Numer Algor (2008) 47:63–80
DOI 10.1007/s11075-007-9147-6

ORIGINAL PAPER

Spectral methods for some singularly perturbed third order ordinary differential equations

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Received: 11 August 2007 / Accepted: 26 November 2007 /
Published online: 10 January 2008

Abstract Spectral methods with interface point are presented to deal with some singularly perturbed third order boundary value problems of reaction-diffusion and convection-diffusion types. First, linear equations are considered and then non-linear equations. To solve non-linear equations, Newton's method of quasi-linearization is applied. The problem is reduced to two systems of ordinary differential equations. And, then, each system is solved using spectral collocation methods. Our numerical experiments show that the proposed methods are produce highly accurate solutions in little computer time when compared with the other methods available in the literature.

Keywords Singular perturbed problem · Third order differential equation · Boundary value problem · Spectral method

PACS 03.40.Gc · 02.70.Hm · 02.70.Jn · 02.60.-x

1 Introduction

Singularly perturbed ordinary differential equations (SPODEs) appear in several branches of applied mathematics. Analytical and numerical treatment of these equations have drawn much attention of many researchers [1, 2, 5–15]. In general, classical numerical methods fail to produce good approximations for these equations. Hence one has to look for non-classical methods. A good number of articles have been appearing in the past two to three decades on non-classical methods which cover mostly second order singularly perturbed

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boundary value problems (SPBVPs). But only a few authors have developed numerical methods for higher-order differential equations [3, 4, 16–24]. Singularly perturbed higher-order problems are classified on the basis that how the order of the original differential equation is affected if one sets $\epsilon = 0$. Here ϵ is a small positive parameter multiplying the highest derivative of the differential equation. We say that the singular perturbation problem is of convection-diffusion type if the order of the differential equation is reduced by one, whereas it is called reaction-diffusion type if the order is reduced by two. By employing of an asymptotic numerical fitted mesh method, Valarmathi and Ramanujam [21] has considered third order SPODEs of reaction-diffusion type. Also Valarmathi and Ramanujam [22] have considered third order SPODEs of convection-diffusion type and applied an asymptotic numerical method. In this paper the linear third order SPODEs of reaction-diffusion and convection-diffusion types are considered and then non-linear equations. To solve non-linear equations, Newton's method of quasi-linearization is applied.

The purpose of this paper is to present two numerical methods for the solutions of SPBVPs.

(P1) The third order SPBVP of convection-diffusion type [22, 23]

$$-\epsilon u'''(x) + a(x)u''(x) + b(x)u'(x) + c(x)u(x) = f(x), \quad 0 \le x \le 1 \tag{1}$$

$$u(0) = p, \quad u'(0) = q, \quad u'(1) = r. \tag{2}$$

where ϵ is a small positive parameter, a(x), b(x), c(x) and f(x) are sufficiently smooth functions, $a(x) \le -\alpha, \alpha > 0$ and $b(x) \ge 0$.

(P2) The third order SPBVP of reaction-diffusion type [21]

$$-\epsilon u'''(x) + b(x)u'(x) + c(x)u(x) = f(x), \quad 0 \le x \le 1 \tag{3}$$

$$u(0) = p, \quad u'(0) = q, \quad u'(1) = r. \tag{4}$$

where ϵ is a small positive parameter, b(x), c(x), and f(x) are sufficiently smooth functions and $b(x) \ge \beta > 0$.

(P3) The nonlinear third order SPBVP of convection-diffusion type [22]

$$\epsilon u''''(x) = F(x, u, u', u''), \quad 0 \le x \le 1 \tag{5}$$

$$u(0) = p, \quad u'(0) = q, \quad u'(1) = r. \tag{6}$$

where F(x, u, u',u") is a smooth functions such that $F_{u''}(x, u, u', u'') \le -\alpha, \alpha > 0$ and $F_{u'}(x, u, u', u'') \ge 0$.

(P4) The nonlinear third order SPBVP of reaction-diffusion type [21]

$$\epsilon u'''(x) = F(x, u, u'), \quad 0 \le x \le 1 \tag{7}$$

$$u(0) = p, \quad u'(0) = q, \quad u'(1) = r. \tag{8}$$

where F(x, u, u') is a smooth functions such that $F_{u'}(x, u, u') \ge \beta > 0$.

Remark 1.1 Problem p1 has a boundary layer at x = 0 which is less severe, problem P2 has less severe layers at x = 0 and x = 1 because the boundary conditions are of the Neumann type [15]. Conditions $a(x) \leq -\alpha, \alpha > 0$ and $b(x) \geq \beta > 0$ for problems P1–P2, respectively, say that these problems are non turning point problems.

Remark 1.2 The authors in [21, 22] have considered problems P1–P2, respectively, and derived stability result, convergence and error estimates.

These methods will be referred to as collocation method I [CM-I] and collocation method II [CM-II]. The collocation method I based on differentiated Chebyshev polynomial and The collocation method II based on integrated Chebyshev polynomial. It is known that the above two methods are not satisfactory for the problems P1, P2, P3 and P4. Therefore we introduce an appropriate interface point $d \in R = [0, 1]$ and hence apply the collocation method I and the collocation method II in the two intervals $R_1 = [0, d]$ and $R_2 = [d, 1]$. The methods are to be compared with that of Valarmathi and Ramanujam [21–23] for the above problems.

The paper is organized as follows: In Sections 2 and 3, The collocation methods I and II with interface point are used to obtain the numerical solutions of the problems P1–P4, the numerical results are presented in Section 4 and Section 5 gives the conclusions.

2 Numerical solutions by the spectral collocation method I with interface point

To illustrate the procedure, four examples related to the problems P1, P2, P3 and P4 are given in the following.

First we consider the problem P1, we now seek solutions $u_k(x)$ defined on R_k, k =1,2 to the problems

$$\bar{P}1 : -\epsilon u_k'''(x) + a(x)u_k''(x) + b(x)u_k'(x) + c(x)u_k(x) = f(x), \quad x \in R_k, k = 1, 2 \tag{9}$$

$$u_1(0) = p, \quad u_1'(0) = q, \quad u_2'(1) = r. \tag{10}$$

subject to the interface conditions

$$u_1(d) = u_2(d), \quad u_1'(d) = u_2'(d), \quad u_1''(d) = u_2''(d) \tag{11}$$

Solutions to $\bar{P}1$ are sought in the form

$$u_k(x) = \sum_{j=0}^{N_k}{}'' a_j^{(k)} \overset{*}{T}_j^{(k)}(x), k = 1, 2 \tag{12}$$

where $\overset{*}{T}_j^{(k)}(x_n) = T_j((2x_n^{(k)} - (x_{N_k}^{(k)} + x_0^{(k)}))/(x_{N_k}^{(k)} - x_0^{(k)}))$ are the jth Chebyshev polynomial of the first kind. A summation symbol with double primes denotes a sum with the first and last term halved, $d = \ell\epsilon \in R = [0, 1]$.

