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An exponential regula falsi method for solving nonlinear equations

Jinhai Chen [a,b,*] and Weiguo Li [b]

[a] *Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong*
E-mail: cjh_maths@yahoo.com.cn, cjh_maths@sina.com
[b] *School of Mathematics and Computational Sciences, University of Petroleum, Dongying 257061 Shandong Province, People's Republic of China*

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A class of modified regula falsi iterative formulae for solving nonlinear equations is presented in this paper. This method is shown to be quadratically convergent for the sequence of diameters and the sequence of iterative points. The numerical experiments show that new method is effective and comparable to well-known methods.

Keywords: regula falsi method, nonlinear equations, quadratic convergence

AMS subject classification: 65H05, 65H10

1. Introduction

We consider a class of iterative method for computing approximate solutions of the nonlinear equation

$$f(x) = 0 \tag{1}$$

where $f(x)$ is real value function.

It is well known that the classical regula falsi method (see [1]) finds a simple root of the nonlinear equation (1) by repeated linear interpolation between the current bracketing estimates. There is a distinct shortcoming, however; one endpoint is retained step after step, whenever a concave or convex region of $f(x)$ has been reached. Furthermore, the asymptotic convergence rate of the sequence $\{(x_n - x^*)\}$ is low as well.

Although some modifications aimed at overcoming these difficulties have been offered (see [4–6]), most of them do not have superlinearly or quadratically

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* Corresponding author.

asymptotic convergence properties for the diameters $\{(b_n - a_n)\}_{n=1}^{\infty}$ (see [2, 3]). In the new modified method given in this paper, we employ an exponential iterative method for accelerating the convergence after using the classical regula falsi method, in order that both the sequence of diameters $\{(b_n - a_n)\}_{n=1}^{\infty}$ and the sequence of iterations $\{(x_n - x^*)\}_{n=1}^{\infty}$ produced by the new method converge quadratically to zero.

2. A class of quadratically convergent exponential iterative formula

The class of iteration formula under consideration has the following form

$$x_{n+1} = x_n \exp\left\{-\frac{f^2(x_n)}{x_n(f(x_n) - f(x_n - f(x_n)))}\right\}, \quad n = 0, 1, 2, \cdots. \tag{2}$$

Theorem 1. Assume that $f(x^*) = 0$ and $U(x^*)$ to be a sufficiently small neighborhood of x^*, $x^* \neq 0$. Let $f''(x)$ exist in $U(x^*)$ and $f'(x) \neq 0$ in $U(x^*)$. Then the sequence $\{x_n\}$ produced by the iterative formula (2) is at least quadratically convergent.

Proof. First, the iterative function of formula (2) can be expressed in the following form

$$\begin{aligned}\varphi(x) &= x \, \exp\left\{-\frac{f^2(x)}{x(+f(x) - f(x - f(x)))}\right\} \\ &= x \, \exp\left\{-\frac{f(x)}{x\left(+\frac{f(x-f(x))-f(x)}{-f(x)}\right)}\right\}.\end{aligned}$$

Because

$$\lim_{x\to x^*} f(x) = 0,$$

$$\lim_{x\to x^*} \frac{f(x - f(x)) - f(x)}{-f(x)} = f'(x^*),$$

we obtain $\varphi(x^*) = x^*$.

This means that x^* is a fixed point of the iterative function $\varphi(x)$.

Secondly, using the Taylor series expansion of $\exp(x)$

$$\exp(x) = 1 + x + \frac{x^2}{2} + o(x^2),$$

and from (2), we have

$$\begin{aligned} x_{n+1} &= x_n \exp\left\{ -\frac{f(x_n)}{x_n\left(\frac{f(x_n-f(x_n))-f(x_n)}{-f(x_n)}\right)} \right\} \\ &= x_n - \frac{f(x_n)}{\frac{f(x_n-f(x_n))-f(x_n)}{-f(x_n)}} + \frac{f^2(x_n)}{2x_n\left(\frac{f(x_n-f(x_n))-f(x_n)}{-f(x_n)}\right)^2} \\ &\quad + o\left(\frac{f^2(x_n)}{2x_n\left(\frac{f(x_n-f(x_n))-f(x_n)}{-f(x_n)}\right)^2} \right). \end{aligned}$$

