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WEAKLY NONLINEAR BOUNDARY-VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS IN A BANACH SPACE IN THE CRITICAL CASE

O. A. Boichuk[1] **and E. V. Panasenko**[2]

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We obtain necessary and sufficient conditions for the existence of solutions of weakly nonlinear boundary-value problems for differential equations in a Banach space. A convergent iterative procedure is proposed for the determination of solutions. We also establish a relationship between necessary and sufficient conditions.

1. Statement of the Problem and Preliminary Results

In a Banach space $\mathbf{B}_1$, we consider a boundary-value problem for a nonlinear differential equation with small nonnegative parameter ε of the form

$$\frac{dx(t)}{dt} = A(t)x(t) + \varepsilon Z(x,t,\varepsilon) + f(t), \tag{1}$$

$$\ell x(\cdot) = \alpha + \varepsilon J(x(\cdot,\varepsilon),\varepsilon), \tag{2}$$

where the vector function $f(t)$ acts from a segment $[a;b]$ into the Banach space $\mathbf{B}_1$, i.e.,

$$f(t) \in C\,([a;b],\mathbf{B}_1) := \left\{ f(\cdot)\colon [a;b] \to \mathbf{B}_1,\ \ |||f||| = \sup_{t\in[a;b]} \|f(t)\| \right\},$$

$C([a;b],\mathbf{B}_1)$ is the Banach space of vector functions continuous on $[a;b]$, the operator function $A(t)$ acts from the Banach space $\mathbf{B}_1$ into itself for every $t \in [a;b]$, is strongly continuous [1, p. 141], and has the norm

$$|||A||| = \sup_{t\in[a;b]} \|A(t)\| < \infty,$$

$Z(x,t,\varepsilon)$ is a nonlinear vector function continuously differentiable with respect to x in the neighborhood of a generating solution and continuous in t and ε, i.e.,

$$Z(\cdot,t,\varepsilon) \in C^1[\|x - x_0\| \le q], \quad Z(x,\cdot,\varepsilon) \in C\,([a;b],\mathbf{B}_1)\,, \quad Z(x,t,\cdot) \in C[0;\varepsilon_0],$$

q and ε_0 are sufficiently small constants, α is an element of the space $\mathbf{B}_2$, i.e., $\alpha \in \mathbf{B}_2$, and $J(x(\cdot,\varepsilon),\varepsilon)$ is a nonlinear bounded vector functional continuously differentiable in the sense of Fréchet with respect to x and

[1] Institute of Mathematics, Ukrainian National Academy of Sciences, Kyiv, Ukraine; e-mail: boichuk@imath.kiev.ua.
[2] Zaporizhzhya National University, Zaporizhzhya, Ukraine; e-mail: innovatory@rambler.ru.

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continuous in ε in the neighborhood of the solution. The operator ℓ is a linear vector functional continuous on $[a;b]$ and acting from the space $C[a;b]$ into the Banach space $\mathbf{B}_2$, i.e.,

$$\ell: C^1([a;b], \mathbf{B}_1) \to \mathbf{B}_2.$$

A solution of Eq. (1) is understood as a solution $x(t) = x(t, \varepsilon)$ of the integral equation

$$x(t,\varepsilon) = x_0 + \int_a^t (A(s)x(s) + \varepsilon Z(x(s,\varepsilon), s, \varepsilon) + f(s))ds$$

that is continuously differentiable at every point $t \in [a;b]$ and for which Eq. (1) is satisfied everywhere on $[a;b]$.

We consider the critical case where the corresponding inhomogeneous generating boundary-value problem has a nontrivial solution $x_0(t,c)$ [2]. We seek a condition for the existence and an algorithm for the construction of a solution $x = x(t,\varepsilon)$, $x(\cdot,\varepsilon) \in C^1([a;b], \mathbf{B}_1)$, $x(t,\cdot) \in C[0;\varepsilon_0]$, of the boundary-value problem (1), (2) that turns into one of solutions $x_0(t,c) = x(t,0)$ of the following boundary-value problem for $\varepsilon = 0$:

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t), \quad t \in [a;b], \tag{3}$$

$$\ell x(\cdot) = \alpha \in \mathbf{B}_2. \tag{4}$$

In what follows, this solution is called a generating solution of the boundary-value problem (1), (2).

