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ORIGINAL PAPER

Convergence acceleration of orthogonal series

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Abstract The aim of this paper is to verify efficiency of two acceleration methods for orthogonal series (more strictly, for series defined at the beginning of Section 1). These methods are quite different although they use the same transform of such a series given there. The first method (Section 3) has some features common with Levin's and Weniger's methods. It may be profitably used in numerical calculations for a vast class of series. The second one (Sections 4 and 5) is somewhat similar to the Euler–Knopp transform of power series. Also this method is numerically realizable but more important is that for a narrower class of series, including some ones having applications in physics, it gives explicit analytic formulae of their transform.

Keywords Orthogonal series · Trigonometric series · Orthogonal polynomials · Convergence acceleration

Mathematics Subject Classifications (2000) 33C45 · 42A32 · 42C10 · 42C20 · 65B10

1 Basic transform of orthogonal series

In the sequel we consider infinite series

$$\sum_{j=l}^{\infty} \alpha_j f_j, \tag{1}$$

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where l is a nonnegative integer (as a rule, $l = 1$) and the sequence $\{f_j\}$ satisfies a linear homogeneous difference equation of the second order

$$f_j + \lambda_j f_{j+1} + \mu_j f_{j+2} = 0 \quad (j = 0, 1, \ldots). \tag{2}$$

Its coefficients λ_j and μ_j, where $\mu_j \neq 0$, as well as initial conditions defining uniquely this sequence are given. We suppose that series (1) converges. Its sum is denoted by s.

It is known that (2) is satisfied if $f_j = W_j(x)$, where W_j is the jth orthogonal polynomial W_j in a given interval I and with a given weight $\omega(x)$. In particular, if W_j is

(1) The jth Chebyshev polynomial of the first kind T_j ($I = [-1, 1]$, $\omega(x) = (1 - x^2)^{-1/2}$) or the jth Chebyshev polynomial of the second kind U_j ($I = [-1, 1]$, $\omega(x) = (1 - x^2)^{1/2}$), then λ_j and μ_j do not depend on j: $\lambda_j = -2x$, $\mu_j = 1$,

(2) The jth Legendre polynomial P_j ($I = [-1, 1]$, $\omega(x) \equiv 1$), then

$$\lambda_j = -\frac{2j+3}{j+1}\,x, \quad \mu_j = \frac{j+2}{j+1} \tag{3}$$

(more generally, W_j can be the jth Jacobi polynomial),

(3) The jth Laguerre polynomial $L_j^{(a)}(x)$ ($I = [0, +\infty)$, $\omega(x) = x^a e^{-x}$, where $a > -1$), then

$$\lambda_j = -\frac{2j - x + a + 3}{j + a + 1}, \quad \mu_j = \frac{j+2}{j+a+1}. \tag{4}$$

Difference (2) is also satisfied when $f_j = \cos jx$ or $f_j = \sin jx$, i.e. for trigonometric series, cosine or sine ones. For both cases is $\lambda_j = -2\cos x$, $\mu_j = 1$. In view of known expressions for the polynomials T_n and U_n these series are in fact identical with *Chebyshev series*, where $f_j := T_j(x)$, $\lambda_j = -2x$ and $\mu_j = 1$.

A special feature of series (1) is that (even for the coefficients α_j depending very regularly on j) partial sums

$$s_n := \sum_{j=0}^{n} \alpha_j f_j \quad (n = 0, 1, \ldots) \tag{5}$$

vary in a capricious manner and it is difficult to estimate the deviations $s_n - s$. In particular, for a series with respect to orthogonal polynomials oscillations of the sums s_n round s depend on distribution (in general, unknown) of zeros of these polynomials. Therefore standard methods of convergence acceleration are rarely sufficiently efficient in case of the series (1). Whereas using (2) in a suitable manner we can find methods which take into account a specific character of such series and therefore are more useful.

The starting point for announced methods is an obvious fact: by (2) we can subtract from (1) an arbitrary multiplicity of $f_j + \lambda_j f_{j+1} + \mu_j f_{j+2}$. The sum of series remains unchanged. More generally, for any factors p_j ($j \geqslant l$) we have, at least formally,

$$\sum_{j=l}^{\infty} \alpha_j f_j - \sum_{j=l}^{\infty} p_j(f_j + \lambda_j f_{j+1} + \mu_j f_{j+2})$$

$$= (\alpha_l - p_l) f_l + (\alpha_{l+1} - \lambda_l p_l - p_{l+1}) f_{l+1} + \sum_{j=l+2}^{\infty} \alpha'_j f_j, \tag{6}$$

where

$$\alpha'_j := \alpha_j - \mu_{j-2} p_{j-2} - \lambda_{j-1} p_{j-1} - p_j \quad (j = l+2, l+3, \ldots). \tag{7}$$

More strictly, if the series on the right in (6) converges, then its sum is identical with the sum of (1). This new series is more useful than the previous one if the coefficients α'_j are much less in modulus from α_j. In the sequel it was explained how we can obtain such result when the sequence $\{\alpha_j\}$ is sufficiently regular.

It is worthwhile to remark that if $f_j = W_j(x)$, where W_j is an orthogonal polynomial, then λ_j and μ_j in (2) depend in general on x. The same is true for the coefficients α'_j of the new series. Thus it is only a *quasiorthogonal* series (cf. *quasipower series*, defined in [12]).

Remark also that a similar reasoning leads to the classical *Euler–Knopp transform* of a power series

$$\sum_{j=0}^{\infty} \alpha_j f_j \quad \left(f_j := x^j\right) \tag{8}$$

having s as its sum into series

$$\frac{1}{1-x} \sum_{k=0}^{\infty} \Delta^k \alpha_0 \left(\frac{x}{1-x}\right)^k. \tag{9}$$

If, namely, $\lambda_j := -x^{-1}$, then $f_j + \lambda_j f_{j+1} = 0$. For any p_j series (8) can be expressed in the form

$$\sum_{j=0}^{\infty} \alpha_j f_j - \sum_{j=0}^{\infty} p_j(f_j + \lambda_j f_{j+1}) = (\alpha_0 - p_0) f_0 + \sum_{j=1}^{\infty} \alpha'_j f_j,$$

where

$$\alpha'_j := \alpha_j - \lambda_{j-1} p_{j-1} - p_j \quad (j = 1, 2, \ldots).$$

This gives, in particular, for $x \neq 1$ and

$$p_j := -\frac{x}{1-x} \alpha_j$$

the relation

$$s = \frac{1}{1-x} \left(\alpha_0 + x \sum_{j=0}^{\infty} \Delta \alpha_j x^j \right).$$

Transforming the obtained series in the same way and repeating such a process we obtain formally (9). It is known that (8) and (9) have the same sum only in a common part of convergence domains of these two series; see, e.g., [3]).

2 Test series

Methods described below were applied, among others, to some orthogonal series such that λ_j, μ_j in (2) depend on j [see (3) and (4)]:

$$\sum_{j=1}^{\infty} \frac{1}{j+1} P_j(x) = \log\left(\sqrt{\frac{2}{1-x}} + 1\right) - 1, \tag{10}$$

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)(2j+3)} P_j(x) = \frac{1}{3} - \sqrt{\frac{1-x}{8}}, \tag{11}$$

$$\sum_{j=1}^{\infty} \log\left(1 + 2j^{-1}\right) P_j(x), \tag{12}$$

$$\sum_{j=1}^{\infty} (2j+1) q^j P_j(x) = \frac{1-q^2}{(1-2qx+q^2)^{3/2}} - 1 \quad (|q| < 1), \tag{13}$$

$$\sum_{j=0}^{\infty} \frac{1}{j+1} L_j^{(a)}(x) = x^{-a} e^x \Gamma(a, x). \tag{14}$$

[NB. Sum of (13) in [4, 8.922.7] is erroneous. Also paper [5] contains an error: in Legendre series (61) given there $2n-1$ should be replaced by $2n+1$.]

The same methods were tested for a few trigonometric series for which λ_j, μ_j do not depend on j. This case is somewhat simpler. A cosine series after a suitable change of variable becomes a Chebyshev one.

$$\sum_{j=1}^{\infty} \frac{q^j}{j+c} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} jx \quad (|q| \leqslant 1), \tag{15}$$

$$\sum_{j=1}^{\infty} \frac{(-1)^j}{j^2} \cos jx = \frac{3x^2 - \pi^2}{12} \quad (-\pi < x < \pi) \tag{16}$$

$$\sum_{j=1}^{\infty} \frac{\left(\frac{1}{2}\right)_j}{(j+1)!} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} jx, \tag{17}$$

$$\sum_{j=1}^{\infty} \log(1 + 2j^{-1}) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} jx. \tag{18}$$

Two series (15) for $q = 1$ represent the indirect interaction energy of a periodic configuration of adsorbed atoms (for details, see [11]).

If the sum of a tested series is known we may verify accuracy of obtained approximants. As a measure of precision of σ approaching the sum $s \neq 0$ we choose

$$\texttt{acc}\ \sigma := -\log_{10}\left|\frac{\sigma}{s} - 1\right|,$$

i.e. the number of exact decimal digits in this approximant. All calculations were performed using Turbo Pascal and variables of the type `Extended`. Thus in the best case we may obtain approximants with about 17–18 exact digits. The accuracy is much worse (as for other convergence acceleration methods), for instance, for series (10) and (11) when x is close to the singular point 1.

3 Recursive optimalization of the factors p_j

Transformation (6) of an orthogonal series (1) is optimal when the factors p_j are such that all the α'_j defined by (7) vanish:

$$\mu_j p_j + \lambda_{j+1} p_{j+1} + p_{j+2} = \alpha_{j+2} \quad (j = 1, 2, \ldots). \tag{19}$$

In other words, the sequence $\{p_j\}$ should verify a linear nonhomogeneous difference equation of second order with parameters λ_j, μ_j appearing in (2) and α_j from the transformed series. For such p_j the sum of this series is equal to

$$\tilde{s} := \alpha_1 - p_1) f_1 + (\alpha_2 - \lambda_1 p_1 - p_2) f_2. \tag{20}$$

Equation 19 has a two-parameter family of solutions. Its form is here important. Consider a simple example, namely the Chebyshev series with the coefficients $\alpha_j := 1/(j)_2$. This equation has then the following form:

$$p_j - 2xp_{j+1} + p_{j+2} = \frac{1}{(j+2)_2} \quad (-1 \leqslant x \leqslant 1). \tag{21}$$

If $x < 1$, then its certain solution p_j can be formally developed, for instance, into a power series

$$\sum_{l=2}^{\infty} \frac{b_l}{(j+1)^l}. \tag{22}$$

Equation 21 permits us determine recursively the coefficients $b_2, b_3, \ldots$ In particular, $b_2 = 1/[2(1-x)]$. The homogeneous equation $q_j - 2xq_{j+1} + q_{j+2} = 0$ has a general solution

$$q_j = c_+\left(x + i\sqrt{1-x^2}\right)^j + c_-\left(x - i\sqrt{1-x^2}\right)^j.$$

Each solution of (21) is a sum of series (22) and some q_j. As $\left|x \pm i\sqrt{1-x^2}\right| = 1$, if $|c_+| + |c_-| > 0$, then $|p_j|$ doesn't converge to 0 and usefulness of transform (6) is questionable. Therefore the factors p_j have to be such that $c_+ = c_- = 0$.