Let $e_n = x_n - x^*$, then we have

$$f(x_n) = f'(x^*)e_n + \frac{f''(x^*)}{2}e_n^2 + o(e_n^2),$$

$$f(x_n - f(x_n)) = f'(x^*)\left[e_n - f'(x^*)e_n - \frac{f''(x^*)}{2}e_n^2\right] + \frac{f''(x^*)\left[e_n - f'(x^*)e_n - \frac{f''(x^*)}{2}e_n^2\right]^2}{2} + o(e_n^2),$$

and

$$f(x_n - f(x_n)) - f(x_n) = -f'^2(x^*)e_n + \frac{(f'(x^*)-3)f'(x^*)f''(x^*)}{2}e_n^2 + o(e_n^2),$$

$$\frac{f(x_n - f(x_n)) - f(x_n)}{-f(x^*)} = \frac{2f'^2(x^*) + (3 - f'(x^*))f'(x^*)f''(x^*)e_n}{2f'(x^*) + f''(x^*)e_n} + o(e_n).$$

So, we obtain

$$\begin{aligned} e_{n+1} = e_n &\left(1 - \frac{f'(x^*) + \frac{f''(x^*)}{2}e_n + o(e_n)}{\frac{2f'^2(x^*) + (3-f'(x^*))f'(x^*)f''(x^*)e_n}{2f'(x^*) + f''(x^*)e_n} + o(e_n)} \right) \\ &+ \frac{\left(f'(x^*)e_n + \frac{f''(x^*)}{2}e_n^2 + o(e_n^2)\right)^2}{2x_n\left(\frac{2f'^2(x^*) + (3-f'(x^*))f'(x^*)f''(x^*)e_n}{2f'(x^*) + f''(x^*)e_n} + o(e_n)\right)} + o(e_n^2) \end{aligned}$$

$$= e_n^2 \frac{\dfrac{(1-f'(x^*))f'(x^*)f''(x^*)}{2f'(x^*)+f''(x^*)e_n} + o(1)}{\dfrac{2f'^2(x^*)+(3-f'(x^*))f'(x^*)f''(x^*)e_n}{2f'(x^*)+f''(x^*)e_n} + o(e_n)}$$

$$+ e_n^2 \frac{\left(f'(x^*)+\frac{f''(x^*)}{2}e_n + o(e_n)\right)^2}{2x_n\left(\dfrac{2f'^2(x^*)+(3-f'(x^*))f'(x^*)f''(x^*)e_n}{2f'(x^*)+f''(x^*)e_n} + o(e_n)\right)} + o(e_n^2).$$

Therefore

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} = \frac{f''(x^*)}{2f'(x^*)} - \frac{f''(x^*)}{2} + \frac{1}{2x^*}, \quad x^* \neq 0.$$

This means that the iterative formula (2) has a convergence of order two. □

Remark 1. If we keep only a first order approximation of the exponential function in formula (2), the following iteration formula is obtained

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f(x_n) - f(x_n - f(x_n))}, \tag{3}$$

which is the well-known Steffensen method [7].

Remark 2. For multiple roots, we can use the transform explained in [10]. Let

$$F(x) = \frac{\mathrm{sign}(f(x))f(x)|f(x)|^{\frac{1}{m}}}{\mathrm{sign}\Big(f(x+\mathrm{sign}(f(x))|f(x)|^{\frac{1}{m}})-f(x)\Big)f(x)|f(x)|^{\frac{1}{m}} + f(x+\mathrm{sign}(f(x))|f(x)|^{\frac{1}{m}}) - f(x)},$$

where the integer m is the number of multiple roots. Then, the multiple roots of $f(x)$ are transformed into simple roots of $F(x)$, and equation $F(x) = 0$ is solved by formula (2),

$$x_{n+1} = x_n \exp\left\{-\frac{F^2(x_n)}{x_n(F(x_n) - F(x_n - F(x_n)))}\right\}. \tag{4}$$

Thus, we obtain a new iteration formula for computing multiple roots of an equation.

