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Boundary integral approximation of a heat-diffusion problem in time-harmonic regime

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In this paper we propose and analyse numerical methods for the approximation of the solution of Helmholtz transmission problems in the half plane. The problems we deal with arise from the study of some models in photothermal science. The solutions to the problem are represented as single layer potentials and an equivalent system of boundary integral equations is derived. We then give abstract necessary and sufficient conditions for convergence of Petrov–Galerkin discretizations of the boundary integral system and show for three different cases that these conditions are satisfied. We extend the results to other situations not related to thermal science and to non-smooth interfaces. Finally, we propose a simple full discretization of a Petrov–Galerkin scheme with periodic spline spaces and show some numerical experiments.

Keywords: thermal waves, Petrov–Galerkin methods, boundary integral equations, Helmholtz transmission problems

AMS subject classifications: 65R20, 65N38

1. Introduction

Thermal waves are time-harmonic solutions to the heat equation. Their study arises in several Physical and Engineering problems. A.J. Ångström in 1861 [1] was the pioneer in the theoretical and experimental study of thermal waves. However, significant progress in their science and technologies has occurred only recently (see [2, 20] and references therein). The most important and popular diffusion-waves methodologies today fall into three branches: photothermal spectroscopies and microscopies [4], photomodulated thermoreflectance [19] and diffuse photon density [35].

The problem we consider arises in the study of composite materials by means of a photothermal technique which consists of illuminating a side of the material by a defocused laser beam modulated at a constant frequency. The heating induces an incident wave which spreads into the material and is scattered by the inhomogeneities placed under the surface. Measurements of the temperature at the thermal-excited side endeavours to determine internal properties of the material. However, here we only deal with the direct problem. This kind of technique has been previously studied from the point of view of physical experimentation and search of analytical solutions in [30, 31].

In this work we deal with an exterior Helmholtz transmission problem (arising from imposing a time-harmonic regime in a heat equation) in a half plane. Such kind of problems has already been widely studied in two and three dimensions. Existence and uniqueness results can be found in [8, 16, 18, 32, 33]. Boundary integral formulations are a powerful tool to solve this kind of problems. We propose an indirect formulation in which the solution is represented as a single layer potential in each domain. Indirect formulations of similar problems to the proposed here can be found in [16, 18] for a domain with smooth boundary or in [32] for Lipschitz interfaces. In all of them the solution is represented as a linear combination of a single and a double layer potential (sometimes referred to as a Brakhage–Werner or Panich potential) in each domain, obtaining an equivalent system of boundary integral equations. Direct formulations have been studied in [16] for smooth domains, in [8] for smooth and polygonal boundaries and in [33] for Lipschitz ones. Mixed direct–indirect formulations of Helmholtz transmission problems for smooth domains have also been analyzed in [16]. Notice that the thermal case allows the use of indirect formulations, since only when there is no adsorption in the medium single layer potentials can fail to give inversible integral operators.

Of the works mentioned above, numerical methods to solve the equivalent system of integral equations are studied only in [8]. The methods considered in that work are: a Galerkin scheme with piecewise polynomial functions for the case with smooth interfaces; a Galerkin method where the discrete spaces include splines and the known functions which characterize the singularities at the corners of the domain for the polygonal case.

Our paper deals with numerical approximation of Helmholtz transmission problems in the half plane with an homogeneous Neumann boundary condition (modelling adiabaticity) on the line that separates the medium from the exterior. We analyse invertibility of the system of integral equations that appears when appropriate single layer potentials are proposed for the solution of the transmission problem. This is done in sections 2 and 3, for parametrizable smooth interfaces separating the inclusions from the homogenous medium occupying the half space. Smoothness hypotheses can be somewhat relaxed: for many results to hold in the natural norm (that of $H^{-1/2}(\Gamma)$), Lyapunov regularity is enough. However, we will deal with the more restrictive situation to be able to go further in the analysis.

In section 4 we give abstract conditions for some Petrov–Galerkin methods to be convergent when applied to the system of boundary integral equations of the preceding section. We then verify those conditions in three situations: trigonometric polynomials; smoothest splines on uniform staggered grids; piecewise constant and linear functions on non-uniform staggered grids. Note that the techniques to prove the abstract conditions vary substantially from one example to another. The next two sections are devoted to generalize the kind of results given before to more general situations unrelated to thermal waves. In section 8 we describe how these methods apply for many Helmholtz transmission problems in two and three dimensions when the interfaces are smooth. In section 9, we show the necessary adjustments to apply these results when the boundaries of the inclusions have only Lipschitz regularity. The question on whether simple couples of spaces satisfy the requirements guaranteeing convergence is left open. Finally, in the last two sections we indicate on detail a numerical implementation of the method using piecewise constant and linear functions. We illustrate with some numerical examples related to thermal waves in the two dimensional setting that this method is simpler and more suitable than trigonometric polynomial approximations when the boundaries of the obstacles are smooth but not $\mathcal{C}^\infty$-curves.

Notation As usual in numerical analysis of integral and partial differential equations, C (also C', C'',...) will denote a general positive constant independent of the discretization parameter N or h and of the quantities it is multiplied by. We will use the Sobolev spaces $H^m(\omega)$, where ω is a general bounded or unbounded domain and m is a non-negative integer. In the final sections, we will also use the Sobolev spaces $H^r(\Gamma)$ for smooth and Lipschitz closed curves and surfaces. We refer to [21] for a very detailed explanation of these spaces in the same setting we will be using them.

2. Description of the problem

Let $\mathbb{R}^2_- := \{(x_1, x_2) \,|\, x_2 < 0\}$ and $\Pi := \{(x_1, 0) \,|\, x_1 \in \mathbb{R}\}$. We consider a finite collection of simply connected open sets Ω_i $(i = 1, \ldots, d)$ with non intersecting closures and such that $\overline{\Omega}_i \cap \Pi = \emptyset$ for all i. We further assume that the boundaries $\Gamma_i := \partial\Omega_i$ are parametrizable $\mathcal{C}^\infty$-curves and denote

$$\Gamma := \cup_{i=1}^d \Gamma_i, \qquad \Omega := \mathbb{R}^2_- \setminus \big(\cup_{i=1}^d \overline{\Omega}_i\big).$$

Normals are directed towards the exterior of Ω_i for each i. The normal derivative on Π is directed upwards (pointing towards the exterior of Ω). Throughout this work if $\mathbf{x} = (x_1, x_2)$, we write $\widetilde{\mathbf{x}} := (x_1, -x_2)$.

We look for time-harmonic solutions

$$T(\mathbf{x}, t) = \mathrm{Re}(v(\mathbf{x}) \exp(-\imath\omega t)),$$

of the heat transfer problem

$$\kappa\,\Delta T = \partial_t T, \qquad \kappa = \begin{vmatrix} \kappa_0, & \text{in } \Omega, \\ \kappa_i, & \text{in } \Omega_i, \quad i = 1,\ldots,d, \end{vmatrix} \tag{1}$$

with transmission conditions in the inner boundaries ($\nu, \nu_i > 0$ are given parameters)

$$T|_{\Gamma_i}^{\text{int}} = T|_{\Gamma_i}^{\text{ext}}, \qquad \nu_i\,\partial_n T|_{\Gamma_i}^{\text{int}} = \nu\,\partial_n T|_{\Gamma_i}^{\text{ext}}, \qquad i = 1,\ldots,d. \tag{2}$$

Let $\lambda := (1+\imath)\sqrt{\omega/2\kappa_0}$ and $u_{\text{inc}} : \mathbb{R}^2_- \to \mathbb{C}$ (which will play the role of an incident wave) a known solution to the Helmholtz equation

$$\Delta u_{\text{inc}} + \lambda^2 u_{\text{inc}} = 0, \qquad \text{in } \mathbb{R}^2_-.$$

We also demand $\partial_n v|_\Pi = \partial_n u_{\text{inc}}|_\Pi$, which corresponds asymptotically in t to a condition on $\partial_n T(\mathbf{x}, t)|_\Pi$ that is periodic in time. Let us take

$$u := \begin{vmatrix} v - u_{\text{inc}}, & \text{in } \Omega, \\ v, & \text{in } \Omega_i, \quad i = 1,\ldots,d, \end{vmatrix}$$

as unknown. Then $u \in H^1(\Omega) \times \prod_{i=1}^d H^1(\Omega_i)$ has to satisfy

$$\Delta u + \lambda^2 u = 0, \qquad \text{in } \Omega, \tag{3}$$

$$\Delta u + \lambda_i^2 u = 0, \qquad \text{in } \Omega_i, \quad i = 1,\ldots,d, \tag{4}$$

$$u|_{\Gamma_i}^{\text{int}} - u|_{\Gamma_i}^{\text{ext}} = g_i^0, \qquad i = 1,\ldots,d, \tag{5}$$

$$\nu_i\,\partial_n u|_{\Gamma_i}^{\text{int}} - \nu\,\partial_n u|_{\Gamma_i}^{\text{ext}} = g_i^1, \qquad i = 1,\ldots,d, \tag{6}$$

$$\partial_n u|_\Pi = 0, \tag{7}$$

being

$$g_i^0 = u_{\text{inc}}|_{\Gamma_i}, \qquad g_i^1 = \nu\,\partial_n u_{\text{inc}}|_{\Gamma_i}, \tag{8}$$

and $\lambda_i = (1+\imath)\sqrt{\omega/2\kappa_i}$.

By $\partial_n u|_\Pi = 0$ we mean that

$$\int_\Omega \nabla u \cdot \nabla\varphi + \int_\Omega \Delta u\,\varphi = 0, \qquad \forall \varphi \in \mathcal{D}(\mathbb{R} \times (-\varepsilon, \varepsilon)),$$

where $\varepsilon < \mathrm{dist}(\Pi, \Gamma)$. Written in this form, this condition implies that the reflected function

$$\widetilde{u}(\mathbf{x}) := \begin{vmatrix} u(\mathbf{x}), & \text{if } x_2 \leq 0, \\ u(\widetilde{\mathbf{x}}), & \text{if } x_2 > 0 \end{vmatrix} \tag{9}$$

is a solution to $\Delta\widetilde{u} + \lambda^2\widetilde{u} = 0$ in the reflected exterior domain, including Π. Therefore $u \in \mathcal{C}^\infty$ in a neighbourhood of Π and equation (7) holds in a classical way.

As incident wave it can be chosen the function $u_{\mathrm{inc}}(\mathbf{x}) := \exp(-\imath\lambda x_2)$. It is also physically feasible to take $u_{\mathrm{inc}}(\mathbf{x}) := H_0^{(1)}(\lambda|\mathbf{x} - \mathbf{x}_0|)$ for $\mathbf{x}_0 \in \Pi$, $H_0^{(1)}$ being the Hankel function of the first kind and order zero.

The forthcoming analysis can be extended to more general values of λ, λ_i, ν and ν_i. In this case, we have to take care of uniqueness issues (see [8, 16, 18, 32, 33]) and to impose a Sommerfeld radiation condition at infinity instead of behaviour in $H^1(\Omega)$ when $\lambda^2 \in \mathbb{R}^+$. Notice also that we will be using single layer potentials in our formulation, and they can fail to provide inversible operators in some of these more general situations. We will deal with this in sections 8 and 9.

Proposition 2.1. The problem (3–7) has at most a solution.

Proof. We just have to prove that the homogeneous problem (i.e., with $g_i^0 = g_i^1 = 0$ for all i) admits only the trivial solution. Considering the reflected domains and boundaries

$$\Omega_{i+d} := \{\mathbf{x} \in \mathbb{R}^2 \,|\, \widetilde{\mathbf{x}} \in \Omega_i\}, \qquad \Gamma_{i+d} := \partial\Omega_{i+d} \;=\; \{\mathbf{x} \in \mathbb{R}^2 \,|\, \widetilde{\mathbf{x}} \in \Gamma_i\}, \quad i = 1, \ldots, d,$$

the reflected function defined in equation (9) solves the following exterior homogeneous transmission problem in the plane

$$\begin{vmatrix} \Delta u + \lambda^2 u = 0, & \text{in } \mathcal{U} := \mathbb{R}^2 \setminus \left(\cup_{i=1}^{2d} \overline{\Omega}_i\right), \\ \Delta u + \lambda_i^2 u = 0, & \text{in } \Omega_i, \quad i = 1, \ldots, 2d, \\ u|_{\Gamma_i}^{\mathrm{int}} - u|_{\Gamma_i}^{\mathrm{ext}} = 0, & i = 1, \ldots, 2d, \\ \nu_i\, \partial_n u|_{\Gamma_i}^{\mathrm{int}} - \nu\, \partial_n u|_{\Gamma_i}^{\mathrm{ext}} = 0, & i = 1, \ldots, d, \end{vmatrix}$$

and belongs to $H^1(\mathcal{U}) \times \prod_{i=1}^{2d} H^1(\Omega_i)$. The result follows now from [18]. □

With a similar reflection technique we can apply the results of [18] to show existence of solution for arbitrary $g_i^0 \in H^{1/2}(\Gamma_i)$ and $g_i^1 \in H^{-1/2}(\Gamma_i)$. We will however prove this as a consequence of the boundary integral formulation we will adopt for numerical purposes. It will also be proved that the solution to equations (3–7) satisfies the Sommerfeld radiation condition at infinity

$$\lim_{r\to\infty} r^{1/2}(\partial_r u - \imath\lambda u) = 0,$$

uniformly in all available directions.

