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Simple geometric constructions of quadratically and cubically convergent iterative functions to solve nonlinear equations

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Abstract In this paper, we derive one-parameter families of Newton, Halley, Chebyshev, Chebyshev-Halley type methods, super-Halley, C-methods, osculating circle and ellipse methods respectively for finding simple zeros of nonlinear equations, permitting $f'(x)=0$ at some points in the vicinity of the required root. Halley, Chebyshev, super-Halley methods and, as an exceptional case, Newton method are seen as the special cases of the family. All the methods of the family and various others are cubically convergent to simple roots except Newton's or a family of Newton's method.

Keywords Nonlinear equations · Iterative methods · One-parameter family · Newton's method · Halley's method · Chebyshev's method · super-Halley method

AMS subject classifications 65H05

1 Introduction

One of the most basic problems in computational mathematics is to find the solution of nonlinear equations

$$f(x) = 0. \tag{1}$$

To solve equation (1), we can use iterative methods such as Newton's method and its variants namely Halley's method [1–10, 12, 14–17], Chebyshev's method [1–3, 5,

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10, 12] and super-Halley method [11, 12, 14–16] respectively. Among these iteration methods, Newton's method is probably the best known and most used algorithm. Its geometric construction consists in considering the straight line

$$y = ax + b\ , \tag{2}$$

and then determining the unknowns a and b by imposing the tangency conditions:

$$y(x_n) = f(x_n),\ y'(x_n) = f'(x_n), \tag{3}$$

thereby obtaining the tangent line

$$y(x) - f(x_n) = f'(x_n)(x - x_n), \tag{4}$$

to the graph of $f(x)$ at $\{x_n, f(x_n)\}$. The point of intersection of this tangent line with *x-axis*, gives the celebrated Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},\ n \geq 0. \tag{5}$$

The tangent line in Newton method can be seen as the first degree polynomial of $f(x)$ at $x{=}x_n$. Though, Newton's formula is simple and fast with quadratic convergence, it may fail miserably if at any stage of computation the derivative of the function $f(x)$ is either zero or very small. Therefore, it has poor convergence and stability problems as it is very sensitive to initial guess. The well-known third-order methods which entail the evaluation of $f''(x)$ are close relatives of Newton's method and can be obtained by admitting geometric derivation from the quadratic curves, e.g. parabola, hyperbola, circle or ellipse.

In this category, the Euler's method [1–3, 6, 12] can be constructed by considering the parabola

$$x^2 + ax + by + c = 0, \tag{6}$$

and imposing the tangency conditions

$$y(x_n) = f(x_n),\ y'(x_n) = f'(x_n) \text{ and } y''(x_n) = f''(x_n), \tag{7}$$

we obtain

$$y(x) - f(x_n) = f'(x_n)(x - x_n) + \frac{f''^2(x_n)}{2}(x - x_n)^2. \tag{8}$$

The point of intersection of (8) with $x - axis$ gives the iteration scheme

$$x_{n+1} = x_n - \left\{\frac{2}{1 + \sqrt{1 - 2L_f(x_n)}}\right\}\frac{f(x_n)}{f'(x_n)},\ n \geq 0, \tag{9}$$

where $L_f(x_n) = \frac{f(x_n)f''(x_n)}{f'^2(x_n)}$.

The well-known Halley's method [1–10, 12, 14–17] admits its geometric derivation from a hyperbola in the form

$$axy + y + bx + c = 0. \tag{10}$$

After the imposition of conditions (7) on (10) and taking the intersection of (10) with $x-axis$ as next iterate, the Halley's iteration process reads as

$$x_{n+1}=x_n-\left\{\frac{2}{2-L_f(x_n)}\right\}\frac{f(x_n)}{f'(x_n)},\ n\geq 0. \tag{11}$$

The classical Chebyshev's method [1–3, 5, 10, 12] admits its geometric derivation from a parabola in the form

$$ay^2+y+bx+c=0. \tag{12}$$

Similarly after imposing the tangency conditions (7) and taking the intersection of (12) with *x-axis* as next iterate, Chebyshev's iteration process reads as

$$x_{n+1}=x_n-\left\{1+\frac{L_f(x_n)}{2}\right\}\frac{f(x_n)}{f'(x_n)},\ n\geq 0. \tag{13}$$