3. The new algorithm combining regula falsi and formula (2)

In what follows, we will always assume that $f(x)$ is continuous on $[a, b]$ and

$$f(a)f(b) < 0, \tag{5}$$

where we suppose that $f(a) < 0, f(b) > 0$ without loss of generality. This assumption guarantees the existence of a zero of $f(x)$ in the interval $[a, b]$. Suppose that the regula falsi method produces a point $c \in [a, b]$. Then, a new interval $[\overline{a}, \overline{b}] \subset [a, b]$ containing at least one zero of $f(x)$ can be constructed by using the following algorithm at the nth step:

Algorithm 1. Regula falsi $(a_n, b_n, c_n, \overline{a}_n, \overline{b}_n)$

$$c_n = a_n - \frac{(b_n - a_n)f(a_n)}{f(b_n) - f(a_n)}.$$

If $f(c_n) = 0$, then the solution is obtained and we stop.

If $f(a_n)f(c_n) < 0$, then $\overline{a}_n = a_n, \overline{b}_n = c_n$.

If $f(b_n)f(c_n) < 0$, then $\overline{a}_n = c_n, \overline{b}_n = b_n$.

Then, we have obtained a new interval $[\overline{a}_n, \overline{b}_n] \subset [a, b]$ with $f(\overline{a}_n)f(\overline{b}_n) < 0$ and $f(\overline{a}_n), f(\overline{b}_n) > 0$. If the values of $f(x)$ at a_n and b_n are known, then each iteration requires only one function evaluation.

Clearly, one bit of precision is gained at each step of the algorithm. Besides, only the convergence rate of $\{(c_n - x^*)\}_{n=1}^{\infty}$, where c_n is the current estimate of x^*, has been studied and not the convergence rate of the diameters $\{(b_n - a_n)\}_{n=1}^{\infty}$. Therefore, in the algorithm to be described in the sequel, after using the previous algorithm, and obtaining c_n and $[\overline{a}, \overline{b}]$, we attempt to obtain a smaller enclosing interval by means of a point $\overline{c}_n$ yielded via the exponential iterative formula (2) for accelerating the convergence at the nth step. Namely, we have the following iteration formula

$$x_{n+1} = x_n \exp\left\{-\frac{h_n f^2(x_n)}{x_n(f(x_n) - f(c_n))}\right\}, \quad n = 0, 1, 2, \cdots, \tag{6}$$

where

$$\begin{cases} h_n = \dfrac{b_n - a_n}{f(b_n) - f(a_n)}, \\ c_n = \dfrac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}. \end{cases} \tag{7}$$

Algorithm 2. EXRF $(a_n, b_n, c_n, \overline{a}_n, \overline{b}_n, x_n, x_{n+1}, a_{n+1}, b_{n+1})$

$\overline{c}_n = x_n \ \exp\left\{-\frac{h_n f^2(x_n)}{x_n(f(x_n) - f(c_n))}\right\}.$

If $\overline{c}_n \in [\overline{a}_n, \overline{b}_n]$, then $x_{n+1} = \overline{c}_n$.

If $f(\overline{a}_n)f(\overline{c}_n) < 0$, then $a_{n+1} = \overline{a}_n, b_{n+1} = \overline{c}_n$ else $a_{n+1} = \overline{c}_n, b_{n+1} = \overline{b}_n$.

If $|f(x_{n+1})| < \varepsilon_1$ or $b_{n+1} - a_{n+1} < \varepsilon_2$, then keep x_{n+1} and stop.

If $\overline{c}_n \notin [\overline{a}_n, \overline{b}_n]$, then $x_{n+1} = \overline{c}_n, a_{n+1} = \overline{a}_n$ and $b_{n+1} = \overline{b}_n$.

An iteration of Algorithm 2 requires function evaluations whenever $\overline{c} \in [\overline{a}, \overline{b}]$. Using iteratively these two algorithms, we obtain the sequences $\{x_n\}$ and $\{(b_n - a_n)\}$. We call this procedure the *exponential regula falsi method* for root finding.

Keeping a first order approximation of the exponentially function in formula (6), the following formula is obtained

$$x_{n+1} = x_n - \frac{\frac{b_n - a_n}{f(b_n) - f(a_n)} f^2(x_n)}{f(x_n) - f\left(\frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}\right)}, \quad n = 0, 1, 2, \cdots. \tag{8}$$

i.e., the method presented in [11].