According to the theorem in [2], the generating boundary-value problem (3), (4) has a family of linearly independent solutions

$$x_0(t,c) = U(t)\mathcal{P}_{N(Q)}c + U(t)Q^{-}\alpha + (G[f])(t) \tag{5}$$

if and only if the inhomogeneities $f(t) \in C([a;b], \mathbf{B}_1)$ in the differential equation and $\alpha \in \mathbf{B}_2$ in the boundary condition satisfy the following relation:

$$\mathcal{P}_{N(Q^*)}\left[\alpha - \ell \int_a^b K(\cdot,\tau) f(\tau) d\tau\right] = 0, \tag{6}$$

where $U(t)$ is the evolution operator of the homogeneous differential equation (3) [1, p. 147], $Q = \ell U(\cdot)$ is the operator obtained by the substitution of the evolution operator into the boundary condition (4), Q^{-} is the generalized inverse of the operator Q [3], $\mathcal{P}_{N(Q)} = I - Q^{-}Q$ and $\mathcal{P}_{N(Q^*)} = I - QQ^{-}$ are the projection operators that project the Banach space $\mathbf{B}_1$ to the kernel $N(Q)$ and the cokernel $N(Q^*)$ of the operator Q, respectively, and $(G[f])(t)$ is the generalized Green operator of problem (3), (4) that acts on a vector function $f(t) \in C([a;b], \mathbf{B}_1)$ as follows:

$$(G[f])(t) := \int_a^b K(t,\tau) f(\tau) d\tau - U(t)Q^{-} \cdot \ell \int_a^b K(\cdot,\tau) f(\tau) d\tau.$$

In the case of the finite-dimensional spaces $\mathbf{B}_1 = \mathbb{R}^n$ and $\mathbf{B}_2 = \mathbb{R}^m$, this problem was solved in [3, 4]. For periodic $(m = n$ and $\ell x = x(a) - x(b) = \alpha = 0)$ and two-point $(m = n$ and $\ell x = M_1 x(a) - M_2 x(b)$, where M_i are $n \times n$ matrices) boundary-value problems, analogous problems were considered in [5–8]. Conditions for the existence and an algorithm for the construction of solutions bounded on the entire axis $\mathbb{R} = (-\infty; +\infty)$ for weakly nonlinear differential equations in a Banach space were considered in [9].

2. Main Result

Consider the problem of finding necessary conditions for the existence of solutions $x(t, \varepsilon)$ of the boundary-value problem (1), (2) in the critical case $(\mathcal{P}_{N(Q^*)} \neq 0)$ that, for $\varepsilon = 0$, turn into a generating solution $x_0(t, c)$ of the generating boundary-value problem (3), (4) that has the form (5) with constant $c = c_0$.

For this purpose, we impose the following restriction on the operator function $Z(x, t, \varepsilon)$:

$$Z(\cdot, \cdot, \cdot) \in C[\|x - x_0\| \leq q] \times C([a; b], \mathbf{B}_1) \times C[0; \varepsilon_0],$$

where q and ε_0 are sufficiently small constants.

Theorem 1 (necessary condition)**.** *Suppose that the solvability condition (6) for the generating boundary-value problem (3), (4) is satisfied and the critical boundary-value problem (1), (2)* $(\mathcal{P}_{N(Q^*)} \neq 0)$ *has a solution* $x(t, \varepsilon)$, $x(\cdot, \varepsilon) \in C^1([a; b], \mathbf{B}_1)$, $x(t, \cdot) \in C[0; \varepsilon_0]$, *that turns into a generating solution* $x_0(t, c_0)$ *of the form (5) with constant* $c = c_0$ *for* $\varepsilon = 0$. *Then the constant* $c_0 \in \mathbf{B}_1$ *satisfies the operator equation*

$$F(c) = \mathcal{P}_{N(Q^*)} \left[J(x_0(\cdot, c), 0) - \ell \int_a^b K(\cdot, \tau) Z(x_0(\tau, c), \tau, 0) d\tau \right] = 0, \tag{7}$$

which is called the equation for generating constants of the boundary-vale problem (1), (2).

Proof. If the boundary-value problem (1), (2) has a solution, then, according to the theorem presented in [2], the solvability condition

$$\mathcal{P}_{N(Q^*)} \left[\alpha + \varepsilon J(x(\cdot, \varepsilon), \varepsilon) - \ell \int_a^b K(\cdot, \tau) \left(\varepsilon Z(x(\tau, \varepsilon), \tau, \varepsilon) + f(\tau) \right) d\tau \right] = 0$$

is satisfied.