Amat et al. [12] studies another type of third-order methods by considering the hyperbola in the form

$$a_ny^2+b_nxy+y+c_nx+d_n=0, \tag{14}$$

and by the similar conditions as before, the iteration process reads as

$$x_{n+1}=x_n-\left\{1+\frac{1}{2}\frac{L_f(x_n)}{1+b_n(f(x_n)/f'(x_n))}\right\},\ n\geq 0, \tag{15}$$

where b_n is a parameter depending on n.

For particular values of b_n, some interesting particular cases of this family are

(1) For b_n=0, the formula (15) corresponds to classical Chebyshev's method.
(2) For $b_n=-\frac{f''(x_n)}{2f'(x_n)}$, the formula (15) corresponds to famous Halley's method.
(3) For $b_n=-\frac{f''(x_n)}{f'(x_n)}$, formula (15) corresponds to super-Halley method.
(4) For $b_n=\pm\infty$, the formula (15) corresponds to Newton's formula.

Osculating circle method [13] and ellipse method [14–16] are more cumbersome from computing point of view as compared to Halley, Chebyshev or super-Halley methods. For a more detailed survey of these most important techniques, some excellent text books [1–3] are available in the literature. The problem with the existing methods is that these techniques may fail to converge in some cases if the initial guess is far from the zero point or if the derivative of the function is small or even zero in the vicinity of the required root. Recently, Kanwar and Tomar [18] using different approach derived new classes of iterative techniques having quadratic and cubic convergence, respectively. These techniques provide an alternative to the failure situation of existing classical techniques.

The purpose of this work is to eliminate the defects of existing classical methods by the simple modification of iteration processes.

2 Preliminary results

The following two theorems establish the existence and uniqueness of the root r of equation (1) in [a,b]. The proof can be found in Wu and Wu [19].

Theorem 1. *Suppose that* $f(x) \in C^1[a,b]$ *and* $f'(x) - pf(x) \neq 0$, *where p is a real number, then equation* (1) *has at most one root in* [a,b].

Theorem 2. *If* $f(x) \in C^1[a,b]$, $f(a)f(b) < 0$ *and* $f'(x) - pf(x) \neq 0$, *where p is a real number, then the equation* (1) *has unique root in* (a,b).

3 Development of iteration scheme

Based on these two theorems, we shall now develop the iteration scheme by fitting the function $f(x)$ of equation (1) in the following form:

$$
\begin{aligned}
&a_n\left\{ye^{-p(x-x_n)} - f(x_n)\right\}^2 + b_n\,(x - x_n)\left\{ye^{-p\,(x-x_n)} - f(x_n)\right\} \\
&\quad + \left\{ye^{-p(x-x_n)} - f(x_n)\right\} + c_n(x - x_n) + d_n = 0,
\end{aligned}
\tag{16}
$$

where $p \in \Re$, $|p| < \infty$. The unknowns a_n, c_n and d_n are now determined in terms of b_n by using the tangency conditions (7) on equation (16). This gives

$$
\left.
\begin{aligned}
&a_n = -\frac{f''(x_n) + p^2 f(x_n) - 2pf'(x_n)}{2\left\{f'(x_n) - pf(x_n)\right\}^2} - \frac{b_n}{f'(x_n) - pf(x_n)} \\
&c_n = -\{f'(x_n) - pf(x_n)\} \text{ and } d_n = 0
\end{aligned}
\right\}.
\tag{17}
$$

Therefore, at a root

$$
y(x_{n+1}) = 0,
\tag{18}
$$

and it follows from (16) that

$$
x_{n+1} = x_n - \frac{\left\{a_n f^2(x_n) - f(x_n)\right\}}{c_n - b_n f(x_n)},\ n \geq 0.
\tag{19}
$$

If (16) is an exponentially fitted straight line, then $a_n = 0 = b_n$ and from (19), we have

$$
x_{n+1} = x_n - \frac{f(x_n)}{f''(x_n) - pf(x_n)},\ n \geq 0.
\tag{20}
$$

This is one-parameter family of Newton's method [18]. In order to obtain quadratic convergence, the entity in the denominator should be largest in magnitude. For $p = 0$, we obtain Newton's method.