3. Indirect formulation

Let $\mathbf{x}_j : \mathbb{R} \to \Gamma_j$ be regular 1−periodic parameterizations of the boundaries of the obstacles, satisfying therefore

$$|\mathbf{x}_j'(t)| \neq 0, \quad \forall t; \qquad \mathbf{x}_j(t) \neq \mathbf{x}_j(s), \quad s-t \notin \mathbb{Z}.$$

To simplify, we will assume that $\mathbf{x}_j \in \mathcal{C}^\infty(\mathbb{R})$.

To a given density $\varphi_j : \mathbb{R} \to \mathbb{C}$ we associate the following single layer potentials

$$\mathcal{S}_j^\rho \varphi_j := \frac{\imath}{4}\int_0^1 H_0^{(1)}(\rho|\cdot - \mathbf{x}_j(t)|)\,\varphi_j(t)\,dt \ : \ \mathbb{R}^2 \longrightarrow \mathbb{C},$$

$$\widetilde{\mathcal{S}}_j^\rho \varphi_j := \frac{\imath}{4}\int_0^1 \Big(H_0^{(1)}(\rho|\cdot - \mathbf{x}_j(t)|) + H_0^{(1)}(\rho|\cdot - \widetilde{\mathbf{x}}_j(t)|)\Big)\varphi_j(t)\,dt \ : \ \mathbb{R}^2 \longrightarrow \mathbb{C}.$$

Given densities $\varphi_j : \mathbb{R} \to \mathbb{C}$, $j = 1, \ldots, d$, which we group in the vector $\varphi = (\varphi_1, \ldots, \varphi_d)^\top$, we consider the potential

$$\widetilde{\mathcal{S}}^\rho \varphi := \sum_{j=1}^d \widetilde{\mathcal{S}}_j^\rho \varphi_j.$$

Since $\widetilde{\mathcal{S}}^\rho\varphi(\widetilde{\mathbf{x}}) = \widetilde{\mathcal{S}}^\rho\varphi(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$ it follows readily that $\partial_n \widetilde{\mathcal{S}}^\rho \varphi|_\Pi = 0$. We also introduce the following integral operators

$$V_{ij}^\rho \varphi_j := \frac{\imath}{4}\int_0^1 H_0^{(1)}(\rho|\mathbf{x}_i(\cdot) - \mathbf{x}_j(t)|)\,\varphi_j(t)\,dt \ : \ \mathbb{R} \longrightarrow \mathbb{C},$$

$$J_{ij}^\rho \varphi_j := \frac{\imath}{4}\int_0^1 |\mathbf{x}_i'(\cdot)|\partial_{n_i(\cdot)} H_0^{(1)}(\rho|\mathbf{x}_i(\cdot) - \mathbf{x}_j(t)|)\,\varphi_j(t)\,dt \ : \ \mathbb{R} \longrightarrow \mathbb{C},$$

($\partial_{n_i(s)}$ is the exterior normal derivative at $\mathbf{x}_i(s)$). We likewise define $\widetilde{V}_{ij}^\rho$ and $\widetilde{J}_{ij}^\rho$.

In this parameterized form, the well-known jump relations of the single layer potential (see [5, Chapter 7] or [25]) read as follows: for all i, j

$$\mathcal{S}_j^\rho \varphi_j|_{\Gamma_i}^{\rm int} \circ \mathbf{x}_i = V_{ij}^\rho \varphi_j, \qquad \widetilde{\mathcal{S}}_j^\rho \varphi_j|_{\Gamma_i}^{\rm ext} \circ \mathbf{x}_i = \widetilde{V}_{ij}^\rho \varphi_j,$$

$$|\mathbf{x}_i'|\,\partial_n \mathcal{S}_j^\rho \varphi_j|_{\Gamma_i}^{\rm int} \circ \mathbf{x}_i = \frac{\delta_{ij}}{2}\varphi_j + J_{ij}^\rho\varphi_j, \qquad |\mathbf{x}_i'|\,\partial_n \widetilde{\mathcal{S}}_j^\rho \varphi_j|_{\Gamma_i}^{\rm ext} \circ \mathbf{x}_i = -\frac{\delta_{ij}}{2}\varphi_j + \widetilde{J}_{ij}^\rho\varphi_j,$$

δ_{ij} being the Kronecker symbol.

We consider the Sobolev spaces (see for instance [17, Chapter 8] or [25, Chapter 5]),

$$H^s := \{\phi \in \mathcal{D}' \mid |\widehat{\phi}(0)|^2 + \sum_{0\neq k\in\mathbb{Z}} |k|^{2s} |\widehat{\phi}(k)|^2 < \infty\},$$

where $\mathcal{D}'$ is the space of 1-periodic distributions at the real line and $\widehat{\phi}(k)$ are the Fourier coefficients of ϕ. The $H^0 = L^2(0,1)$ inner product extends to the duality bracket between H^s and H^{-s} for all $s \in \mathbb{R}$. We will denote it by $(\cdot,\cdot)$. Finally we consider the product spaces

$$\mathbf{H}^s := \underbrace{H^s \times \ldots \times H^s}_{d \text{ times}}.$$

We will denote by $\|\cdot\|_s$ the norm in H^s and in $\mathbf{H}^s$ as well.

We consider the Bessel operator

$$V_0\varphi := -\frac{1}{4\pi}\int_0^1 \log(4e^{-1}\sin^2\pi\,(\,\cdot\, - t))\,\varphi(t)\,dt \tag{10}$$

which satisfies

$$4\pi(V_0\varphi,\varphi) = \|\varphi\|^2_{-1/2}, \qquad \forall\varphi \in H^{-1/2}, \tag{11}$$

and defines a bounded isomorphism from H^s into H^{s+1} for all $s \in \mathbb{R}$ (see [25, Chapter 5]). We also consider the diagonal operator $\mathcal{V}_0 := \mathrm{diag}(V_0,\ldots,V_0) : \mathbf{H}^s \to \mathbf{H}^{s+1}$.

We summarize here some properties of the boundary integral operators defined above. Most of them are well-known and the other ones can be derived from standard results of periodic integral operators (see [25]).

Proposition 3.1. For all $s \in \mathbb{R}$,

(a) for all i, the bounded operator $V^\rho_{ii} : H^s \to H^{s+1}$ is invertible if and only if $-\rho^2$ is not a Dirichlet eigenvalue of the Laplace operator in Ω_i,

(b) for all i, the operators $V^\rho_{ii} - V_0 : H^s \to H^{s+1}$ and $\widetilde{V}^\rho_{ii} - V^\rho_{ii} : H^s \to H^{s+1}$ are compact,

(c) for $i \neq j$, the operators V^ρ_{ij}, $\widetilde{V}^\rho_{ij} : H^s \to H^{s+1}$ are compact,

(d) for all i,j, the operators J^ρ_{ij}, $\widetilde{J}^\rho_{ij} : H^s \to H^s$ are compact.

Let us rename the one-periodic data functions

$$g^0_i := g^0_i \circ \mathbf{x}_i, \qquad g^1_i := |\mathbf{x}'_i|\; g^1_i \circ \mathbf{x}_i, \tag{12}$$

and set

$$g_0 = (g^0_1,\ldots,g^0_d)^\top, \qquad g_1 = (g^1_1,\ldots,g^1_d)^\top. \tag{13}$$

As a proposal to solve the transmission problem we consider the function

$$u := \begin{vmatrix} \widetilde{\mathcal{S}}^{\lambda}\psi, & \text{in } \Omega, \\ \mathcal{S}_i^{\lambda_i}\varphi_i, & \text{in } \Omega_i, \quad i = 1, \ldots, d, \end{vmatrix} \tag{14}$$

for densities

$$\psi = (\psi_1, \ldots, \psi_d)^\top, \quad \varphi = (\varphi_1, \ldots, \varphi_d)^\top \quad \in \mathbf{H}^{-1/2}$$

to be determined. By definition u satisfies equations (3), (4) and (7) and belongs to $H^1(\Omega) \times \prod_{i=1}^d H^1(\Omega_i)$. Condition (5) is equivalent to

$$V_{ii}^{\lambda_i}\varphi_i - \sum_{j=1}^d \widetilde{V}_{ij}^{\lambda}\psi_j = g_i^0, \qquad i = 1, \ldots, d,$$

whereas equation (6) is equivalent to

$$\nu_i\left(\frac{1}{2}\varphi_i + J_{ii}^{\lambda_i}\varphi_i\right) + \nu\left(\frac{1}{2}\psi_i - \sum_{j=1}^d \widetilde{J}_{ij}^{\lambda}\psi_j\right) = g_i^1, \qquad i = 1, \ldots, d.$$

These equations can be collected as

$$\mathcal{H}\begin{bmatrix} \varphi \\ \psi \end{bmatrix} := \begin{bmatrix} \mathcal{V}^{\Lambda} & -\widetilde{\mathcal{V}}^{\lambda} \\ \mathcal{N}(\frac{1}{2}\mathcal{I} + \mathcal{J}^{\Lambda}) & \nu(\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{J}}^{\lambda}) \end{bmatrix}\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix}, \tag{15}$$

with

$$\widetilde{\mathcal{V}}^{\lambda} := (\widetilde{V}_{ij}^{\lambda}), \quad \widetilde{\mathcal{J}}^{\lambda} := (\widetilde{J}_{ij}^{\lambda}), \quad \mathcal{V}^{\Lambda} := \mathrm{diag}(V_{ii}^{\lambda_i}), \quad \mathcal{J}^{\Lambda} := \mathrm{diag}(J_{ii}^{\lambda_i}), \quad \mathcal{N} := \mathrm{diag}(\nu_i I).$$

Lemma 3.2. For all $s \in \mathbb{R}$, $\widetilde{\mathcal{V}}^{\lambda} : \mathbf{H}^s \to \mathbf{H}^{s+1}$ is an isomorphism.

Proof. By Proposition 3.1 it is clear that $\widetilde{\mathcal{V}}^{\lambda} - \mathcal{V}_0$ is compact, which implies that $\widetilde{\mathcal{V}}^{\lambda}$ is Fredholm of index zero. We then prove injectivity of $\widetilde{\mathcal{V}}^{\lambda}$. Indeed it is sufficient to prove it for just a single value of s (see [25, Chapter 6]). If $\psi \in H^{-1/2}$ and $\widetilde{\mathcal{V}}^{\lambda}\psi = 0$, we define $u := \widetilde{\mathcal{S}}^{\lambda}\psi$, which satisfies

$$\begin{vmatrix} \Delta u + \lambda^2 u = 0, & \text{in } \Omega \text{ and } \cup_{i=1}^d \Omega_i, \\ u|_{\Gamma_i}^{\text{int}} = u|_{\Gamma_i}^{\text{ext}} = 0, & i = 1, \ldots, d, \\ \partial_n u|_{\Pi} = 0, & \\ \lim_{r\to\infty} r^{1/2}(\partial_r u - \imath\lambda u) = 0. & \end{vmatrix}$$

By uniqueness of solution of the interior problems it follows that $u = 0$ in Ω_i for all i. A reflection argument and the uniqueness of the exterior Dirichlet problem (see [5, Theorem 7.5.5]) prove that also $u = 0$ in Ω. Finally, by the jump relations $\psi_i = (\partial_n u\,|^{\text{int}}_{\Gamma_i} \circ \mathbf{x}_i - \partial_n u\,|^{\text{ext}}_{\Gamma_i} \circ \mathbf{x}_i)|\mathbf{x}_i'| = 0$ for all i. □

Proposition 3.3. For all $s \in \mathbb{R}$, $\mathcal{H} : \mathbf{H}^s \times \mathbf{H}^s \to \mathbf{H}^{s+1} \times \mathbf{H}^s$ is an isomorphism.

Proof. Boundedness of the operator is straightforward. We first remark that $\mathcal{V}^\Lambda$ is invertible by Proposition 3.1, which justifies the decomposition

$$\begin{bmatrix} \mathcal{V}^\Lambda & -\widetilde{\mathcal{V}}^\lambda \\ \mathcal{N}(\frac{1}{2}\mathcal{I} + \mathcal{J}^\Lambda) & \nu(\frac{1}{2}\mathcal{I} - \widetilde{\mathcal{J}}^\lambda) \end{bmatrix} = \begin{bmatrix} \mathcal{I} & 0 \\ \mathcal{N}(\frac{1}{2}\mathcal{I} + \mathcal{J}^\Lambda)(\mathcal{V}^\Lambda)^{-1} & \mathcal{I} \end{bmatrix} \begin{bmatrix} \mathcal{V}^\Lambda & -\widetilde{\mathcal{V}}^\lambda \\ 0 & \mathcal{H}_{\lambda\Lambda} \end{bmatrix} \tag{16}$$

where

$$\mathcal{H}_{\lambda\Lambda} := \nu(\tfrac{1}{2}\mathcal{I} - \widetilde{\mathcal{J}}^\lambda) + \mathcal{N}(\tfrac{1}{2}\mathcal{I} + \mathcal{J}^\Lambda)(\mathcal{V}^\Lambda)^{-1}\,\widetilde{\mathcal{V}}^\lambda.$$

Since $V_{ii}^{\lambda_i} - V_0$ and $\widetilde{V}_{ii}^\lambda - V_0$ are compact (as all off-diagonal operators are), it follows readily that $(\mathcal{V}^\Lambda)^{-1}\,\widetilde{\mathcal{V}}^\lambda - \mathcal{I}$ is compact in $\mathbf{H}^s$. Therefore $\mathcal{H}_{\lambda\Lambda} - \frac{1}{2}(\nu\mathcal{I} + \mathcal{N})$ is compact in $\mathbf{H}^s$ and thus $\mathcal{H}_{\lambda\Lambda}$ is Fredholm of index zero.