The first three derivatives of the Chebyshev functions are given from the following:

$$\overset{*}{T'}{}_j^{(k)}(x) = \lambda_k T'_j(y^{(k)}) \tag{13}$$

$$\overset{*}{T''}{}_j^{(k)}(x) = \lambda_k^2 T''_j(y^{(k)}) \tag{14}$$

$$\overset{*}{T'''}{}_j^{(k)}(x) = \lambda_k^3 T'''_j(y^{(k)}) \tag{15}$$

where $\lambda_k = 2/(x_{N_k}^{(k)} - x_0^{(k)})$, $y^{(k)} = (2x - (x_{N_k}^{(k)} + x_0^{(k)}))/(x_{N_k}^{(k)} - x_0^{(k)})$ and the derivatives $\mathrm{T}'_j(y^{(k)})$, $T'_j(y^{(k)})$ and $T'_j(y^{(k)})$ are given from the following recurrence relations,

$$T'_{j+1}\left(y^{(k)}\right) = 2y^{(k)} T'_j\left(y^{(k)}\right) - T'_{j-1}\left(y^{(k)}\right) + 2T_j\left(y^{(k)}\right),$$
$$T''_{j+1}\left(y^{(k)}\right) = 2y^{(k)} T''_j\left(y^{(k)}\right) - T''_{j-1}\left(y^{(k)}\right) + 4T'_j\left(y^{(k)}\right),$$
$$T'''_{j+1}\left(y^{(k)}\right) = 2y^{(k)} T'''_j\left(y^{(k)}\right) - T'''_{j-1}\left(y^{(k)}\right) + 6T''_j\left(y^{(k)}\right),$$

$j{=}2, 3, ...,$ $T'_0\left(y^{(k)}\right){=}0,$ $T'_1\left(y^{(k)}\right){=}1,$ $T''_0\left(y^{(k)}\right){=}0,$ $T''_1\left(y^{(k)}\right){=}0,$ $T'''_0\left(y^{(k)}\right){=}0$ and $T'''_1\left(y^{(k)}\right) = 0$

From (13)–(15) and by the differentiated the series in (12) term by term, we get

$$u'_k(x_i) = \sum_{j=0}^{N_k} \bar{A}_{ij}^{(k)} a_j^{(k)}, \tag{16}$$

$$u''_k(x_i) = \sum_{j=0}^{N_k} \bar{\bar{A}}_{ij}^{(k)} a_j^{(k)} \tag{17}$$

and

$$u'''_k(x_i) = \sum_{j=0}^{N_k} \bar{\bar{\bar{A}}}_{ij}^{(k)} a_j^{(k)}. \tag{18}$$

where, $A_{ij}^{(k)} = c_j \overset{*}{T}{}_j^{(k)}(x_i)$, $\bar{A}_{ij}^{(k)} = c_j \overset{*}{T'}{}_j^{(k)}(x_i)$, $\bar{\bar{A}}_{ij}^{(k)} = c_j \overset{*}{T''}{}_j^{(k)}(x_i)$, $\bar{\bar{\bar{A}}}_{ij}^{(k)} = c_j \overset{*}{T'''}{}_j^{(k)}(x_i)$, $k = 1, 2$, $c_0 = c_{N_k} = 1/2$ and $c_j = 1$ for $j = 1, 2, ..., N_k - 1$ and the collocation points are given by

$$x_j^{(1)} = \frac{d}{2} - \frac{d}{2}\cos\left(\frac{\pi j}{N_1}\right) \in R_1, \quad j = 0, 1, \ldots, N_1 \tag{19}$$

and

$$x_j^{(2)} = \frac{1+d}{2} - \frac{1-d}{2}\cos\left(\frac{\pi j}{N_2}\right) \in R_2, \quad j = 0, 1, \ldots, N_2 \tag{20}$$

where N_k, $k = 1, 2$ are the number of mesh points in R_1 and R_2 respectively.

Using boundary conditions (10), we get

$$U_k = E^k + C^{(k)} a^{(k)} \tag{21}$$

$$U'_k = \bar{E}^k + \bar{C}^{(k)} a^{(k)} \tag{22}$$

$$U''_k = \bar{\bar{E}}^k + \bar{\bar{C}}^{(k)} a^{(k)} \tag{23}$$

$$U'''_k = \bar{\bar{\bar{E}}}^k + \bar{\bar{\bar{C}}}^{(k)} a^{(k)} \tag{24}$$

where

$$C_{ij}^{(k)} = A_{ij+2}^{(k)} - \alpha_{j+2} A_{i1}^{(k)} - \beta_{j+2} A_{i2}^{(k)} - \gamma_{j+2},$$

$$\bar{C}_{ij}^{(k)} = \bar{A}_{ij+2}^{(k)} - \alpha_{j+2} \bar{A}_{i1}^{(k)} - \beta_{j+2} \bar{A}_{i2}^{(k)},$$

$$\bar{\bar{C}}_{ij}^{(k)} = \bar{\bar{A}}_{ij+2}^{(k)} - \alpha_{j+2} \bar{\bar{A}}_{i1}^{(k)} - \beta_{j+2} \bar{\bar{A}}_{i2}^{(k)},$$

$$\bar{\bar{\bar{C}}}_{ij}^{(k)} = \bar{\bar{\bar{A}}}_{ij+2}^{(k)} - \alpha_{j+2} \bar{\bar{\bar{A}}}_{i1}^{(k)} - \beta_{j+2} \bar{\bar{\bar{A}}}_{i2}^{(k)},$$

$$E_i^k = u_k\left(x_0^{(k)}\right) + p_i^k u'_k\left(x_0^{(k)}\right) + q_i^k u'_k\left(x_{N_k}^{(k)}\right),$$

$$\bar{E}_i^k = \bar{p}_i^k u'_k\left(x_0^{(k)}\right) + \bar{q}_i^k u'_k\left(x_{N_k}^{(k)}\right),$$