In most cases is rather impossible to investigate in detail solutions of (19) but in all tests numerical methods described below distinguish solution which is suitable in a sense.

If $x = 1$, then (21) has the form $\Delta^2 p_j = 1/(j+2)_2$ and each its solution p_j is such that

$$p_j = fj + g - \sum_{l=1}^{j+1} \frac{1}{l} \quad (f,\ g \text{ arbitrary constants}),$$

i.e. $|p_j|$ is at least of order $\log j$ and transformation (6) lost a sense.

For orthogonal series useful in practice the parameters λ_j, μ_j of (2) are rational in j. If the coefficients α_j behave also sufficiently regularly, then a solution $\{p_j\}$ of (19) interesting for us has the same property and its founding should be easier. By contrast, partial sums (5) of an orthogonal series do not have such property. Therefore assumption, as in [8, (8)], that $s_n - s$ behaves similarly to $\omega_n g_n$, where $\{g_n\}$ is a solution of difference equation (2) and ω_n equals for instance α_n, seems to be rather unjustified; cf. also [6–8], where a direct transferring onto orthogonal series of Levin's methods bases on some expansions of $s_n - s$.

Bearing in mind remarks concerning model example (19) we assume, similarly as for numerous known convergence acceleration methods, that in the method $\mathfrak{L}(r, \omega_j)$ defined below is, for some $\omega_j \neq 0$ and an integer r at least formally

$$\frac{p_j}{\omega_j} = \sum_{l=r}^{\infty} \frac{b_l}{(j+1)^l} \quad (j = 1, 2, \ldots) \tag{23}$$

(convergence of this series is not supposed). In Levin's methods [2, Section 2.7]) used to accelerate convergence of a series such an assumption concerns its remainders.

For the second method, denoted by $\mathfrak{W}(r, \omega_j)$, we assume that, also at least formally,

$$\frac{p_j}{\omega_j} = \sum_{l=r}^{\infty} \frac{a_l}{(j+1)_l} \quad (j = 1, 2, \ldots). \tag{24}$$

Instead of a power series we have here a *factorial* series. A similar assumption concerns remainders of a series in Weniger's methods [14, Section 8].

Power series from (23) can be formally transformed into factorial series from (24) and conversely but each of these two relations are used in a different manner and therefore two methods give different results. However, remark that factorial series is in a sense more natural. In fact, it is easy, e.g., to transform p_{j+1} into a series with terms $a'_l/(j+1)_l$ and we may use it in some analytic manipulations. In both announced methods $j+1$ is often replaced by more general sum $j+\beta$, but choosing of $\beta \neq 1$ rarely affects results in an essential manner or even leaves theirs without change; cf. a comment given below in this section about series (14).

In general, using announced methods is justified if we may choose such parameters r and ω_j that initial coefficients a_j or b_j can be calculated recurrently with the aid of (19). This is the case for very many series (1). Both the definitions take into account that for all interesting orthogonal series the coefficients λ_j, μ_j in (19) are rational in j and are such that

$$\lambda_j = \lambda + c_1 j^{-1} + \dots, \qquad \mu_j = \mu + d_1 j^{-1} + \dots \tag{25}$$

(1) Let the coefficients α_j be such that (formally)

$$\frac{\alpha_{j+1}}{\alpha_j} = q \sum_{l=0}^{\infty} \pi_l (j+1)_l \quad (\pi_0 = 1,\ q \neq 0). \tag{26}$$

In particular this is the case when above quotient is rational in j, with the numerator and the denominator of the same degree. For series (17) this quotient equals $\left(j+\frac{1}{2}\right)/(j+2)$. Write (19) in the form

$$\mu_j \frac{p_j}{\omega_j} + \lambda_{j+1} \frac{p_{j+1}}{\omega_{j+1}} \frac{\omega_{j+1}}{\omega_j} + \frac{p_{j+2}}{\omega_{j+2}} \frac{\omega_{j+1}}{\omega_j} \frac{\omega_{j+2}}{\omega_{j+1}} = \frac{\alpha_{j+2}}{\alpha_{j+1}} \frac{\alpha_{j+1}}{\omega_j}. \tag{27}$$

If $r := 0$ and $\omega_j := \alpha_{j+1}$, then right-hand side of this relation equals $\alpha_{j+2}/\alpha_{j+1}$, i.e. is similar to (26). The quotients of quantities $\omega_.$ can be expressed in the same manner. Thus, suppose that p_j/ω_j has form (24). Then also the left-hand side of (27) can be transform into a series similar to (26). Equating coefficients of two sides of (27) of $(j+1)_l$ for $l = 0, 1, \dots$ we may compute successively $a_0, a_1, \dots$. More strictly, this is possible if

$$\mu + \lambda q + q^2 \neq 0 \tag{28}$$

because it should be $a_0 = q/(\mu + \lambda q + q^2)$. One can verify that the same assumption permits to determine uniquely all the next a_l. This is the case for series (12). Putting $r = -1$, $\omega_j = q^j$, $x = \frac{1}{2}$ and $q = \frac{2}{3}$ we obtain

$$a_{-1} = \frac{8}{7},\ a_0 = \frac{76}{49},\ a_1 = -\frac{120}{343},\ a_2 = \frac{60}{2401}.$$

These a_l give some approximants of p_1 and p_2, and as a consequence some approximant of the sum of (12), but the last one has only one exact decimal digit. Methods $\mathfrak{L}$ i $\mathfrak{W}$ are more efficient.

(2) Assumption (26) is fulfilled, among others, if $\alpha_j = q^j R(j)$ where $q \neq 0$ and, at least formally,

$$R(j) = \sum_{l=r}^{\infty} \rho_l (j+1)_l \quad (\rho_r \neq 0).$$

In particular this is the case if R is a rational function in j, with the numerator of degree l and the denominator of degree m and if $r := m - l$ [cf. series (10–12), (15), (16) and (18)]. For such r we put $\omega_j := q^j$. It should be

$$\mu_j \frac{p_j}{q^j} + q\lambda_{j+1} \frac{p_{j+1}}{q^{j+1}} + \frac{p_{j+2}}{q^{j+2}} = q^2 \sum_{l=r}^{\infty} \rho_l (j+3)_l.$$

If (28) holds, then for selected r and ω_j one can evaluate successively the coefficients a_l from (24).

If assumption (28) is not fulfilled, then the above definitions of the parameters r and ω_j are incorrect. One can prove that for $f_j = P_j(1)$ and $\alpha_j = 1/(j+1)_3$; series (1) evidently converges to $\frac{1}{12}$ and (26) holds for $q = 1$. As we have also $\lambda = -2$ and $\mu = 1$, (25) is false. System (19) is satisfied by the factors $p_j = (j+1)/[4(j+2)_2]$, then $p_j/\alpha_{j+1} = \frac{1}{4}(j+1)(j+4)$, i.e. in the variant (1) it should be $r = -2$, and not $r = 0$, whereas for $\omega_j \equiv 1$ [variant (2)] it should be $r = 1$, and not $r = 3$.

Now, methods $\mathfrak{W}$ and $\mathfrak{L}$ will be described in details. In both cases for given natural numbers n and k we restrict system (19) to k equations for $j = n$, $n+1, \ldots, n+k-1$ containing the factors

$$p_n, p_{n+1}, \ldots, p_{n+k+1}. \tag{29}$$

To fix ideas, at first we discuss method $\mathfrak{W}$. For $\mathfrak{L}$ a considerable part of arguments is very similar. We truncate each series in (24) to the sum of k initial terms, i.e. we assume that (approximately)

$$\frac{p_j}{\omega_j} = \sum_{l=r}^{k+r-1} \frac{a_l}{(j+1)_l}.$$

In this case

$$(j+1)_{k+r-1}\frac{p_j}{\omega_j} = \sum_{l=r}^{k+r-1} a_l(j+l+1)_{k+r-l-1}.$$

As usual, Pochhammer symbol $(l)_m$ for $m < 0$ (possibly $k+r-1 < 0$) is defined by the formula $(l)_m := 1/(l+m)_{-m}$. The right-hand side of above relation is a polynomial in j, of degree $\leqslant k-1$ and consequently

$$(-1)^k \Delta^k \left[(j+1)_{k+r-1}\frac{p_j}{\omega_j}\right] = 0$$

(the sign $(-1)^k$ affects only auxiliary quantities). Factors (29) appear only in two such equations which we adjoin to k selected equations (19):

$$\mu_j p_j + \lambda_{j+1} p_{j+1} + p_{j+2} = \alpha_{j+2} \quad (j = n, n+1, \ldots, n+k-1), \tag{30}$$

$$(-1)^k \Delta^k \left[(j+1)_{k+r-1}\frac{p_j}{\omega_j}\right]\Bigg|_{j=n,n+1} = 0. \tag{31}$$

Considering method $\mathfrak{L}$ instead of $\mathfrak{W}$ it should be replaced equations (31) by

$$(-1)^k \ \Delta^k \left[(j+1)^{k+r-1}\frac{p_j}{\omega_j}\right]\Bigg|_{j=n,n+1} = 0. \tag{32}$$

Let $\mathcal{S}_n^k$ denote the system composed of k linear equations (30) as well as two equations (31) or (32) and containing $k+2$ unknowns p_j.

Construction of some approximants with the aid of *truncated* systems of equations relative to *truncated* sequence of unknowns is typical for convergence acceleration methods. Systems $\mathcal{S}_n^k$ are however untypical and they require a special treatment.