Hence, to prove the result we only have to show that $\mathcal{H}_{\lambda\Lambda}$ is one-to-one. If $\psi \in \mathbf{H}^{-1/2}$ satisfies $\mathcal{H}_{\lambda\Lambda}\psi = 0$ it can be seen that

$$u := \left| \begin{array}{ll} \widetilde{\mathcal{S}}^\lambda \psi, & \text{in } \Omega, \\ \mathcal{S}_i^{\lambda_i}(V_{ii}^{\lambda_i})^{-1}\Big(\sum_{j=1}^{d} \widetilde{V}_{ij}^\lambda \psi_j\Big), & \text{in } \Omega_i, \quad i = 1, \ldots, d, \end{array} \right.$$

is a solution to the homogeneous transmission problem in the half-plane, i.e., to (3–7) with $g_i^0 = g_i^1 = 0$, and thus has to vanish everywhere. By taking exterior traces it follows that $\widetilde{\mathcal{V}}^\lambda \psi = 0$, and hence $\psi = 0$ by Lemma 3.2. □

Obviously, this result implies that taking expression (14) such that (15) holds, we have the unique solution to equations (3–7).

4. Numerical approximation

We consider two families of finite dimensional spaces depending on the parameter $N \to \infty$

$$X_N \subset H^{-1/2}, \qquad Y_N \subset H^{1/2}, \qquad \dim X_N = \dim Y_N,$$

and the product spaces

$$\mathbf{X}_N := \underbrace{X_N \times \ldots \times X_N}_{d\text{ times}}, \qquad \mathbf{Y}_N := \underbrace{Y_N \times \ldots \times Y_N}_{d\text{ times}}.$$

We consider a Petrov–Galerkin scheme for the system of integral equations (15):

$$\left| \begin{array}{lll} \text{Find } \varphi_N, \psi_N \in \mathbf{X}_N \text{ such that} & & \\ (\mathcal{V}^\Lambda \varphi_N, r_N) - (\widetilde{\mathcal{V}}^\lambda \psi_N, r_N) & = (g_0, r_N), & \forall r_N \in \mathbf{X}_N, \\ (\mathcal{N}(\tfrac{1}{2}\mathcal{I} + \mathcal{J}^\Lambda)\varphi_N, t_N) + (\nu(\tfrac{1}{2}\mathcal{I} - \widetilde{\mathcal{J}}^\lambda)\psi_N, t_N) & = (g_1, t_N), & \forall t_N \in \mathbf{Y}_N. \end{array} \right. \tag{17}$$

Remark. As we have taken them here, the dimensions of all the components are the same, but we can easily create families $\mathbf{X}_N := X_{N_1} \times \ldots \times X_{N_d}$ and $\mathbf{Y}_N := Y_{N_1} \times \ldots \times Y_{N_d}$ directed on the parameter $N = (N_1, \ldots, N_d)$ which has to diverge componentwise. We stay on the original situation for notational simplicity.

Before giving necessary and sufficient conditions for convergence of this class of schemes we recall some general concepts on Petrov–Galerkin schemes for operator equations (see also [15, 17, 34]). Given two Hilbert spaces V and W, an isomorphism $\mathcal{A} : V \to W'$, a Petrov–Galerkin method consists of two sequences of finite dimensional subspaces

$$V_N \subset V, \qquad W_N \subset W, \qquad \dim V_N = \dim W_N,$$

and the discretization scheme:

$$\left| \begin{array}{l} v_N \in V_N, \\ (\mathcal{A} v_N, w_N) = (\mathcal{A} v, w_N), \qquad \forall w_N \in W_N. \end{array} \right.$$

For easy reference we will call this method the Petrov–Galerkin $\{V_N; W_N\}$ scheme for $\mathcal{A} : V \to W'$. The method is said to be stable if we can find $\gamma > 0$ independent of N (at least for N large enough) such that the discrete Babuška–Brezzi condition

$$\sup_{0 \neq w_N \in W_N} \frac{|(\mathcal{A} v_N, w_N)|}{\|w_N\|} \geq \gamma \|v_N\|, \qquad \forall v_N \in V_N, \tag{18}$$

holds. This condition is equivalent to unique solvability of the discrete equations plus the inequality

$$\|v_N\| \leq (\| \mathcal{A} \| / \gamma) \|v\|$$

and to the Céa estimate

$$\|v - v_N\| \leq (\| \mathcal{A} \| / \gamma) \inf_{u_N \in V_N} \|v - u_N\|.$$

Then convergence is equivalent to stability plus the approximation property:

$$\inf_{u_N \in V_N} \|v - u_N\| \to 0, \qquad \forall v \in V.$$

Hypothesis 1. The $\{X_N; Y_N\}$ method for $I : H^{-1/2} \to H^{-1/2}$ converges. Equivalently X_N satisfies the approximation property in $H^{-1/2}$ and we have the stability estimate

$$\sup_{0 \neq y_N \in Y_N} \frac{|(x_N, y_N)|}{\|y_N\|_{1/2}} \geq \gamma \|x_N\|_{-1/2}, \qquad \forall x_N \in X_N.$$

Theorem 4.1. Hypothesis 1 is equivalent to convergence of method (17) in $\mathbf{H}^{-1/2} \times \mathbf{H}^{-1/2}$.

Proof. Notice that inequality (11) (that is, ellipticity of $V_0 : H^{-1/2} \to H^{1/2}$) implies that the Galerkin $\{X_N; X_N\}$ scheme for V_0 is $H^{-1/2}-$stable. Hence, Hypothesis 1 is equivalent to convergence of the $\{X_N \times X_N; X_N \times Y_N\}$ method for the operator

$$\begin{bmatrix} V_0 & 0 \\ 0 & I \end{bmatrix} : H^{-1/2} \times H^{-1/2} \to H^{1/2} \times H^{-1/2}.$$

Convergence of this method is, in its turn, equivalent to convergence of the same scheme for the operator

$$\begin{bmatrix} V_0 & -V_0 \\ \alpha I & \beta I \end{bmatrix} : H^{-1/2} \times H^{-1/2} \to H^{1/2} \times H^{-1/2}, \tag{19}$$

with $\alpha, \beta > 0$, since

$$\begin{bmatrix} V_0 & -V_0 \\ \alpha I & \beta I \end{bmatrix} = \begin{bmatrix} V_0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -I \\ \alpha I & \beta I \end{bmatrix}$$

and the right-most matrix in this expression defines a uniformly bounded isomorphism in $X_N \times X_N$. Moreover, convergence of the $\{X_N \times X_N; X_N \times Y_N\}$ method for operator (19) is equivalent to convergence of the $\{\mathbf{X}_N \times \mathbf{X}_N; \mathbf{X}_N \times \mathbf{Y}_N\}$ method for

$$\mathcal{H}_p := \begin{bmatrix} \mathcal{V}_0 & -\mathcal{V}_0 \\ \frac{1}{2}\mathcal{N} & \frac{\nu}{2}\mathcal{I} \end{bmatrix} : \mathbf{H}^{-1/2} \times \mathbf{H}^{-1/2} \to \mathbf{H}^{1/2} \times \mathbf{H}^{-1/2},$$

since we can decouple this problem in d smaller ones. We have obtained so far that Hypothesis 1 is equivalent to convergence of this last method. However $\mathcal{H} - \mathcal{H}_p$ is compact, which finishes the result since convergence of Petrov–Galerkin schemes is preserved by compact perturbations (see [17, Theorem 13.7]). □

If we want to prove convergence of method (17) in $\mathbf{H}^s \times \mathbf{H}^s$, a correct statement of the method requires first that the discrete subspaces satisfy

$$X_N \subset H^s \cap H^{-s-1} \subset H^{-1/2}, \qquad Y_N \subset H^{-s}.$$

Hypothesis 2. For a given value of s:

(a) The $\{X_N; Y_N\}$ method is stable for $I : H^s \to H^s$.

(b) The $\{X_N; X_N\}$ method is stable for $V_0 : H^s \to H^{s+1}$.

(c) X_N satisfies the approximation property in H^s.

Equivalently, we have simultaneous convergence of the $\{X_N; Y_N\}$ scheme for $I : H^s \to H^s$ and of the $\{X_N; X_N\}$ scheme for $V_0 : H^s \to H^{s+1}$.

In terms of Babuška–Brezzi conditions (a) and (b) read respectively

$$\sup_{0 \neq y_N \in Y_N} \frac{|(x_N, y_N)|}{\|y_N\|_{-s}} \geq \gamma \|x_N\|_s, \qquad \forall x_N \in X_N$$

and

$$\sup_{0 \neq z_N \in X_N} \frac{|(V_0 x_N, z_N)|}{\|z_N\|_{-s-1}} \geq \gamma \|x_N\|_s, \qquad \forall x_N \in X_N,$$

for some $\gamma > 0$ independent of N (possibly different in both formulas). When $s = -1/2$, this hypothesis coincides with Hypothesis 1. Following step by step the proof of Theorem 4.1, we can prove the next result.

Theorem 4.2. Hypothesis 2 is equivalent to convergence of method (17) in $\mathbf{H}^s \times \mathbf{H}^s$.

5. Spectral approximation

A particular example where the scheme works is taking a pure Galerkin method with $X_N = Y_N = T_N := \text{span}\{\phi_n \mid -N/2 \leq n < N/2\}$, where $\phi_n(t) := \exp(2\pi\imath n t)$. We recall that the truncation operator for the Fourier series

$$S_N u := \sum_{-\frac{N}{2} \leq n < \frac{N}{2}} (u, \phi_n) \phi_n$$

satisfies for arbitrary $t \geq s$ (see [25, Theorem 8.2.1])

$$\|u - S_N u\|_s \leq (2/N)^{t-s} \|u\|_t, \qquad \forall u \in H^t, \tag{20}$$

and $S_N u \to u$ in H^s for all $u \in H^s$. Notice that $S_N u$ is characterized by the problem

$$\left| \begin{array}{l} S_N u \in T_N, \\ (S_N u, v_N) = (u, v_N), \qquad \forall v_N \in T_N. \end{array} \right.$$

Therefore, for all $s \in \mathbb{R}$, the Galerkin $\{T_N; T_N\}$ method for $I : H^s \to H^s$ is stable: one just has to take $t = s$ in inequality (20).

Proposition 5.1. The $\{T_N; T_N\}$ method is stable for $V_0 : H^s \to H^{s+1}$ for all $s \in \mathbb{R}$.

Proof. Notice first that

$$(V_0 u, v) = (u, V_0 v), \qquad \forall u \in H^s,\ v \in H^{-s-1},$$

and that, being V_0 a periodic convolution operator, $V_0 v_N \in T_N$ for all $v_N \in T_N$. Therefore the solution to

$$\left| \begin{array}{l} u_N \in T_N, \\ (V_0 u_N, v_N) = (V_0 u, v_N), \qquad \forall v_N \in T_N, \end{array} \right.$$

is simply $u_N = S_N u$. This fact and inequality (20) prove the result. □

Proposition 5.2. For all $s \in \mathbb{R}$, the scheme (17) is convergent in $\mathbf{H}^s \times \mathbf{H}^s$. Moreover, if $\varphi, \psi \in \mathbf{H}^t$ we have the error estimate

$$\|\varphi - \varphi_N\|_s + \|\psi - \psi_N\|_s \le C(1/N)^{t-s}\big(\|\varphi\|_t + \|\psi\|_t\big) \tag{21}$$

for arbitrary $s \le t$, with C depending on s and t.

Proof. It is a straightforward consequence of the theory in the preceding section plus the observations above. □

When data are given by the incident waves (see definitions (8, 12, 13)) we can ensure that $\varphi, \psi \in \mathbf{H}^t$ for all $t \in \mathbb{R}$.