$$\bar{\bar{E}}_i^k = \bar{\bar{p}}_i^k u'_k\left(x_0^{(k)}\right) + \bar{\bar{q}}_i^k u'_k\left(x_{N_k}^{(k)}\right),$$

$$\bar{\bar{\bar{E}}}_i^k = \bar{\bar{\bar{p}}}_i^k u'_k\left(x_0^{(k)}\right) + \bar{\bar{\bar{q}}}_i^k u'_k\left(x_{N_k}^{(k)}\right),$$

$$p_i^k = \left(5 + 4A_{i1}^{(k)} - A_{i2}^{(k)}\right)/8\lambda_k,\ q_i^k = \left(3 + 4A_{i1}^{(k)} + A_{i2}^{(k)}\right)/8\lambda_k,$$

$$\bar{p}_i^k = \left(4\bar{A}_{i1}^{(k)} - \bar{A}_{i2}^{(k)}\right)/8\lambda_k,\ \bar{q}_i^k = \left(4\bar{A}_{i1}^{(k)} + \bar{A}_{i2}^{(k)}\right)/8\lambda_k,$$

$$\bar{\bar{p}}_i^k = \left(4\bar{\bar{A}}_{i1}^{(k)} - \bar{\bar{A}}_{i2}^{(k)}\right)/8\lambda_k,\ \bar{\bar{q}}_i^k = \left(4\bar{\bar{A}}_{i1}^{(k)} + \bar{\bar{A}}_{i2}^{(k)}\right)/8\lambda_k,$$

$$\bar{\bar{\bar{p}}}_i^k = \left(4\bar{\bar{\bar{A}}}_{i1}^{(k)} - \bar{\bar{\bar{A}}}_{i2}^{(k)}\right)/8\lambda_k,\ \bar{\bar{\bar{q}}}_i^k = \left(4\bar{\bar{\bar{A}}}_{i1}^{(k)} + \bar{\bar{\bar{A}}}_{i2}^{(k)}\right)/8\lambda_k,$$

$$\alpha_j = c_j j^2\left(1 + (-1)^{j+1}\right)/2,\ \beta_j = c_j j^2\left(1 - (-1)^{j+1}\right)/8,$$

$$\gamma_j = c_j(-1)^j + \alpha_j - \beta_j,$$

$$U_k = \left[u_k\left(x_1^{(k)}\right), ..., u_k\left(x_{N_k-2}^{(k)}\right)\right]^T,$$

$$U'_k = \left[u'_k\left(x_1^{(k)}\right), ..., u'_k\left(x_{N_k-2}^{(k)}\right)\right]^T,\ a^{(k)} = \left[a_3^{(k)}, ..., a_{N_k}^{(k)}\right]^T,$$

$$U''_k = \left[u''_k\left(x_1^{(k)}\right), ..., u''_k\left(x_{N_k-2}^{(k)}\right)\right]^T,\ U'''_k = \left[u'''_k\left(x_1^{(k)}\right), ..., u'''_k\left(x_{N_k-2}^{(k)}\right)\right]^T.$$

The problem $\bar{P}1$ becomes

$$S^{(1)}a^{(1)} = Z^1 \tag{25}$$

$$S^{(2)}a^{(2)} = Z^2 \tag{26}$$

$$u_1(d) = u_1(0) + p^1_{N_1}u'_1(0) + q^1_{N_1}u'_1(d) + \sum_{j=1}^{N_1-2} C^{(1)}_{N_1 j}a^{(1)}_{j+2} \tag{27}$$

$$\left[\bar{\bar{q}}^1_{N_1} - \bar{\bar{p}}^2_0\right]u'_1(d) = \bar{\bar{q}}^2_0 u'_2(1) - \bar{\bar{p}}^1_{N_1}u'_1(0) + \sum_{j=1}^{N_2-2} \bar{\bar{C}}^{(2)}_{0j}a^{(2)}_{j+2} - \sum_{j=1}^{N_1-2} \bar{\bar{C}}^{(1)}_{N_1 j}a^{(1)}_{j+2} \tag{28}$$

where

$$S^{(k)} = -\epsilon \bar{\bar{\bar{C}}}^{(k)} + [a(x)]\bar{\bar{C}}^{(k)} + [b(x)]\bar{C}^{(k)} + [c(x)]C^{(k)}$$

$$Z^k_i = \epsilon \bar{\bar{\bar{E}}}^k_i - a(x_i)\bar{\bar{E}}^k_i - b(x_i)\bar{E}^k_i - c(x_i)E^k_i + f(x_i)$$

$i = 1, 2, ..., N_k - 2$, $k = 1, 2$, $[a(x)]$, $[b(x)]$ and $[c(x)]$ are diagonal matrices,

The system (25) is used to eliminate the coefficients $a^{(1)}_j$, $j = 3, 4, ..., N_1$ in (27)–(28). The resulting equations in conjunction with system (26) are used to determine the values of $a^{(2)}_j$, $j = 3, 4, ..., N_2$, $u_1(d)$ and $u'_1(d)$. Hence on using system (25), $a^{(1)}$ is determined. Also the values of $u_k(x^{(k)}_i)$, $i = 0, 1, ..., N_k$, $k = 1, 2$ are determined from

$$u_k\left(x^{(k)}_i\right) = E^k_i + \sum_{j=1}^{N_k-2} C^{(k)}_{ij}a^{(k)}_{j+2}, k = 1, 2. \tag{29}$$

We repeat this process by increasing the value of ℓ until the solution profiles do not differ much from iteration to iteration.

Second we consider the problem p2, problem P2 is of the form problem P1 when a(x)=0. Hence it can be solved by the spectral collocation method I with interface point, described above.

