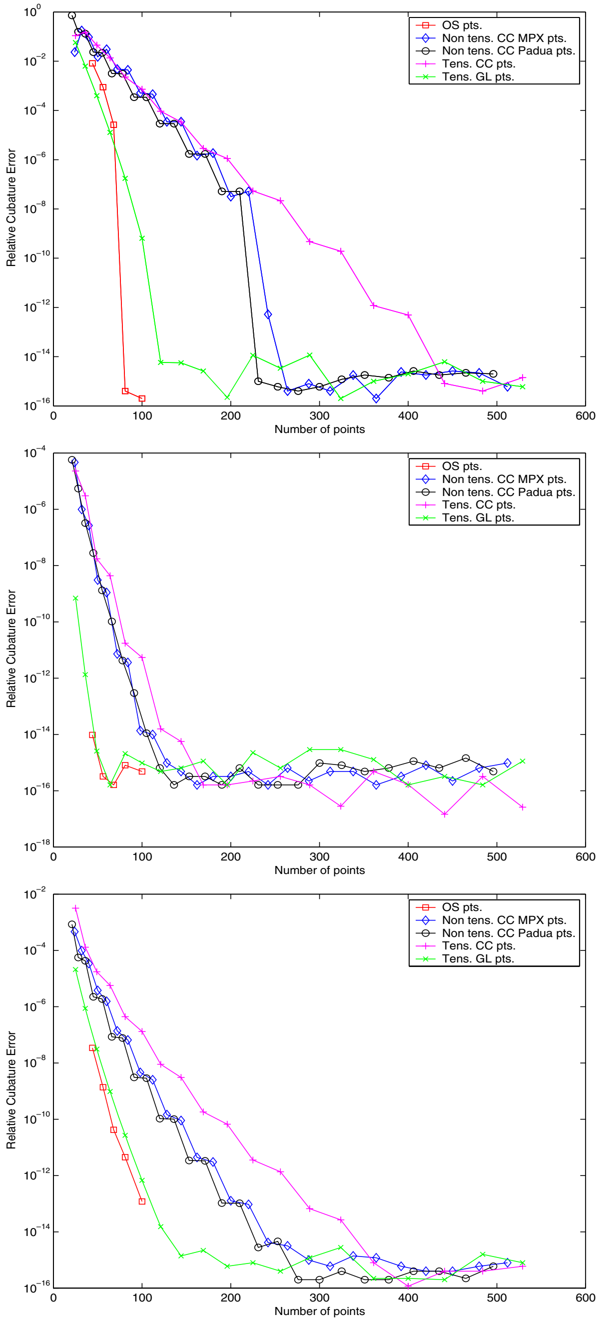

Fig. 1 Relative cubature errors versus the number of cubature points (*CC* Clenshaw–Curtis, *GL* Gauss–Legendre, *MPX* Morrow–Patterson–Xu, *OS* Omelyan–Solovyan) for the three test functions: $f(\boldsymbol{x}) = (x_1 + x_2)^{20}$ (*top*), $f(\boldsymbol{x}) = \exp(x_1 + x_2)$ (*center*), $f(\boldsymbol{x}) = \exp(-|\boldsymbol{x}|^2)$ (*bottom*); the integration domain is $[-1, 1]^2$, the integrals up to machine precision are, respectively 18157.16017316017, 5.524391382167263 and 2.230985141404135

In order to test the accuracy of nontensorial Clenshaw–Curtis cubature over the square, we have considered comparisons with the tensor-product versions of Clenshaw–Curtis (CC) and Gauss–Legendre (GL) quadrature formulas on some test functions. Such functions are the bivariate analogous to those used