6. Periodic smoothest splines in uniform meshes

Let $h := 1/N$, $x_i := i\,h$ and $x_{i+1/2} := (i + 1/2)\,h$ for all $i \in \mathbb{Z}$. Given a non-negative integer m we consider the spaces of periodic smoothest splines on staggered grids

$$\begin{aligned} \widehat{S}_h^m &:= \{u_h \in \mathcal{C}^{m-1} \mid u_h|_{[x_{i-1/2}, x_{i+1/2}]} \in \mathbb{P}_m, \quad \forall i\}, \\ S_h^m &:= \{v_h \in \mathcal{C}^{m-1} \mid u_h|_{[x_{i-1}, x_i]} \in \mathbb{P}_m, \quad \forall i\}, \end{aligned}$$

where $\mathcal{C}^k$ is the space of 1-periodic complex valued functions with continuous derivatives up to order k (when $k = -1$, we just demand periodicity and drop any continuity assumption) and $\mathbb{P}_m$ is the space of polynomials of degree less than or equal to m. The method we consider here consists of taking

$$X_N = \widehat{S}_h^m, \qquad Y_N = S_h^{m+1}.$$

Lemma 6.1. Let $u_h \in \widehat{S}_h^m$ and $v_h \in S_h^{m+1}$. Then,

$$(u_h, v_h) = \sum_{\mu \in \Lambda_N} C^N_{\mu,m}\, \widehat{u}_h(\mu)\, \overline{\widehat{v}_h(\mu)}$$

where

$$C^N_{\mu,m} := \left| \begin{array}{ll} 1, & \mu = 0, \\ \displaystyle\sum_{k\in\mathbb{Z}} (-1)^k \left(\frac{\mu}{\mu + kN}\right)^{2m+3}, & 0 \neq \mu \in \Lambda_N, \end{array}\right. \tag{22}$$

and $\Lambda_N := \{\mu \in \mathbb{Z} \mid -\frac{N}{2} \leq \mu < \frac{N}{2}\}$.

Proof. First notice that $w_h \in S_h^m$ if and only if for all $\mu, k \in \mathbb{Z}$ there holds

$$\widehat{w}_h(\mu)\, \mu^{m+1} = \widehat{w}_h(\mu + kN)\, (\mu + kN)^{m+1}, \tag{23}$$

(see for instance [3] or [25, Lemmas 13.2.1 and 13.2.2]). By using that $z_h \in \widehat{S}_h^m$ if and only if $z_h(\cdot - h/2) \in S_h^m$, it can be easily proven that $z_h \in \widehat{S}_h^m$ if and only if for all $\mu, k \in \mathbb{Z}$ there holds

$$\widehat{z}_h(\mu)\, \mu^{m+1} = \widehat{z}_h(\mu + kN)\, (-1)^k (\mu + kN)^{m+1}. \tag{24}$$

Now the result is direct from identities (23–24) and the equality

$$(u_h, v_h) = \widehat{u}_h(0)\, \overline{\widehat{v}_h(0)} + \sum_{0 \neq \mu \in \Lambda_N} \sum_{k\in\mathbb{Z}} \widehat{u}_h(\mu + kN)\, \overline{\widehat{v}_h(\mu + kN)}.$$

□

Lemma 6.2. Let $m \geq 0$. The function

$$C_m(\xi) := \xi^{2m+3} \sum_{k\in\mathbb{Z}} (-1)^k \frac{1}{(k+\xi)^{2m+3}}, \qquad \xi \in [-1/2, 1/2], \tag{25}$$

satisfies

(a) $C_m(0) = 1$,

(b) $C_m(-\xi) = C_m(\xi)$,

(c) $C_m(\mu/N) = C^N_{\mu,m}$, $C^N_{\mu,m}$ being the coefficients defined in equation (22),

(d) $\min_{\xi\in[-1/2,1/2]} C_m(\xi) \geq 1 - \pi^2/24 > 0$.

Proof. Properties (a), (b) and (c) are straightforward. To prove (d), note that it is enough to study the minimum on $[0, 1/2]$. When $\xi \in [0, 1/2]$, eliminating the term with $k = -1$ and reordering the sums, we obtain

$$C_m(\xi) \geq 1 + \sum_{k=1}^{\infty} (-1)^k \left(\frac{\xi}{k+\xi}\right)^{2m+3} + \sum_{k=2}^{\infty} (-1)^{k+1} \left(\frac{\xi}{k-\xi}\right)^{2m+3}.$$

Now, the (very rough) bounds

$$\left| \sum_{k=1}^{\infty} (-1)^k \left(\frac{\xi}{k+\xi}\right)^{2m+3} \right| \leq \xi^{2m+3} \sum_{k=1}^{\infty} \frac{1}{k^{2m+3}} \leq \xi^3 \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{\pi^2}{48}$$

and

$$\left| \sum_{k=2}^{\infty} (-1)^{k+1} \left(\frac{\xi}{k-\xi}\right)^{2m+3} \right| \leq \xi^{2m+3} \sum_{k=2}^{\infty} \frac{1}{(k-1/2)^{2m+3}} \leq \xi^3 \sum_{k=1}^{\infty} \frac{1}{k^2} \leq \frac{\pi^2}{48}$$

imply (d). □

Proposition 6.3. The discrete problem

$$\left| \begin{array}{l} u_h \in \widehat{S}_h^m, \\ (u_h, v_h) = (u, v_h), \qquad \forall v_h \in S_h^{m+1}, \end{array} \right. \tag{26}$$

has a unique solution and there exists $C > 0$, independent of h, such that

$$\|u_h\|_0 \leq C \|u\|_0.$$

Hence the Petrov–Galerkin $\{\widehat{S}_h^m; S_h^{m+1}\}$ method for $I : H^0 \to H^0$ is stable.

Proof. Let us consider the basis of S_h^{m+1}, denoted $\{\varphi_h^\mu\}_{\mu \in \Lambda_N}$, characterized by the relations

$$\widehat{\varphi}_h^\mu(\nu) = \delta_{\mu\nu}, \qquad \forall \nu, \mu \in \Lambda_N,$$

(see [3, 10]). Then by Lemmas 6.1 and 6.2 (c),

$$(u, \varphi_h^\mu) = C_m(\mu/N)\, \widehat{u}_h(\mu).$$

Applying Lemma 6.2 (d) and equality (23), we obtain for all $\mu \in \Lambda_N$

$$\begin{aligned} |\widehat{u}_h(\mu)| \leq C|(u, \varphi_h^\mu)| &= C \Big| \sum_{\nu \in \Lambda_N} \sum_{k \in \mathbb{Z}} \widehat{u}(\nu + kN)\, \overline{\widehat{\varphi}_h^\mu(\nu + kN)} \Big| \\ &= C \Big| \sum_{k \in \mathbb{Z}} \widehat{u}(\mu + kN) \Big(\frac{\mu}{\mu + kN}\Big)^{m+2} \Big| \leq C' \Big(\sum_{k \in \mathbb{Z}} |\widehat{u}(\mu + kN)|^2 \Big)^{1/2}. \end{aligned}$$

This and the inequality $\|u_h\|_0^2 \le C\sum_{\mu\in\Lambda_N} |\widehat{u}_h(\mu)|^2$ (which can be easily proven using (24)) yield

$$\|u_h\|_0^2 \le C \sum_{\mu\in\Lambda_N}\sum_{z\in\mathbb{Z}} |\widehat{u}(\mu+kN)|^2 = C\|u\|_0^2,$$

and the result is proven. □

Proposition 6.4. The $\{\widehat{S}_h^m; S_h^{m+1}\}$ method is stable for the operator $I : H^s \to H^s$ for all $-m-3/2 < s < m+1/2$.

Proof. The result follows from Proposition 6.3 by rather standard arguments. For $s \in (0, m+1/2)$, we use the existence of a projection $\pi_h : H^0 \to \widehat{S}_h^m$ that is stable (uniformly bounded) simultaneously in H^0 and H^s and satisfies the inequality

$$\|v - \pi_h v\|_0 \le Ch^s \|v\|_s, \qquad \forall v \in H^s,$$

with C depending only on s (see [25, Lemma 13.3.1]). Then, if u_h is the solution to problem (26) it follows that

$$\begin{aligned}\|u_h\|_s &\le \|u_h - \pi_h u\|_s + \|\pi_h u\|_s \le C\big[h^{-s}\|u_h - \pi_h u\|_0 + \|u\|_s\big]\\ &\le Ch^{-s}(\|u - u_h\|_0 + \|u - \pi_h u\|_0) + C\|u\|_s \le C'\|u\|_s\end{aligned}$$

where we have applied the inverse inequality (see [25, Lemma 13.3.2] or [23, Theorem 2.11])

$$\|v_h\|_s \le Ch^{-s}\|v_h\|_0, \qquad \forall v_h \in \widehat{S}_h^m.$$

Instead of proving stability for $s \in (-m-3/2, 0)$ we can show that the $\{S_h^{m+1}; \widehat{S}_h^m\}$ method for $I : H^{-s} \to H^{-s}$ is stable. To do this we notice that stability with $s = 0$ has already been proven in Proposition 6.3, since we can always reverse inf-sup conditions for discrete spaces of the same dimension and that we can repeat step by step the proof given above using now similar properties of the discrete space S_h^{m+1}. □

Proposition 6.5. The $\{\widehat{S}_h^m; \widehat{S}_h^m\}$ method is stable for the operator $V_0 : H^s \to H^{s+1}$ for all $-m-3/2 < s < m+1/2$.

Proof. This result is well-known, see for instance [23, Theorem 13.27]. □

Proposition 6.6. The scheme (17) is convergent in $\mathbf{H}^s \times \mathbf{H}^s$ for all $-m-3/2 < s < m+1/2$. Furthermore for couples (s,t) satisfying

$$-m-2 \le s < m+1/2, \qquad -m-3/2 < t \le m+1, \qquad s \le t,$$

we have the estimate

$$\|\varphi - \varphi_h\|_s + \|\psi - \psi_h\|_s \leq Ch^{t-s}\big(\|\varphi\|_t + \|\psi\|_t\big) \tag{27}$$

provided that $\varphi, \psi \in \mathbf{H}^t$.

Proof. The first assertion follows from Propositions 6.4 and 6.5 by the theory given in section 4. Once stability has been shown, convergence estimates for $s \in (-m-3/2, m+1/2)$ follow from Céa's lemma and approximation properties of splines (see [25, Chapter 13] or [23, Chapter 2]). For $s \in (-m-2, -m-3/2)$ the bounds can be verified using the Aubin–Nitsche duality technique. For notational simplicity, we group $\xi := (\varphi, \psi)$ and $\xi_h := (\varphi_h, \psi_h)$. For $s_1, s_2 \in \mathbb{R}$ we denote by $\|\cdot\|_{s_1,s_2}$ the norm in $\mathbf{H}^{s_1} \times \mathbf{H}^{s_2}$. Let $\mathcal{H}^*$ be the adjoint operator of $\mathcal{H}$ defined in equation (15). Then, by Proposition 3.3, $\mathcal{H}^* : \mathbf{H}^{-r-1} \times \mathbf{H}^{-r} \to \mathbf{H}^{-r} \times \mathbf{H}^{-r}$ is an isomorphism for all $r \in \mathbb{R}$. By standard duality arguments, if the $\{\widehat{\mathbf{S}}_h^m \times \widehat{\mathbf{S}}_h^m; \widehat{\mathbf{S}}_h^m \times \mathbf{S}_h^{m+1}\}$ method for $\mathcal{H}$ is $\mathbf{H}^r \times \mathbf{H}^r$-stable for $r \in \mathbb{R}$, then the transposed method $\{\widehat{\mathbf{S}}_h^m \times \mathbf{S}_h^{m+1}; \widehat{\mathbf{S}}_h^m \times \widehat{\mathbf{S}}_h^m\}$ for $\mathcal{H}^*$ is $\mathbf{H}^{-r-1} \times \mathbf{H}^{-r}$-stable for the same value of r. Thus,

$$\|\xi - \xi_h\|_{s,s} = \sup_{v \in \mathbf{H}^{-s} \times \mathbf{H}^{-s}} \frac{|(\xi - \xi_h, v)|}{\|v\|_{-s,-s}} \leq C \sup_{w \in \mathbf{H}^{-s-1} \times \mathbf{H}^{-s}} \frac{|(\xi - \xi_h, \mathcal{H}^* w)|}{\|w\|_{-s-1,-s}}. \tag{28}$$

Now, for $w \in \mathbf{H}^{-s-1} \times \mathbf{H}^{-s}$ we take the solution to the problem

$$\left| \begin{array}{l} w_h \in \widehat{\mathbf{S}}_h^m \times \mathbf{S}_h^{m+1}, \\ (\zeta_h, \mathcal{H}^* w_h) = (\zeta_h, \mathcal{H}^* w), \qquad \forall \zeta_h \in \widehat{\mathbf{S}}_h^m \times \widehat{\mathbf{S}}_h^m. \end{array} \right.$$

Note that this is the $\{\widehat{\mathbf{S}}_h^m \times \mathbf{S}_h^{m+1}; \widehat{\mathbf{S}}_h^m \times \widehat{\mathbf{S}}_h^m\}$ method for $\mathcal{H}^*$. Setting $r := \min\{t, -1/2\}$, which satisfies $r \in (-m-3/2, -1/2)$,

$$\|\xi - \xi_h\|_{r,r} \leq Ch^{t-r}\|\xi\|_{t,t}, \tag{29}$$

$$\|w - w_h\|_{-r-1,-r} \leq Ch^{-s+r}\|w\|_{-s-1,-s}. \tag{30}$$

By bound (28), the inequality

$$\begin{aligned} |(\xi - \xi_h, \mathcal{H}^* w)| = |(\mathcal{H}(\xi - \xi_h), w - w_h)| &\leq \|\mathcal{H}(\xi - \xi_h)\|_{r+1,r}\|w - w_h\|_{-r-1,-r} \\ &\leq C\|\xi - \xi_h\|_{r,r}\|w - w_h\|_{-r-1,-r}, \end{aligned}$$

and estimates (29–30), we obtain bound (27). □

7. A method with non-uniform meshes

Let now $0 = x_0 < x_1 < x_2 < \ldots < x_N = 1$ and consider the following notations:

$$h_i := x_i - x_{i-1}, \qquad \widehat{h}_i := \tfrac{1}{2}(h_i + h_{i+1}), \qquad x_{i+\frac{1}{2}} := \tfrac{1}{2}(x_i + x_{i+1}),$$

and $h := \max_i h_i$. Notice that $h/2 \le \max_i \widehat{h}_i \le h$. We define the spaces

$$\begin{aligned} \widehat{S}_h^0 &:= \{u_h \in H^0 \,|\, u_h|_{[x_{i-1/2}, x_{i+1/2}]} \in \mathbb{P}_0, \quad \forall i\}, \\ S_h^1 &:= \{u_h \in \mathcal{C}^0 \,|\, v_h|_{[x_{i-1}, x_i]} \in \mathbb{P}_1, \quad \forall i\}, \end{aligned}$$

and consider their respective elementary bases defined as follows: χ_i will denote the periodized characteristic function of $(x_{i-1/2}, x_{i+1/2})$ and η_i the function of S_h^1 such that $\eta_i(x_j) = \delta_{ij}$.