Third we consider problem P3. In order to obtain numerical solution of P3, the Newton's method of quasilinearization [1] is applied. Consequently, we get a sequence $\{u^{(m)}\}^{\infty}_{m=0}$ of successive approximations with a proper choice of initial guess $u^{(0)}$. Then, define $u^{(m+1)}$, for each fixed nonnegative integer m, to be the solution of the following linear problem:

$$\bar{P}3 : -\epsilon \left(u'''\right)^{(m+1)}(x) + a^m(x)\left(u''\right)^{(m+1)}(x) + b^m(x)\left(u'\right)^{(m+1)}(x)$$

$$+ c^m(x)u^{(m+1)}(x) = f^m(x), \quad 0 \le x \le 1 \tag{30}$$

$$u^{(m+1)}(0) = p, \quad \left(u'\right)^{(m+1)}(0) = q, \quad \left(u'\right)^{(m+1)}(1) = r. \tag{31}$$

where

$$a^m(x) = F_{u''}\left(x, u^{(m)}, \left(u'\right)^{(m)}, \left(u''\right)^{(m)}\right) \leq -\alpha, \quad \alpha > 0$$

$$b^m(x) = F_{u'}\left(x, u^{(m)}, \left(u'\right)^{(m)}, \left(u''\right)^{(m)}\right) \geq 0$$

$$c^m(x) = F_u\left(x, u^{(m)}, \left(u'\right)^{(m)}, \left(u''\right)^{(m)}\right)$$

$$f^m = F\left(x, u^{(m)}, (u')^{(m)}, \left(u''\right)^{(m)}\right) - u^m F_u\left(x, u^{(m)}, \left(u'\right)^{(m)}, \left(u''\right)^{(m)}\right) - \left(u'\right)^{(m)} F_{u'}\left(x, u^{(m)}, \left(u'\right)^{(m)}, \left(u''\right)^{(m)}\right) - \left(u''\right)^{(m)} F_{u''}\left(x, u^{(m)}, \left(u'\right)^{(m)}, \left(u''\right)^{(m)}\right)$$

Problem $\bar{P}3$ for each fixed m, is a linear third order SPBVP of convection-diffusion type and is of form problem P1. Hence, it can be solved by the spectral collocation method I with interface point, presented in this section.

Lastly we consider problem P4. Using the Newton's method of quasilinearization [1], problem P4 be replaced by

$$\bar{P}4: -\epsilon\left(u'''\right)^{(m+1)}(x) + b^m(x)\left(u'\right)^{(m+1)}(x) + c^m(x)u^{(m+1)}(x) = f^m(x), \quad 0 \leq x \leq 1 \tag{32}$$

$$u^{(m+1)}(0) = p, \quad (u')^{(m+1)}(0) = q, \quad (u')^{(m+1)}(1) = r. \tag{33}$$

where

$$b^m(x) = F_{u'}\left(x, u^{(m)}, \left(u'\right)^{(m)}\right) \geq \beta > 0$$

$$c^m(x) = F_u\left(x, u^{(m)}, \left(u'\right)^{(m)}\right)$$

$$f^m = F\left(x, u^{(m)}, \left(u'\right)^{(m)}\right) - u^m F_u\left(x, u^{(m)}, \left(u'\right)^{(m)}\right) - \left(u'\right)^{(m)} F_{u'}\left(x, u^{(m)}, \left(u'\right)^{(m)}\right)$$

Problem $\bar{P}4$ for each fixed m, is a linear third order SPBVP of reaction-diffusion type and is of form problem P2. Hence, it can be solved by the spectral collocation method I with interface point described above.

Remark 2.1 Analytical results such as existence, uniqueness, and asymptotic behaviour of the solutions P3 and P4 can be found in [3, 4, 24].

Remark 2.2 If the initial guess $u^{(0)}$ is sufficiently close to the solution u(x) of problem P3 or P4 , then, following the method of proof given in [1], one can prove that the sequence $\{u^{(m)}\}_{m=0}^{\infty}$ converges to u(x).

Remark 2.3 For the above Newton's quasi-linearisation process the following convergence criterion is used,

$$\left|u^{(m+1)}(x_j) - u^{(m)}(x_j)\right| \leq \tau, m \geq 0, x_j \in R.$$

Remark 2.4 In [5–8] it came out that the best choice for the interface point d should be chosen in dependence of N, i.e., $d = N\epsilon$ where N = N1 = N2.

3 Numerical solutions by the spectral collocation method II with interface point

We consider to apply the spectral collocation method II with interface point for the problem $\bar{P}1$. The third derivatives of the solution $u_k(x)$ are sought in the form

$$u_k'''(x) = \sum_{j=0}^{N_k}{}'' a_j^{(k)} \overset{*}{T}_j^{(k)}(x), k = 1, 2 \tag{34}$$

let $u_k'''(x) = \Phi_k(x)$, and by successive integration, we get

$$u_k''(x) = \int_{x_0^{(k)}}^{x} \Phi_k dx + c_1 \tag{35}$$

$$u_k'(x) = \int_{x_0^{(k)}}^{x} \int_{x_0^{(k)}}^{x} \Phi_k dx dx + c_1(x - x_0^{(k)}) + c_2 \tag{36}$$

$$u_k(x) = \int_{x_0^{(k)}}^{x} \int_{x_0^{(k)}}^{x} \int_{x_0^{(k)}}^{x} \Phi_k dx dx dx + \frac{1}{2} c_1 \left(x - x_0^{(k)}\right)^2 + (x - x_0^{(k)}) c_2 + c_3 \tag{37}$$

where

$$c_1 = \frac{\left(u_k'\left(x_{N_k}^{(k)}\right) - u_k'\left(x_0^{(k)}\right)\right)}{\left(x_{N_k}^{(k)} - x_0^{(k)}\right)} - \frac{1}{\left(x_{N_k}^{(k)} - x_0^{(k)}\right)} \int_{x_0^{(k)}}^{x_{N_k}^{(k)}} \int_{x_0^{(k)}}^{x} \Phi_k dx dx,$$

$c_2 = u_k'(x_0^{(k)})$, $c_3 = u_k(x_0^{(k)})$ and the collocation points are given by (16, 17).