Further, making use of a_n and c_n in (19), one can obtain

$$x_{n+1} = x_n - \left\{ 1 + \frac{1}{2} \frac{L_f^*(x_n)}{1 + b_n \left\{ f(x_n) / \{ f'(x_n) - pf(x_n) \} \right\}} \right\} \frac{f(x_n)}{f'(x_n) - pf(x_n)}, \ n \geq 0 \tag{21}$$

where b_n is a parameter and

$$L_f^*(x_n) = \frac{f(x_n) \{ f''(x_n) + p^2 f(x_n) - 2pf'(x_n) \}}{\{ f'(x_n) - pf(x_n) \}^2} \tag{22}$$

This is the modification over the formula (15) of Amat et al. [12] and do not fail if $f'(x_n)$ is very small or zero in the vicinity of the root. For particular values of b_n, some interesting particular cases of (21) are

1. For b_n=0, we get

$$x_{n+1} = x_n - \left\{ 1 + \frac{1}{2} L_f^*(x_n) \right\} \frac{f(x_n)}{f'(x_n) - pf(x_n)}, \ n \geq 0. \tag{23}$$

2. For $b_n = -\frac{f''(x_n) + p^2 f(x_n) - 2pf'(x_n)}{2\{ f'(x_n) - pf(x_n) \}}$, we get

$$x_{n+1} = x_n - \frac{2}{2 - L_f^*(x_n)} \frac{f(x_n)}{f'(x_n) - pf(x_n)}, \ n \geq 0. \tag{24}$$

3. For $b_n = -\frac{f''(x_n) + p^2 f(x_n) - 2pf'(x_n)}{f'(x_n) - pf(x_n)}$, we obtain

$$x_{n+1} = x_n - \left\{ 1 + \frac{1}{2} \frac{L_f^*(x_n)}{1 - L_f^*(x_n)} \right\} \frac{f(x_n)}{f'(x_n) - pf(x_n)}, \ n \geq 0. \tag{25}$$

4. For $b_n = -\alpha \frac{f''(x_n) + p^2 f(x_n) - 2pf'(x_n)}{f'(x_n) - pf(x_n)}$, $\alpha \in \Re$, we obtain

$$x_{n+1} = x_n - \left\{ 1 + \frac{1}{2} \frac{L_f^*(x_n)}{1 - \alpha L_f^*(x_n)} \right\} \frac{f(x_n)}{f'(x_n) - pf(x_n)}, \ n \geq 0. \tag{26}$$

5. For $b_n = \pm\infty$, we obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n) - pf(x_n)}, \ n \geq 0. \tag{27}$$

It can be verified that for p=0, the formulae (23), (24), (25), (26) and (27) reduce to Chebyshev, Halley, super-Halley, Chebyshev–Halley type methods [5] and

Newton formula, respectively. The parameter p is chosen so as to give the largest value of the denominator. We can further derive some new families of multi-point iterative methods free from second derivative by discretizing the second-order derivative involved in family (21) (similar to Kanwar and Tomar [20]).

By using the same ideas, we can obtain other iterative processes by considering different exponentially fitted osculating curves. For instance, if we take the cubic function (similar to Amat et al. [12]) namely

$$a_n\left\{ye^{-p(x-x_n)}-f(x_n)\right\}^3+b_n\left\{ye^{-p(x-x_n)}-f(x_n)\right\}^2+\left\{ye^{-p(x-x_n)}-f(x_n)\right\}+d_n(x-x_n)=0, \tag{28}$$

and then using the tangency conditions (7) on (28), we get

$$\left.\begin{array}{l} a_n = C\frac{f''(x_n)+p^2f(x_n)-2pf'(x_n)}{\{f'(x_n)-pf(x_n)\}^2}, \\ b_n = -\frac{f''(x_n)+p^2f(x_n)-2pf'(x_n)}{2\{f'(x_n)-pf(x_n)\}^2}, \\ d_n = -\{f'(x_n)-pf(x_n)\}. \end{array}\right\}. \tag{29}$$