All the systems

$$\mathcal{S}_n^1,\ \mathcal{S}_{n-1}^2, \ldots, \mathcal{S}_1^n \tag{33}$$

lying on nth antidiagonal of the two-dimensional array contain the unknowns up to p_{n+2}. We describe now a method of constructing and solving such systems from consecutive antidiagonals (33), on each of them from left to right.

The simplest system $\mathcal{S}_n^1$ contains only one (30) which implies the relation

$$p_{n+2} = \alpha_{n+2} - \mu_n p_n - \lambda_{n+1} p_{n+1}.$$

Using in succession such equalities for $n = 1, 2, \ldots$ we express $p_3, p_4, \ldots$ by p_1 i p_2:

$$p_n = \gamma_n + \delta_n p_1 + \zeta_n p_2 \quad (n = 3, 4, \ldots). \tag{34}$$

The coefficients $\gamma_n, \delta_n, \zeta_n$ satisfy recurrence formulae

$$\left.\begin{aligned} \gamma_n &= \alpha_n - \mu_{n-2}\gamma_{n-2} - \lambda_{n-1}\gamma_{n-1}, \\ \delta_n &= -\mu_{n-2}\delta_{n-2} - \lambda_{n-1}\delta_{n-1}, \\ \zeta_n &= -\mu_{n-2}\zeta_{n-2} - \lambda_{n-1}\zeta_{n-1} \end{aligned}\right\} \quad (n = 3, 4, \ldots), \tag{35}$$

where

$$\gamma_1 = \gamma_2 = \delta_2 = \zeta_1 = 0, \quad \delta_1 = \zeta_2 = 1. \tag{36}$$

Quantities $\gamma_n, \delta_n, \zeta_n$ do not depend on chosen method and only γ_n depends on coefficients of series (1). It is worthwhile to remark that these quantities, similarly to elements f_j, depend very irregularly on n. This property carry over into coefficients of system (43) but this is rather unimportant.

By (34) each system $\mathcal{S}_n^k$ reduces to two equations (31) [or (32)] with the unknowns p_1 and p_2; cf. (43). Due to (20) solving these equations leads to certain approximate value $s_n^{(k)}$ of the sum of series (1).

Two mentioned equations can be constructed recursively in a manner similar to that used in Levin's and Weniger's methods. For (31) we define $\varphi_{nl}^{(k)}$ such that

$$(-1)^k\ \Delta^k \left[(j+1)_{k+r-1} \frac{p_j}{\omega_j} \right] \bigg|_{j=n} = (n+k+1)_{r-1} \sum_{l=0}^{k} \varphi_{nl}^{(k)} \frac{p_{n+l}}{\omega_{n+l}}. \tag{37}$$

Thus

$$\varphi_{nl}^{(k)} = (-1)^l \binom{k}{l} (n+l+1)_{k-l} (n+k+r)_l \quad (l = 0, 1, \ldots, k).$$

Let

$$\varphi_{n0}^{(0)} = 1, \quad \varphi_{n,-1}^{(k)} = \varphi_{n,k+1}^{(k)} = 0 \quad (k \geqslant 0) \tag{38}$$

(the first relation results of course from general definition of $\varphi_{nl}^{(k)}$). It is easy to prove that coefficients $\varphi_{nl}^{(k)}$ for $k > 0$ satisfy a recurrence formula:

$$\varphi_{nl}^{(k)} = (n+k)\varphi_{nl}^{(k-1)} - (n+2k+r-1)\varphi_{n+1,l-1}^{(k-1)} \quad (l = 0, 1, \ldots, k); \tag{39}$$

cf. [14, (8.3–8.5)] where $r = 0$.

Two equations (32) correspond to method $\mathfrak{L}$. Therefore the quantities $\varphi_{nl}^{(k)}$ should be now defined differently:

$$(-1)^k \, \Delta^k \left[(j+1)^{k+r-1} \frac{p_j}{\omega_j} \right] \Bigg|_{j=n} = \sum_{l=0}^{k} \varphi_{nl}^{(k)} \frac{p_{n+l}}{\omega_{n+l}}, \tag{40}$$

i.e.

$$\varphi_{nl}^{(k)} = (-1)^l \binom{k}{l} (n+l+1)^{k+r-1} \quad (l = 0, 1, \ldots, k).$$

If

$$\varphi_{n0}^{(0)} = (n+1)^{r-1}, \quad \varphi_{n,-1}^{(k)} = \varphi_{n,k+1}^{(k)} = 0 \quad (k \geqslant 0), \tag{41}$$

then for $k = 1, 2, \ldots$

$$\varphi_{nl}^{(k)} = (n+1)\varphi_{nl}^{(k-1)} - (n+k+1)\varphi_{n+1,l-1}^{(k-1)} \quad (l = 0, 1, \ldots, k). \tag{42}$$

For both the defined methods substitution of (34) into (31) or (32) leads to the equations

$$\gamma_n^{(k)} + \delta_n^{(k)} p_1 + \zeta_n^{(k)} p_2 = 0, \quad \gamma_{n+1}^{(k)} + \delta_{n+1}^{(k)} p_1 + \zeta_{n+1}^{(k)} p_2 = 0 \tag{43}$$

where by virtue of (37) or (40) we have

$$\eta_n^{(k)} := \sum_{l=0}^{k} \varphi_{nl}^{(k)} \frac{\eta_{n+l}}{\omega_{n+l}} \quad (\eta \equiv \gamma, \delta, \zeta).$$

We conclude from here and from (38) or (41) that recurrence formulae (39) and (42) lead to similar formulae for $\eta_n^{(k)}$ used respectively in methods $\mathfrak{W}$ and $\mathfrak{L}$:

$$\eta_n^{(k)} = (n+k)\eta_n^{(k-1)} - (n+2k+r-1)\eta_{n+1}^{(k-1)}, \tag{44}$$

$$\eta_n^{(k)} = (n+1)\eta_n^{(k-1)} - (n+k+1)\eta_{n+1}^{(k-1)}. \tag{45}$$

Initial conditions result from (38) and (41), respectively, and from (36):

$$\eta_n^{(0)} = \varphi_{n0}^{(0)} \frac{\eta_n}{\omega_n} \tag{46}$$

where $\varphi_{n0}^{(0)}$ equals $(n+1)^{r-1}$ (method $\mathfrak{L}$) or 1 (method $\mathfrak{W}$). The remaining quantities $\varphi_{nl}^{(k)}$ are not calculated.

As usual in convergence acceleration methods we construct successively antidiagonals (33) for $n = 1, 2, \ldots$, and for each of them corresponding approximants $s_n^{(1)}, s_{n-1}^{(2)}, \ldots, s_1^{(n)}$ of the sum of series (1). We begin by quantities

$\eta_1, \eta_1^{(0)}, \eta_2, \eta_2^{(0)}$ (from here $\eta \equiv \gamma, \delta, \zeta$) evaluated from (36) and (46). Next actions can be described as follows:

for $n = 1, 2, \ldots$

for $k = 1, 2, \ldots, n$ and $m := n - k + 1$

1. if $k = 1$, then computing η_{n+2} from (35) and $\eta_{n+2}^{(0)}$ from (46),
2. if $k = n$, then computing $\eta_1^{(n)}$ by means of (45) (method $\mathfrak{L}$) or (44) (method $\mathfrak{W}$),
3. computing $\eta_{m+1}^{(k)}$ by means of (45) (method $\mathfrak{L}$) or (44) (method $\mathfrak{W}$),
4. finding solution p_1, p_2 of system $\mathcal{S}_m^k$ reduced to equations such as (43),
5. expressing value $s_m^{(k)}$ by p_1, p_2 according to (20).

To avoid a confusion we emphasize that e.g. for $n = k = 1$ it is necessary to execute *all* the steps 1–5 and for $n = 2, k = 1$ only 2 should be omitted.

The above algorithm has a remarkable feature: in the steps 1–4 is needed, apart from coefficients α_j of series (1), only *indirect* information about sequence $\{f_j\}$, namely coefficients λ_j and μ_j of difference equation (2). Only the step 5 requires values f_1 and f_2 of the two initial elements of sequence $\{f_j\}$, the remaining ones are superfluous. This is important at least for trigonometric series because after the steps 1–4 a few additional arithmetic operations give approximate values of the sum for *two* series, cosine one and sine one:

$$\sum_{j=1}^{\infty} \alpha_j \cos jx, \qquad \sum_{j=1}^{\infty} \alpha_j \sin jx.$$

Such pairs of series are applied in physics; see [11].

If we execute calculations for $n \leqslant n_{\max}$, then a program uses three one-dimensional arrays `sg`, `sd`, `sz`$[1\,..\,n_{\max}+1]$. Quantities $\gamma_n^{(k)}, \delta_n^{(k)}, \zeta_n^{(k)}$ for each k are stored as `sg`$[n]$, `sd`$[n]$, `sz`$[n]$, respectively (the order of steps 1–3 is essential). The cost of all calculations is little greater than for Levin's and Weniger's algorithms (where only two auxiliary arrays are needed). Here we have additional steps 4 and 5, but we calculate only two initial elements of sequence $\{f_j\}$.

For both methods $\mathfrak{L}$ and $\mathfrak{W}$ some properties of the table of approximants $s_n^{(k)}$ are such as for many known convergence acceleration methods. In particular on each antidiagonal (33) (depending on the same initial coefficients α_j) successively evaluated approximants are in general more and more accurate. Therefore in the sequel we analyse only the final approximants: $s^{(k)} := s_1^{(k)}$. Accuracy `acc` $s^{(k)}$ in general grow with k, at least up to certain limit quantity dependent of properties of a series. This growth may be however irregular. Example: for series (11), $x = 0.97$, for method $\mathfrak{W}(0, \alpha_{j+1})$ and $k = 5, 6, \ldots, 21$ accuracies `acc` $s^{(k)}$ are equal to

$$\begin{gathered} 5.31,\ 5.80,\ 5.31,\ 5.85,\ 7.08,\ 7.86,\ 7.91,\ 8.16,\ 8.99, \\ 10.42,\ 10.85,\ 10.60,\ 11.20,\ 12.09,\ 12.09,\ 11.52,\ 10.74, \end{gathered} \tag{47}$$

and the next ones are less and less. These are other corollaries arising from tests:

- Methods $\mathfrak{W}$ are in general only a little more efficient than $\mathfrak{L}$.
- For $\alpha_j = q^j R(j)$ (cf. above) method $\mathfrak{W}(r, q^j)$ is a little more efficient than $\mathfrak{W}(0, \alpha_{j+1})$. Further informations concern method $\mathfrak{W}$ with reasonably chosen parameters.
- For series such that (10), i.e. having a singularity at a point (here, at the point 1), both the methods are the less efficient the x is closer to this point. Consider, as an example, method $\mathfrak{W}(1, 1)$. For (10) and $x \leqslant 0.5$ it gives accuracies acc $s^{(k)}$ growing with k (local exceptions to the rule are possible) and attaining at least level of 17 digits for $k = 11, 12, 13, 15, 15$ when $x = -0.7, -0.3, 0.1, 0.3, 0.5$, respectively. This limit accuracy is attained at least for a few next k and one can accept stability of the method. If $x > 0.5$, then the greatest acc $s^{(k)}$ is less and equals, say, 16.7, 16.1, 13.4, 12.2, 8.9 for $x = 0.75, 0.8, 0.9, 0.95, 0.99$ and $k = 17, 16, 16, 18, 22$, respectively. For greater k this accuracy decreases and the method is unstable.
- Series (11) has at the point 1 a weaker singularity (it converges there but its first derivative is infinite) and probably for this reason near this point maximal accuracies are greater than for (10); they equal 14.0, 13.2, 10.3 for $x = 0.9, 0.95, 0.99$, respectively.
- Series (13) depends on two parameters, q and x. It converges obviously very slowly when at least one parameter is close to 1. Also two methods described above are then not much efficient. The table below concerns results of calculation with the aid of method $\mathfrak{W}(-1, q^j)$. It contains for many pairs (q, x) the greatest accuracy obtained for the sum of the series. In each case it suffices to calculate $s^{(k)}$ for $k \leqslant 19$.