Now, we give approximations to the integrals (35)–(37) based on the Chebyshev expansion (34) in the matrix format as follows.

$$U_k = R_k + M_k\Phi_k \tag{38}$$

$$U'_k = \bar{R}_k + \bar{M}_k\Phi_k \tag{39}$$

$$U''_k = \bar{\bar{R}}_k + \bar{\bar{M}}_k\Phi_k \tag{40}$$

where

$$[M_k]_{ij} = \left[B^3{}_k\right]_{ij} - \left[\frac{\left(x - x_0^{(k)}\right)^2}{2\left(x_{N_k}^{(k)} - x_0^{(k)}\right)}\right]\left[B^2{}_k\right]_{N_k j},$$

$$\left[\bar{M}_k\right]_{ij} = \left[B^2{}_k\right]_{ij} - \left[\frac{\left(x - x_0^{(k)}\right)}{\left(x_{N_k}^{(k)} - x_0^{(k)}\right)}\right]\left[B^2{}_k\right]_{N_k j},$$

$$\left[\bar{\bar{M}}_k\right]_{ij} = [B_k]_{ij} - \left[\frac{1}{\left(x_{N_k}^{(k)} - x_0^{(k)}\right)}\right]\left[B^2{}_k\right]_{N_k j},$$

$$[R_k]_i = u_k\left(x_0^{(k)}\right) - \left[\frac{\left(x_i - x_0^{(k)}\right)\left(x_i + x_0^{(k)} - 2x_{N_k}^{(k)}\right)}{2\left(x_{N_k}^{(k)} - x_0^{(k)}\right)}\right]u'_k\left(x_0^{(k)}\right)$$
$$+ \left[\frac{\left(x_i - x_0^{(k)}\right)^2}{2\left(x_{N_k}^{(k)} - x_0^{(k)}\right)}\right]u'_k\left(x_{N_k}^{(k)}\right),$$

$$\left[\bar{R}_k\right]_i = \left[\frac{\left(x_i - x_0^{(k)}\right)}{\left(x_{N_k}^{(k)} - x_0^{(k)}\right)}\right]u'_k\left(x_{N_k}^{(k)}\right) - \left[\frac{\left(x_i - x_{N_k}^{(k)}\right)}{\left(x_{N_k}^{(k)} - x_0^{(k)}\right)}\right]u'_k\left(x_0^{(k)}\right),$$

$$\left[\bar{\bar{R}}_k\right]_i = \left[\frac{1}{\left(x_{N_k}^{(k)} - x_0^{(k)}\right)}\right]u'_k\left(x_{N_k}^{(k)}\right) - \left[\frac{1}{\left(x_{N_k}^{(k)} - x_0^{(k)}\right)}\right]u'_k\left(x_0^{(k)}\right),$$

$$B_k\Phi_k = \left[\int_{x_0^{(k)}}^{x^{(k)}} \phi_k(x)dx\right]$$

$$U_k = \left[u_k\left(x_0^{(k)}\right), ..., u_k\left(x_{N_k}^{(k)}\right)\right]^T, U'_k = \left[u'_k\left(x_0^{(k)}\right), ..., u'_k\left(x_{N_k}^{(k)}\right)\right]^T,$$

$$U''_k = \left[u''_k\left(x_0^{(k)}\right), ..., u''_k\left(x_{N_k}^{(k)}\right)\right]^T, \Phi_k = \left[u'''_k\left(x_0^{(k)}\right), ..., u'''_k\left(x_{N_k}^{(k)}\right)\right]^T,$$

$i;\ j = 0, 1, ..., N_k, k = 1, 2.$

The problem $\bar{P}1$ becomes

$$H_1\Phi_1 = F_1 \tag{41}$$

$$H_2\Phi_2 = F_2 \tag{42}$$

and

$$u_1(d) = u_1(0) + \frac{d}{2}u_1'(0) + \frac{d}{2}u_1'(d) + \sum_{j=0}^{N_1}[M_1]_{N_1 j}\phi_1\left(x_j^{(1)}\right) \tag{43}$$

$$\left[\frac{1}{d}+\frac{1}{1-d}\right]u_1'(d) = \left[\frac{1}{d}u_1'(0)+\frac{1}{1-d}u_2'(1)\right] - \sum_{j=0}^{N_1}\left[\bar{\bar{M}}_1\right]_{N_1 j}\phi_1\left(x_j^{(1)}\right)+\sum_{j=0}^{N_2}\left[\bar{\bar{M}}_2\right]_{0j}\phi_2\left(x_j^{(2)}\right) \tag{44}$$

where

$$H_k = -\epsilon I_k + [a(x)]\bar{\bar{M}}_k + [b(x)]\bar{M}_k + [c(x)]M_k$$

$$[F_k]_i = f(x_i) - a(x_i)\left[\bar{\bar{R}}_k\right]_i - b(x_i)[\bar{R}_k]_i - c(x_i)[R_k]_i \; i = 0, 1, ..., N_k, k=1, 2,$$

$[a(x)]$, $[b(x)]$ and $[c(x)]$ are diagonal matrices, I_k, $k = 1, 2$ are unit matrices of order $N_1 + 1$ and $N_2 + 1$ respectively.

The system (41) is used to eliminate the coefficients $\phi_1(x_j^{(1)})$, $j = 0, 1, ..., N_1$ in (43)–(44). The resulting equations in conjunction with system (42) are used to determine the values of $\phi_2(x_j^{(2)})$, $j = 0, 1, ..., N_2$, $u_1(d)$ and $u_1'(d)$. Hence on using system (41), Φ_1 is determined. Also the values of U_1 and U_2 are determined from

$$U_k = R_k + M_k\Phi_k, k = 1, 2. \tag{45}$$

We repeat this process by increasing the value of ℓ until the solution profiles do not differ much from iteration to iteration.

Similarly, we can obtain the numerical solutions of the problems $\bar{P}2$, $\bar{P}3$ and $\bar{P}4$ via the spectral collocation method II with interface point, described above.

4 Numerical results

Nine test examples were solved using the spectral collocation method I with interface point(CM-I) and the spectral collocation method II with interface point(CM-II). The accuracy of these two methods are measured by computing the difference between the exact and numerical solutions at each mesh point and use these to compute the L_∞ error norm. We choice the initial guess $u^{(0)} = 1$ to generate the successive approximations $u^{(m)}$ for nonlinear problems.