Proposition 7.1. For arbitrary grids

$$\sup_{0 \ne u_h \in \widehat{S}_h^0} \frac{|(u_h, v_h)|}{\|u_h\|_0} \ge \frac{1}{2}\, \|v_h\|_0, \qquad \forall v_h \in S_h^1.$$

Proof. Let $T_h : S_h^1 \to \widehat{S}_h^0$ be defined by $T_h(\sum_{i=1}^N v_i \eta_i) := \sum_{i=1}^N v_i \chi_i$. Elementary computations show that

$$(T_h v_h, v_h) \; = \frac{3}{4} \sum_{i=1}^N \widehat{h}_i\, |v_i|^2 + \frac{1}{4} \sum_{i=1}^N h_i \,\mathrm{Re}\,(v_i \overline{v_{i-1}}) \; \ge \frac{1}{2} \sum_{i=1}^N \widehat{h}_i\, |v_i|^2.$$

Besides, it is simple to verify that

$$\|T_h v_h\|_0 = \left[\sum_{i=1}^N \widehat{h}_i\, |v_i|^2\right]^{1/2}$$

and

$$\frac{1}{\sqrt{3}} \left[\sum_{i=1}^N \widehat{h}_i\, |v_i|^2\right]^{1/2} \le \|v_h\|_0 \le \left[\sum_{i=1}^N \widehat{h}_i\, |v_i|^2\right]^{1/2},$$

which completes the result. □

Lemma 7.2. Assume that

$$\frac{1}{C_1} \le \frac{h_{i-1}}{h_i} \le C_1, \tag{31}$$

$$6 - \left(\frac{\widehat{h}_{i-1}}{\widehat{h}_i} + \frac{\widehat{h}_i}{\widehat{h}_{i-1}} \right) \ge C_2, \tag{32}$$

with constants C_1, $C_2 > 0$ independent of h. Then,

$$\sum_{i=1}^N h_i^2 \int_{x_{i-1}}^{x_i} |u_h|^2 \le C \sum_{i=1}^N \left(\frac{|(u_h, \eta_i)|}{\|\eta_i\|_1} \right)^2, \qquad \forall u_h \in \widehat{S}_h^0.$$

Proof. Let $u_h = \sum_{i=1}^N u_i \chi_{\widehat{i}} \in \widehat{S}_h^0$. Without lost of generality we can assume that $u_i \in \mathbb{R}$ for all i. As h_i, $h_{i+1} \le 2\widehat{h}_i$,

$$\sum_{i=1}^N h_i^2 \int_{x_{i-1}}^{x_i} u_h^2 = \sum_{i=1}^N \frac{h_i^3 + h_{i+1}^3}{2} \, u_i^2 \le 8 \sum_{i=1}^N \widehat{h}_i^3 u_i^2. \tag{33}$$

Easy computations and inequality (32) show that

$$\begin{aligned}
\sum_{i=1}^N (u_h, \eta_i) \, \widehat{h}_i^2 u_i &= \frac{1}{8} \sum_{i=1}^N (h_i u_{i-1} + 3(h_i + h_{i+1}) u_i + h_{i+1} u_{i+1}) \, \widehat{h}_i^2 u_i \\
&= \frac{1}{8} \sum_{i=1}^N h_i \left(3\widehat{h}_i^2 u_i^2 + 3\widehat{h}_{i-1}^2 u_{i-1}^2 + \left(\frac{\widehat{h}_i}{\widehat{h}_{i-1}} + \frac{\widehat{h}_{i-1}}{\widehat{h}_i} \right) \widehat{h}_{i-1} \widehat{h}_i u_{i-1} u_i \right) \\
&\ge \frac{1}{16} \sum_{i=1}^N h_i \left(6 - \left(\frac{\widehat{h}_i}{\widehat{h}_{i-1}} + \frac{\widehat{h}_{i-1}}{\widehat{h}_i} \right) \right) (\widehat{h}_i^2 u_i^2 + \widehat{h}_{i-1}^2 u_{i-1}^2) \\
&\ge \frac{C_2}{8} \sum_{i=1}^N \widehat{h}_i^3 u_i^2.
\end{aligned} \tag{34}$$

From inequalities (31) we deduce now that

$$\|\eta_i\|_1^2 = \frac{2}{3} \, \widehat{h}_i + \frac{1}{h_i} + \frac{1}{h_{i+1}} \le C \widehat{h}_i^{-1}$$

and hence,

$$\sum_{i=1}^{N}(u_h,\eta_i)\,\widehat{h}_i^2 u_i \le \left(\sum_{i=1}^{N}(u_h,\eta_i)^2\,\widehat{h}_i\right)^{1/2}\left(\sum_{i=1}^{N}\widehat{h}_i^3 u_i^2\right)^{1/2}$$
$$\le C\left(\sum_{i=1}^{N}\left(\frac{(u_h,\eta_i)}{\|\eta_i\|_1}\right)^2\right)^{1/2}\left(\sum_{i=1}^{N}\widehat{h}_i^3 u_i^2\right)^{1/2}.$$

Finally, using equation (34),

$$\sum_{i=1}^{N}\left(\frac{(u_h,\eta_i)}{\|\eta_i\|_1}\right)^2 \ge C\sum_{i=1}^{N}\widehat{h}_i^3\, u_i^2,$$

and inequality (33) proves the result. □

Proposition 7.3. Assume that there exist $C_1, C_2 > 0$, independent of h, such that conditions (31) and (32) hold. Then, the $\{\widehat{S}_h^0; S_h^1\}$ method is stable for the operator $I : H^s \times H^s$ for all $s \in [-1, 0]$.

Proof. Let $G_h : H^{-1} \to \widehat{S}_h^0$ be the operator defined by

$$\left|\begin{array}{l} G_h u \in \widehat{S}_h^0, \\ (G_h u, v_h) = (u, v_h), \qquad \forall v_h \in S_h^1. \end{array}\right.$$

Firstly we prove that

$$\|G_h u\|_{-1} \le C\|u\|_{-1}, \qquad \forall u \in H^{-1}. \tag{35}$$

By the well-known properties (see [25, Chapter 5]) of the Bessel operator $\Lambda := 4\pi V_0$ (see definition (10)) we can define $\Pi_h : H^1 \to \widehat{S}_h^0$ by

$$\left|\begin{array}{l} \Pi_h u \in \widehat{S}_h^0, \\ (\Pi_h u, v_h) = (\Lambda^{-2}u, v_h) = (u, v_h)_1, \qquad \forall v_h \in S_h^1. \end{array}\right.$$

Notice that if we show that

$$\|\Pi_h u\|_{-1} \le C\|u\|_1, \qquad \forall u \in H^1, \tag{36}$$

then, as

$$(G_h u, v_h) = (u, v_h) = (\Lambda^{-2}\Lambda^2 u, v_h) = (\Pi_h \Lambda^2 u, v_h), \qquad \forall v_h \in S_h^1,$$

that is, $G_h = \Pi_h \Lambda^2$, this gives bound (35).

Let $P_h : H^1 \to S_h^1$ be the interpolation operator on the points x_i (see [23, Chapter 2] for properties of this operator). Then,

$$\|\Pi_h u\|_{-1} = \sup_{v\in H^1} \frac{|(\Pi_h u, v)|}{\|v\|_1} \leq \sup_{v\in H^1} \frac{|(\Pi_h u, P_h v)|}{\|v\|_1} + \sup_{v\in H^1} \frac{|(\Pi_h u, v - P_h v)|}{\|v\|_1}. \quad (37)$$

On the one hand,

$$|(\Pi_h u, P_h v)| = |(u, P_h v)_1| \leq \|u\|_1 \|P_h v\|_1 \leq C\|u\|_1\|v\|_1, \qquad \forall v \in H^1. \quad (38)$$

On the other hand, by Lemma 7.2,

$$\begin{aligned}\sum_{i=1}^N h_i^2 \int_{x_{i-1}}^{x_i} |\Pi_h u|^2 \leq C \sum_{i=1}^N \left(\frac{|(\Pi_h u, \eta_i)|}{\|\eta_i\|_1}\right)^2 &= C \sum_{i=1}^N \left(\frac{|(u, \eta_i)_1|}{\|\eta_i\|_1}\right)^2 \\ \leq C \sum_{i=1}^N \int_{x_{i-1}}^{x_{i+1}} (|u|^2 + |u'|^2) &\leq C' \|u\|_1^2,\end{aligned}$$

and therefore

$$\begin{aligned}|(\Pi_h u, v - P_h v)| &\leq \left(\sum_{i=1}^N h_i^2 \int_{x_{i-1}}^{x_i} |\Pi_h u|^2\right)^{1/2} \left(\sum_{i=1}^N h_i^{-2} \int_{x_{i-1}}^{x_i} |v - P_h v|^2\right)^{1/2} \\ &\leq C\|u\|_1\|v\|_1. \end{aligned} \quad (39)$$

Now estimates (37–39) yield bound (36) and consequently the $\{\widehat{S}_h^0, S_h^1\}$ method is stable for $I : H^{-1} \to H^{-1}$, i.e., inequality (35) holds. Proposition 7.1 proves stability of the same method for $I : H^0 \to H^0$. For $s \in (-1, 0)$ the result follows by interpolation. □

8. A simple extension

We have restricted the exposition to the two dimensional parameterized equations proceeding from time-harmonic solutions to the heat equation with the additional constraint $\partial_n u|_\Pi = 0$. We briefly explain how these results apply to more general situations and what modifications have to be performed in order to analyse the methods.

Let Γ_i ($i = 1, \ldots, d$) be the boundaries of $\Omega_i \subset \mathbb{R}^n$, $n = 2$ or 3. We assume that each Γ_i is simply connected and smooth. We denote

$$\Gamma := \cup_{i=1}^d \Gamma_i, \qquad \Omega := \mathbb{R}^n \backslash \left(\cup_{i=1}^d \overline{\Omega}_i\right),$$

(note that now we consider the whole plane or space). For $s \in \mathbb{R}$ we set $\mathbf{H}^s(\Gamma) := H^s(\Gamma_1) \times \ldots \times H^s(\Gamma_d)$.