The point of intersection of (28) with *x-axis*, gives the iteration scheme

$$x_{n+1}=x_n-\left\{1+\frac{1}{2}L_f^*(x_n)+CL_f^*(x_n)^2\right\}\frac{f(x_n)}{f'(x_n)-pf(x_n)},\ n\geq 0. \tag{30}$$

This is the modification over Amat et al. C-methods [11–12] and do not fail if $f'(x_n)$ is very small or zero in the vicinity of the required root.

Similarly, if we consider the following quadratic equation:

$$(x-x_n)^2+a_n\left(ye^{-p(x-x_n)}-f(x_n)\right)^2+b_n(x-x_n)+c_n\left(ye^{-p(x-x_n)}-f(x_n)\right)+d_n=0, \tag{31}$$

and determine the unknowns b_n, c_n and d_n in terms of parameter a_n by the tangency conditions (7), we have

$$\left.\begin{array}{l} b_n = \frac{2\left[1+a_n\{f'(x_n)-pf(x_n)\}^2\right]\{f'(x_n)-pf(x_n)\}}{f''(x_n)+p^2f(x_n)-2pf'(x_n)}, \\ c_n = -\frac{2\left[1+a_n\{f'(x_n)-pf(x_n)\}^2\right]}{f''(x_n)+p^2f(x_n)-2pf'(x_n)}, \\ d_n = 0. \end{array}\right\} \tag{32}$$

Taking the next iterate x_{n+1} as the point of intersection of (31) with $x-axis$, we get another two-parameter family of iteration scheme as

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)-pf(x_n)}\times\frac{D_1}{D_2} \tag{33}$$

where

$$D_1=2+a_n\{f'(x_n)-pf(x_n)\}^2\left(2+L_f^*(x_n)\right),$$

and

$$D_2=1+a_n\{f'(x_n)-pf(x_n)\}^2 \pm\sqrt{\left\{1+a_n\{f'(x_n)-pf(x_n)\}^2\right\}^2-L_f^*(x_n)\left\{2+a_n\{f'(x_n)-pf(x_n)\}^2\left(2+L_f^*(x_n)\right)\right\}}.$$

This is the modification over the formula (20) of Sharma [14] and do not fail if $f'(x_n)$ is very small or zero. Iterative family (33) requires the computation of square root at each step and sometimes it is not convenient for practical use. Therefore, it can further be reduced to two-parameter rational iterative family (similar to Chun [15]), or by using the well known approximation (similar to Jiang and Han [16])

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x, \ |x| < 1.$$

Alternatively, all these modified techniques can be obtained by using the idea of Ben-Israel [17]. By applying the existing classical techniques to a modified function, say, $\widehat{f}(x) := e^{-\int a(x)\,dx} f(x)$, for some suitable function $a(x)$, that has better behavior as well as has the same root as $f(x)$, we shall get the corresponding modified iterative schemes.

4 Order of convergence

We shall present the mathematical proof for the order of convergence of formula (21) and the order of convergence for the remaining formulae can be proved on similar lines.

Theorem 4.1 *Let $f: I \to \Re$ be a sufficiently differentiable real valued function defined on I, where I is a neighborhood of a simple root r of f(x). Let x_0 be an initial guess sufficiently close to r and $f'(x_n) - pf(x_n) \neq 0$ in I. Then the sequence of iterates generated by* (21) *is cubically convergent.*