Example 1 Consider the third order SPBVP of reaction-diffusion type [21]

$$-\epsilon u'''(x) + 4u'(x) + u(x) = f(x), \quad 0 \le x \le 1 \tag{46}$$

$$u(0) = 1, \quad u'(0) = 1, \quad u'(1) = 1. \tag{47}$$

where

$$f(x) = 2 + \frac{x}{4} + \frac{3\sqrt{\epsilon}}{8(1 + e^{-2/\sqrt{\epsilon}})}\left[1 - e^{-2/\sqrt{\epsilon}} - e^{-2x/\sqrt{\epsilon}} + e^{-2(1-x)/\sqrt{\epsilon}}\right]$$

Example 2 Consider the third order SPBVP of reaction-diffusion type [21]

$$-\epsilon u'''(x) + 4u'(x) - u(x) = f(x), \quad 0 \le x \le 1 \tag{48}$$

$$u(0) = 1, \quad u'(0) = 1, \quad u'(1) = 1. \tag{49}$$

where

$$f(x) = -\frac{x}{4} - \frac{3\sqrt{\epsilon}}{8(1 + e^{-2/\sqrt{\epsilon}})}\left[1 - e^{-2/\sqrt{\epsilon}} - e^{-2x/\sqrt{\epsilon}} + e^{-2(1-x)/\sqrt{\epsilon}}\right]$$

Example 3 Consider the non-linear third order SPBVP of reaction-diffusion type [21]

$$-\epsilon u'''(x) + 4u'(x) - (1/2)u^2(x) = f(x), \quad 0 \le x \le 1 \tag{50}$$

$$u(0) = 1, \quad u'(0) = 1, \quad u'(1) = 1. \tag{51}$$

where

$$f(x) = 1 - \frac{1}{2}\left[1 + \frac{x}{4} + \frac{3\sqrt{\epsilon}}{8(1 + e^{-2/\sqrt{\epsilon}})}\left[1 - e^{-2/\sqrt{\epsilon}} - e^{-2x/\sqrt{\epsilon}} + e^{-2(1-x)/\sqrt{\epsilon}}\right]\right]^2$$

In Tables 1, 2 and 3 , we give the maximum absolute error (MAER) between the exact solution of the problems considered in Examples 1–3, respectively, and the results obtained by CM-I and CM-II for different values of ϵ, compared with the results given by Valarmathi and Ramanujam [21] (VR-I) with $N = 256$. It is seen that the results of CM-I and CM-II give good accuracy

Table 1 (Example 1)

ϵ	N_1	N_2	CM-I		CM-II		VR-I
			ℓ	MAER	ℓ	MAER	MAER
10^{-1}	15	35	2	5.76E-7	2	4.77E-7	
10^{-2}	15	35	1	3.58E-7	5	7.15E-7	
10^{-3}	15	35	2	5.96E-7	1	2.38E-6	
10^{-4}	15	35	3	3.93E-6	2	1.32E-5	7.50E-3
10^{-5}	15	35	2	1.65E-4	2	1.56E-4	
10^{-6}	15	35	1	3.24E-4	3	4.40E-3	
10^{-8}	15	35	1	6.49E-4			
10^{-10}	15	35	1	7.01E-4			

Table 2 (Example 2)

ϵ	N_1	N_2	CM-I		CM-II		VR-I
			ℓ	MAER	ℓ	MAER	MAER
10^{-1}	15	35	3	3.58E-7	1	4.77E-7	
10^{-2}	15	35	4	4.76E-7	4	5.96E-7	
10^{-3}	15	35	2	4.77E-7	2	3.22E-6	
10^{-4}	15	35	2	4.17E-6	2	1.10E-5	7.50E-3
10^{-5}	15	35	2	1.74E-4	2	1.30E-4	
10^{-6}	15	35	1	3.34E-4	1	3.81E-4	
10^{-8}	15	35	1	6.49E-4			
10^{-10}	15	35	1	7.01E-4			

in little computer time as compared with VR-I. Also The maximum absolute errors from CM-I are smaller than those from CM-II since the latter did not yield an accurate approximate solution, especially for $\epsilon < 10^{-6}$.

In Fig. 1, the difference between the exact solution and the approximate solutions obtained via the CM-I and CM-II is plotted for Example 1 for $N_1 = 15, N_2 = 35, \epsilon = 10^{-8}, d = 10^{-8}$.

In Fig. 2, the difference between the exact solution and the approximate solutions obtained via the CM-I and CM-II is plotted for Example 3 for $N_1 = 15, N_2 = 35, \epsilon = 10^{-8}, d = 10^{-8}$.

In Fig. 3, the difference between the exact solution and the approximate solutions obtained via the CM-I and CM-II is plotted for Example 3 for $N_1 = 15, N_2 = 35, \epsilon = 10^{-10}, d = 10^{-10}$.

Figures 1, 2 and 3 show that the errors from CM-II exhibit oscillations, but the errors from CM-I are much smaller and contain no oscillations.

Example 4 Consider the third order SPBVP of convection-diffusion type [22, 23]

$$-\epsilon u'''(x) - 2u''(x) + 4u'(x) + u(x) = f(x), \quad 0 \leq x \leq 1 \tag{52}$$

$$u(0) = 1, \quad u'(0) = 1, \quad u'(1) = 1. \tag{53}$$

Table 3 (Example 3)

ϵ	N_1	N_2	CM-I		CM-II		VR-I
			ℓ	MAER	ℓ	MAER	MAER
10^{-1}	15	35	4	9.54E-7	2	4.77E-7	
10^{-2}	15	35	3	7.15E-7	4	4.77E-7	
10^{-3}	15	35	2	4.77E-7	1	2.74E-6	
10^{-4}	15	35	2	3.93E-6	1	1.32E-5	8.86E-3
10^{-5}	15	35	2	1.74E-4	1	1.37E-4	
10^{-6}	15	35	1	3.36E-4	3	5.58E-4	
10^{-8}	15	35	1	5.63E-4			
10^{-10}	15	35	1	7.14E-4			

where

$$f(x)=2-\frac{\epsilon\left(1-e^{-2x/\epsilon}\right)}{4\left(1-e^{-2/\epsilon}\right)}+\left(1+\frac{1}{2(1+e^{-2/\epsilon})}\right)x-\frac{x^2}{4}+4\left[1-\frac{x}{2}+\frac{\left(1-e^{-2x/\epsilon}\right)}{2\left(1-e^{-2/\epsilon}\right)}\right]$$

Example 5 Consider the third order SPBVP of convection-diffusion type [22, 23]

$$-\epsilon u'''(x)-2u''(x)+4u'(x)-u(x)=f(x),\quad 0\le x\le 1 \tag{54}$$

$$u(0)=1,\quad u'(0)=1,\quad u'(1)=1. \tag{55}$$

where

$$f(x)=\frac{\epsilon\left(1-e^{-2x/\epsilon}\right)}{4\left(1-e^{-2/\epsilon}\right)}-\left(1+\frac{1}{2(1+e^{-2/\epsilon})}\right)x+\frac{x^2}{4}+4\left[1-\frac{x}{2}+\frac{\left(1-e^{-2x/\epsilon}\right)}{2\left(1-e^{-2/\epsilon}\right)}\right]$$

