Numer Algor (2007) 46:141–151
DOI 10.1007/s11075-007-9132-0

ORIGINAL PAPER

A generalization of Euler's constant

Alina Sîntămărian

Received: 28 February 2007 / Accepted: 29 September 2007 /
Published online: 31 October 2007

Abstract The purpose of this paper is to evaluate the limit $\gamma(a)$ of the sequence $\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right)_{n\in\mathbb{N}}$, where $a\in(0,+\infty)$.

Keywords Sequence · Convergence · Euler's constant · Approximation · Series

Mathematics Subject Classifications (2000) 11Y60 · 40A05

1 Introduction

Let $(D_n)_{n\in\mathbb{N}}$ be the sequence defined by $D_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n$, for each $n\in\mathbb{N}$. It is well-known that the sequence $(D_n)_{n\in\mathbb{N}}$ is convergent and its limit, usually denoted by γ, is called Euler's constant.

For $D_n-\gamma$, $n\in\mathbb{N}$, many lower and upper estimates have been obtained in the literature. We remind some of them:

$$\frac{1}{2(n+1)} < D_n-\gamma < \frac{1}{2(n-1)},\ \text{for each } n\in\mathbb{N}\setminus\{1\}\ ([14]);$$

$$\frac{1}{2(n+1)} < D_n-\gamma < \frac{1}{2n},\ \text{for each } n\in\mathbb{N}\ ([10, 20]);$$

$$\frac{1}{2n+1} < D_n-\gamma < \frac{1}{2n},\ \text{for each } n\in\mathbb{N}\ ([17]);$$

A. Sîntămărian (✉)
Department of Mathematics, Technical University of Cluj-Napoca,
Str. C. Daicoviciu nr. 15, 400020 Cluj-Napoca, Romania
e-mail: Alina.Sintamarian@math.utcluj.ro

$$\frac{1}{2n+\frac{2}{5}} < D_n - \gamma < \frac{1}{2n+\frac{1}{3}}, \text{ for each } n \in \mathbb{N} \text{ ([15, 16])};$$

$$\frac{1}{2n+\frac{2\gamma-1}{1-\gamma}} \le D_n - \gamma < \frac{1}{2n+\frac{1}{3}}, \text{ for each } n \in \mathbb{N}$$ ([16, Editorial comment], [1]).

We notice that some approximation formulae, which lead to geometric convergence to γ, were given in [3, 5, 6].

In Section 2 we present a generalization of Euler's constant as limit of the sequence $\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right)_{n\in\mathbb{N}}$ and we denote this limit by $\gamma(a)$. In Section 3 we give some sequences that converge quickly to $\gamma(a)$.

The following theorem, which we shall need further on, is a Stolz–Cesaro theorem proved by I. Rizzoli in [11].

Theorem 1.1 *Let* $(a_n)_{n\in\mathbb{N}}$ *and* $(b_n)_{n\in\mathbb{N}}$ *be two sequences of real numbers with the following properties*:

(i) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$;
(ii) *the sequence* $(b_n)_{n\in\mathbb{N}}$ *is strictly decreasing* (*or strictly increasing*);
(iii) *there exists* $\lim\limits_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = l \in \overline{\mathbb{R}}$.

Then there exists the limit of the sequence $\left(\frac{a_n}{b_n}\right)_{n\in\mathbb{N}}$ *and* $\lim\limits_{n\to\infty}\frac{a_n}{b_n} = l$.

2 The number $\gamma(a)$

It is known that the sequence $\left(\frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}\right)_{n\in\mathbb{N}}$, where $a\in(0,+\infty)$, is convergent (see for example Knopp [7, p. 453] and Nedelcu [9], where problems in this sense were proposed; [4, 8, 13]).

The results contained in the following theorem were given by A. Sîntămărian (submitted for publication).

Theorem 2.1 *Let* $a\in(0,+\infty)$. *We consider the sequences* $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ *and* $(z_n)_{n\in\mathbb{N}}$ *defined by*

$$x_n = \frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n}{a},$$

$$y_n = \frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{a+n-1}{a}$$

and

$$z_n = \frac{x_n+y_n}{2} = \frac{1}{a}+\frac{1}{a+1}+\cdots+\frac{1}{a+n-1}-\ln\frac{\sqrt{(a+n-1)(a+n)}}{a},$$

for each $n\in\mathbb{N}$.