Theorem 2. Assume that $f(x^*) = 0$ and $U(x^*)$ to be a sufficiently small neighborhood of $x^*, x^* \neq 0$. Let $f''(x)$ exist in $U(x^*)$ and $f'(x) \neq 0$ in $U(x^*)$. Then the sequence $\{x_n\}$ produced by the iterative formula (6) is at least quadratically convergent for $h_n > 0$.

The proof of Theorem 2 is similar to Theorem 1 and will be omitted.

Remark 3. From the Regula Falsi Method, with the assumptions of Theorem 2, we know that either a zero of $f(x)$ is obtained in a finite number of steps, or the sequence $\{(b_n - a_n)\}$ enclosing a zero of $f(x)$, and generated by algorithm 2 (EXRF), converges to zero.

Before giving the convergence theorem for the Algorithm 2 (EXRF), we present two lemmas which will be used for its proof.

Lemma 1. Assume $f(x) \in C^1_{[a,b]}$ and $f(a) < 0, f(b) > 0$. $\{q_n\}$ is a real sequence with $0 < r < q_n < q < 1$. Then either the nonzero root x^* of equation (1) in $[a, b]$ is obtained in a finite number of steps, or the sequence of diameters $\{(b_n - a_n)\}_{n=1}^{\infty}$ generated by the following Algorithm 3 converges to zero and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = x^*, f(x^*) = 0$.

Algorithm 3.

Let $v_n = q_n a_n + (1 - q_n) b_n$.

If $f(v_n) = 0$, then keep v_n and stop.

If $f(a_n)f(v_n) < 0$, then $\overline{a}_n = a_n, \overline{b}_n = v_n$, else $\overline{a}_n = v_n, \overline{b}_n = b_n$.

Let $z_n = x_n \exp\left\{\frac{-q_n(b_n - a_n)|f(x_n)|}{x_n(f(x_n) - f(v_n))}\right\}$.

If $z_n \in [\overline{a}_n, \overline{b}_n]$, then $x_{n+1} = z_n$.
If $f(\overline{a}_n)f(z_n) < 0$, then $a_{n+1} = \overline{a}_n, b_{n+1} = z_n$ else $a_{n+1} = z_n, b_{n+1} = \overline{b}_n$.
If $|f(z_n)| < \varepsilon_1$ or $b_{n+1} - a_{n+1} < \varepsilon_2$, then keep z_n and stop.
If $z_n \notin [\overline{a}_n, \overline{b}_n]$, then $x_{n+1} = v_n, a_{n+1} = \overline{a}_n$ and $b_{n+1} = \overline{b}_n$.

Proof. From Algorithm 3 and

$$z_n = x_n \exp\left\{\frac{-q_n(b_n - a_n)|f(x_n)|}{x_n(f(x_n) - f(v_n))}\right\},$$

$$v_n = q_n a_n + (1 - q_n)b_n.$$

We can see that Algorithm 3 produces a sequence of $\{(b_n - a_n)\}_{n=1}^{\infty}$, and a sequence of iterates (x_n) such that we have

$$x^* \in [a_{n+1}, b_{n+1}] \subset [a_n, b_n] \subset \cdots \subset [a, b],$$

$$x_n \in [a_{n+1}, b_{n+1}] \subset [a_n, b_n] \subset \cdots \subset [a, b],$$

$$f(a_n)f(b_n) < 0, \qquad n = 0, 1, 2, \cdots,$$

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n,$$

$$b_{n+1} - a_{n+1} \leq q_n(b_n - a_n) < q(b_n - a_n).$$

Since $0 < q < 1$, we obtain that $b_n - a_n \leq q^n(b - a)$. This means that

$$\lim_{n\to\infty}(b_n - a_n) = 0$$

and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x^*.$$

So $\lim_{n\to\infty} x_n = x^*, f(x^*) = 0$. □

Lemma 2. Under the hypothesis of Lemma 1, and assuming that there exists a positive integer N_0 such that $|f(v_n)| < q_n|f(x_n)|$ whenever $n > N_0$, and that Algorithm 3 does not terminate after a finite number of steps, then the sequence of diameters $\{(b_n - a_n)\}_{n=1}^{\infty}$ converges Q-quadratically [8] to zero. Namely, there is a constant C such that