$q \diagdown x$	0.5	0.7	0.8	0.9	0.95	0.99
0.8	17.3	15.6	15.7	14.1	14.0	13.4
0.9	17.2	15.5	13.6	12.9	11.8	9.4
0.95	17.3	16.8	14.6	12.0	12.0	10.0
0.99	16.8	15.7	14.1	12.9	11.5	7.3

- Methods of Section 5 do not apply to the Laguerre and Hermite series (cf. Th. 6 where for them $\sigma = 0$). To verify efficiency of the above described methods in such case series (14) was used. It is exceptional in a sense, namely its sum can be expressed by a continued fraction:

$$\frac{1|}{|x} + \mathop{\mathbf{K}}_{k=1}^{\infty} \left[\frac{k-a|}{|1} + \frac{k|}{|x} \right]$$

[10, p. 576]. It is then evident that for any natural integer a this sum is a rational function (equal, for example, to $(x^2 + 2x + 2)/x^3$ if $a = 3$). For such a also method $\mathfrak{W}(0, 1)$ gives this result in a finite number of steps. This is effect of particularly simple expression of factors p_j. If $a = 3$, then

$$p_j = \frac{1}{x} + \frac{2(1+x)}{x^2(j+2)} + \frac{2(2+x)}{x^3(j+2)_2}. \tag{48}$$

So, in this case series (24) with $j+2$ instead of $j+2$ and for $r=0$, $\omega_j \equiv 1$ reduces to three initial terms. Since

$$\frac{1}{(j+2)_1} = \frac{1}{(j+1)_1} - \frac{1}{(j+1)_2}, \quad \frac{1}{(j+2)_2} = \frac{1}{(j+1)_2} - \frac{2}{(j+1)_3},$$

the original series (24) contains four terms. Of course, in the considered example p_j doesn't have finite expansion of type (23). From (19) results that some $\{\alpha_j\}$ correspond to any $\{p_j\}$. Then such expansions are possible only for artificial orthogonal series.

For $a \neq 1, 2, \ldots$ continued fraction (48) is infinite. It converges very slowly for a and x close to 0, but in practice it always permits to evaluate its value with maximal accuracy. Efficiency of method $\mathfrak{W}(0, 1)$ for such a, x is also low. For, e.g., $a = 0.5$ and $x = 2, 4, 6$ the method gives 13.2, 14.6, 14.6 accurate digits, and for $a = 2.5$ and $x = 2, 4, 6$ it gives 12.5, 14.5, 16.4 accurate digits.

- For cosine series (16) method $\mathfrak{W}(0, \alpha_{j+1})$ was used. If $0 \leqslant x \leqslant 0.5\pi$, then accuracies $\texttt{acc}\ s^{(k)}$ grow enough regularly and at least from $k = 16$ onwards stabilize on a level exceeding 17. When x changes from 0.5π to π, efficiency of the method diminishes. In particular the greatest $\texttt{acc}\ s^{(k)}$ equals 16.9, 15.5, 13.6, 13.5, 10.2 for $x/\pi = 0.7, 0.8, 0.85, 0.9, 0.95$ and $k = 19, 19, 18, 21, 18$, respectively. If however $x \geqslant 0.5\pi$, then an initial Oleksy's transform, cited in the next section, permits us to obtain the sum of a series with greater accuracy. The same transform is recommended for cosine series (15), if $|x| < 0.5\pi$ and q is close or equal to 1. Methods applied directly to this series give its sum for $x = 0.1\pi, 0.2\pi, 0.3\pi$ with 11.2, 13.2, 15.0 exact digits.
- Method $\mathfrak{W}(0, \alpha_{j+1})$ applied to series (17) and (18) with irrational coefficients gives results with accuracy depending from x as for (16).
- It is rather surprising that for all investigated series it suffices to evaluate approximants $s^{(k)}$ for $k \leqslant k_{\max} := 30$, because the next ones do not give additional informations about the sum of series. A reasonable value of $k_{\max}$ probably depends, among others, on accuracy of variables applied in the algorithm.

A model sequence (47) of accuracies $\texttt{acc}\ s^{(k)}$ testifies that for a series with unknown sum is difficult to choose optimal approximant $s^{(k)}$ and to estimate its accuracy. In [1, Section 6.6.2] the following advice is formulated about a sequence $\{x_n\}$ of approximants of a root of equation: x_n equals this root with (probably) the best attainable accuracy if at the same time two inequalities are satisfied, namely $|x_{n+1} - x_n| \geqslant |x_n - x_{n-1}|$ and $|x_n - x_{n-1}| < \delta$. The last parameter is a *rough tolerance* protecting us against a too early stopping of computations. In the case of summing of the Legendre series a reasonable tolerance depends, in a manner difficult to predict, on coefficients α_j and variable x. The difference $s^{(k)} - s^{(k-1)}$ is often *locally* small even when these two approximants are very inaccurate. Therefore, taking into account many tests executed for series with known sum, we adopt a modified criterion of *probably* the optimal $s^{(k)}$:

(1) For a certain $k_{\max}$ (in the tests $k_{\max} = 30$ was a reasonable value) we evaluate $s^{(1)}, s^{(2)}, \ldots, s^{(k_{\max})}$.
(2) We define $\varkappa$ as that k, for which the sum $\sigma_k := |\Delta s^{(k-1)}| + |\Delta s^{(k)}|$ is smallest.
(3) We accept $s^{(\varkappa)}$ as a reasonable approximant of the sum of series and $\sigma_\varkappa$ as (rough) estimate of its absolute error.

Of course also this criterion, as each other, is disputable. Sum in (2) smooths local irregularities of the sequence $\{s^{(k)}\}$ and in general the quantities σ_k decrease and attain their minimum for such $\varkappa$ that $s^{(\varkappa)}$ is the best or almost the best approximant of s. A little other situation is also possible: for series (16), $x = 0.74$ and $k = 16, 17, \ldots, 22$ the quantities $10^{18}\sigma_k$ and acc $s^{(k)}$ are as follows:

514	**111**	149	300	263	**58**	378
16.00	16.21	15.97	15.66	15.63	15.71	15.58

In spite of inequality $149 > 111$ the above criterion tells us to examine further σ_k and to put $\varkappa = 21$ although the approximant $s^{(17)}$ is a little more accurate than $s^{(21)}$.

A connection between $\sigma_\varkappa$ and acc $s^{(\varkappa)}$ was also examined. More strictly, the quantities $-\log_{10}|\sigma_\varkappa/s|$ and acc $s^{(\varkappa)}$ were compared. For (16) and $x = 0.60, 0.61, \ldots, 0.96$ in 30 cases the estimate is correct, i.e. the first number is less than the second one (usually slightly and in the extreme case by 1.22). On the other hand, for $x = 0.60, 0.62, 0.69, 0.72,\ 0.74, 0.76, 0.85$ the estimate is over-optimistic: the first number is greater than the second one by 0.18, 0.45, 0.11, 0.27, 0.25, 0.13, 0.22, respectively.

4 Equation 2 with constant coefficients

Relation (6) serves also as a basis for transform of a given orthogonal series into other infinite series which converges more rapidly than the first one. To this end one should choose p_j in a special manner. For a restricted class of series repeating such transform leads finally to a series of quite different type which converges very rapidly and whose coefficients have a simple analytic expression.

In this section we suppose that λ_j, μ_j in (2) do not depend on j: $\lambda_j = \lambda$, $\mu_j = \mu$ where $1 + \lambda + \mu \neq 0$. This is the case for the Chebyshev and trigonometric series. Methods proposed below are useful at least if $\alpha_j = q^j R(j)$ where $|q| \leqslant 1$ and R is a rational function in j. For $|q| < 1$ such a series converges (but slowly for $|q|$ close to 1) even if the degree of the numerator of R is not less than the degree of its denominator. For $|q| = 1$ this series converges very slowly or even diverges at certain points. Only for some q and R the sum of such a series is known. Examples: series (16) and

$$\sum_{j=1}^{\infty} \frac{q^j}{j} T_j(x) = -\frac{1}{2} \log\left(1 - 2qx + q^2\right).$$

Cosine and sine series with the coefficients $1/(j+c)$, i.e. (15), can be expressed by a hypergeometric series ${}_2F_1(c, 1; c+1; e^{ix})$, which in turn can be expanded into a Gauss continued fraction [10, pp. 295–296]. Also this fraction converges slowly for x close to 1.

Now we consider several variants of choice of factors p_j occurring in (6). Transform (6) may be useful if the new coefficients α'_j are small in comparison with α_j. If the last ones are defined as above this is the case only for $q = 1$. Otherwise, however, the series $\sum \alpha_j f_j$ can be expressed in the form $\sum R(j) \tilde{f}_j$ where $\tilde{f}_j := q^j f_j$. Equation 2 implies that the sequence $\{\tilde{f}_j\}$ verifies the equation $\tilde{f}_j + \tilde{\lambda} f_{j+1} + \tilde{\mu} f_{j+2} = 0$ where $\tilde{\lambda} := \lambda q^{-1}$, $\tilde{\mu} := \mu q^{-2}$. Thus also for $q \neq 1$ the announced methods are applicable; however, as we will see, change of the parameters λ and μ has some important consequences.