Consider the fundamental solution to the Helmholtz equation

$$\phi_\rho(\mathbf{x},\mathbf{y}) := \begin{cases} \dfrac{\imath}{4} H_0^{(1)}(\rho\,|\mathbf{x}-\mathbf{y}|), & \text{in two dimensions,} \\ \dfrac{\exp(\imath\,\rho|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}, & \text{in three dimensions,} \end{cases}$$

and the associated single layer potentials

$$\mathcal{S}_j^\rho \varphi_j := \int_{\Gamma_j} \phi_\rho(\,\cdot\,,\mathbf{y})\,\varphi_j(\mathbf{y})\,d\gamma_{\mathbf{y}} \;:\; \mathbb{R}^n \longrightarrow \mathbb{C}.$$

Consider also the operators

$$V_{ij}^\rho \varphi_j := \int_{\Gamma_j} \phi_\rho(\,\cdot\,,\mathbf{y})\,\varphi_j(\mathbf{y})\,d\gamma_{\mathbf{y}} \;:\; \Gamma_i \longrightarrow \mathbb{C},$$
$$J_{ij}^\rho \varphi_j := \int_{\Gamma_j} \partial_{n(\,\cdot\,)} \phi_\rho(\,\cdot\,,\mathbf{y})\,\varphi_j(\mathbf{y})\,d\gamma_{\mathbf{y}} \;:\; \Gamma_i \longrightarrow \mathbb{C}.$$

We collect them as in section 3 and consider the system of boundary integral equations

$$\mathcal{H}\begin{bmatrix}\varphi\\ \psi\end{bmatrix} := \begin{bmatrix} \mathcal{V}^\Lambda & -\mathcal{V}^\lambda \\ \mathcal{N}(\frac{1}{2}\mathcal{I}+\mathcal{J}^\Lambda) & \nu(\frac{1}{2}\mathcal{I}-\mathcal{J}^\lambda) \end{bmatrix} \begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \begin{bmatrix} g_0 \\ g_1 \end{bmatrix}. \tag{40}$$

If $(\varphi,\psi) \in \mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\Gamma)$ solves equation (40), then

$$u := \left| \begin{array}{ll} \mathcal{S}^\lambda \psi, & \text{in } \Omega, \\ \mathcal{S}_i^{\lambda_i} \varphi_i, & \text{in } \Omega_i, \quad i = 1,\ldots,d, \end{array} \right. \tag{41}$$

satisfies equations (3–6). If $\lambda^2 \notin \mathbb{R}^+$, then $u \in H^1(\Omega)$. Otherwise u is only locally in $H^1(\Omega)$, up to the boundary, and satisfies the Sommerfeld radiation condition at infinity

$$\lim_{r\to\infty} r^{\frac{n-1}{2}}(\partial_r u - \imath\lambda u) = 0,$$

uniformly in all directions. Then we can easily prove the following results.

Proposition 8.1. The operator $\mathcal{H} : \mathbf{H}^s(\Gamma) \times \mathbf{H}^s(\Gamma) \longrightarrow \mathbf{H}^s(\Gamma) \times \mathbf{H}^{s+1}(\Gamma)$ is bounded for all $s \in \mathbb{R}$. Moreover, if $\nu_i \neq -\nu$ for all i, then $\mathcal{H}$ is Fredholm of index zero.

Proof. The first assertion is direct from the well-known properties of the boundary integral operators in $\mathcal{H}$ (see [5, Chapter 7] for instance). To see the second one, we introduce the operators

$$V_i \varphi := \int_{\Gamma_i} \phi_0^i(\,\cdot\,,\mathbf{y})\,\varphi(\mathbf{y})\,d\gamma_{\mathbf{y}} \;:\; \Gamma_i \longrightarrow \mathbb{C}$$

where

$$\phi_0^i(\mathbf{x},\mathbf{y}) := \begin{cases} -\dfrac{1}{2\pi}\log|r_i(\mathbf{x}-\mathbf{y})|, & \text{in two dimensions,} \\ \dfrac{1}{4\pi|\mathbf{x}-\mathbf{y}|}, & \text{in three dimensions.} \end{cases} \tag{42}$$

Here $r_i < 1/\operatorname{diam}\Omega_i$, which ensures ellipticity of $V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ in the two-dimensional case (see [21, Theorem 8.16]). In all cases, V_i is invertible from $H^s(\Gamma_i)$ to $H^{s+1}(\Gamma_i)$ for all $s \in \mathbb{R}$. Furthermore, $V_{ii}^\rho - V_i : H^s(\Gamma_i) \to H^{s+1}(\Gamma_i)$ is compact. The operators $J_{ij}^\rho : H^s(\Gamma_i) \to H^s(\Gamma_j)$ for all i,j and $V_{ij}^\rho : H^s(\Gamma_i) \to H^{s+1}(\Gamma_j)$ for $i \neq j$ are also compact for all $s \in \mathbb{R}$ (see [8] and references therein). We set $\mathcal{V}_0 := \operatorname{diag}(V_i)$. Then, the principal part of the operator $\mathcal{H}$ is

$$\begin{bmatrix} \mathcal{V}_0 & -\mathcal{V}_0 \\ \frac{1}{2}\mathcal{N} & \frac{\nu}{2}\mathcal{I} \end{bmatrix} = \begin{bmatrix} \mathcal{I} & 0 \\ \frac{1}{2}\mathcal{N}\mathcal{V}_0^{-1} & \mathcal{I} \end{bmatrix} \begin{bmatrix} \mathcal{V}_0 & -\mathcal{V}_0 \\ 0 & \frac{1}{2}(\nu\mathcal{I}+\mathcal{N}) \end{bmatrix}.$$

Since $\nu_i \neq -\nu$ for all i, we deduce its invertibility from the decomposition given above. □

Proposition 8.2. The operator $\mathcal{H}$ is invertible if

(a) $-\lambda_i^2$ is not a Dirichlet eigenvalue of the Laplace operator in Ω_i;

(b) $-\lambda^2$ is not a Dirichlet eigenvalue of the Laplace operator in Ω_i for all i;

(c) $\nu_i \neq -\nu$ for all i;

(d) the exterior homogeneous transmission problem

$$\left|\begin{array}{ll} \Delta u + \lambda^2 u = 0, & \text{in } \Omega, \\ \Delta u + \lambda_i^2 u = 0, & \text{in } \Omega_i, \quad i = 1,\ldots,d, \\ u|_{\Gamma_i}^{\text{int}} - u|_{\Gamma_i}^{\text{ext}} = 0, & i = 1,\ldots,d, \\ \nu_i\,\partial_n u|_{\Gamma_i}^{\text{int}} - \nu\,\partial_n u|_{\Gamma_i}^{\text{ext}} = 0, & i = 1,\ldots,d, \\ \lim_{r\to\infty} r^{\frac{n-1}{2}}(\partial_r u - \imath\lambda u) = 0, & \end{array}\right.$$

has no solution except for the trivial one.

Proof. Exactly the same as in the two dimensional parameterized problem. □

Conditions guaranteeing that (d) holds can be found in [8, 16, 18, 32, 33]. When ν, $\nu_i > 0$, $\lambda_j^2, \lambda^2 \notin \mathbb{R}^+$ and $\lambda_j/\lambda \in \mathbb{R}$, all the conditions above hold (see [18]). This case is of interest when we want to solve a transmission problem for the heat equation via Laplace transform methods (see [14, 28, 29]).

A parallel approximation for Petrov–Galerkin methods for subspaces $\mathbf{X}_N := X_N^1 \times \ldots \times X_N^d$ and $\mathbf{Y}_N := Y_N^1 \times \ldots \times Y_N^d$ with $X_N^i \subset H^{-1/2}(\Gamma_i)$, $Y_N^i \subset H^{1/2}(\Gamma_i)$ and $\dim X_N^i = \dim Y_N^i$ can be likewise performed. The abstract conditions are now:

- convergence of the global method in $\mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\Gamma)$ is equivalent to the convergence of the Petrov–Galerkin $\{X_N^i; Y_N^i\}$ method for $I : H^{-1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ for all i,
- convergence of the global method in $\mathbf{H}^s(\Gamma) \times \mathbf{H}^s(\Gamma)$ is equivalent to the simultaneous convergence of the Petrov–Galerkin $\{X_N^i; Y_N^i\}$ method for $I : H^s(\Gamma_i) \to H^s(\Gamma_i)$ and of the Galerkin $\{X_N^i; X_N^i\}$ method for $V_i : H^s(\Gamma_i) \to H^{s+1}(\Gamma_i)$.

In general it will be more difficult to find adequate pairs satisfying the new hypotheses. In three dimensions, if all the objects are sphere-like and can be parameterized in polar coordinates or are smooth deformations of the sphere, then spherical harmonics play the role of trigonometric polynomials and give a method with superalgebraic convergence order.

9. Lipschitz boundaries

We finish with some words for the non-smooth case. We keep the notations of section 8. Now

$$\mathcal{H} := \begin{bmatrix} \mathcal{V}^\Lambda & -\mathcal{V}^\lambda \\ \mathcal{N}\left(\frac{1}{2}\mathcal{I} + \mathcal{J}^\Lambda\right) & \nu\left(\frac{1}{2}\mathcal{I} - \mathcal{J}^\lambda\right) \end{bmatrix} : \mathbf{H}^s(\Gamma) \times \mathbf{H}^s(\Gamma) \longrightarrow \mathbf{H}^{s+1}(\Gamma) \times \mathbf{H}^s(\Gamma)$$

is bounded for all $s \in (-1, 0)$, see [7]. To deal with the loss of compactness of the operators J_{ii}^ρ, we introduce

$$J_i\varphi := \int_{\Gamma_i} \partial_{n(\,\cdot\,)}\phi_0^i(\,\cdot\,, \mathbf{y})\,\varphi(\mathbf{y})\,d\gamma_{\mathbf{y}} \; : \; \Gamma_i \longrightarrow \mathbb{C},$$

ϕ_0^i being as in expression (42).

Lemma 9.1. The operators

$$\nu_i\left(\tfrac{1}{2}I + J_i\right) + \nu\left(\tfrac{1}{2}I - J_i\right) \; : \; H^{-1/2}(\Gamma_i) \longrightarrow H^{-1/2}(\Gamma_i)$$

are Fredholm of index zero.

Proof. Since $V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ is elliptic and bounded, we are allowed to define in $H^{-1/2}(\Gamma_i)$ an equivalent norm induced by it,

$$\|v\|_{V_i} := |\langle V_i v, v\rangle|^{1/2}, \qquad \forall v \in H^{-1/2}(\Gamma_i).$$

Firstly we assume that $\nu_i \le 2\nu$ and decompose

$$\nu_i \left(\tfrac{1}{2}I + J_i\right) + \nu \left(\tfrac{1}{2}I - J_i\right) = \nu\Big(I + \big(\nu_i/\nu - 1\big)\big(\tfrac{1}{2}I + J_i\big)\Big). \tag{43}$$

Then by [27, Theorem 3.1]

$$C_i := \sup_{v\in H^{-1/2}(\Gamma_i)} \frac{\|\big(\tfrac{1}{2}I + J_i\big)v\|_{V_i}}{\|v\|_{V_i}} < 1, \tag{44}$$

and as $|\nu_i/\nu - 1| \le 1$, the operator $(\nu_i/\nu - 1)(\tfrac{1}{2}I + J_i)$ is a contraction in $H^{-1/2}(\Gamma_i)$ with respect to the norm $\|\cdot\|_{V_i}$. Thus the operator in identity (43) is invertible by the Neumann series. For $\nu_i \ge 2\nu$ we introduce the space

$$H_0^{-1/2}(\Gamma_i) := \{v \in H^{-1/2}(\Gamma_i) \mid \langle v, 1\rangle = 0\}$$

and the projection $P_i : H^{-1/2}(\Gamma_i) \to H_0^{-1/2}(\Gamma_i)$ given by

$$\left|\begin{array}{l} P_i u \in H_0^{-1/2}(\Gamma_i), \\ \langle V_i P_i u, v\rangle \;=\; \langle V_i u, v\rangle, \qquad \forall v \in H_0^{-1/2}(\Gamma_i). \end{array}\right.$$

Obviously $\|P_i\|_{V_i} = 1$ and $(\tfrac{1}{2}I - J_i)(I - P_i) : H^{-1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is compact since the image space of $I - P_i$ is one-dimensional. By [27, Theorem 3.2]

$$C_i' := \sup_{v\in H_0^{-1/2}(\Gamma_i)} \frac{\|\big(\tfrac{1}{2}I - J_i\big)v\|_{V_i}}{\|v\|_{V_i}} < 1. \tag{45}$$

Indeed, $C_i' = C_i$ for all i. Finally, we decompose

$$\nu_i\big(\tfrac{1}{2}I + J_i\big) + \nu\left(\tfrac{1}{2}I - J_i\right) = \nu_i\big(I + (\nu/\nu_i - 1)\big(\tfrac{1}{2}I - J_i\big)P_i\big) + (\nu_i - \nu)\big(\tfrac{1}{2}I - J_i\big)(I - P_i).$$

Proceeding as in the previous case, the first operator is invertible and as we have said before the second one is compact, which finishes the result. □

Proposition 9.2. $\mathcal{H} : \mathbf{H}^{-1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\Gamma) \longrightarrow \mathbf{H}^{1/2}(\Gamma) \times \mathbf{H}^{-1/2}(\Gamma)$ is Fredholm of index zero.