Using the solvability condition (6), we obtain the following equality for $\varepsilon \neq 0$:

$$\mathcal{P}_{N(Q^*)} \varepsilon \left[J(x(\cdot, \varepsilon), \varepsilon) - \ell \int_a^b K(\cdot, \tau) Z(x(\tau, \varepsilon), \tau, \varepsilon) \, d\tau \right] = 0.$$

Taking into account that the operator function $Z(x, t, \varepsilon)$ is continuous in t and ε and passing to the limit as $\varepsilon \to 0$ and $x(t, \varepsilon) \to x_0(t, c_0)$, we get

$$F(c_0) = \mathcal{P}_{N(Q^*)}\left[J(x_0(\cdot, c_0), 0) - \ell \int_a^b K(\cdot, \tau) Z(x_0(t, c_0), \tau, 0)\, d\tau \right] = 0, \tag{8}$$

which proves the theorem.

If Eq. (8) has a solution, then the element c_0 realizes a generating solution $x_0(t, c_0)$ to which a solution $x(t, \cdot) \in C[0; \varepsilon_0]$, $x(t, 0) = x_0(t, c_0)$, of the boundary-value problem (1), (2) may correspond. If Eq. (8) does not have a real solution, then the boundary-value problem (1), (2) does not have the required solution in the space $C^1([a, b], \mathbf{B}_1)$.

In the case of the periodic boundary-value problem (1), (2) ($m = n$, $d = r$, $\ell x = x(a) - x(b) = \alpha = 0$, $\mathbf{B}_1 = \mathbb{R}^n$, and $\mathbf{B}_2 = \mathbb{R}^n$), the constant c_0 has a physical meaning, namely, it is the amplitude of a generating solution. For this reason, in the classic theory of nonlinear oscillations [7, 8], this equation is called the "equation for generating amplitudes."

To obtain a sufficient condition for the existence of a solution we perform the following change of variables in the boundary-value problem (1), (2):

$$x(t, \varepsilon) = x_0(t, c_0) + y(t, \varepsilon),$$

where $x_0(t, c_0)$ is the generating solution (5) and c_0 is an arbitrary element of the Banach space $\mathbf{B}_1$ that satisfies the operator equation for constants (8).

In addition, we assume that the operator function $Z(x, t, \varepsilon)$ is Fréchet-differentiable in the neighborhood of a generating solution $(Z(\cdot, t, \varepsilon) \in C^1[\|x - x_0\| \leq q])$.

Let us find conditions for the existence of a solution $y(t, \varepsilon)$, $y(\cdot, \varepsilon) \in C^1([a; b], \mathbf{B}_1)$, $y(t, \cdot) \in C[0; \varepsilon_0]$, that, for $\varepsilon = 0$, turns into the trivial solution of the boundary-value problem

$$\frac{dx_0(t, c_0)}{dt} + \frac{dy(t, \varepsilon)}{dt} = A(t)x_0(t, c_0) + A(t)y(t, \varepsilon) + \varepsilon Z(x_0(t, c_0) + y(t, \varepsilon), t, \varepsilon) + f(t), \tag{9}$$

$$\ell x_0(\cdot, c_0) + \ell y(\cdot, \varepsilon) = \alpha + \varepsilon J(x_0(\cdot, c_0) + y(\cdot, \varepsilon), \varepsilon), \tag{10}$$

and let us construct this solution.

Taking into account that $x_0(t, c_0)$ is a solution of the boundary-value problem (3), (4), we obtain the following boundary-value problem from (9), (10):

$$\frac{dy(t, \varepsilon)}{dt} = A(t)y(t, \varepsilon) + \varepsilon Z(x_0(t, c_0) + y(t, \varepsilon), t, \varepsilon), \tag{11}$$

$$\ell y(\cdot, \varepsilon) = \varepsilon J(x_0(\cdot, c_0) + y(\cdot, \varepsilon), \varepsilon). \tag{12}$$