Variant u**.** First, we verify for which u the definition

$$p_j := u\alpha_{j+1}$$

is reasonable. In this case (7) implies the formula

$$\alpha'_j = -\mu u \alpha_{j-1} + (1 - \lambda u)\alpha_j - u\alpha_{j+1}.$$

If

$$u := \frac{1}{1 + \lambda + \mu}, \tag{49}$$

then α'_j depends only on differences of the coefficients α_j:

$$\alpha'_j = u(\mu \Delta\alpha_{j-1} - \Delta\alpha_j).$$

Remark also that for $\mu = 1$ (this is the case for the Chebyshev and trigonometric series, of course without the above mentioned transform of $\{f_j\}$) this formula is better still:

$$\alpha'_j = -\frac{1}{2 + \lambda} \Delta^2 \alpha_{j-1}.$$

For such μ transform (6) is expressed as follows:

$$\sum_{j=l}^{\infty} \alpha_j f_j = \frac{1}{2 + \lambda} \left\{ [(\lambda + 1)\alpha_l - \Delta\alpha_l] f_l + (\alpha_{l+1} - \Delta\alpha_{l+1}) f_{l+1} - \sum_{j=l+2}^{\infty} \Delta^2 \alpha_{j-1} f_j \right\}.$$

Definition (49) of the factor u is reasonable also for p_j chosen a little differently. Let, for example, $p_j := u\alpha_{j+2}$. Then, however, for $\mu = 1$ and $u = 1/(2 + \lambda)$ we have $\alpha'_j = -u[(1 + \lambda)\Delta\alpha_j + \Delta\alpha_{j+1}]$ and we can't express α'_j by the second differences $\Delta^2\alpha_j$.

Variant uv**.** The definition

$$p_j := u(\alpha_{j+1} + v\Delta\alpha_{j+1}),$$

with u as in (49) and at the moment arbitrary v, may be more efficient. Then

$$\alpha'_j = u\left[\mu(1-v)\Delta\alpha_{j-1} - (1+\lambda v)\Delta\alpha_j - v\Delta\alpha_{j+1}\right]$$

and for $v := u(\mu - 1)$ this α'_j depends on the second differences, even if $\mu \neq 1$:

$$\alpha'_j = -u\left[\mu(1-v)\Delta^2\alpha_{j-1} + v\Delta^2\alpha_j\right].$$

If $\mu = 1$, then $v = 0$ and variant uv reduces to the former one. For arbitrary μ it follows from (6) that

$$\sum_{j=l}^{\infty}\alpha_j f_j = u\Bigg\{[(\lambda+\mu)\alpha_l - \Delta\alpha_l - v\Delta\alpha_{l+1}]\, f_l + [\mu\alpha_{l+1} - (1+\lambda v)\Delta\alpha_{l+1} - v\Delta\alpha_{l+2}]\, f_{l+1} - \sum_{j=l+2}^{\infty}\left[\mu(1-v)\Delta^2\alpha_{j-1} + v\Delta^2\alpha_j\right] f_j\Bigg\}. \tag{50}$$

Passing from series (1), i.e. the left-hand side of (50), to its right-hand side may be iterated. This is especially simple for $\mu = 1$ because then the result of this procedure (for $l = 1$) is the following:

$$\sum_{j=1}^{\infty}\alpha_j f_j = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2+\lambda)^k}\left\{\left[(\lambda+1)\Delta^{2k-2}\alpha_k - \Delta^{2k-1}\alpha_k\right] f_{2k-1} + \left(\Delta^{2k-2}\alpha_{k+1} - \Delta^{2k-1}\alpha_{k+1}\right) f_{2k}\right\}. \tag{51}$$

m-th partial sum of this series depends on $f_1, f_2, \ldots, f_{2m}$ and on $\alpha_1, \alpha_2, \ldots, \alpha_{3m}$. The above transform (as well as other transform of this type given below) is useful first of all if an analytic form of the differences $\Delta^m\alpha_k$ is known and the number of used coefficients α_j is immaterial. On the numerical point of view is important that (51) contains only such differences $\Delta^m\alpha_k$ for which $m \approx 2k$. In fact, if e.g. $\alpha_j = 1/(j+c)$, then

$$\Delta^m\alpha_k = \frac{(-1)^m m!}{(k+c)_{m+1}} \tag{52}$$

and $|\Delta^m\alpha_k|$ decreases rapidly to 0, but only when *both* indices m and k increase. More generally, this is the case if α_j is any rational function in j. If the degree of its numerator is less than degree of its denominator, then the Chebyshev series with coefficients α_j converges (at least for $-1 \leqslant x < 1$), but very slowly.

Example 1 Let

$$\alpha_j := \frac{1}{j+c} \quad (c \neq -1, -2, \ldots).$$

Then (51) and (52) imply that

$$\sum_{j=1}^{\infty} \frac{1}{j+c} f_j = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-2)!}{(2+\lambda)^k (k+c+1)_{2k-1}} \times \left\{ \frac{(3\lambda+5)k+\lambda c-\lambda+c-2}{k+c} f_{2k-1} + \frac{5k+c-1}{3k+c} f_{2k} \right\}. \quad (53)$$

Let β_k denote the factor multiplied on the right-hand side by the linear combination in { }. As in the most interesting cases (the Chebyshev and trigonometric series) $|f_j| \leqslant 1$, just properties of β_k determine speed of convergence of the new series. One can prove that

$$|\beta_k| \sim \sqrt{\frac{3\pi}{4k}}\, 3^{-c} \varphi^k, \quad \text{gdzie} \quad \varphi := \frac{4}{27(2+\lambda)}. \quad (54)$$

Then the series with coefficients β_k behaves a little better than a geometric series with common ratio $\pm\varphi$. A similar property is typical also for series resulting from (60) by means of a modified Euler–Knopp transform (as well as series obtained in [9] by transformations of some hypergeometric series). However a value of φ is essential. Let $f_j = T_j(x)$ and| $|x| \leqslant 1$. Then $|f_j| \leqslant 1$ and $\lambda = -2x$. Thus $\varphi = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{10}, \frac{1}{20}$ for x non greater than $\frac{25}{27}, \frac{23}{27}, \frac{19}{27}, \frac{7}{27}, -\frac{13}{27}$, respectively. Hence, though the series

$$\sum_{j=1}^{\infty} \frac{1}{j+c} T_j(x) \quad (55)$$

(diverging for $x = 1$) converges very slowly for each $x \in [-1, 1)$, the transformed series converges at least geometrically already, e.g., if $x = 0.9$, and for $x \leqslant \frac{7}{27}$ joining a term to is partial sum (i.e. growing k by 1) gives one additional exact decimal digit. It was verified for $c = 0$ when series (55) has the sum $-\frac{1}{2}\log(2-2x)$. Below is given the number of initial terms of the transformed series needed for computing its sum with at least 17 exact significant digits.

x :	-1.0	-0.5	-0.25	0	$\frac{7}{27}$	0.6	$\frac{19}{27}$	$\frac{23}{27}$	0.9
max k :	12	13	14	15	17	24	28	55	123

For $x = \frac{25}{27}$ the new series is useless as its partial sums behave irregularly and the most accurate from them gives only two significant digits of its sum. Finally, we can regard using this series and another ones of the same type as worthwhile for $x \leqslant 0.7$.

Let now μ in identity (50) be arbitrary. Then passing from the left-hand side of (50) to its right-hand side is more difficult to repeat. Even in a series resulting from first iteration each coefficient depends on three fourth differences of α_j. Next series are more and more complicated. To avoid this difficulty we regroup terms of the series appearing (50) on the right so that each term depends on only one difference $\Delta^2\alpha_j$. Furthermore we eliminate there the terms $-v\Delta\alpha_{l+1} f_l$, $-\lambda v\Delta\alpha_{l+1} f_{l+1}$ and $\mu v\Delta\alpha_{l+1} f_{l+2}$ whose sum equals 0:

$$\sum_{j=l}^{\infty} \alpha_j f_j = u\left\{[(\lambda+\mu)\alpha_l - \Delta\alpha_l] f_l + [\mu\alpha_{l+1} - \Delta\alpha_{l+1} - v\Delta\alpha_{l+2}] f_{l+1} + \mu[\Delta\alpha_{l+1} - (1-v)\Delta\alpha_{l+2}] f_{l+2} - \sum_{j=l+2}^{\infty} \Delta^2\alpha_j f_j'\right\}, \tag{56}$$

where $f_j' := v f_j + \mu(1-v) f_{j+1}$. The sequence $\{f_j'\}$ satisfies, as $\{f_j\}$, relation (2). The transform resulting from (56) can be easily iterated:

$$\sum_{j=1}^{\infty} \alpha_j f_j = \sum_{k=1}^{\infty} (-1)^{k-1} u^k \times \left\{\left[(\lambda+\mu)\Delta^{2k-2}\alpha_{2k-1} - \Delta^{2k-1}\alpha_{2k-1}\right] f_{2k-1}^{(k-1)} + \left[\mu\Delta^{2k-2}\alpha_{2k} - \Delta^{2k-1}\alpha_{2k} - v\Delta^{2k-1}\alpha_{2k+1}\right] f_{2k}^{(k-1)} + \mu\left[\Delta^{2k-1}\alpha_{2k} - (1-v)\Delta^{2k-1}\alpha_{2k+1}\right] f_{2k+1}^{(k-1)}\right\}, \tag{57}$$

where

$$f_j^{(0)} := f_j, \quad f_j^{(m)} := v f_j^{(m-1)} + \mu(1-v) f_{j+1}^{(m-1)} \quad (m = 1, 2, \ldots).$$

Remark that mth partial sum of a series after transform depends on $f_1, f_2, \ldots, f_{3m}$ (for $v = 1$ only on $f_1, f_2, \ldots, f_{2m+1}$) and on $\alpha_1, \alpha_2, \ldots, \alpha_{4m}$, i.e. this sum uses larger information that analogous sum in (51). This is immaterial if this transform is executed analytically.

Transform (57) is the simplest one for $v = 0$ and $v = 1$; cf. respectively Examples 2 and 3.