Proof. Omitting compact parts and decoupling the resulting operator in d smaller ones, we can equivalently show that the operators

$$H_i := \begin{bmatrix} V_i & -V_i \\ \nu_i\left(\tfrac{1}{2}I + J_i\right) & \nu\left(\tfrac{1}{2}I - J_i\right) \end{bmatrix}$$

are Fredholm of index zero. Furthermore, by the decomposition

$$H_i = \begin{bmatrix} I & 0 \\ 0 & \frac{1}{\nu+\nu_i} I \end{bmatrix} \begin{bmatrix} V_i & 0 \\ 2\,\nu_i \nu J_i & \nu_i \left(\frac{1}{2}I + J_i\right) + \nu \left(\frac{1}{2}I - J_i\right) \end{bmatrix} \begin{bmatrix} I & -I \\ \nu_i I & \nu I \end{bmatrix}$$

and the invertibility of V_i, the result is a consequence of Lemma 9.1. □

We have injectivity of $\mathcal{H}$ for the same values of the parameters λ, λ_i, ν, ν_i as in the smooth case since in the proof we do not use the regularity of the boundaries: see Proposition 3.3 and the comments thereafter. In this situation, by the Fredholm alternative, we can assure uniqueness. The analysis of numerical methods, even in the natural $H^{-1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ norm is more involved, since we have to stabilize the $\{X_N^i; Y_N^i\}$ methods for

$$I + (\nu_i/\nu - 1)\left(\tfrac{1}{2}I + J_i\right), \quad \text{if } \nu_i \leq 2\nu,$$
$$I + (\nu/\nu_i - 1)\left(\tfrac{1}{2}I - J_i\right)P_i, \quad \text{if } \nu_i \geq 2\nu,$$

which appear now in the study of the principal part of the operator $\mathcal{H}$. This can be accomplished provided that

$$\sup_{0 \neq y_N \in Y_N^i} \frac{|\langle x_N, y_N \rangle|}{\|y_N\|_{1/2,\Gamma_i}} \geq \gamma_i \|x_N\|_{-1/2,\Gamma_i}, \qquad \forall x_N \in X_N^i,$$

for $\gamma_i > 0$ independent of N suffiently large. Explicitly, a condition that guarantees this is

$$\gamma_i > |\nu_i/\nu - 1|\, C_i \sqrt{M_i/\alpha_i}, \quad \text{if } \nu_i \leq 2\nu, \tag{46}$$

$$\gamma_i > |\nu/\nu_i - 1|\, C_i \sqrt{M_i/\alpha_i}, \quad \text{if } \nu_i \geq 2\nu, \tag{47}$$

C_i being the constant in (44) or in (45), and α_i, M_i the ellipticity and continuity constants of $V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ with the usual norms. The convergence and stability analysis follows in this case by results on small perturbations of convergent methods (see [17, Theorem 13.7]).

For the three dimensional problem, we can give a generalization of the method proposed in section 7 when Γ_i are polyhedra. In this case, following the ideas of [26], Y_N^i can be chosen as the standard space of continuous piecewise linear functions defined with respect to a locally quasi-uniform mesh Γ_h^i of Γ_i and X_N^i as the space of piecewise constant functions with respect to the dual mesh $\widehat{\Gamma}_h^i$ of Γ_h^i (the dual mesh is constructed as it is usually done in finite volume methods, see for instance [11]). When the mesh sizes satisfy some local conditions uniformly (similar to conditions (31) and (32)), the analogous result to Proposition 7.3 holds (see [26]). It remains, however, to verify that the stability constants for the identity operator are large enough to satisfy condition (46) or (47).

For this case, with Lipschitz boundaries, there are some simpler alternatives, at least in terms of numerical analysis. They are related to mixed direct–indirect formulations and are the aim of future work.

10. A full discretization

In this section we propose a fully discrete version of the numerical method with periodic splines of degree zero and one on staggered uniform meshes. In order to keep this work in a reasonable size, we will just write its convergence properties omitting all proofs. For a detailed analysis we refer to [24].

We write $h = 1/N$, $t_k = kh$, $t_{k+1/2} = (k+1/2)h$ and consider the discrete spaces

$$\begin{aligned} \widehat{S}^0_h &:= \{u_h \in H^0 \,|\, u_h|_{[t_{k-1/2},\, t_{k+1/2}]} \in \mathbb{P}_0, \quad \forall k\}, \\ S^1_h &:= \{u_h \in \mathcal{C}^0 \,|\, v_h|_{[t_{k-1},\, t_k]} \in \mathbb{P}_1, \quad \forall k\}. \end{aligned}$$

Our aim now is to approximate the integrals appearing in the implementation of (17) for $\mathbf{X}_N = \widehat{S}^0_h \times \widehat{S}^0_h$ and $\mathbf{Y}_N = \widehat{S}^0_h \times S^1_h$.

We denote

$$Q_{k\ell} := [t_{k-1/2}, t_{k+1/2}] \times [t_{\ell-1/2}, t_{\ell+1/2}], \qquad P_{k\ell} := [t_{k-1}, t_{k+1}] \times [t_{\ell-1/2}, t_{\ell+1/2}].$$

Approximations for $\mathcal{V}^{\boldsymbol{\Lambda}}$ and $\widetilde{\mathcal{V}}^{\boldsymbol{\lambda}}$. We propose a generalization of the Galerkin-collocation method (see [9, 12, 13]) to compute numerically the integrals

$$v^{k\ell}_{ii}(\lambda_i) := \int_{Q_{k\ell}} V^{\lambda_i}_{ii}(s,t)\, ds\, dt, \quad \widetilde{v}^{k\ell}_{ij}(\lambda) := \int_{Q_{k\ell}} \widetilde{V}^{\lambda}_{ij}(s,t)\, ds\, dt,$$

$V^{\lambda_i}_{ii}(\,\cdot\,,\,\cdot\,)$ and $\widetilde{V}^{\lambda}_{ij}(\,\cdot\,,\,\cdot\,)$ being the kernels of the integral operators $V^{\lambda_i}_{ii}$ and $\widetilde{V}^{\lambda}_{ij}$ introduced at the beginning of section 3. Following [6, section 3.5], we consider the splitting

$$\begin{aligned} V^{\rho}_{ii}(s,t) \;=\; & -\frac{1}{4\pi}\log(s-t)^2 - \frac{1}{4\pi}\log\left(\frac{|\mathbf{x}_i(s)-\mathbf{x}_i(t)|^2}{(s-t)^2}\right) \\ & + \left(A^{\rho}_i(s,t) + \frac{1}{4\pi}\right)\log(|\mathbf{x}_i(s)-\mathbf{x}_i(t)|^2) + B^{\rho}_i(s,t), \end{aligned} \tag{48}$$

where $A^{\rho}_i(\,\cdot\,,\,\cdot\,)$ and $B^{\rho}_i(\,\cdot\,,\,\cdot\,)$ are 1-periodic with respect to each variable and smooth satisfying

$$A^{\rho}_i(s,s) = -\frac{1}{4\pi}, \;\; \nabla A^{\rho}_i(s,s) = 0, \quad B^{\rho}_i(s,s) = -\frac{1}{4}\left(\imath + \frac{\log(2/\rho)^2 - 2\gamma}{\pi}\right), \;\; \forall s \in \mathbb{R}.$$

Here γ is the Euler constant. Note that

$$\int_{Q_{k\ell}} \log(s-t)^2\,ds\,dt \;=\; \left| \begin{array}{ll} h^2(\log h^2 - 3), & k=\ell, \\ h^2(\log h^2(k-\ell)^2 + \Theta(k-\ell)), & k \neq \ell, \end{array} \right.$$

with

$$\Theta(m) \;:=\; \int_{[-1/2,1/2]^2} \log\Big(1+\frac{u-v}{m}\Big)^2 du\,dv.$$

We want to point out that to avoid round-off errors, we compute $\Theta(m)$ explicitly for small values of $|m|$ and use a truncation of its power expansion as $|m|$ increases (see [13]). This has to be done once and then the result can be stored in a file. For the remaining terms in identity (48) we use the simple midpoint rule. Taking into account that $V_{ii}^{\lambda_i}(\,\cdot\,,\,\cdot\,)$ and $\widetilde{V}_{ij}^{\lambda}(\,\cdot\,,\,\cdot\,)$ are one-periodic with respect to each variable, we finally propose the following approximations:

- if $i=j,\;\; k=\ell$,

$$\begin{aligned} v_{ii}^{kk}(\lambda_i) &\approx a_0 h^2(\log h^2 - 3) + a_0 \log|\mathbf{x}_i'(t_k)|^2 + h^2 B_i^{\rho}(t_k,t_k) \\ &= \frac{h^2}{4\pi}\Big(-\log|\mathbf{x}_i'(t_k)|^2 + \log(2/(h\lambda_i))^2 + 3 - 2\gamma + \pi\imath\Big) \;=:\; \beta_{ii}^{kk}(\lambda_i), \\ \widetilde{v}_{ii}^{kk}(\lambda) &\approx \beta_{ii}^{kk}(\lambda) + \; h^2(\widetilde{V}_{ii}^{\lambda} - V_{ii}^{\lambda})(t_k,t_k) \;=:\; \widetilde{\beta}_{ii}^{kk}(\lambda), \end{aligned}$$

- if $i=j,\;\; k\neq\ell,\;\; |k-\ell|\le N/2$,

$$\begin{aligned} v_{ii}^{k\ell}(\lambda_i) &\approx -\frac{h^2}{4\pi}\,\Theta(k-\ell) + h^2 V_{ii}^{\lambda_i}(t_k,t_\ell) \;=:\; \beta_{ii}^{k\ell}(\lambda_i), \\ \widetilde{v}_{ii}^{k\ell}(\lambda) &\approx \beta_{ii}^{k\ell}(\lambda) + \; h^2(\widetilde{V}_{ii}^{\lambda} - V_{ii}^{\lambda})(t_k,t_\ell) \;=:\; \widetilde{\beta}_{ii}^{k\ell}(\lambda), \end{aligned}$$

- if $i\neq j,\;\; |k-\ell|\le N/2$,

$$\widetilde{v}_{ij}^{k\ell}(\lambda_i) \;\approx\; h^2 \widetilde{V}_{ij}^{\lambda}(t_k,t_\ell) \;=:\; \widetilde{\beta}_{ij}^{k\ell}(\lambda_i),$$

- if $|k-\ell| > N/2$,

$$\begin{array}{lll} \beta_{ii}^{k\ell}(\lambda_i) \;:=\; \beta_{ii}^{k,\ell+N}(\lambda_i), & \widetilde{\beta}_{ij}^{k\ell}(\lambda) \;:=\; \beta_{ij}^{k,\ell+N}(\lambda), & k-\ell > N/2, \\ \beta_{ii}^{k\ell}(\lambda_i) \;:=\; \beta_{ii}^{k,\ell-N}(\lambda_i), & \widetilde{\beta}_{ij}^{k\ell}(\lambda) \;:=\; \widetilde{\beta}_{ij}^{k,\ell-N}(\lambda), & \ell-k > N/2. \end{array}$$

Approximations for $\mathcal{J}^{\Lambda}$ and $\widetilde{\mathcal{J}}^{\lambda}$. We deal now with the numerical computation of

$$r_{ii}^{k\ell}(\lambda_i) := \int_{P_{k\ell}} J_{ii}^{\lambda_i}(s,t)\,\eta_k(s)\,ds\,dt, \quad \widetilde{r}_{ij}^{k\ell}(\lambda) := \int_{P_{k\ell}} \widetilde{J}_{ij}^{\lambda}(s,t)\,\eta_k(s)\,ds\,dt,$$

η_k being the function of S_h^1 such that $\eta_k(x_\ell) = \delta_{k\ell}$. Here $J_{ii}^{\lambda_i}(\,\cdot\,,\,\cdot\,)$ and $\widetilde{J}_{ij}^{\lambda}(\,\cdot\,,\,\cdot\,)$ are the kernels of the integral operators $J_{ii}^{\lambda_i}$ and $\widetilde{J}_{ij}^{\lambda}$, which are continuous functions. We decompose

$$J_{ii}^{\rho}(s,t) \;=\; C_i^{\rho}(s,t)\log|\mathbf{x}_i(s)-\mathbf{x}_i(t)|^2 + D_i^{\rho}(s,t),$$

C_i^{ρ} and D_i^{ρ} being smooth and 1-periodic functions with respect of each variable satisfying

$$C_i^{\rho}(s,s)=0, \qquad \nabla C_i^{\rho}(s,s)=0, \qquad D_i^{\rho}(s,s)=\frac{1}{4\pi}\,\frac{\mathbf{x}_i''(s)\cdot\mathbf{n}_i(s)}{|\mathbf{x}_i'(s)|}, \qquad \forall s\in\mathbb{R}.$$

Again we simply apply midpoint rules:

$$r_{ii}^{k\ell}(\lambda_i) \;\approx\; h^2 J_{ii}^{\lambda_i}(t_k,t_\ell) \;=:\; \rho_{ii}^{k\ell}(\lambda_i), \qquad \widetilde{r}_{ij}^{k\ell}(\lambda) \;\approx\; h^2\widetilde{J}_{ij}^{\lambda}(t_k,t_\ell) \;=:\; \widetilde{\rho}_{ij}^{k\ell}(\lambda).$$

These quantities can be computed directly from the definitions of $J_{ii}^{\lambda_i}(\,\cdot\,,\,\cdot\,)$ and $\widetilde{J}_{ij}^{\lambda}(\,\cdot\,,\,\cdot\,)$ except for $i=j$ and $k=\ell$, in which case we use the expressions

$$J_{ii}^{\lambda_i}(t_k,t_k) = \frac{1}{4\pi}\,\frac{\mathbf{x}_i''(t_k)\cdot\mathbf{n}_i(t_k)}{|\mathbf{x}_i'(t_k)|},$$
$$\widetilde{J}_{ii}^{\lambda}(t_k,t_k) = \frac{1}{4\pi}\,\frac{\mathbf{x}_i''(t_k)\cdot\mathbf{n}_i(t_k)}{|\mathbf{x}_i'(t_k)|} + h^2(\widetilde{J}_{ii}^{\lambda}-J_{ii}^{\lambda})(t_k,t_k).$$