Using the continuous differentiability of the operator function $Z(x, t, \varepsilon)$ and vector functional $J(x(\cdot, \varepsilon), \varepsilon)$ with respect to x in the neighborhood of the point $\varepsilon = 0$, we separate the linear part with respect to y and the zero-order terms with respect to ε in the operator function $Z(x_0(t, c_0) + y(t, \varepsilon), t, \varepsilon)$ and vector functional $J(x_0(t, c_0) + y(t, \varepsilon), \varepsilon)$. As a result, we obtain the following decompositions:

$$Z(x_0(t, c_0) + y(t, \varepsilon), t, \varepsilon) = \varphi_0(t, c_0) + A_1(t)y(t, \varepsilon) + R(y(t, \varepsilon), t, \varepsilon), \tag{13}$$

$$J(x_0(\cdot, c_0) + y(\cdot, \varepsilon), \varepsilon) = J_0(x_0(\cdot, c_0)) + \ell_1 y(\cdot, \varepsilon) + R_1(y(\cdot, \varepsilon), \varepsilon), \tag{14}$$

where

$$\varphi_0(t, c_0) = Z(x_0(t, c_0), t, 0) \in C([a; b], \mathbf{B}_1),$$

$$J_0(x_0(\cdot, c_0)) = J(x_0(\cdot, c_0), 0),$$

$$A_1(t) = A_1(t, c_0) = \left.\frac{\partial Z(x, t, 0)}{\partial x}\right|_{x = x_0(t, c_0)} \in C([a; b], \mathbf{B}_1),$$

the derivative is understood in the Fréchet sense, and $\ell_1 y(\cdot, \varepsilon)$ is the linear part of the vector functional $J(x_0(\cdot, c_0) + y(\cdot, \varepsilon), \varepsilon)$. The nonlinear operator function $R(y(t, \varepsilon), t, \varepsilon)$ belongs to the class $C^1[\|y\| \le q]$, $C([a; b], \mathbf{B}_1)$, $C[0; \varepsilon_0]$. Furthermore,

$$R(0, t, 0) = 0, \qquad \frac{\partial R(0, t, 0)}{\partial y} = 0,$$

$$R_1(0, 0) = 0, \qquad \frac{\partial R_1(0, 0)}{\partial y} = 0.$$

We apply the theorem presented in [2] to the boundary-value problem (11), (12), formally considering the nonlinearities $Z(x_0(t, c_0) + y(t, \varepsilon), t, \varepsilon)$ and $J(x_0(\cdot, c_0) + y(\cdot, \varepsilon), \varepsilon)$ as inhomogeneities. Furthermore, a solution of the boundary-value problem (11), (12) can formally be rewritten in the form $y(t, \varepsilon) = U(t)\mathcal{P}_{N(Q)}c + y^{(1)}(t, \varepsilon)$. The unknown element $c = c(\varepsilon) \in \mathbf{B}_1$ and unknown function $y^{(1)}(t, \varepsilon)$ are determined, respectively, from the solvability condition of the boundary-value problem (11), (12) and from the representation $y^{(1)}(t, \varepsilon)$ of a particular solution of the boundary-value problem

$$\mathcal{P}_{N(Q^*)}\varepsilon\Bigg[J_0(x_0(\cdot, c_0)) + \ell_1 y(\cdot, \varepsilon) + R_1(y(\cdot, \varepsilon), \varepsilon)$$

$$- \ell \int_a^b K(\cdot, \tau)\,(\varphi_0(t, c_0) + A_1(t) y(t, \varepsilon) + R(y(t, \varepsilon), t, \varepsilon))\, d\tau\Bigg] = 0, \quad \varepsilon \neq 0,$$

or

$$\mathcal{P}_{N(Q^*)}\Bigg[J_0(x_0(\cdot, c_0)) + \ell_1\left(U(t)\mathcal{P}_{N(Q)}c + y^{(1)}(\cdot, \varepsilon)\right) + R_1(y(\cdot, \varepsilon), \varepsilon)$$

$$-\ell \int_a^b K(\cdot, \tau)\left(\varphi_0(t, c_0) + A_1(t)\left\{U(t)\mathcal{P}_{N(Q)}c + y^{(1)}(\tau, \varepsilon)\right\} + R(y(t, \varepsilon), t, \varepsilon)\right) d\tau\Bigg] = 0,$$

and

$$y^{(1)}(t,\varepsilon) = \varepsilon\Big(G\Big[\varphi_0(\tau,c_0) + A_1(\tau)y(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)\Big]\Big)(t)$$

$$+ \varepsilon U(t)Q^{-}\{J_0(x_0(\cdot,c_0)) + \ell_1 y(\cdot,\varepsilon) + R_1(y(\cdot,\varepsilon),\varepsilon)\}.$$