Example 2 Consider again the series from Example 1. We have then $\mu = 1$, $u = 1/(2+\lambda)$, $v = 0$ and for this reason $f_j^{(m)} = f_{m+j}$. One can prove that

$$\sum_{j=1}^{\infty} \frac{1}{j+c} f_j = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-2)!}{(2+\lambda)^k (2k+c)_{2k}} \times \left\{\frac{(4k+c-1)[(1+\lambda)(4k+c-2)+2k-1]}{2k+c-1} f_{3k-2} + (6k+c-2) f_{3k-1} - \frac{(2k-1)_2}{4k+c} f_{3k}\right\}, \tag{58}$$

As

$$\frac{(2k-2)!}{(2k+c)_{2k}} \sim \sqrt{\frac{\pi}{2k^3}}\, 2^{-c} 16^{-k},$$

one can say that the above series is better than that in the previous example ($16 > \frac{27}{4}$). It is true at least when the sum in $\{\}$ simplifies in a substantial manner. Suppose that $f_j = T_j\left(-\frac{1}{2}\right)$ in (58). Then due to the equalities $f_{3k-2} = f_{3k-1} = -\frac{1}{2}$, $f_{3k} = 1$ we have

$$-\frac{1}{2}\log 3 = \sum_{j=1}^{\infty} \frac{1}{j} T_j\left(-\frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{2(-1)^k(7k-2)(2k-2)!}{3^k(2k)_{2k}}.$$

Series (53) doesn't simplify as much as above. However, it may be important how a consecutive term of the sequence $\{f_j\}$ improves the accuracy of the new series. In this sense series (58) is better than (53) only if $3^9(2+\lambda) < 2^{14}$, i.e. (for the Chebyshev series) if $x \gtrapprox 0.584$.

Example 3 Section 4 explains how we should process when $\alpha_j = q^j R(j)$, where $|q| \leqslant 1$ and R is a rational function. According to that if we wish accelerate convergence of the series

$$\sum_{j=1}^{\infty} \frac{q^j}{j+c} T_j(x) \quad (c \neq -1, -2, \ldots) \tag{59}$$

[cf. (15)] we use the equation $f_j - 2q^{-1}xf_{j+1} + q^{-2}f_{j+2} = 0$ satisfied by the quantities $f_j := q^j T_j(x)$. Then

$$\lambda = -2q^{-1}x, \quad \mu = q^{-2}, \quad u = \frac{q^2}{1-2qx+q^2}, \quad v = \frac{1-q^2}{1-2qx+q^2}.$$

In particular, for $q = x$ we have $u = q^2/(1-q^2)$, $v = 1$, and then $f_j^{(m)}$ doesn't depend on m: $f_j^{(m)} = q^j T_j(q)$. In such case it is worth simplifying the right-hand side of (57). To this end we combine two terms containing f_j with odd j. For $c = 0$ and $x = q$, when series (59) has the sum $-\frac{1}{2}\log(1-q^2)$, this leads to the series

$$\frac{2q-3q^3}{2(1-q^2)} T_1(q) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2k-2)!}{(2k+1)_{2k+1}} \left(\frac{q^4}{1-q^2}\right)^k$$
$$\times \left\{ (4k+1)[q^{-2}(8k-2)+6k-3]T_{2k}(q) \right.$$
$$\left. -(2k-1)\left[\frac{2q-3q^3}{1-q^2}k + 2q^{-1}(4k+1)\right] T_{2k+1}(q) \right\}.$$

Coefficients of $T_{2k}(q)$ and $T_{2k+1}(q)$ are of order

$$\frac{1}{\sqrt{k}} \left(\frac{q^4}{16(1-q^2)}\right)^k.$$

For q very close to 1 (strictly, for $q \geqslant \sqrt{\sqrt{80}-8} \approx 0.9717$) the above series may be divergent, for $q < 4/\sqrt{17} \approx 0.9701$ it converges more rapidly than the previous one, and for $q \leqslant \sqrt{\sqrt{2.24}-0.8} \approx 0.8347$ adding a term to a partial sum increases its accuracy at least by one decimal digit.

Remark also that for $x \neq q$ passing from $\{f_j^{(m-1)}\}$ to $\{f_j^{(m)}\}$ can increase these elements about by the factor σ, where

$$\sigma := v + \mu(1-v) = \frac{-2q^{-1}x + 3 - q^2}{1 - 2qx + q^2}, \quad \sigma - 1 = \frac{2\left(q^{-1} - q\right)(q - x)}{1 - 2qx + q^2}.$$

It is important when we estimate advantages of transform (57).

Variant uvw**.** Finally, we consider a third definition of the auxiliary factors p_j:

$$p_j := u\left(\alpha_j + v\Delta\alpha_j + w\Delta^2\alpha_j\right).$$

Here u is defined as in (49) and parameters $v,\ w$ should be chosen in this way that α'_j depends only on differences $\Delta^k\alpha_j$ of possibly high order (as we may verify, the definition $p_j := u(\alpha_{j+1} + v\Delta\alpha_{j+1} + w\Delta^2\alpha_{j+1})$ is not so profitable). Elementary manipulations lead us to the formula

$$\alpha'_j = u\{\mu(1 - v + w)\Delta\alpha_{j-2} + [\lambda(1 - v + w) + \mu(1 - w)]\Delta\alpha_{j-1} \\ - (\lambda w + v - w)\Delta\alpha_j - w\Delta\alpha_{j+1}\}.$$

α'_j depends only on the differences $\Delta^2\alpha_j$ if the coefficients of this linear combination sum up to 0. This is the case for $v = u(\lambda + 2\mu)$. Then α'_j can be expressed by the differences $\Delta^3\alpha_j$ if

$$3\mu(1 - v + w) + 2[\lambda(1 - v + w) + \mu(1 - w)] - (\lambda w + v - w) = 0,$$

i.e. if $w = -(\lambda + 3\mu - \mu^2)u^2$. For such $u,\ v,\ w$

$$\alpha'_j = u\left[\mu(1 - v + w)\Delta^3\alpha_{j-2} - w\Delta^3\alpha_{j-1}\right].$$

Under known assumption $\mu = 1$ we have

$$u = \frac{1}{2+\lambda}, \quad v = 1, \quad w = -\frac{1}{2+\lambda}, \quad \alpha'_j = \frac{1}{(2+\lambda)^2}\,\Delta^4\alpha_{j-2},$$

$$p_j = \frac{1}{2+\lambda}\left(\alpha_{j+1} - \frac{1}{2+\lambda}\Delta^2\alpha_j\right)$$

and finally

$$\sum_{j=l}^{\infty}\alpha_j f_j = \frac{1}{(2+\lambda)^2}\left\{\left[(1+\lambda)_2\alpha_l - (2+\lambda)\Delta\alpha_l + \Delta^2\alpha_l\right] f_l \right. \\ \left. + \left[\lambda\alpha_l + 2\alpha_{l+1} - 2\Delta\alpha_{l+1} + \Delta^2\alpha_{l+1}\right] f_{l+1} + \sum_{j=l+2}^{\infty}\Delta^4\alpha_{j-2} f_j\right\}.$$

Remark that on the right coefficient of f_{l+2} depends among others on α_l. The same dependence concerns series which result by natural iterating the above relation. Hence a final transform, e.g. for the series with coefficients $\alpha_j = 1/(j+c)$ is equally inefficient as the original Euler–Knopp transform of the power series with the same coefficients [cf. Section 1, (8) and (9)]. For this reason in each iteration we should adjoin at least one term of the series on the right to the terms with f_l and f_{l+1} [transform (57) is constructed in the same way and even for $\mu = 1$ it differs from (51)]. A similar modification of aforementioned classical transform has in some particular cases following form:

$$\frac{1}{1-x}\sum_{k=0}^{\infty}\left(\frac{x^2}{1-x}\right)^k\left(\Delta^k\alpha_k+\Delta^{k+1}\alpha_k x\right),$$

$$\frac{1}{1-x}\sum_{k=0}^{\infty}\left(\frac{x^3}{1-x}\right)^k\left(\Delta^k\alpha_{2k}+\Delta^{k+1}\alpha_{2k}x+\Delta^{k+1}\alpha_{2k+1}x^2\right).$$

The Euler–Knopp transform and such its modifications for the series

$$\sum_{j=0}^{\infty}\frac{1}{j+1}\,x^j \tag{60}$$

converging to $-x^{-1}\log(1-x)$ for $-1 \leqslant x < 1$ give respectively the series

$$\frac{1}{1-x}\sum_{k=0}^{\infty}\frac{1}{k+1}\left(-\frac{x}{1-x}\right)^k,\quad \frac{2-x}{1-x}\sum_{k=0}^{\infty}\frac{k!}{(k+2)_{k+1}}\left(-\frac{x^2}{1-x}\right)^k,$$

$$\frac{1}{3(1-x)}\sum_{k=0}^{\infty}\frac{k!}{(2k+1)_{k+2}}\left(-\frac{x^3}{1-x}\right)^k\left\{3[3k+2-(k+1)x]-(2k+1)x^2\right\}.$$

The first of them converges for each $x \leqslant \frac{1}{2}$ (i.e. also for $x < -1$ although series (60) is there divergent), for $-1 \leqslant x < 0$ converges more rapidly than the original series and diverges for $x > \frac{1}{2}$. The second series converges for $-\sqrt{8}-2 \leqslant x \leqslant \sqrt{8}-2$ roughly as a geometrical series with the common ratio $-x^2/[4(1-x)]$ and for $-1 \leqslant x \leqslant \frac{1}{2}$ its each term gives one additional octal digit. The third series converges for $-3 \leqslant x \leqslant \xi$ where $\xi \approx 0.89410745$ is the unique real zero of the polynomial $4x^3+27x-27$.

Also for the orthogonal series distinct procedures lead to series having different properties. In particular a procedure recommended above gives the following result:

$$\begin{aligned}\sum_{j=1}^{\infty}\alpha_j f_j = \sum_{k=1}^{\infty}&\frac{1}{(2+\lambda)^{2k}}\\ &\times\Big\{\left[(1+\lambda)_2\Delta^{4k-4}\alpha_k-(2+\lambda)\Delta^{4k-3}\alpha_k+\Delta^{4k-2}\alpha_k\right]f_{3k-2}\\ &\quad+\left[\lambda\Delta^{4k-4}\alpha_k+2\Delta^{4k-4}\alpha_{k+1}-2\Delta^{4k-3}\alpha_{k+1}+\Delta^{4k-2}\alpha_{k+1}\right]f_{3k-1}\\ &\quad+\Delta^{4k}\alpha_k f_{3k}\Big\}.\end{aligned}$$

It is worth comparing it with (51). Here mth partial sum of the new series depends on $f_1, f_2, \ldots, f_{3m}$ and $\alpha_1, \alpha_2, \ldots, \alpha_{5k-1}$.