Approximation of the right hand side. We consider the approximations

$$g_{0i}^{k} := \int_{t_{k-1/2}}^{t_{k+1/2}} g_i^0(s)\,ds \;\approx\; h\,g_i^0(t_k) \;=:\; \widehat{g}_{0i}^{k},$$

$$g_{1i}^{k} := \int_{t_{k-1}}^{t_{k+1}} g_i^1(s)\,\eta_k(s)\,ds \;\approx\; h\,g_i^1(t_k) \;=:\; \widehat{g}_{1i}^{k}.$$

Fully discrete method. We take the basis $\{\eta_k\}$ of S_h^1 and the basis of $\widehat{S}_h^0$ formed by the 1-periodization of the characteristic functions associated to the intervals $[t_{k-1/2}, t_{k+1/2}]$. Then, identifying each element in the periodic spline spaces with its coordinates in the corresponding basis, the approximation is reduced to solving the following linear system of equations in $\mathbb{C}^{2Nd}$ for the unknowns $\widehat{\varphi}_h := (\widehat{\varphi}_1^1, \ldots, \widehat{\varphi}_1^N, \ldots, \widehat{\varphi}_d^1, \ldots, \widehat{\varphi}_d^N)$ and $\widehat{\psi}_h := (\widehat{\psi}_1^1, \ldots, \widehat{\psi}_1^N, \ldots, \widehat{\psi}_d^1, \ldots, \widehat{\psi}_d^N)$,

$$\begin{bmatrix} \mathcal{V}_h^\Lambda & -\widetilde{\mathcal{V}}_h^\lambda \\ \mathcal{N}_h\left(\frac{1}{2}\mathcal{I}_h + \mathcal{J}_h^\Lambda\right) & \nu\left(\frac{1}{2}\mathcal{I}_h - \widetilde{\mathcal{J}}_h^\lambda\right) \end{bmatrix} \begin{bmatrix} \widehat{\varphi}_h \\ \widehat{\psi}_h \end{bmatrix} = \begin{bmatrix} \widehat{g}_h^0 \\ \widehat{g}_h^1 \end{bmatrix}. \tag{49}$$

The entries of $\mathcal{V}_h^\Lambda := \mathrm{diag}(V_{ii}^h)$ are the matrices $V_{ii}^h := (\beta_{ii}^{k\ell}(\lambda_i))_{k\ell}$ and $\widetilde{\mathcal{V}}_h^\lambda := (\widetilde{V}_{ij}^h)_{ij}$ where $\widetilde{V}_{ij}^h := (\widetilde{\beta}_{ij}^{k\ell}(\lambda))_{k\ell}$. With the elements $\rho_{ii}^{k\ell}(\lambda_i)$ and $\widetilde{\rho}_{ij}^{k\ell}(\lambda)$ we define $\mathcal{J}_h^\Lambda$ and $\widetilde{\mathcal{J}}_h^\lambda$ likewise. $\mathcal{I}_h$ contains the exact values of the integrals related to the identity operator, $\mathcal{N}_h := \mathrm{diag}(\nu_1, \ldots, \nu_1, \ldots, \nu_d, \ldots, \nu_d)$ and $\widehat{g}_h^\alpha := (\widehat{g}_{\alpha 1}^1, \ldots, \widehat{g}_{\alpha 1}^N, \ldots, \widehat{g}_{\alpha d}^1, \ldots, \widehat{g}_{\alpha d}^N)$ for $\alpha = 0, 1$.

Once equation (49) is solved, we can approximate the solution u defined in expression (14) by

$$\widehat{u}_h(\mathbf{z}) := \left| \begin{array}{ll} \dfrac{\imath}{4N} \displaystyle\sum_{i=1}^{d} \sum_{\ell=1}^{N} \Big(H_0^{(1)}(\lambda|\mathbf{z} - \mathbf{x}_i(t_\ell)|) + H_0^{(1)}(\lambda|\mathbf{z} - \widetilde{\mathbf{x}}_i(t_\ell)|) \Big) \widehat{\psi}_i^\ell, & \text{if } \mathbf{z} \in \Omega, \\ \dfrac{\imath}{4N} \displaystyle\sum_{\ell=1}^{N} H_0^{(1)}(\lambda_i|\mathbf{z} - \mathbf{x}_i(t_\ell)|)\, \widehat{\varphi}_i^\ell, & \text{if } \mathbf{z} \in \Omega_i. \end{array} \right. \tag{50}$$

This function is a discrete version of the single layer potentials obtained by substituting the real densities by the approximate ones and applying simple midpoint rules.

The proposed method has the following convergence properties: if $g_0, g_1 \in \mathcal{C}^2$, then for $-1 \le s < 1/2$,

$$\|\varphi - \varphi_h\|_s + \|\psi - \psi_h\|_s = \mathcal{O}_s(h^{1-s}), \tag{51}$$

$$|u(\mathbf{z}) - \widehat{u}_h(\mathbf{z})| = \mathcal{O}_{\mathbf{z}}(h^2), \qquad \mathbf{z} \notin \cup_{i=1}^d \Gamma_i. \tag{52}$$

Notice that for the considered range of s, both estimates (27) and (51) are of the same order.

11. Numerical experiments

We compute the numerical solution to three test problems for which the exact solution is known. The aim is to illustrate that, when the boundaries of the obstacles have $\mathcal{C}^2-$regularity, the method proposed in section 10 has quadratic convergence order. Although in [24] is proved quadratic convergence for $g_0, g_1 \in \mathcal{C}^2$, numerical experiments show that this remains valid for $g_1 \in \mathcal{C}^1$.

We also compute the solution to the test problems with a fully discrete version of the spectral method proposed in section 5, approximating the involved integrals by

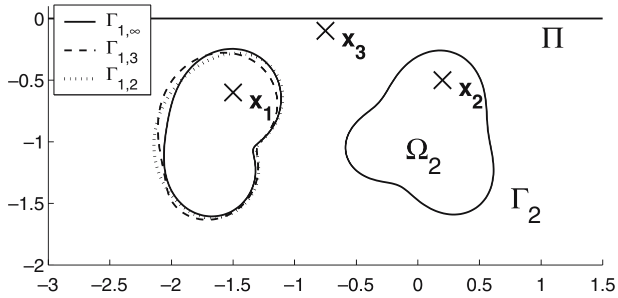

Figure 1. Geometry of the test problems.

using the ideas of [6, section 3.5]. For a complete description of the discretization we refer to [22].

In order to show how both methods work when considering obstacles with different regularity properties, we consider three different problems which are almost the same. Firstly we choose the obstacles $\Omega_{1,\infty}$ and Ω_2 represented in figure 1: $\Gamma_{1,\infty}$ and Γ_2 are defined by the $\mathcal{C}^\infty$- parameterizations

$$\begin{aligned}\mathbf{y}_{1,\infty}(t) &= (-1.5 + r_{1,\infty}(t)\cos(2\pi t), -1 + r_{1,\infty}(t)\sin(2\pi t)),\\ \mathbf{y}_2(t) &= (r_2(t)\cos(2\pi t), -1 + r_2(t)\sin(2\pi t)),\end{aligned}$$

with

$$\begin{aligned}r_{1,\infty}(t) &= 25/48 + 25/128\sin(2\pi(t-1/6)) + 5/32\cos(4\pi(t-5/6)),\\ r_2(t) &= 2 + 3/4\sin(2\pi t)\cos(4\pi(t-1/6)).\end{aligned}$$

Table 1
Relative errors and estimated convergence rates for $\mathcal{C}^\infty$-boundaries.

N	E_1^{spline}	e.c.r.	E_2^{spline}	e.c.r.	E_3^{spline}	e.c.r.
32	8.889(−3)		7.441(−3)		9.177(−5)	
64	2.110(−3)	2.07	2.013(−3)	1.88	2.044(−5)	2.16
128	5.178(−4)	2.02	5.263(−4)	1.93	4.904(−6)	2.05
256	1.284(−4)	2.01	1.347(−4)	1.96	1.206(−6)	2.02
N	E_1^{spec}	e.c.r.	E_2^{spec}	e.c.r.	E_3^{spec}	e.c.r.
32	7.593(−7)		1.109(−5)		1.342(−7)	
64	2.899(−13)	21.32	5.973(−12)	20.83	1.077(−13)	20.24
128	1.642(−15)	7.46	8.243(−16)	12.82	1.662(−17)	12.66
256	2.166(−15)	−0.39	1.571(−15)	−0.93	3.324(−17)	−1.00

Table 2
Relative errors and estimated convergence rates for $\mathcal{C}^3$-boundaries.

N	E_1^{spline}	e.c.r.	E_2^{spline}	e.c.r.	E_3^{spline}	e.c.r.
32	1.216(−2)		7.434(−3)		7.222(−5)	
64	2.972(−3)	2.03	2.011(−3)	1.88	1.631(−5)	2.14
128	7.372(−4)	2.01	5.260(−4)	1.93	3.936(−6)	2.05
256	1.837(−4)	2.00	1.346(−4)	1.96	9.705(−6)	2.02
N	E_1^{spec}	e.c.r.	E_2^{spec}	e.c.r.	E_3^{spec}	e.c.r.
32	4.027(−6)		1.108(−5)		1.309(−7)	
64	4.380(−7)	3.20	7.326(−10)	13.88	5.099(−10)	8.00
128	5.089(−8)	3.10	8.354(−11)	3.13	6.197(−11)	3.04
256	6.096(−9)	3.06	9.935(−12)	3.07	7.514(−12)	3.04

Because of the fact that we take the same parameters for Ω, $\Omega_{1,\infty}$ and Ω_2, ($\lambda = \lambda_j = 4(1+\imath)$ and $\nu = \nu_j = 1$), the solution to the transmission problem (equations (3–7)) is the incident wave inside the obstacles and zero in Ω (recall that our unknown for the unbounded domain is the scattered wave $u - u_{inc}$). We choose $u_{inc} = \frac{\imath}{4}H_0^{(1)}(\lambda|\cdot - (-0.5,0)|)$.

Table 1 shows the relative errors E_i^{spline} and E_i^{spec}, $i = 1,2,3$, at $\mathbf{x}_1 = (-1.5, -0.6)$, $\mathbf{x}_2 = (0.2, -0.5)$ and $\mathbf{x}_3 = (-0.75, -0.1)$ for a sequence of uniform grids, each consisting of N nodes per domain. The estimated convergence rates (e.c.r.) are computed by comparing the errors on consecutive grids in the usual way.

Secondly we replace the obstacle $\Omega_{1,\infty}$ by $\Omega_{1,3}$ (see figure 1) obtained by approximating $r_{1,\infty}$ by the 1-periodic spline of degree four that interpolates $r_{1,\infty}$ at $1/4$, $9/20$, $3/5$ and 1. We take the same values for λ, λ_j, ν and ν_j as before. The computed relative errors and the estimated convergence rates are shown in table 2.

Table 3
Relative errors and estimated convergence rates for $\mathcal{C}^2$-boundaries.

N	E_1^{spline}	e.c.r.	E_2^{spline}	e.c.r.	E_3^{spline}	e.c.r.
32	4.156(−3)		7.441(−3)		9.407(−5)	
64	1.037(−3)	2.00	2.013(−3)	1.88	2.081(−5)	2.17
128	2.727(−4)	1.92	5.264(−4)	1.93	5.009(−6)	2.05
256	6.949(−5)	1.97	1.347(−4)	1.96	1.229(−6)	2.02
N	E_1^{spec}	e.c.r.	E_2^{spec}	e.c.r.	E_3^{spec}	e.c.r.
32	4.139(−3)		2.200(−5)		4.005(−5)	
64	2.084(−3)	0.99	5.860(−6)	1.90	1.997(−5)	1.00
128	1.033(−3)	1.01	2.903(−6)	1.01	9.904(−6)	1.01
256	5.165(−4)	0.99	1.451(−6)	0.99	4.952(−6)	1.00

Finally, we consider a new approximation of $r_{1,\infty}$ using the 1-periodic cubic spline interpolating $r_{1,\infty}$ at $1/6$, $1/3$, $2/3$ and 1, and repeat the experiment. The results are written in table 3.

It is clear here that using spectral methods is only useful for very smooth curves. In the other cases, the additional implementation difficulty is wasted due to lack of smoothness and simpler finite element type spaces are to be preferred. The interest of using smooth but not $\mathcal{C}^\infty$-curves is not purely academic. When solving inverse problems via iteration schemes one is obliged to use parametric representations of the obstacles. Spline curves or the type of curves in polar coordinates as the ones given in the examples are simple to handle and more flexible to approximate difficult points of the boundaries than trigonometric polynomial representations.

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