For the determination of a solution $y(t,\cdot) \in C[0;\varepsilon_0]$, $y(t,0) = 0$, of the boundary-value problem (11), (12), we obtain the equivalent operator system

$$y(t,\varepsilon) = U(t)\mathcal{P}_{N(Q)}c + y^{(1)}(t,\varepsilon),$$

$$B_0 c = -\mathcal{P}_{N(Q^*)}\left[\ell_1 y^{(1)}(\cdot,\varepsilon) + R_1(y(\cdot,\varepsilon),\varepsilon) - \ell\int_a^b K(\cdot,\tau)\left\{A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)\right\}d\tau\right], \quad (15)$$

and

$$y^{(1)}(t,\varepsilon) = \varepsilon\Big(G\Big[\varphi_0(\tau,c_0) + A_1(\tau)\left(U(t)\mathcal{P}_{N(Q)}c + y^{(1)}(\tau,\varepsilon)\right) + R(y(\tau,\varepsilon),\tau,\varepsilon)\Big]\Big)(t)$$

$$+ \varepsilon U(t)Q^{-}\left\{J_0(x_0(\cdot,c_0)) + \ell_1\left(U(t)\mathcal{P}_{N(Q)}c + y^{(1)}(\cdot,\varepsilon)\right) + R_1(y(\cdot,\varepsilon),\varepsilon)\right\},$$

where the operator B_0 has the form

$$B_0 = \mathcal{P}_{N(Q^*)}\left[\ell_1 U(\cdot)\mathcal{P}_{N(Q)} - \ell\int_a^b K(\cdot,\tau)A_1(\tau)U(\tau)\mathcal{P}_{N(Q)}d\tau\right]. \quad (16)$$

Let $B_0: \mathbf{B}_1 \to \mathbf{B}_2$ be a generalized inverse operator [4, p. 39]. As shown in [10], it is normally solvable and there exist bounded projectors $\mathcal{P}_{N(B_0)}: \mathbf{B}_1 \to N(B_0)$ and $\mathcal{P}_Y: \mathbf{B}_2 \to Y$ that induce decompositions of $\mathbf{B}_1$ and $\mathbf{B}_2$ into direct topological sums of closed subspaces:

$$\mathbf{B}_1 = N(B_0) \oplus X,$$

$$\mathbf{B}_2 = Y \oplus R(B_0).$$

By virtue of the normal solvability of the operator B_0, Eq. (15) is solvable [11] if and only if its right-hand side satisfies the condition

$$\mathcal{P}_{N(B_0^*)}\mathcal{P}_{N(Q^*)}\Big[\ell_1 y^{(1)}(\cdot,\varepsilon) + R_1(y(\cdot,\varepsilon),\varepsilon)$$

$$- \ell\int_a^b K(\cdot,\tau)\left\{A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)\right\}d\tau\Big] = 0.$$

The last condition is satisfied if

$$\mathcal{P}_{N(B_0^*)}\mathcal{P}_{N(Q^*)} = 0. \tag{17}$$

The operator equation (15) is solvable under condition (17). As a result, an operator system equivalent to the boundary-value problem (11), (12) in the space of functions $y(\cdot,\varepsilon) \in C^1([a;b],\mathbf{B}_1)$, $y(t,\cdot) \in C[0;\varepsilon_0]$, has the form

$$y(t,\varepsilon) = U(t)\mathcal{P}_{N(Q)}c + y^{(1)}(t,\varepsilon),$$

$$c = -B_0^-\mathcal{P}_{N(Q^*)}\left[\ell_1 y^{(1)}(\cdot,\varepsilon) + R_1(y(\cdot,\varepsilon),\varepsilon) - \ell\int_a^b K(\cdot,\tau)\left\{A_1(\tau)y^{(1)}(\tau,\varepsilon) + R(y(\tau,\varepsilon),\tau,\varepsilon)\right\}d\tau\right], \tag{18}$$

$$y^{(1)}(t,\varepsilon) = \varepsilon\left(G\left[\varphi_0(\tau,c_0) + A_1(\tau)\left(U(t)\mathcal{P}_{N(Q)}c + y^{(1)}(\tau,\varepsilon)\right) + R(y(\tau,\varepsilon),\tau,\varepsilon)\right]\right)(t)$$

$$+\,\varepsilon U(t)Q^-\left\{J_0(x_0(\cdot,c_0)) + \ell_1\left(U(t)\mathcal{P}_{N(Q)}c + y^{(1)}(\cdot,\varepsilon)\right) + R_1(y(\cdot,\varepsilon),\varepsilon)\right\}.$$

The operator system (18) belong to the class of systems described in [7]. For its solution, we use the method of simple iterations.