Then:

(i) *the sequences* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are convergent to the same number, which we denote by* $\gamma(a)$*, and satisfy the inequalities* $x_n < x_{n+1} < \gamma(a) < y_{n+1} < y_n$*, for each* $n \in \mathbb{N}$;

(ii) $0 < \frac{1}{a} - \ln\left(1+\frac{1}{a}\right) < \gamma(a) < \frac{1}{a}$;

(iii) $\lim\limits_{n\to\infty} n(\gamma(a) - x_n) = \frac{1}{2}$, $\lim\limits_{n\to\infty} n(y_n - \gamma(a)) = \frac{1}{2}$ *and* $\lim\limits_{n\to\infty} n^2(z_n - \gamma(a)) = \frac{1}{6}$.

Remark 2.1 The sequence $(y_n)_{n\in\mathbb{N}}$ from Theorem 2.1, for $a = 1$, becomes the sequence $(D_n)_{n\in\mathbb{N}}$, so $\gamma(1) = \gamma$.

Theorem 2.2 *The function* $\gamma : (0, +\infty) \to (0, +\infty)$*, defined by*

$$\gamma(a) = \lim_{n\to\infty}\left(\frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\frac{a+n-1}{a}\right),$$

for each $a \in (0, +\infty)$*, is decreasing.*

Proof Let $n \in \mathbb{N}$. We consider the function $h : (0, +\infty) \to (0, +\infty)$, defined by

$$h(a) = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\frac{a+n-1}{a},$$

for each $a \in (0, +\infty)$. We have

$$\begin{aligned} h'(a) &= -\sum_{k=1}^{n}\frac{1}{(a+k-1)^2} + \frac{n-1}{a(a+n-1)} \\ &< -\sum_{k=1}^{n}\frac{1}{(a+k-1)(a+k)} + \frac{n-1}{a(a+n-1)} \\ &= -\sum_{k=1}^{n}\left(\frac{1}{a+k-1} - \frac{1}{a+k}\right) + \frac{n-1}{a(a+n-1)} \\ &= -\left(\frac{1}{a} - \frac{1}{a+n}\right) + \frac{n-1}{a(a+n-1)} = -\frac{1}{(a+n-1)(a+n)} < 0, \end{aligned}$$

for each $a \in (0, +\infty)$. It follows that the function h is strictly decreasing, so the function $\gamma(\cdot)$ is decreasing. □

In a similar way as in the proof of Theorem 3.2, which we shall give in Section 3, the following theorem can be obtained.

Theorem 2.3 *Let* $a \in (0, +\infty)$*. We consider the sequence* $(u_n)_{n\in\mathbb{N}}$ *defined by* $u_n = y_n - \frac{1}{2(a+n-1)+\frac{1}{3}}$*, for each* $n \in \mathbb{N}$*, where* $(y_n)_{n\in\mathbb{N}}$ *is the sequence from the enunciation of Theorem* 2.1*. Also, we specify that* $\gamma(a)$ *is the limit of the sequence* $(y_n)_{n\in\mathbb{N}}$.

Then:

(i) $u_n < u_{n+1} < \gamma(a)$, *for each* $n \in \mathbb{N} \setminus \{1\}$, *and*

$$\lim_{n\to\infty} n^3(\gamma(a) - u_n) = \frac{1}{72};$$

(ii) $\frac{1}{2(a+n-1)+\frac{11}{28}} < y_n - \gamma(a) < \frac{1}{2(a+n-1)+\frac{1}{3}}$, *for each* $n \in \mathbb{N} \setminus \{1\}$.

Remark 2.2 The lower estimate from part (ii) of Theorem 2.3 holds for $n = 1$ too.

Remark 2.3 The second limit from part (iii) of Theorem 2.1 follows from part (ii) of Theorem 2.3 as well.

Let $a \in (0, +\infty)$. In the following theorem we give a representation of $\gamma(a) - x_n$, for each $n \in \mathbb{N}$. One of the reviewers of the paper indicated to the author that this representation was mentioned by C. Elsner [4, p. 446], using different notations than ours.