$$b_{n+1} - a_{n+1} \leq C(b_n - a_n)^2, \quad n = 0, 1, 2, \cdots. \tag{9}$$

Proof. From Theorem 2, it follows that

$$\lim_{n\to\infty} \frac{e_{n+1}}{e_n^2} = K,$$

and

$$\lim_{n\to\infty}\frac{e_{n+1}-e_n}{(e_n-e_{n-1})^2}=\lim_{n\to\infty}\frac{\frac{e_{n+1}}{e_n}-1}{e_n-2e_{n-1}+\frac{e_{n-1}^2}{e_n}}=-K.$$

So there exists an integer N_1 such that

$$\left|\frac{e_{n+1}-e_n}{(e_n-e_{n-1})^2}\right|<|K|+1 \text{ and } \left|\frac{x_{n+2}-x_{n+1}}{(x_{n+1}-x_n)^2}\right|<|K|+1,$$

for all $n>N_1$. From the assumption, for all $n>N_0$ we have

$$|f(v_n)|<q_n|f(x_n)|<q|f(x_n)|, \qquad 0<q<1. \tag{10}$$

From this inequality and Lemma 1, we can deduce that $z_n\in[\overline{a}_n,\overline{b}_n]$, for all $n>\max\{N_0,N_1\}$, so $x_{n+1}=z_n$, and the above inequality (10) means that

$$\frac{f(x_n)-f(v_n)}{f(x_n)}>0. \tag{11}$$

Using the Taylor series expansion of formula (6), we have

$$\begin{aligned}x_{n+1} \quad = \quad & x_n-\frac{q_n(b_n-a_n)|f(x_n)|}{2(f(x_n)-f(v_n))}+\frac{q_n^2(b_n-a_n)^2f^2(x_n)}{4x_n(f(x_n)-f(v_n))^2}\\ &+o\left(\frac{q_n(b_n-a_n)^2f^2(x_n)}{4x_n(f(x_n)-f(v_n))^2}\right).\end{aligned}$$

Then we obtain

$$b_n-a_n+o(b_n-a_n)=\frac{2(x_n-x_{n+1})(f(x_n)-f(v_n))}{q_n|f(x_n)|},$$

and it follows that

$$\frac{b_{n+1}-a_{n+1}+o(b_{n+1}-a_{n+1})}{(b_n-a_n+o(b_n-a_n))^2}=\frac{q_n^2(x_{n+1}-x_{n+2})(f(x_{n+1})-f(v_{n+1}))f^2(x_n)}{2q_{n+1}(x_{n+1}-x_n)^2(f(x_n)-f(v_n))^2|f(x_{n+1})|}.$$

From formulae (10) and (11), we know that

$$1-q\le\left|\frac{f(x_n)-f(v_n)}{f(x_n)}\right|\le 1+q. \tag{12}$$

So

$$\frac{\left|\frac{f(x_{n+1})-f(v_{n+1})}{f(x_{n+1})}\right|}{\frac{(f(x_n)-f(v_n))^2}{f^2(x_n)}}<\frac{1+q}{(1-q)^2},$$

Table 1
The numerical comparison of examples.

	Algorithm 2 (EXRF)			Algorithm 1 (Regula Falsi)			Steffensen	Newton		
	n	x_n	$\|f(x_n)\|$	n	x_n	$\|f(x_n)\|$	n x_n $\|f(x_n)\|$	n	x_n	$\|f(x_n)\|\}$
1	6	1.00000e+00	0.00000e+00	27	1.00000e+00	8.88178e−16	Failure	Divergent		
2	5	1.69681e+00	4.44089e−16	32	1.69681e+00	4.44089e−16	Failure	Not convergent to x^*}}		
3	7	8.04133e−01	1.22124e−15	101	8.04133e−01	1.25422e−13	Divergent	7	8.04133e−01	4.44089e−16
4	6	1.11833e−01	0.00000e+00	15	1.11833e−01	7.49401e−16	Failure	Failure		

and if $n > \max\{N_0, N_1\}$, then

$$\left|\frac{b_{n+1} - a_{n+1} + o(b_{n+1} - a_{n+1})}{(b_n - a_n + o(b_n - a_n))^2}\right| \leq \frac{q(1+q)(|K|+1)}{2r(1-q)^2}.$$