Example 6 Consider the nonlinear third order SPBVP of convection-diffusion type [22]

$$-\epsilon u'''(x)-2u''(x)+4u'(x)-(1/2)u^2(x)=f(x),\quad 0\le x\le 1 \tag{56}$$

$$u(0)=1,\quad u'(0)=1,\quad u'(1)=1. \tag{57}$$

where

$$f(x)=1-\frac{1}{2}\left[1-\frac{\epsilon\left(1-e^{-2x/\epsilon}\right)}{4\left(1-e^{-2/\epsilon}\right)}+\left(1+\frac{1}{2\left(1+e^{-2/\epsilon}\right)}\right)x-\frac{x^2}{4}\right]^2 +4\left[1-\frac{x}{2}+\frac{\left(1-e^{-2x/\epsilon}\right)}{2\left(1-e^{-2/\epsilon}\right)}\right]$$

In Tables 4, 5 and 6, we give the maximum absolute error[MAER] between the exact solution of the problems considered in Examples 4–6, respectively,

Fig. 1 Errors of CM-I and CM-II solutions for Example 1 for $N_1=15$, $N_2=35$, $\epsilon=10^{-8}$, $d=10^{-8}$ (*solid line* diff. of exact and CM-I solutions, *dotted line* diff. of exact and CM-II solutions)

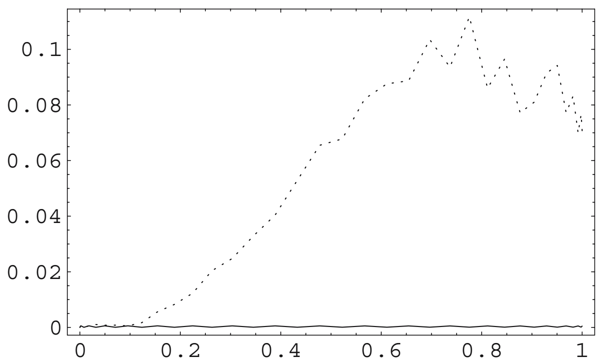
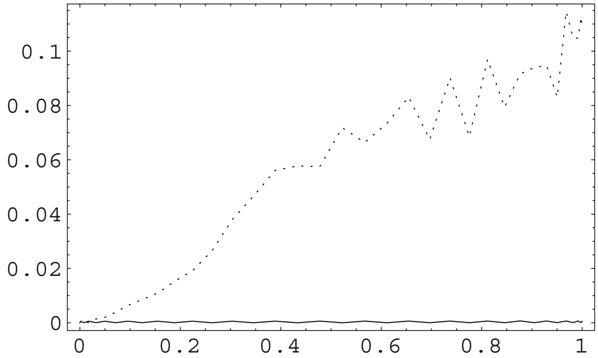


Fig. 2 Errors of CM-I and CM-II solutions for Example 3 for $N_1 = 15$, $N_2 = 35$, $\epsilon = 10^{-8}$, $d = 10^{-8}$ (*solid line* diff. of exact and CM-I solutions, *dotted line* diff. of exact and CM-II solutions)



and the approximate solutions obtained via CM-I and CM-II for different values of ϵ , compared with the results given by Valarmathi and Ramanujam [22](VR-II) and Valarmathi and Ramanujam [23] (VR-III). It is seen that the results of CM-II give good accuracy as compared with VR-II, VR-III and CM-I. Also VR-II and VR-III need more computer time than CM-I and CM-II.

In Fig. 4, the difference between the exact solution and the approximate solutions obtained via the CM-I and CM-II is plotted for Example 5 for $N_1 = 35$, $N_2 = 15$, $\epsilon = 10^{-5}$, $d = 8 \times 10^{-5}$. Figure 4 shows that the errors from CM-II are smaller than those from CM-I.

In Tables 7, 8 and 9, we give the maximum absolute error (MAER) between the exact solution of the problems considered in Examples 1, 3 and 5, respectively, and the results obtained by CM-I and CM-II for different values of ϵ and N where $N = N_1 = N_2$ with interface point $d = \ell\epsilon$ and $d = N\epsilon$, we observe that $d = N\epsilon$ yields a good choice since the choice $d = \ell\epsilon$ need more computer time than the choice $d = N\epsilon$. Also Heinrichs [5–8] found out that the interface point $d = N\epsilon$ is the best choice. From Table 9, we observe that the errors from CM-I method become smaller for increasing N.

Fig. 3 Errors of CM-I and CM-II solutions for Example 3 for $N_1 = 15$, $N_2 = 35$, $\epsilon = 10^{-10}$, $d = 10^{-10}$ (*solid line* diff. of exact and CM-I solutions, *dotted line* diff. of exact and CM-II solutions)

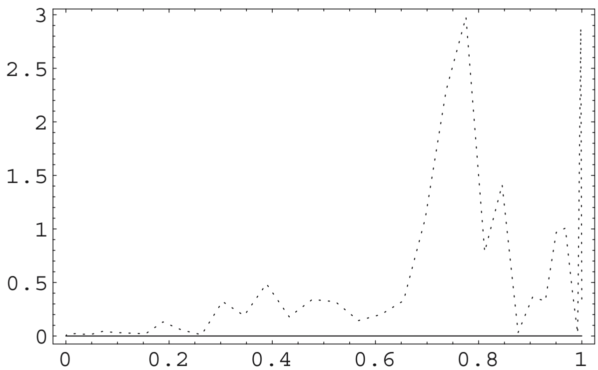


Table 4 (Example 4)

ϵ	N_1	N_2	CM-I		CM-II		VR-II	VR-III
			ℓ	MAER	ℓ	MAER	MAER	MAER
10^{-1}	30	15	2	4.77E-7	2	4.77E-7		
10^{-2}	30	15	9	1.07E-6	9	3.58E-6		
10^{-3}	30	15	9	5.98E-5	9	1.03E-5		
10^{-4}	35	15	9	9.87E-4	9	8.71E-5	2.46E-3	1.47E-3
10^{-5}	35	15	9	5.17E-3	9	4.47E-4		

Table 5 (Example 5)

ϵ	N_1	N_2	CM-I		CM-II		VR-II	VR-III
			ℓ	MAER	ℓ	MAER	MAER	MAER
10^{-1}	30	15	3	4.77E-7	3	4.77E-7		
10^{-2}	30	15	9	5.01E-6	9	9.54E-7		
10^{-3}	30	15	9	3.62E-5	9	2.98E-6		
10^{-4}	35	15	8	9.03E-4	9	6.60E-5		1.47E-3
10^{-5}	35	15	8	6.68E-3	9	8.40E-4	2.50E-3	