Theorem 2.4 *Let* $a \in (0, +\infty)$ *and* $b_m = -\frac{1}{m}\int_0^1 (0-t)(1-t)\cdots(m-1-t)\,\mathrm{d}t$, *for each* $m \in \mathbb{N}$. *We specify that* $\gamma(a)$ *is the limit of the sequence* $(x_n)_{n\in\mathbb{N}}$ *from the enunciation of Theorem* 2.1.

Then

$$\gamma(a) - x_n = \sum_{k=1}^{\infty} \frac{b_k}{(a+n)(a+n+1)\cdots(a+n+k-1)},$$

for each $n \in \mathbb{N}$.

Remark 2.4 Taking $a = 1$ in Theorem 2.4, we obtain that

$$\gamma - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n+1)\right) = \sum_{k=1}^{\infty} \frac{b_k}{(n+1)(n+2)\cdots(n+k)},$$

for each $n \in \mathbb{N}$. This is a result proved by J. Ser [12]. See also Yingying [19].

Corollary 2.1 *Let* $b_m = -\frac{1}{m}\int_0^1 (0-t)(1-t)\cdots(m-1-t)\,\mathrm{d}t$, *for each* $m \in \mathbb{N}$. *We specify that* $\gamma : (0, +\infty) \to (0, +\infty)$ *is the function from Theorem* 2.2.

Then

$$\gamma(n) = \sum_{k=1}^{\infty} \frac{b_k}{n(n+1)\cdots(n+k-1)},$$

for each $n \in \mathbb{N} \setminus \{1\}$.

Proof From Theorem 2.4 we have that

$$\gamma(n) = \gamma(1) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1} - \ln n\right) = \sum_{k=1}^{\infty} \frac{b_k}{n(n+1)\cdots(n+k-1)},$$

for each $n \in \mathbb{N} \setminus \{1\}$. □

3 Sequences convergent to $\gamma(a)$

D. W. DeTemple [2] proved that

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2},$$

for each $n \in \mathbb{N}$, where $R_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right)$, for each $n \in \mathbb{N}$. This result was refined by E. A. Karatsuba [6].

Having in view the results presented in DeTemple [2], we give the following two theorems for the generalization of Euler's constant $\gamma(a)$. The proof of the first theorem can be made in a similar way as the proof of the second one.

Theorem 3.1 *Let $a \in (0, +\infty)$. We consider the sequences $(\lambda_n)_{n\in\mathbb{N}}$ and $(\mu_n)_{n\in\mathbb{N}}$ defined by*

$$\lambda_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + \frac{1}{2a}\right)$$

and

$$\mu_n = \lambda_n - \frac{1}{24\left(a+n-\frac{1}{2}\right)^2},$$

for each $n \in \mathbb{N}$. Also, we specify that $\gamma(a)$ is the limit of the sequence $(y_n)_{n\in\mathbb{N}}$ from the enunciation of Theorem 2.1.

Then:

(i) $\gamma(a) < \lambda_{n+1} < \lambda_n$, *for each $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} n^2(\lambda_n - \gamma(a)) = \frac{1}{24};$$

(ii) $\mu_n < \mu_{n+1} < \gamma(a)$, *for each $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} n^4(\gamma(a) - \mu_n) = \frac{7}{960};$$

(iii) $\frac{1}{24(a+n)^2} < \lambda_n - \gamma(a) < \frac{1}{24\left(a+n-\frac{1}{2}\right)^2}$, *for each $n \in \mathbb{N}$.*

Remark 3.1 For $a = 1$, the lower estimate from part (iii) of Theorem 3.1 is the lower estimate given by D. W. DeTemple [2, Theorem] and the upper estimate from part (iii) of Theorem 3.1 is finer than that given in DeTemple [2, Theorem].

Remark 3.2 The following limits:

$$\lim_{n\to\infty} n^6 \left(\mu_n + \frac{7}{960\left(a+n-\frac{1}{2}\right)^4} - \gamma(a) \right) = \frac{31}{8064}$$

and

$$\lim_{n\to\infty} n^8 \left(\gamma(a) - \left(\mu_n + \frac{7}{960\left(a+n-\frac{1}{2}\right)^4} - \frac{31}{8064\left(a+n-\frac{1}{2}\right)^6} \right) \right) = \frac{127}{30720},$$

where $(\mu_n)_{n\in\mathbb{N}}$ is the sequence from the enunciation of Theorem 3.1, can be obtained in a similar way as in the proof of Theorem 3.2, which we shall give next.