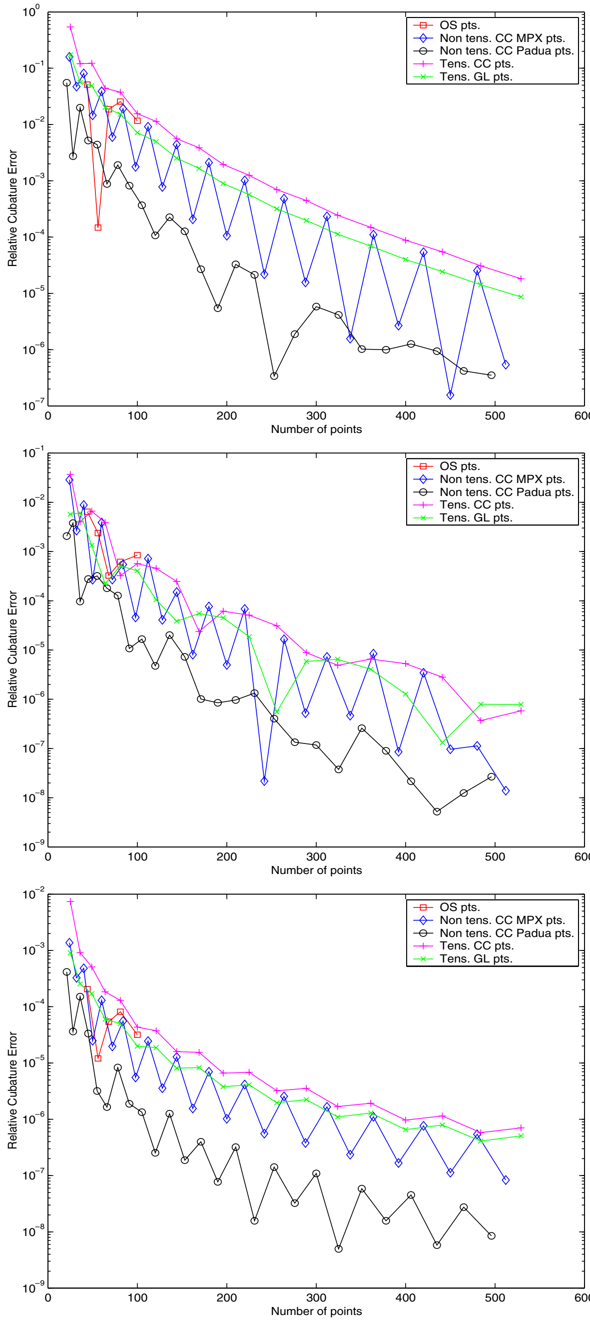

Fig. 2 Relative cubature errors versus the number of cubature points (*CC* Clenshaw–Curtis, *GL* Gauss–Legendre, *MPX* Morrow–Patterson–Xu, *OS* Omelyan–Solovyan) for the three test functions: $f(\boldsymbol{x}) = 1/(1 + 16|\boldsymbol{x}|^2)$ (*top*), $f(\boldsymbol{x}) = \exp(-1/|\boldsymbol{x}|^2)$ (*center*), $f(\boldsymbol{x}) = |\boldsymbol{x}|^3$ (*bottom*); the integration domain is $[-1, 1]^2$, the integrals up to machine precision are, respectively 0.597388947274307, 0.853358758654305 and 2.508723139534059

by L.N. Trefethen in [43] to compare the univariate CC and GL formulas at the same interpolation degree (i.e., at the same number of quadrature points). They are six, with diverse regularity, namely three entire functions (i.e., analytic in the whole $\mathbb{C}^2$: a polynomial, an exponential and a Gaussian),

Table 2 Relative errors of minimal formulas (Omelyan–Solovyan) and of nontensorial CC cubature at the Padua points, at two sequences of cardinalities (in parentheses: the theoretical algebraic degree of exactness)

Number of OS pts. Number of Padua pts.		44 (15) 45 (8)	56 (17) 55 (9)	68 (19) 66 (10)	81 (21) 78 (11)	100 (23) 81 (12)
$f(\boldsymbol{x}) = 1/(1 + 16\lvert\boldsymbol{x}\rvert^2)$	$\mathcal{E}_{\text{OS}}$	5.1E-2	1.5E-4	1.9E-2	2.6E-2	1.2E-2
	$\mathcal{E}_{\text{Pad}}$	5.2E-3	4.4E-3	8.8E-4	1.9E-3	8.2E-4
$f(\boldsymbol{x}) = \exp(-1/\lvert\boldsymbol{x}\rvert^2)$	$\mathcal{E}_{\text{OS}}$	6.4E-3	2.4E-3	3.3E-4	6.2E-4	8.4E-4
	$\mathcal{E}_{\text{Pad}}$	2.8E-4	3.2E-4	1.8E-4	1.3E-4	1.1E-5
$f(\boldsymbol{x}) = \lvert\boldsymbol{x}\rvert^3$	$\mathcal{E}_{\text{OS}}$	2.1E-4	1.2E-5	5.4E-5	8.1E-5	3.2E-5
	$\mathcal{E}_{\text{Pad}}$	3.3E-5	3.2E-6	1.7E-6	9.0E-6	1.9E-6

one analytic nonentire (a bivariate analogous to Runge's function), one C^∞ nonanalytic and one C^2.