In particular for $c \neq -1, -2, \ldots$

$$\sum_{j=1}^{\infty} \frac{1}{j+c}\, f_j = \sum_{k=1}^{\infty} \frac{(4k-4)!}{(2+\lambda)^{2k}(k+c)_{4k}}$$

$$\times \Bigg\{ (5k+c-1)[(1+\lambda)_2(5k+c-3)_2$$

$$+ (2+\lambda)(4k-3)(5k+c-2) + (4k-3)_2]\, f_{3k-2}$$

$$+ [\lambda(5k+c-3)_3 + 2(k+c)(5k+c-2)_2$$

$$+ 2(4k-3)(k+c)(5k+c-1) + (4k-3)_2(k+c)]\, f_{3k-1}$$

$$+ \frac{(4k-3)_4}{5k+c}\, f_{3k} \Bigg\}.$$

Factor appearing here before {} is equal asymptotically to

$$\frac{5^{1-c}}{64k^3}\sqrt{\frac{\pi}{10k}}\left(\frac{256}{3125(2+\lambda)^2}\right)^k,$$

i.e. the last series is roughly a geometrical series with common ratio equal about to 0.0091 and 0.082 respectively for $\lambda = 1, -1$.

Example 4 The last relation implies that

$$\sum_{j=1}^{\infty} \frac{1}{j}\, T_j\left(-\frac{1}{2}\right) = -\frac{1}{10} \sum_{k=1}^{\infty} \frac{(6{,}293k^3 - 7{,}527k^2 + 2{,}698k - 282)(k-1)!}{9^k(4k-3)_{k+3}},$$

$$\sum_{j=1}^{\infty} \frac{1}{j}\, T_j\left(\frac{1}{2}\right) = -\frac{1}{10} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(483k^3 - 797k^2 + 358k - 42)(k-1)!}{(4k-3)_{k+3}}.$$

The first series (where $\lambda = 1$) has the sum $-\frac{1}{2}\log 3$. The partial sums of the series on the right with $k \leqslant 1, 2, 3$ give this value respectively with 2.42, 4.59, 6.71 significant decimal digits. The second series ($\lambda = -1$) has the sum 0. In this case analogous partial sums of the new series are equal to $\frac{1}{120}$, $-\frac{1}{1{,}680}$, $\frac{31}{720{,}720}$. Their behaviour agrees with the general informations given before the example.

As is mentioned in Section 1, summing of orthogonal series is troublesome because even the signs of f_j vary irregularly. This is the case for example if $f_j = T_j(x)$ and x is close to 1. The above defined transforms for such x and even very regular coefficients α_j (as $1/(j+c)$) are also weakly efficient. Oleksy [11, (4–7)] proposed a *preliminary* transform which for a trigonometrical series removes this drawback:

$$\sum_{j=1}^{\infty} \alpha_j \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} jx = 2\sum_{j=1}^{\infty} \alpha_{2j} \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} j(2x) - \sum_{j=1}^{\infty} \alpha_j \begin{Bmatrix}\cos\\ \sin\end{Bmatrix} j(x+\pi) \qquad (61)$$

(the sum $x+\pi$ may be replaced by the difference $x-\pi$). The essence of this transform is then passing from an argument x, for which summing of a series is particularly difficult, to "better" arguments $2x$ and $x+\pi$. For the Chebyshev series an analogous transform

$$\sum_{j=1}^{\infty} \alpha_j T_j(x) = 2\sum_{j=1}^{\infty} \alpha_{2j} T_j\big(T_2(x)\big) - \sum_{j=1}^{\infty} \alpha_j T_j(-x) \tag{62}$$

results from the relations $T_j\big(T_2(x)\big) = T_{2j}(x)$, $T_j(-x) = (-1)^j T_j(x)$. Identities (61) and (62) are trivial if all α_{2j} vanish. Otherwise the first series on the right in these identities may be transformed in the same way. After k iterations of such procedure we obtain a linear combination of $k+1$ series. Oleksy [11] suggests using a standard convergence acceleration method (e.g. the ε-algorithm or Levin's t-transform) to each of these series and not to the original series. He informs also for which k this process is the most reasonable [op. cit., (15)]. For the Chebyshev series with coefficients $\alpha_j = 1/(j+c)$ and x close to 1 optimal k is very large:

$$k := \left[\log_2 \frac{\pi}{\arccos x}\right] - 1. \tag{63}$$

The same is true for cosine or sine series with identical coefficients and for some x; these series are important in physics [op. cit., (27), (28)].

Series resulting from k-fold application of Oleksy's transform may be of course evaluated with the aid of reasonably chosen numerical methods from Section 3. For the simplest rational coefficients more recommended are analytical methods from this section. Let

$$S(c; x) := \sum_{j=1}^{\infty} (j+c)^{-1} T_j(x).$$

Identity (62) is here particularly simple:

$$S(c; x) = S\big(c/2; T_2(x)\big) - S(c; -x). \tag{64}$$

Series S converges very slowly for $x \in [-1, 1)$ and is divergent if $x = 1$. For each $x \in [-1, 1)$ the quasiorthogonal series $Q(c; x)$ defined as the right-hand side of (53) for $f_j := T_j(x)$ and $\lambda = -2x$ corresponds formally to S. This new series diverges for $x \geqslant \frac{25}{27} \approx 0.926$ and for another values of x it behaves roughly as a geometrical series with the common ratio $\varphi := 2/[27(1-x)]$. Therefore series $Q(c; x)$ converges very slowly for, say, $x = 0.9$, but, say, for $x = 0.5$ and first of all for $x < 0$ it converges rapidly. As $T_2(0.9) = 0.62$, for $x = 0.9$ the quasiorthogonal series $Q(c/2; T_2(x))$ and $Q(c; -x)$ converge sufficiently rapidly (we have respectively $\varphi = \frac{100}{513}$ and $\varphi = \frac{20}{513}$). Thus, by (64), in place to $S(c; 0.9)$ it is worthwhile to compute $Q(c/2; 0.62) - Q(c; -0.9)$.

Now let $x = 0.95$, from where $x \geqslant \frac{25}{27}$. In view of (64) one has $S(c; 0.95) = S(c/2; 0.805) - S(c; -0.95)$. Series $Q(c/2; 0.805)$ converges very slowly but the same identity gives us

$$S(c/2; 0.805) = S(c/4; 0.29605) - S(c/2; -0.805).$$

For both the series on the right corresponding quasiorthogonal series converge sufficiently rapidly. Therefore we use finally the relation

$$S(c; 0.95) = Q(c/4; 0.29605) - Q(c/2; -0.805) - Q(c; -0.95).$$

More generally, the value of $S(c, x)$ can be calculated as follows:

(1) If $x < 1/\sqrt{2}$, then we calculate $Q(c; x)$.
(2) Otherwise we iterate (as above) identity (64) k times, where k is defined in (63) and we replace each series S by the corresponding Q:

$$S(c; x) = Q\big(c/2^m; -T_{2^m}(x)\big) - \sum_{m=0}^{k-1} Q\big(c/2^m; -T_{2^m}(x)\big),$$

where $T_{2^k}(x) \leqslant 1/\sqrt{2}$ and all the $-T_{2^m}(x)$ are negative. Values $T_{2^m}(x)$ obviously are computed iteratively: $T_1(x) = x$, $T_{2^m}(x) = 2T_{2^{m-1}}(x) - 1$ $(m = 1, 2, \ldots)$.

Parameter k in (63) for $x = 1 - 10^{-l}$ and $l = 1, 2, \ldots, 6$ is equal respectively to $1, 3, 5, 6, 8, 10$. Then we know how series Q should be summed for such x close to 1, for which direct evaluating $S(c; x)$ is the most expensive. Suppose we wish evaluate the sum of each series Q with the absolute error less than 10^{-17}. In this case for $c = 0$ in each series $Q\big(c/2^m; -T_{2^m}(x)\big)$ it suffices to take 13 initial terms and in $Q\big(c/2^m; -T_{2^m}(x)\big)$ at most 27 terms. In view of (54) the cost of computation is still less for $c > 0$.

The above arguments remain in force when a series Q results from (58).

5 Equation (2) with variable coefficients

Suppose now that the parameters λ_j and μ_j of (2) are rational functions in j such that (25) holds and $\sigma := 1 + \lambda + \mu \neq 0$. This is the case for the Jacobi orthogonal series provided that $x \neq 1$; it is important also that for them $\mu = 1$. For the Laguerre and Hermite series λ_j and μ_j are rational but $\sigma = 0$.

It turns out that if the λ_j and μ_j depend on j, then factors p_j chosen similarly as in Section 4 do not decrease the coefficients α'_j (in comparison with α_j) as well as earlier. Furthermore, analytical iterating of transform (6) is here rather unfeasible even for the simplest p_j.

Variant u**.** We may reason as in Section 4 but in a definition of a factor p_j in (6) and (7) a quantity u should be now dependent of j. We take then $p_j := u_j\alpha_{j+1}$. Therefore

$$\alpha'_j = -\mu_{j-2}u_{j-2}\alpha_{j-1} + (1 - \lambda_{j-1}u_{j-1})\alpha_j - u_j\alpha_{j+1} \quad (j \geqslant 3). \tag{65}$$

If

$$1 - \mu_{j-2}u_{j-2} - \lambda_{j-1}u_{j-1} - u_j = 0,$$

i.e. the sequence $\{u_j\}$ satisfies a nonhomogeneous difference equation of the second order, then α'_j is expressed by the differences $\Delta\alpha_{j-1}$ and $\Delta\alpha_j$, and hence is small compared with α_j. Generally it is impossible to solve exactly this equation (identical with (19) for $\alpha_j = 1$). An approximate solution is as follows [cf. (49)]:

$$u_j = \frac{1}{1 + \lambda_j + \mu_j}. \tag{66}$$

Lemma 5 *If u_j is defined by (66), then*

$$\alpha'_j = \mu_{j-2} u_{j-2} \Delta\alpha_{j-1} - u_j \Delta\alpha_j - [\Delta u_{j-2} + \Delta u_{j-1} + \Delta(\lambda_{j-2} u_{j-2})]\alpha_j. \tag{67}$$

Thus, in comparison with the case of constant λ_j and μ_j, a combination of the differences $\Delta\alpha_{j-1}$ and $\Delta\alpha_j$ should be now decreased by the sum of differences of expressions dependent only of the parameters of difference equation (2), multiplied by α_j. Remark that α'_j depends on $\alpha_{j-1}, \alpha_j, \alpha_{j+1}$.