Theorem 3.2 *Let* $a \in (0, +\infty)$. *We consider the sequences* $(\alpha_n)_{n\in\mathbb{N}}$ *and* $(\beta_n)_{n\in\mathbb{N}}$ *defined by*

$$\alpha_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + \frac{1}{2a}\right) - \frac{1}{24\left(a+n-\frac{1}{2}\right)^2} + \frac{7}{960\left(a+n-\frac{1}{2}\right)^4} - \frac{31}{8064\left(a+n-\frac{1}{2}\right)^6} + \frac{127}{30720\left(a+n-\frac{1}{2}\right)^8}$$

and

$$\beta_n = \alpha_n - \frac{511}{67584\left(a+n-\frac{1}{2}\right)^{10}},$$

for each $n \in \mathbb{N}$. *Also, we specify that* $\gamma(a)$ *is the limit of the sequence* $(y_n)_{n\in\mathbb{N}}$ *from the enunciation of Theorem* 2.1.

Then:

(i) $\gamma(a) < \alpha_{n+1} < \alpha_n$, *for each* $n \in \mathbb{N}$, *and*

$$\lim_{n\to\infty} n^{10}(\alpha_n - \gamma(a)) = \frac{511}{67584};$$

(ii) $\beta_n < \beta_{n+1} < \gamma(a)$, *for each* $n \in \mathbb{N}$, *and*

$$\lim_{n\to\infty} n^{12}(\gamma(a) - \beta_n) = \frac{1414477}{67092480};$$

(iii) $\frac{511}{67584(a+n)^{10}} < \alpha_n - \gamma(a) < \frac{511}{67584\left(a+n-\frac{1}{2}\right)^{10}}$, *for each* $n \in \mathbb{N}$.

Proof (i) We have

$$\alpha_{n+1}-\alpha_n=\frac{1}{a+n}-\ln\left(\frac{a+n}{a}+\frac{1}{2a}\right)+\ln\left(\frac{a+n-1}{a}+\frac{1}{2a}\right)$$
$$-\frac{1}{24\left(a+n+\frac{1}{2}\right)^2}+\frac{1}{24\left(a+n-\frac{1}{2}\right)^2}+\frac{7}{960\left(a+n+\frac{1}{2}\right)^4}$$
$$-\frac{7}{960\left(a+n-\frac{1}{2}\right)^4}-\frac{31}{8064\left(a+n+\frac{1}{2}\right)^6}+\frac{31}{8064\left(a+n-\frac{1}{2}\right)^6}$$
$$+\frac{127}{30720\left(a+n+\frac{1}{2}\right)^8}-\frac{127}{30720\left(a+n-\frac{1}{2}\right)^8},$$

for any $n\in\mathbb{N}$.

We consider the function $f:[1,+\infty)\to\mathbb{R}$, defined by

$$f(x)=\frac{1}{a+x}-\ln\left(\frac{a+x}{a}+\frac{1}{2a}\right)+\ln\left(\frac{a+x-1}{a}+\frac{1}{2a}\right)$$
$$-\frac{1}{24\left(a+x+\frac{1}{2}\right)^2}+\frac{1}{24\left(a+x-\frac{1}{2}\right)^2}+\frac{7}{960\left(a+x+\frac{1}{2}\right)^4}$$
$$-\frac{7}{960\left(a+x-\frac{1}{2}\right)^4}-\frac{31}{8064\left(a+x+\frac{1}{2}\right)^6}+\frac{31}{8064\left(a+x-\frac{1}{2}\right)^6}$$
$$+\frac{127}{30720\left(a+x+\frac{1}{2}\right)^8}-\frac{127}{30720\left(a+x-\frac{1}{2}\right)^8},$$

for each $x\in[1,+\infty)$. We have

$$f'(x)=-\frac{1}{(a+x)^2}-\frac{2}{2(a+x)+1}+\frac{2}{2(a+x)-1}+\frac{2}{3[2(a+x)+1]^3}$$
$$-\frac{2}{3[2(a+x)-1]^3}-\frac{14}{15[2(a+x)+1]^5}+\frac{14}{15[2(a+x)-1]^5}$$
$$+\frac{62}{21[2(a+x)+1]^7}-\frac{62}{21[2(a+x)-1]^7}$$
$$-\frac{254}{15[2(a+x)+1]^9}+\frac{254}{15[2(a+x)-1]^9},$$

for any $x\in[1,+\infty)$.