Here the comparisons have been made plotting the relative cubature error versus the number of cubature points, which is more meaningful than comparing versus the underlying (hyper)interpolation degree, since different polynomial spaces are involved. On the other hand, the number of function evaluations is typically the important parameter in the frame of numerical integration.

The results are collected in Figs. 1 (the first three test functions) and 2 (the other three). For tensor-product cubature formulas, the sequence of cardinalities N of the point sets corresponds to underlying (hyper)interpolation degrees $n = 4, 5, \ldots, 23$, and for the nontensorial CC formulas the sequence is $n = 5, 6, \ldots, 30$. We recall that $N = (n + 1)^2$ for tensor-product formulas, whereas see (35)–(36) for the nontensorial CC formulas.

Moreover, we have also compared with the "best on the market" known algebraic formulas, namely the minimal ones recently numerically determined by Omelyan and Solovyan in [29], which have improved previous results [16]. Such formulas correspond to algebraic degree of exactness 15, 17, 19, 21, 23 with corresponding cardinalities $N = 44, 56, 68, 81, 100$. Computational difficulties in solving the relevant nonlinear systems have till now prevented the construction of high order formulas. On the contrary, tensor-product CC and GL, as well as nontensorial CC formulas, can be easily obtained at high degrees.

Table 3 Relative errors of Sparse-Grids nontensorial cubature formulas based on univariate CC rules, at a sequence of cardinalities

Number of pts.	29	65	145	321	705
$f(\boldsymbol{x}) = \exp(-\lvert\boldsymbol{x}\rvert^2)$	4.1E-3	1.8E-4	2.5E-6	7.0E-10	5.0E-14
$f(\boldsymbol{x}) = 1/(1 + 16\lvert\boldsymbol{x}\rvert^2)$	1.4E+00	7.0E-1	2.3E-1	4.7E-2	6.0E-3

Table 4 Relative errors of Sparse-Grids nontensorial cubature formulas based on univariate GL rules, at a sequence of cardinalities

Number of pts.	22	75	224	613
$f(\boldsymbol{x}) = \exp(-\lvert\boldsymbol{x}\rvert^2)$	2.3E-3	1.1E-5	1.1E-10	2.6E-15
$f(\boldsymbol{x}) = 1/(1 + 16\lvert\boldsymbol{x}\rvert^2)$	1.2E+00	4.0E-1	7.0E-2	8.7E-3

It is interesting to notice that, differently from the one-dimensional case, tensor-product CC is not competitive with tensor-product GL. Moreover, nontensorial CC formulas are more accurate than tensor-product CC in all the tests, and less accurate than tensor-product GL and minimal formulas on the entire functions.

The situation changes for the less regular test functions, where nontensorial CC cubature at the Padua points gives the best error curve (up to 2–3 orders of magnitude below the tensorial error curves). As an example, to have an error of 10^{-6} in the integration of the less regular test function (Fig. 2, bottom), we need around 100 Padua points, whereas the required number of tensor-product GL points is more than 500. In order to clarify the comparison between nontensorial CC cubature at the Padua points and minimal formulas, we present also Table 2, where we report the relative errors in the integration of the three nonentire functions of Fig. 2, at two sequences of cubature point sets. If we take into account that the Omelyan–Solovyan formulas are minimal, the performance of the nontensorial cubature formulas considered here is surprisingly good.

It should be recalled that there are other ways of obtaining useful nontensorial cubature formulas from Clenshaw–Curtis rules, like the so-called sparse grids; cf., e.g., [7, 28, 30, 31, 40] and references therein. In Tables 3 and 4 we report the relative cubature errors of two test functions versus the number of nodes, obtained by the Sparse-Grids Matlab codes in [8], where the sequence of cardinalities is determined by successive “levels” of the sparse grid. The comparison with Fig. 1 (bottom) for the Gaussian function and with Fig. 2 (top) for the Runge function shows that nontensorial CC cubature generated by (hyper)interpolation appears more accurate in dimension 2 than CC-like and GL-like cubature with sparse grids.

The very good observed behavior of nontensorial CC cubature at the Padua points is in some respect similar to the one-dimensional phenomenon discussed in [43], where a sophisticated analysis explains the experimental fact that CC quadrature has an accuracy close to GL, for univariate functions that are not analytic in a large region of the complex plane surrounding the integration interval.

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