We introduce an auxiliary column vector that belongs to the Cartesian product of three copies of the space $\mathbf{B}_1$, namely

$$u = \begin{pmatrix} y \\ c \\ y^{(1)} \end{pmatrix} \in \mathbf{B}_1 \times \mathbf{B}_1 \times \mathbf{B}_1,$$

and the auxiliary operator

$$L_1\psi(t) = -B_0^-\mathcal{P}_{N(Q^*)}\left[\ell_1\psi(\cdot) - \ell\int_a^b K(\cdot,\tau)A_1(\tau)\psi(\tau)d\tau\right]. \tag{19}$$

Then the operator system (18) can be rewritten in the form

$$u = \begin{bmatrix} 0 & U(t)\mathcal{P}_{N(Q)} & I \\ 0 & 0 & L_1 \\ 0 & 0 & 0 \end{bmatrix} u + F(u,t,\varepsilon), \tag{20}$$

where I is the identity operator and

$$F(u,t,\varepsilon)=\operatorname{col}\Bigg(0,-R_1(y(\cdot,\varepsilon),\varepsilon)+\ell\int_a^b K(\cdot,\tau)R(y(\tau,\varepsilon),\tau,\varepsilon)\,d\tau,$$

$$\varepsilon(G[\varphi_0(\tau,c_0)+A_1(\tau)(U(t)\mathcal{P}_{N(Q)}c+y^{(1)}(\tau,\varepsilon))+R(y(\tau,\varepsilon),\tau,\varepsilon)])(t)$$

$$+\,\varepsilon U(t)Q^-\Big\{J_0(x_0(\cdot,c_0))+\ell_1\big(U(t)\mathcal{P}_{N(Q)}c+y^{(1)}(\cdot,\varepsilon)\big)+R_1(y(\cdot,\varepsilon),\varepsilon)\Big\}\Bigg),$$

$$F(0,t,0)=0,\qquad \frac{\partial F(0,t,0)}{\partial y}=0.$$

The operator system (20) is equivalent to the system

$$\begin{bmatrix} I & -U(t)\mathcal{P}_{N(Q)} & -I \\ 0 & I & -L_1 \\ 0 & 0 & I \end{bmatrix} u = F(u,t,\varepsilon).$$

Denote

$$L=\begin{bmatrix} I & -U(t)\mathcal{P}_{N(Q)} & -I \\ 0 & I & -L_1 \\ 0 & 0 & I \end{bmatrix},\qquad F=F(u,t,\varepsilon).$$

We prove that the operator L is invertible and its inverse is bounded. To this end, we determine the inverse L^{-1} in the explicit form. This operator exists due to the structure of the upper-triangular cell operator L, whose principal diagonal consists of the identity operators. For this reason, we seek it in the form of an operator matrix, i.e.,

$$L^{-1}=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

where each component a_{ij} is an operator acting in the Banach space $\mathbf{B}_1$ $(a_{ij}\colon \mathbf{B}_1\to\mathbf{B}_1)$.

One can directly verify that the operator L has a bounded inverse L^{-1}:

$$L^{-1}=\begin{bmatrix} I & U(t)\mathcal{P}_{N(Q)} & U(t)\mathcal{P}_{N(Q)}L_1+I \\ 0 & I & L_1 \\ 0 & 0 & I \end{bmatrix}.$$

Let us prove that the operator L^{-1} is bounded. We show that there exists a constant $c_1>0$ such that

$$\|L^{-1}u\|_{\mathbf{B}_1^3}\le c_1\|u\|_{\mathbf{B}_1^3}\quad\text{for all}\quad u\in\mathbf{B}_1^3.$$

This inequality is equivalent to the following statement: There exists a constant $c_2 > 0$ such that the following inequality holds for all $y, c, y^{(1)} \in \mathbf{B}_1^3$:

$$\left|\left|\left| L^{-1}\begin{pmatrix} y \\ c \\ y^{(1)} \end{pmatrix} \right|\right|\right|_{\mathbf{B}_1^3} \le c_2\left(|||y|||_{\mathbf{B}_1} + |||c|||_{\mathbf{B}_1} + \left|\left|\left| y^{(1)} \right|\right|\right|_{\mathbf{B}_1} \right),$$

$$L^{-1}\begin{pmatrix} y \\ c \\ y^{(1)} \end{pmatrix} = \begin{pmatrix} y + U(t)\mathcal{P}_{N(Q)}c + U(t)\mathcal{P}_{N(Q)}L_1 y^{(1)} + I y^{(1)} \\ c + L_1 y^{(1)} \\ y^{(1)} \end{pmatrix}.$$