Set $y := a + x$. We have

$$F(y) := -\frac{1}{y^2} - \frac{2}{2y+1} + \frac{2}{2y-1} + \frac{2}{3(2y+1)^3} - \frac{2}{3(2y-1)^3}$$
$$-\frac{14}{15(2y+1)^5} + \frac{14}{15(2y-1)^5} + \frac{62}{21(2y+1)^7}$$
$$-\frac{62}{21(2y-1)^7} - \frac{254}{15(2y+1)^9} + \frac{254}{15(2y-1)^9}$$
$$= \frac{22892800y^8 + 6767872y^6 + 518368y^4 - 368y^2 + 105}{105y^2(2y-1)^9(2y+1)^9}.$$

Taking into account that $f'(x) = F(a+x)$, for each $x \in [1, +\infty)$, it follows that $f'(x) > 0$, for any $x \in [1, +\infty)$. So, the function f is strictly increasing on $[1, +\infty)$. Also, we have that $\lim_{x\to\infty} f(x) = 0$. Therefore $f(x) < 0$, for any $x \in [1, +\infty)$, which means that $\alpha_{n+1} - \alpha_n < 0$, for each $n \in \mathbb{N}$, i.e. the sequence $(\alpha_n)_{n\in\mathbb{N}}$ is strictly decreasing. We have $\lim_{n\to\infty} \alpha_n = \gamma(a)$. Now we can write that $\gamma(a) < \alpha_{n+1} < \alpha_n$, for any $n \in \mathbb{N}$.

We have

$$\lim_{n\to\infty} \frac{\alpha_{n+1} - \gamma(a) - (\alpha_n - \gamma(a))}{\frac{1}{(n+1)^{10}} - \frac{1}{n^{10}}} = \lim_{n\to\infty} \frac{\alpha_{n+1} - \alpha_n}{\frac{1}{(n+1)^{10}} - \frac{1}{n^{10}}} = \frac{511}{67584},$$

taking into account that

$$\lim_{x\to\infty} \frac{f(x)}{\frac{1}{(x+1)^{10}} - \frac{1}{x^{10}}} = \lim_{x\to\infty} \frac{f'(x)}{-\frac{10}{(x+1)^{11}} + \frac{10}{x^{11}}} = \lim_{x\to\infty} \frac{x^{11}(x+1)^{11} f'(x)}{10[(x+1)^{11} - x^{11}]} = \frac{511}{67584}.$$

Now, according to the Stolz–Cesaro theorem (Theorem 1.1), we can write that

$$\lim_{n\to\infty} n^{10}(\alpha_n - \gamma(a)) = \lim_{n\to\infty} \frac{\alpha_n - \gamma(a)}{\frac{1}{n^{10}}}$$
$$= \lim_{n\to\infty} \frac{\alpha_{n+1} - \gamma(a) - (\alpha_n - \gamma(a))}{\frac{1}{(n+1)^{10}} - \frac{1}{n^{10}}} = \frac{511}{67584}.$$

(ii) We have

$$\beta_{n+1} - \beta_n = \alpha_{n+1} - \alpha_n - \frac{511}{67584\left(a+n+\frac{1}{2}\right)^{10}} + \frac{511}{67584\left(a+n-\frac{1}{2}\right)^{10}},$$

for any $n \in \mathbb{N}$.

We consider the function $g : [1, +\infty) \to \mathbb{R}$, defined by

$$g(x) = f(x) - \frac{511}{67584\left(a+x+\frac{1}{2}\right)^{10}} + \frac{511}{67584\left(a+x-\frac{1}{2}\right)^{10}},$$

for each $x \in [1, +\infty)$. We have

$$g'(x) = f'(x) + \frac{5110}{33[2(a+x)+1]^{11}} - \frac{5110}{33[2(a+x)-1]^{11}},$$

for any $x \in [1, +\infty)$.