Proof Let $e_n = x_n - r$ be the error in approximating r by the nth iterate x_n. Using the Taylor series expansion about r and taking into account that $f(r)=0$ and $f'(r) \neq 0$, one can obtain

$$f(x_n) = f'(r)\left[e_n + C_2 e_n^2 + C_3 e_n^3 + O\left(e_n^4\right)\right], \tag{34}$$

$$f'(x_n) = f'(r)\left[1 + 2C_2 e_n + 3C_3 e_n^2 + O\left(e_n^3\right)\right], \tag{35}$$

where $C_k = (1/k!) f^{(k)}(r)/f'(r)$, k=2,3…. Furthermore, we have

$$\frac{f(x_n)}{f'(x_n) - pf(x_n)} = e_n - (C_2 - p)e_n^2 - \left(2C_3 - 2C_2^2 + 2pC_2 - p^2\right) e_n^3 + O\left(e_n^4\right), \tag{36}$$

$$\frac{1}{2}\frac{L_f^*(x_n)}{1 + b_n\{f(x_n)/(f'(x_n) - pf(x_n))\}} = (C_2 - p)\,e_n - \left(3C_2^2 - 3C_3 - 3pC_2 + \frac{3}{2}p^2 + b_n(C_2 - p)\right)e_n^2. \tag{37}$$

Using (36) and (37) in (21), we finally obtain the error equation

$$e_{n+1} = \left\{ 2C_2^2 - C_3 - 3pC_2 + \frac{1}{2}p^2 + b_n(C_2 - p) \right\} e_n^3 + O\left(e_n^4\right). \tag{38}$$

This completes the proof of the theorem.

5 Extension of the proposed scheme to a system of nonlinear equations

To illustrate the extension of proposed scheme, consider the following two nonlinear equations in two unknowns x and y:

$$\left.\begin{array}{l} f(x,y) = 0 \\ g(x,y) = 0 \end{array}\right\}. \tag{39}$$

Consider the auxiliary equations

$$\left.\begin{array}{l} e^{-p\,(x-x_0)} f(x,y) = 0 \\ e^{-p\,(y-y_0)} g(x,y) = 0 \end{array}\right\}, \tag{40}$$

where $p \in \Re$ and $|p| < +\infty$. Roots of equation (39) are also the roots of (40) and vice-versa. Let $(x_0,\ y_0)$ be an initial approximation to the root of a system (40). If $(x_0+h,\ y_0+k)$ is the root of the system (40), then we must have

$$\left.\begin{array}{l} e^{-ph} f(x_0 + h, y_0 + k) = 0 \\ e^{-pk} g(x_0 + h, y_0 + k) = 0 \end{array}\right\}. \tag{41}$$

Assuming that $f(x, y)$ and $g(x, y)$ are sufficiently differentiable, we expand (41) by Taylor's series about the point (x_0, y_0) to obtain

$$\left.\begin{array}{l} h\left(\frac{\partial f}{\partial x_0} - pf_0\right) + k\frac{\partial f}{\partial y_0} + f_0 + \text{—} = 0 \\ h\frac{\partial g}{\partial x_0} + k\left(\frac{\partial g}{\partial y_0} - pg_0\right) + g_0 + \text{—} = 0 \end{array}\right\}, \tag{42}$$

where $\frac{\partial f}{\partial x_0} = \left[\frac{\partial f}{\partial x_0}\right]_{x=x_0}$, $f_0 = f(x_0,\ y_0)$ etc.

Neglecting the second and higher order terms in (42), we obtain the following system of linear equations

$$\left.\begin{array}{l} h\left(\frac{\partial f}{\partial x_0} - pf_0\right) + k\frac{\partial f}{\partial y_0} + f_0 = 0 \\ h\frac{\partial g}{\partial x_0} + k\left(\frac{\partial g}{\partial y_0} - pg_0\right) + g_0 = 0 \end{array}\right\}. \tag{43}$$

If the Jacobian

$$J(f,g) = \begin{vmatrix} \frac{\partial f}{\partial x_0} - pf_0 & \frac{\partial f}{\partial y_0} \\ \frac{\partial g}{\partial x_0} & \frac{\partial g}{\partial y_0} - pg_0 \end{vmatrix}, \tag{44}$$

does not vanish, then the linear system of equations (43) possesses a unique solution given by

$$\left.\begin{aligned} h &= \frac{g_0\frac{\partial f}{\partial y_0} - f_0\left[\frac{\partial g}{\partial y_0} - pg_0\right]}{J(f,g)} \\ k &= \frac{f_0\frac{\partial g}{\partial x_0} - g_0\left[\frac{\partial f}{\partial x_0} - pf_0\right]}{J(f,g)} \end{aligned}\right\}. \tag{45}$$