From $\lim_{n\to\infty}(b_n - a_n) = 0$, we know that there exists an integer N_2 such that, for all $n > N_2$,

$$\frac{b_{n+1} - a_{n+1}}{(b_n - a_n)^2} \leq 2\left|\frac{b_{n+1} - a_{n+1} + o(b_{n+1} - a_{n+1})}{(b_n - a_n + o(b_n - a_n))^2}\right|.$$

Let $N = \max\{N_0, N_1, N_2\}$,

$$C \geq \max\left\{\frac{q(1+q)(|K|+1)}{r(1-q)^2}, \frac{q}{(b_N - a_N)}\right\},$$

then we have $b_{n+1} - a_{n+1} \leq C(b_n - a_n)^2, \quad n = 0, 1, 2, \cdots.$ □

Let us now give the convergence theorem for Algorithm 2 (EXRF).

Theorem 3. Assume $f(x) \in C^1_{[a,b]}$ and $f(a) < 0, f(b) > 0$. x^* is a simple nonzero zero of $f(x)$ in $[a, b]$. Then either the root x^* of (1) in $[a, b]$ is obtained in a finite number of steps, or the sequence of diameters $\{(b_n - a_n)\}_{n=1}^{\infty}$ generated by Algorithm 2(EXRF) is Q-quadratically convergent to zero.

Proof. Let $q_n = \frac{|f(x_n)|}{f(b_n)-f(a_n)}$ in Algorithm 3, then 0. From Lemma 1, if n is large enough, then $|f(x_n)| \simeq |f(a_n)| \simeq |f(b_n)|$ and $\frac{|f(x_n)|}{f(b_n)-f(a_n)} \simeq \frac{1}{2}$. Therefore there exist r, q such that 0 and it leads to Algorithm 2 (EXRF). Thus, from Lemmas 1 to 2, the conclusion of Theorem 3 is obviously true. □

4. Numerical experiments

Example 1. $f(x) = \ln x$, $[a, b] = [0.5, 5]$.

Example 2. $f(x) = x - e^{\sin x} + 1$, $[a, b] = [1, 4]$.

Example 3. $f(x) = 11x^{11} - 1$, $[a, b] = [0.1, 1]$.

Example 4. $f(x) = xe^{-x} - 0.1$, $[a, b] = [0, 1]$.

In the numerical experiments, the initial value of the iterations is $x_0 = b$, the errors are taken as 1×10^{-15}, and the maximum number of iterations is limited to 100,

The results of the examples $1 \sim 4$ are given in table 1.

From table 1, we can see that the new exponentially regula falsi iterative method is more efficient than the classical regula falsi method, and have larger convergence fields, faster convergence, and a higher convergence rate than Newton's method and Steffensen's method (see [7–9]).

5. Conclusions

Combining the classical regula falsi method and a class of exponential iterative method, we presented a class of exponential regula falsi methods with high order convergence for finding simple zeros of nonlinear equations. The iterations $\{x_n\}$ of the new method have a quadratic convergence, and the sequence of diameters $\{(b_n - a_n)\}$ also. Numerical experiments show that the new method is effective and comparable to well-known methods, such as the classical regula falsi, Newton and Steffensen's methods.

In fact, this work proposes a general framework for iterative methods for solving nonlinear equations. In this paper, we choose the parameters in formula (6) as $h_n = \frac{b_n - a_n}{f(b_n) - f(a_n)}$, and combine it with the classical regula falsi method for establishing the new method. Of course, we can take different parameters in the iterative formula (6), and employ another methods, such as the bisection method etc., to build other new algorithms whose sequence of iterations $\{x_n\}$ and the sequence of diameters $\{(b_n - a_n)\}$ converge. The convergence theories for the corresponding algorithms can be established following an analysis analogous to the analysis of the method of this paper with slight and technical modifications.

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