Table 6 (Example 6): converges after two iterations

ϵ	N_1	N_2	CM-I		CM-II		VR-I
			ℓ	MAER	ℓ	MAER	MAER
10^{-1}	30	15	3	4.77E-7	2	4.77E-7	
10^{-2}	30	15	9	1.19E-6	8	4.77E-7	
10^{-3}	30	15	8	7.87E-6	8	3.34E-6	
10^{-4}	30	15	8	8.30E-4	9	1.95E-4	
10^{-5}	30	15	8	7.36E-3	9	3.06E-4	2.50E-3

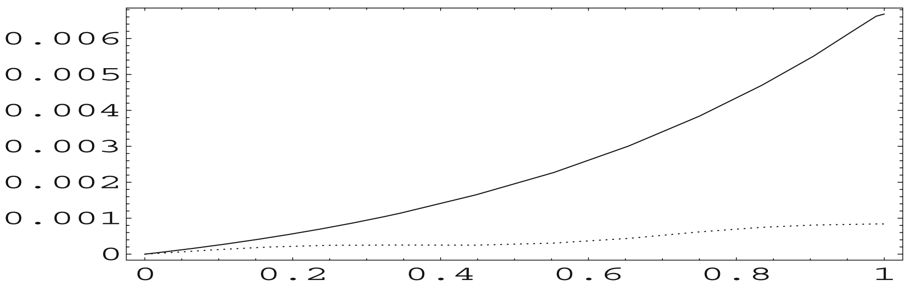


Fig. 4 Errors of CM-I and CM-II solutions for Example 5 for $N_1 = 35$, $N_2 = 15$, $\epsilon = 10^{-5}$, $d = 8 \times 10^{-5}$ (*solid line* diff. of exact and CM-I solutions, *dotted line* diff. of exact and CM-II solutions)

Table 7 (Example 1), $N = N_1 = N_2$

ϵ	N	CM-I			CM-II		
		ℓ	MAER		ℓ	MAER	
			$d = \ell\epsilon$	$d = N\epsilon$		$d = \ell\epsilon$	$d = N\epsilon$
10^{-2}	15	1	3.58E-6	5.01E-6	1	3.58E-7	2.38E-7
	25	1	2.38E-7	4.77E-7	2	5.96E-7	4.77E-7
	35	1	4.77E-7	4.77E-7	5	5.96E-7	4.77E-7
10^{-5}	15	1	2.29E-3	2.31E-3	3	2.58E-3	2.34E-3
	25	9	4.59E-4	4.58E-4	4	5.19E-4	4.64E-4
	35	9	1.53E-4	1.13E-4	1	1.39E-4	1.80E-4
10^{-8}	15	1	3.29E-3	3.29E-3			
	25	1	1.16E-3	1.16E-3			
	35	1	5.75E-4	5.75E-4			

Table 8 (Example 3) $N = N_1 = N_2$, converges after two iterations

ϵ	N	CM-I			CM-II		
		ℓ	MAER		ℓ	MAER	
			$d = \ell\epsilon$	$d = N\epsilon$		$d = \ell\epsilon$	$d = N\epsilon$
10^{-2}	15	1	3.10E-6	4.65E-6	2	5.96E-7	5.96E-7
	25	2	7.17E-7	7.17E-7	2	5.96E-7	7.17E-7
	35	1	7.17E-6	4.77E-7	3	5.96E-7	5.96E-7
10^{-5}	15	1	2.48E-3	2.50E-3	2	3.35E-3	2.91E-3
	25	3	4.88E-4	4.64E-4	2	5.28E-4	4.82E-4
	35	3	1.73E-4	1.66E-4	1	1.54E-4	1.93E-4

Table 9 (Example 5) $N = N_1 = N_2$

ϵ	N	CM-I			CM-II		
		ℓ	MAER		ℓ	MAER	
			$d = \ell\epsilon$	$d = N\epsilon$		$d = \ell\epsilon$	$d = N\epsilon$
10^{-2}	16	6	2.94E-3	4.34E-1	5	4.77E-7	4.77E-7
	26	7	3.58E-7	8.30E-4	7	4.77E-7	9.54E-7
	36	7	1.91E-6	6.68E-6	2	7.15E-7	9.54E-7
	46	4	2.38E-6	1.19E-6	3	7.15E-7	1.91E-6
10^{-3}	16	5	1.97E-2	1.36E-0	8	1.26E-5	3.98E-5
	26	9	9.75E-5	4.04E-1	7	8.35E-6	4.29E-6
	36	9	3.41E-5	6.45E-3	6	1.67E-6	1.19E-5
	46	7	8.82E-6	4.93E-4	5	8.11E-6	2.86E-6
10^{-4}	16	5	1.92E-1	1.43E-0	8	4.32E-5	9.82E-5
	26	7	5.95E-4	1.23E-0	5	8.30E-5	1.17E-4
	36	7	5.48E-4	1.06E-1	5	7.75E-5	1.62E-4
	46	8	1.20E-4	4.97E-3	9	6.94E-5	3.97E-4

5 Conclusions

In this paper, we presented spectral collocation methods with interface point to solve third order SPODEs of reaction-diffusion and convection-diffusion types. As is expected these methods are advantageous over the conventional methods (e.g., classical spectral methods, standard finite difference methods, classical finite element methods, etc). As mentioned in the introduction, the second order SPODEs have been extensively studied but only few results on higher order equations are reported in the literature. The obtained approximate numerical solution maintains a good accuracy in little computer time compared with other methods [21–23]. Importance of the spectral collocation methods with interface point can be seen from the respective tabular results. Based on the experimental results presented in this article, we conclude that, first for SPODEs of reaction-diffusion type, the results of CM-I and CM-II give good accuracy in little computer time as compared with VR-I. Also The maximum absolute errors from CM-I are smaller than those from CM-II, since the latter may contain oscillations, especially for $\epsilon < 10^{-6}$, next for SPODEs of convection-diffusion type, the CM-II scheme produces better results than VR-II, VR-III and CM-I. Also VR-II and VR-III need more computer time than CM-I and CM-II. The interface point $d = N\epsilon$ yields a good choice for third order SPODEs. The problems discussed in this paper belong to the category known as non-turning point problems. In future, we plan to extend our methods to turning point problems, various boundary conditions and fourth order SPODEs.

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