Therefore, the new approximations are given by

$$\left.\begin{aligned} x_1 &= x_0 + h \\ y_1 &= y_0 + k \end{aligned}\right\}. \tag{46}$$

The process is to be repeated till we obtain the roots of desired accuracy. The parameter p in (45) is chosen so as to give the largest value of the denominator. The method (46) will work even if $\frac{\partial f}{\partial x_0}\frac{\partial g}{\partial y_0} - \frac{\partial f}{\partial y_0}\frac{\partial g}{\partial x_0} = 0$ unlike Newton's method. In a similar fashion, the proposed scheme can further be extended to a system of n nonlinear equations in n unknowns.

6 Numerical experimentation with some notorious examples and conclusion

In this section, we present some examples in Table 1 to illustrate the performance of new one-parameter family (21). There are many bad examples for the Newton method [17] as well as for the method (20). Examples 1, 2 and 4 are taken from Ben-Israel's paper [17]. We compare Newton method (NM), method (20), Halley method (HM), modified Halley method (24) (MHM), Chebyshev method (CM), modified Chebyshev's method (MCM) (23), super-Halley method (SHM) and modified super-Halley (MSHM) (25) to solve following nonlinear equations. Here, for simplicity, the formulae are tested for $|p|=1$. Computations have been performed using C^{++} in double precision arithmetic. We use $\in=10^{-15}$. The following stopping criteria are used for computer programs:

$$(i)|x_{n+1} - x_n| <\in . \qquad (ii)|f(x_{n+1})| <\in .$$

The behaviors of the existing classical iterative methods and the proposed modifications can be compared by their corresponding correction factors. The factor $\frac{f(x_n)}{f'(x_n)}$, which appears in the classical iterative schemes, is now modified by $\frac{f(x_n)}{f'(x_n)-pf(x_n)}$, where $p \in \Re$, $0<|p|<+\infty$ and p is chosen such that the corresponding functions values $f'(x_n)$ and $pf(x_n)$ have the opposite sign. As is well known that, if initial guess x_n is close enough to the solution r with $f'(r)\neq 0$, the classical iterates are well defined and converge to the required root. Now this modified factor has two major advantages over its predecessor. The first one is that it is always well defined, even if $f'(x_n)=0$. The second one is that the absolute value of the denominator in the formula is greater than $|f'(x_n)|$ provided x_n is not accepted as an approximation to the required root r. This means that the numerical stability of iterative schemes having this factor is better than the classical iterates. Moreover, if x_n is sufficiently close to r (simple root), then $f(x_n)$ will be sufficiently close to zero, and in particular the considered denominator $f'(x_n) - pf(x_n)$ will also be close to zero if $f'(x_n)\approx 0$. In these problems where

Table 1 Comparison of various quadratically and cubically convergent iterative schemes: test problems, initial guess and number of iterations

Examples	NM	Method (20)	HM	MHM	CM	MCM	SHM	MSHM	Root
$f_1(x)$, $x_0=10$	Divergent	13	7	7	Divergent	9	Divergent	Divergent	1.000000000000000
$f_2(x)$, $x_0=2$	Divergent	1	Divergent	1	Divergent	1	Divergent	1	0.000000000000000
$f_3(x)$, $x_0=1.5$	Converges to undesired root	7	6	5	Converges to undesired root	4	Converges to undesired root	4	0.000000000000000
$f_4(x)$, $x_0=5$	Converges to undesired root	6	4	4	Converges to undesired root	4	4	3	6.285049438476562
$f_5(x)$, $x_0=2$	Divergent	8	4	5	Divergent	6	6	6	0.000000000000000
$f_6(x)$, $x_0=0$	Fails	5	Fails	3	Fails	3	Fails	3	1.365229964256287
$=0.1$	9	5	5	3	74	3	4	3	
$f_7(x)$, $x_0=0$	8	5	1	8	21	4	5	4	2.000000000000000
$=1$	Fails	1	Fails	5	Fails	5	Fails	Fails	
$=1.1$	13	4	6	4	22	4	5	4	

$f'(x_n) \to 0$, a small value of the parameter p may lead to divergence or failure. Such situations can be handled by choosing p in such a way such that the correction factor is as small as possible, i.e. denominator $f'(x_n) - pf(x_n)$ is sufficiently large in magnitude.