Set $y := a + x$. We have

$$\begin{aligned} G(y) &:= F(y) + \frac{5110}{33(2y+1)^{11}} - \frac{5110}{33(2y-1)^{11}} \\ &= -\big(15932668928y^{10} + 10828992768y^{8} + 1859890560y^{6} \\ &\quad +72941088y^{4} + 370988y^{2} - 1155\big)/\big[1155y^{2}(2y-1)^{11}(2y+1)^{11}\big]. \end{aligned}$$

Taking into account that $g'(x) = G(a+x)$, for each $x \in [1, +\infty)$, it follows that $g'(x) < 0$, for any $x \in [1, +\infty)$. So, the function g is strictly decreasing on $[1, +\infty)$. Also, we have that $\lim_{x\to\infty} g(x) = 0$. Therefore $g(x) > 0$, for any $x \in [1, +\infty)$, which means that $\beta_{n+1} - \beta_n > 0$, for each $n \in \mathbb{N}$, i.e. the sequence $(\beta_n)_{n\in\mathbb{N}}$ is strictly increasing. We have $\lim_{n\to\infty} \beta_n = \gamma(a)$. Now we can write that $\beta_n < \beta_{n+1} < \gamma(a)$, for any $n \in \mathbb{N}$.

We have

$$\lim_{n\to\infty} \frac{\gamma(a) - \beta_{n+1} - (\gamma(a) - \beta_n)}{\frac{1}{(n+1)^{12}} - \frac{1}{n^{12}}} = \lim_{n\to\infty} \frac{-(\beta_{n+1} - \beta_n)}{\frac{1}{(n+1)^{12}} - \frac{1}{n^{12}}} = \frac{1414477}{67092480},$$

taking into account that

$$\lim_{x\to\infty} \frac{-g(x)}{\frac{1}{(x+1)^{12}} - \frac{1}{x^{12}}} = \lim_{x\to\infty} \frac{-g'(x)}{-\frac{12}{(x+1)^{13}} + \frac{12}{x^{13}}} = \lim_{x\to\infty} \frac{-x^{13}(x+1)^{13}g'(x)}{12[(x+1)^{13} - x^{13}]} = \frac{1414477}{67092480}.$$

Now, according to the Stolz–Cesaro theorem (Theorem 1.1), we can write that

$$\begin{aligned} \lim_{n\to\infty} n^{12}(\gamma(a) - \beta_n) &= \lim_{n\to\infty} \frac{\gamma(a) - \beta_n}{\frac{1}{n^{12}}} \\ &= \lim_{n\to\infty} \frac{\gamma(a) - \beta_{n+1} - (\gamma(a) - \beta_n)}{\frac{1}{(n+1)^{12}} - \frac{1}{n^{12}}} = \frac{1414477}{67092480}. \end{aligned}$$

(iii) From part (ii) we have that $\alpha_n - \gamma(a) < \frac{511}{67584\left(a+n-\frac{1}{2}\right)^{10}}$, for any $n \in \mathbb{N}$.

Let $(\delta_n)_{n\in\mathbb{N}}$ be the sequence defined by $\delta_n = \alpha_n - \frac{511}{67584(a+n)^{10}}$, for each $n \in \mathbb{N}$. In a similar way as in the proof of part (i) we obtain that $\gamma(a) < \delta_n$, for any $n \in \mathbb{N}$. We specify that, in this case, the corresponding function F is the function H, defined as follows

$$H(y) := F(y) + \frac{2555}{33792(y+1)^{11}} - \frac{2555}{33792y^{11}}.$$

It can be written in the form

$$H(y) = \frac{y(y-1)\,P_{25}(y) + 692581276177794y + 89425}{1182720y^{11}(y+1)^{11}(2y-1)^9(2y+1)^9},$$

where $P_{25}(y)$ is a polynomial of degree 25 in y, with positive coefficients. □

Remark 3.3 The limit from part (i) of Theorem 3.2 follows from part (iii) of Theorem 3.2 as well.

Remark 3.4 The general term of the sequence $(\beta_n)_{n\in\mathbb{N}}$, from Theorem 3.2, can be written in the form

$$\beta_n = \frac{1}{a} + \frac{1}{a+1} + \cdots + \frac{1}{a+n-1} - \ln\left(\frac{a+n-1}{a} + \frac{1}{2a}\right)$$

$$-\sum_{k=1}^{5} \frac{(2^{2k-1}-1)\,B_{2k}}{2^{2k}k\left(a+n-\frac{1}{2}\right)^{2k}},$$

where B_{2k} is the Bernoulli number of index $2k$.