Proof (65) implies than

$$\alpha'_j = \mu_{j-2} u_{j-2} \Delta\alpha_{j-1} - u_j \Delta\alpha_j + (1 - \mu_{j-2} u_{j-2} - \lambda_{j-1} u_{j-1} - u_j)\alpha_j.$$

For u_j from (66) the factor multiplied by α_j may be expressed as in (67). □

For the Legendre series $\lambda_j u_j = \text{const}$ and formula (67) simplifies into

$$\alpha'_j = \frac{1}{1-x}\left[\frac{j}{2j-1}\,\Delta\alpha_{j-1} - \frac{j+1}{2j+3}\Delta\alpha_j - \frac{2}{(2j-1)(2j+3)}\,\alpha_j\right].$$

These new coefficients are rather complicated even for $\alpha_j := 1/(j+c)$:

$$\alpha'_j = -\frac{8j^2 + (6c+5)j + 2c^2 + c - 3}{(1-x)(2j-1)(2j+3)(j+c-1)_3}. \tag{68}$$

However, due to Lemma 5 one can prove that for some series (1) the coefficients α'_j from (7) are small compared to α_j. In such cases it is worth iterating transform (6).

Theorem 6 *Let $\sigma := 1 + \lambda + \mu \neq 0$, $\tau := (c_1 + d_1)/\sigma$. If α_j is such rational function in j that $\alpha_j = a_0 j^{-m} + a_1 j^{-m-1} + \ldots$ (m natural, $a_0 \neq 0$), that for the factors u_j defined in (66) α'_j is also a rational function in j and*

$$\begin{aligned}\alpha'_j = {} & \frac{1}{\sigma} m a_0 (1-\mu) j^{-m-1} \\ & + \frac{1}{\sigma}\Big\{[(m+1)a_1 - m\tau a_0](1-\mu) - m a_0 d_1 - \frac{1}{2}(m)_2 a_0 (1+\mu) \\ & \quad - a_0[(\lambda+2)\tau - c_1]\Big\} j^{-m-2} + \mathcal{O}\left(j^{-m-3}\right).\end{aligned}$$

If in (1) f_j is the jth Jacobi polynomial, then

$$\alpha'_j = -\frac{a_0}{2(1-x)}\,(m+1)^2 j^{-m-2} + \mathcal{O}(j^{-m-3}). \tag{69}$$

It was to be expected that also for λ_j, μ_j dependent of j and rational α_j the transformed series has the best properties for $\mu = 1$, at least in the sense that then $\alpha'_j = \mathcal{O}(j^{-m-2})$ (remark however that the coefficient of j^{-m-2} in α'_j is of order m^2).

An elementary proof can be omitted.

As in Section 4, passing from a series $\sum_{j=l}^{\infty} \alpha_j f_j$ to the series on the right in (6) can be repeated:

Algorithm 7 At least formally the equality

$$\sum_{j=1}^{\infty} \alpha_j f_j = \sum_{l=1}^{\infty} \beta_l f_l,$$

holds, where for $\alpha_j^{(0)} := \alpha_j$ $(j = 1, 2, \ldots)$ the coefficients β_l are recurrently computed by the following formulae used for $k = 1, 2, \ldots$:

$$\begin{aligned}
p_j^{(k-1)} &:= \frac{\alpha_{j+1}^{(k-1)}}{1 + \lambda_j + \mu_j} \quad (j \geqslant 2k-1),\\
\alpha_j^{(k)} &:= \alpha_j^{(k-1)} - \mu_{j-2} p_{j-2}^{(k-1)} - \lambda_{j-1} p_{j-1}^{(k-1)} - p_j^{(k-1)} \quad (j \geqslant 2k+1),\\
\beta_{2k-1} &:= \alpha_{2k-1}^{(k-1)} - p_{2k-1}^{(k-1)},\\
\beta_{2k} &:= \alpha_{2k}^{(k-1)} - \lambda_{2k-1} p_{2k-1}^{(k-1)} - p_{2k}^{(k-1)}.
\end{aligned}$$

To find the coefficients β_l for $l \leqslant 2k_{\max}$ we compute successively for $k = 1, 2, \ldots, k_{\max}$ row by row, from left to right, quantities

$$\begin{array}{llllllll}
\alpha_{3k-2}^{(0)} & p_{3k-3}^{(0)} & \alpha_{3k-3}^{(1)} & p_{3k-4}^{(1)} & \cdots & p_{2k-1}^{(k-2)} & \alpha_{2k-1}^{(k-1)} & \\
\alpha_{3k-1}^{(0)} & p_{3k-2}^{(0)} & \alpha_{3k-2}^{(1)} & p_{3k-3}^{(1)} & \cdots & p_{2k}^{(k-2)} & \alpha_{2k}^{(k-1)} & p_{2k-1}^{(k-1)} \ \beta_{2k-1} \\
\alpha_{3k}^{(0)} & p_{3k-1}^{(0)} & \alpha_{3k-1}^{(1)} & p_{3k-2}^{(1)} & \cdots & p_{2k+1}^{(k-2)} & \alpha_{2k+1}^{(k-1)} & p_{2k}^{(k-1)} \ \beta_{2k}
\end{array}$$

(among them there are the coefficients α_j of the original series for $j \leqslant 3k_{\max}$). They are stored in the arrays

$$\texttt{alpha}[1\,..\,3 \times k_{\max}], \quad \texttt{p}[0\,..\,2, 0\,..\,k_{\max} - 1]$$

($\alpha_m^{(k)} = \texttt{alpha}[m]$, $p_m^{(k)} = \texttt{p}[m \bmod 3, k]$). Algorithm 7 uses many times the coefficients λ_j, μ_j for $j \leqslant 3k_{\max} - 1$ and it is worth storing them. The quantities f_j for $j \leqslant 2k_{\max}$ are also needed.

The algorithm was verified, among others, for series (10–12). In three different cases, namely for $x \leqslant 0.1$, $x \leqslant 0.2$ and $x \leqslant -0.2$, the sum of each series after its transform was computed with at least 17 accurate digits. To this end coefficients β_l for at most $l \leqslant 32$ were used. Generally, evaluating this sum with maximal attainable accuracy requires more coefficients α_j than for methods from Section 3. On the other hand, the above algorithm has an important advantage: accuracy of results is more easily controllable. In fact, the coefficients β_l quite rapidly converge to 0 until they are perturbed by rounding errors. As $|P_j(x)| \leqslant 1$ [13, Section 7.21] one can recommend the following procedure:

(1) We finish computations when for some k

$$|\beta_{2k+1}| + |\beta_{2k+2}| \geqslant |\beta_{2k-1}| + |\beta_{2k}|,$$

(2) We accept the quantity $\sum_{l=1}^{2k} \beta_l f_l$ as the best approximant of the sum of a given series,
(3) We assume that left-hand side of above inequality gives us error of this approximant.

As for methods from Section 3, efficiency of Algorithm 7 decreases when x tends to a singular point of our series. For series (10) (diverging at $x = 1$) and $x = 0.2, 0.3, \ldots, 0.8$ probably the best approximant of its sum has respectively 15.5, 13.9, 14.3, 12.9, 11.3, 9.4, 7.5 exact digits. For series (11) (which very slowly converges at $x = 1$) accuracy is a little greater. Also for these x methods from Section 3 are more efficient.

Variant uv**.** Passing of the variant u to the uv consists in adjoining to α'_j from (65) the sum

$$S_j := -\mu_{j-2} u_{j-2} v_{j-2} \Delta\alpha_{j-1} - \lambda_{j-1} u_{j-1} v_{j-1} \Delta\alpha_j - u_j v_j \Delta\alpha_{j+1}.$$

If for certain v and integer n we have $v_j = v j^{-n} + \mathcal{O}(j^{-n-1})$, then

$$S_j \approx m v a_0 j^{-m-n-1}.$$

Variant uv is a priori reasonable if leads to a qualitative reducing of the coefficients α'_j. If f_j is the jth Jacobi polynomial one must then remove in expression (69) the term with j^{-m-2}, i.e. put

$$n = 1, \quad v = \frac{1}{2(1-x)} \frac{(m+1)^2}{m}.$$

In this case, however, iterating variant uv requires knowledge in each step of the value m (assumption that m systematically decreases by 3 not always is reasonable). The same concerns the case when $\mu \neq 1$. The after-mentioned (untypical) definition of v_j doesn't have this drawback.

We suppose again that f_j is the jth Jacobi polynomial, i.e. (69) holds. Let

$$p_j := u_j \left\{ \alpha_{j+1} + \frac{1}{\sigma^2} \left[\Delta \left(\frac{\alpha_j}{j+c} \right) - \Delta^2 \alpha_j \right] \right\} \quad \big(\sigma := 2(1-x),\ c > 0\big). \tag{70}$$

Denote by v_j the term added above to α_{j+1}. Its introducing increases α'_j from Th. 6 by $-\mu_{j-2} u_{j-2} v_{j-2} - \lambda_{j-1} u_{j-1} v_{j-1} - v_j$. As

$$\Delta \left(\frac{\alpha_j}{j+c} \right) = -(m+1) a_0 j^{-m-2} + \ldots, \quad \Delta^2 \alpha_j = (m)_2 a_0 j^{-m-2} + \ldots,$$

α'_j increases by $(a_0/\sigma)(m+1)^2 j^{-m-2} + \ldots$ and now is equal (for any c) to $\mathcal{O}(j^{-m-3})$. Effects of this modification can be estimated for the Legendre series

$$\sum_{j=1}^{\infty} \frac{1}{j+1} P_j\left(\frac{1}{2}\right).$$

The coefficients α'_j in the variant u result here from (68) for $c = 1$ and $x = \frac{1}{2}$; two values of them are given below in the first row. The second one contains the analogous coefficients in variant uv:

$$\alpha'_6 = -\frac{59}{4620} \approx -0.01277, \quad \alpha'_7 = -\frac{67}{7956} \approx -0.008421,$$
$$\alpha'_6 = \frac{73}{20790} \approx 0.003511, \quad \alpha'_7 = \frac{37}{41580} \approx 0.0008899.$$

However, please note that in the second variant α'_j depends, by (70), on five coefficients of the original series, namely $\alpha_{j-2}, \ldots, \alpha_{j+2}$.

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