Example 1 (Large step). Let

$$f_1(x) = e^{1-x} - 1 = 0.$$

If initial guess $x_0 = 1 + \ln a$, where $a > 0$ is large, then the correction factor for the Newton method, $h = -\frac{f(x_n)}{f'(x_n)} = -(a-1)$, is large and negative, as is $x_1 = \ln a - a$. Many successive steps are ≈1. For example x_0=10, x_1=−8,092.08, x_2=−8,091.08,--- and thousands of iterations are required to approach the required root r=1 due to large h. On the other hand, method (20) converges more rapidly to the required root due to small correction factor $h = -\frac{f(x_n)}{f'(x_n) - pf(x_n)}$.

Example 2 (Wrong direction). Let

$$f_2(x) = x e^{-x} = 0.$$

This problem is a cautionary one and has a unique root at 0. The derivative of $f_2(x)$ is zero at x=1, and negative for $x > 1$. For initial guess $x_0 < 0$, Newton's method and method (20) converges to the root efficiently. For any initial guess $x_0 > 1$, the Newton iterates move away from the zero. For example, x_0=2, x_1=4, x_2=5.3333 and so on. On the other hand, method (20) and its higher order variants can give the required root if p is chosen suitably, i.e. if $p \in I$ and the functions $f'(x_n)$ and $pf(x_n)$ have opposite signs for any arbitrary positive initial guess.

Example 3 (Jumping problem for Newton, Chebyshev and super-Halley methods). Let

$$f_3(x) = \sin x = 0.$$

This equation has infinite number of roots. It can be seen that Newton's method does not necessarily converge to the root that is nearest to the starting value. For example, Newton's method with x_0=1.5 is unsuitable because of numerical instability, which occurs due to large value of the correction factor and it converges to -4π away from the required root. Also Chebyshev's method converges to 1,388.583984375000000 and super-Halley method converges to -2π respectively. Therefore, care must be taken to ensure that the root obtained is the desired one.

Example 4 (Jumping problem for Newton and Chebyshev methods). Let

$$f_4(x) = e^{-x} - \sin\ x = 0.$$

This equation has infinite number of roots lying close to π, 2π, 3π,... It can be seen that Newton's and Chebyshev's methods do not necessarily converge to the root that is nearest to the starting value. For example, Newton's method with x_0=5 converges to the root closest to 3π, i.e. the nearest root 6.285049438476562 is skipped. Chebyshev's method converges to the smallest root 0.5885327458365 in ten iterations which is far away from the required root.

Example 5 Let

$$f_5(x) = \tan^{-1} x = 0.$$

For x_0=2, Newton's method diverges because correction factor increases beyond limit. Method (20) gives the root because its correction factor is small throughout the iteration process.

Example 6

$$f_6(x) = x^3 + 4x^2 - 10 = 0.$$

Example 7

$$f_7(x) = (x-1)^3 - 1 = 0.$$

These numerical examples show that the Halley's method is still attractive among the third-order iterative schemes. However, second-order iteration method (20), modified Chebyshev method (23) and modified super-Halley method (25) appear to be more attractive than their classical counterparts. Also the computational efficiency of the proposed methods is the same as that of existing classical methods. The proposed scheme also has been extended to a system of nonlinear equations. Further, using different finite difference approximations to $f''(x)$, one can further produce several multi-point iterative methods free from second derivative. Finally, we conclude that by using the same ideas, one can derive several iterative processes by considering different exponentially fitted osculating curves.

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