Corollary 3.1 *We consider the sequences $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ defined by*

$$v_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right) - \frac{1}{24\left(n+\frac{1}{2}\right)^2}$$

$$+\frac{7}{960\left(n+\frac{1}{2}\right)^4} - \frac{31}{8064\left(n+\frac{1}{2}\right)^6} + \frac{127}{30720\left(n+\frac{1}{2}\right)^8}$$

and

$$w_n = v_n - \frac{511}{67584\left(n+\frac{1}{2}\right)^{10}},$$

for each $n \in \mathbb{N}$. Also, we specify that γ is the limit of the sequence $\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)_{n\in\mathbb{N}}$.

Then:

(i) $\gamma < v_{n+1} < v_n$, *for each $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} n^{10}(v_n - \gamma) = \frac{511}{67584};$$

(ii) $w_n < w_{n+1} < \gamma$, *for each $n \in \mathbb{N}$, and*

$$\lim_{n\to\infty} n^{12}(\gamma - w_n) = \frac{1414477}{67092480};$$

(iii) $\frac{511}{67584(n+1)^{10}} < v_n - \gamma < \frac{511}{67584\left(n+\frac{1}{2}\right)^{10}}$, *for each $n \in \mathbb{N}$.*

Proof We take $a = 1$ in Theorem 3.2. □

Acknowledgements The author expresses her gratitude to the referees for the remarks made after the careful reading of the manuscript.

References

1. Alzer, H.: Inequalities for the gamma and polygamma functions. Abh. Math. Semin. Univ. Hamb. **68**, 363–372 (1998)
2. DeTemple, D.W.: A quicker convergence to Euler's constant. Am. Math. Mon. **100**(5), 468–470 (1993)
3. Elsner, C.: On a sequence transformation with integral coefficients for Euler's constant. Proc. Am. Math. Soc. **123**(5), 1537–1541 (1995)
4. Elsner, C.: On a sequence transformation with integral coefficients for Euler's constant, II. J. Number Theory **124**, 442–453 (2007)
5. Hessami Pilehrood, Kh., Hessami Pilehrood, T.: Arithmetical properties of some series with logarithmic coefficients. Math. Z. **255**, 117–131 (2007)
6. Karatsuba, E.A.: On the computation of the Euler constant γ. Numer. Algorithms **24**, 83–97 (2000)
7. Knopp, K.: Theory and Application of Infinite Series. Blackie, London (1951)
8. Lehmer, D.H.: Euler constants for arithmetical progressions. Acta Arith. **27**, 125–142 (1975)
9. Nedelcu, I.: Problem 21753. Gaz. Mat. Ser. B **94**(4), 136 (1989)
10. Rippon, P.J.: Convergence with pictures. Am. Math. Mon. **93**(6), 476–478 (1986)
11. Rizzoli, I.: O teoremă Stolz–Cesaro (A Stolz–Cesaro theorem). Gaz. Mat. Ser. B **95**(10-11-12), 281–284 (1990)
12. Ser, J.: Questions 5561, 5562, L'Intermédiaire des Mathématiciens. Série 2, **4**, 126–127 (1925)
13. Tasaka, T.: Note on the generalized Euler constants. Math. J. Okayama Univ. **36**, 29–34 (1994)
14. Tims, S.R., Tyrrell, J.A.: Approximate evaluation of Euler's constant. Math. Gaz. **55**(391), 65–67 (1971)
15. Tóth, L.: Problem E3432. Am. Math. Mon. **98**(3), 264 (1991)
16. Tóth, L.: Problem E3432 (Solution). Am. Math. Mon. **99**(7), 684–685 (1992)
17. Vernescu, A.: Ordinul de convergenţă al şirului de definiţie al constantei lui Euler (The convergence order of the definition sequence of Euler's constant). Gaz. Mat. Ser. B **88** (10–11), 380–381 (1983)
18. Whittaker, E.T., Watson, G.N.: A Course of Modern Analysis. Cambridge University Press, Cambridge (1996)
19. Yingying, Li: On Euler's constant—calculating sums by integrals. Am. Math. Mon. **109**(9), 845–850 (2002)
20. Young, R.M.: Euler's constant. Math. Gaz. **75**(472), 187–190 (1991)