We prove that the norm of each component of a vector in the Banach space $\mathbf{B}_1$ is bounded, i.e.,

$$|||U|||_{\mathbf{B}_1} = \sup_{t\in[a;b]} \|U(t)\| < \infty.$$

Let

$$|||B_0^-|||_{\mathbf{B}_1} = b_0, \quad \left|\left|\left| \mathcal{P}_{N(Q)} \right|\right|\right|_{\mathbf{B}_1} = \tilde{p}, \quad \left|\left|\left| \mathcal{P}_{N(Q^*)} \right|\right|\right|_{\mathbf{B}_1} = \tilde{p}^*,$$

$$\|L_1\psi(t)\| = \left\| -B_0^-\mathcal{P}_{N(Q^*)}\left[\ell_1\psi(\cdot) - \ell \int_a^b K(\cdot,\tau)A_1(\tau)\psi(\tau)d\tau \right] \right\|$$

$$\le b_0\tilde{p}^*\,|||\ell_1|||\cdot|||\psi||| + \tilde{a}\,|||\ell|||\cdot|||A_1|||\cdot|||\psi||| = |||L_1|||\,.$$

Therefore,

$$\left|\left|\left| y + U(\cdot)\mathcal{P}_{N(Q)}c + U(\cdot)\mathcal{P}_{N(Q)}L_1 y^{(1)} + I y^{(1)} \right|\right|\right|_{\mathbf{B}_1}$$

$$\le |||y|||_{\mathbf{B}_1} + \left|\left|\left| U(\cdot)\mathcal{P}_{N(Q)} \right|\right|\right|_{\mathbf{B}_1} |||c|||_{\mathbf{B}_1} + \left|\left|\left| U(\cdot)\mathcal{P}_{N(Q)}L_1 \right|\right|\right|_{\mathbf{B}_1} \left|\left|\left| y^{(1)} \right|\right|\right|_{\mathbf{B}_1} + \left|\left|\left| I y^{(1)} \right|\right|\right|_{\mathbf{B}_1}$$

$$\le |||y|||_{\mathbf{B}_1} + c_3\,|||c|||_{\mathbf{B}_1} + c_4 \left|\left|\left| y^{(1)} \right|\right|\right|_{\mathbf{B}_1}.$$

By analogy, we get

$$\left|\left|\left| c + L_1 y^{(1)} \right|\right|\right|_{\mathbf{B}_1} \le |||c|||_{\mathbf{B}_1} + |||L_1|||_{\mathbf{B}_1} \left|\left|\left| y^{(1)} \right|\right|\right|_{\mathbf{B}_1} \le |||c|||_{\mathbf{B}_1} + c_5 \left|\left|\left| y^{(1)} \right|\right|\right|_{\mathbf{B}_1}.$$

Thus,

$$\left|\left|\left| L^{-1}\begin{pmatrix} y \\ c \\ y^{(1)} \end{pmatrix} \right|\right|\right|_{\mathbf{B}_1^3} \le |||y|||_{\mathbf{B}_1} + (c_3+1)\,|||c|||_{\mathbf{B}_1} + (c_4+c_5+1)\left|\left|\left| y^{(1)} \right|\right|\right|_{\mathbf{B}_1}$$

$$\le c_2\left(|||y|||_{\mathbf{B}_1} + |||c|||_{\mathbf{B}_1} + \left|\left|\left| y^{(1)} \right|\right|\right|_{\mathbf{B}_1} \right),$$

where $c_2 = \max\{1, c_3+1, c_4+c_5+1\}$. Thus, the boundedness of the operator L^{-1} is proved.

In the notation introduced, the operator system (18) takes the form

$$u = L^{-1}F = L^{-1}S(\varepsilon)u,$$

where the operator $S(\varepsilon)$ is nonlinear in the general case. Properly choosing ε and using the boundedness of the operator L^{-1}, we can make the operator $L^{-1}S(\varepsilon)$ contracting. By virtue of the contracting-mapping principle [12], the operator system (18) has a unique fixed point, which is a solution of the boundary-value problem (1), (2).

3. Iterative Procedure

On the basis of the operator system (18), we construct an iterative procedure for the construction of a solution $y(t,\cdot) \in C[0;\varepsilon_0]$, $y(t,0) = 0$, of the boundary-value problem (11), (12). We choose the first approximation $y_1^{(1)}(t,\varepsilon)$ of $y(t,\varepsilon)$ as follows:

$$y_1^{(1)}(t,\varepsilon) = \varepsilon\left(G\left[\varphi_0(\tau,c_0)\right]\right)(t) + \varepsilon U(t)Q^{-}J_0(x_0(\cdot,c_0)).$$

The operator function $y_1^{(1)} = y_1^{(1)}(t,\varepsilon)$ is a particular solution of the boundary-value problem

$$\dot{y}_1 = A(t)y_1 + \varepsilon\varphi_0(t,c_0), \quad \ell y_1 = \varepsilon J_0(x_0(\cdot,c_0));$$

it exists by virtue of the choice of $c_0 \in \mathbf{B}_1$ from the equation for generating constants (8). We assume that the first approximation $y_1(t,\varepsilon)$ of the required solution $y(t,\varepsilon)$ of the boundary-value problem (11), (12) is equal to $y_1^{(1)}(t,\varepsilon)$. The second approximation $y_2^{(1)}(t,\varepsilon)$ of $y(t,\varepsilon)$ is assumed to be a particular solution of the boundary-value problem

$$\dot{y}_2 = A(t)y_2 + \varepsilon\left\{\varphi_0(t,c_0) + A_1(t)\left[U(t)\mathcal{P}_{N(Q)}c_1 + y_1^{(1)}(t,\varepsilon)\right] + R(y_1(t,\varepsilon),t,\varepsilon)\right\},$$

$$\ell y_2 = \varepsilon\left\{J_0(x_0(\cdot,c_0)) + \ell_1\left[U(\cdot)\mathcal{P}_{N(Q)}c_1 + y_1^{(1)}(\cdot,\varepsilon)\right] + R_1(y_1(\cdot,\varepsilon),\varepsilon)\right\},$$

which has the form

$$y_2^{(1)}(t,\varepsilon) = \varepsilon\left(G\left[\varphi_0(t,c_0) + A_1(t)\left[U(t)\mathcal{P}_{N(Q)}c_1 + y_1^{(1)}(t,\varepsilon)\right] + R(y_1(t,\varepsilon),t,\varepsilon)\right]\right)(t)$$

$$+\,\varepsilon U(t)Q^{-}\left\{J_0(x_0(\cdot,c_0)) + \ell_1\left[U(\cdot)\mathcal{P}_{N(Q)}c_1 + y_1^{(1)}(\cdot,\varepsilon)\right] + R_1(y_1(\cdot,\varepsilon),\varepsilon)\right\}.$$

Using the condition for the solvability of this problem, we get

$$\mathcal{P}_{N(Q^*)}\varepsilon\Bigg[J_0(x_0(\cdot,c_0)) + \ell_1\left[U(\cdot)\mathcal{P}_{N(Q)}c_1 + y_1^{(1)}(\cdot,\varepsilon)\right] + R_1(y_1(\cdot,\varepsilon),\varepsilon)$$

$$-\ell\int_a^b K(\cdot,\tau)\left(\varphi_0(\tau,c_0) + A_1(\tau)\left[U(\tau)\mathcal{P}_{N(Q)}c_1 + y_1^{(1)}(\tau,\varepsilon)\right] + R(y_1(\tau,\varepsilon),\tau,\varepsilon)\right)d\tau\Bigg] = 0, \quad \varepsilon \neq 0.$$

Taking into account that the element c_0 satisfies Eq. (8), we obtain the system

$$B_0c_1 = -\mathcal{P}_{N(Q^*)}\Bigg[\ell_1 y_1^{(1)}(\cdot,\varepsilon) + R_1(y_1(\cdot,\varepsilon),\varepsilon)$$

$$-\ell\int_a^b K(\cdot,\tau)\left(A_1(\tau)y_1^{(1)}(\tau,\varepsilon) + R(y_1(\tau,\varepsilon),\tau,\varepsilon)\right)d\tau\Bigg], \tag{21}$$

where the operator B_0 has the form (16). We determine the first approximation c_1 of $c(\varepsilon)$.

By virtue of the normal solvability of the operator B_0, Eq. (21) is solvable [11] if and only if its right-hand side satisfies the condition

$$\mathcal{P}_{N(B_0^*)}\mathcal{P}_{N(Q^*)}\Bigg[\ell_1 y_1^{(1)}(\cdot,\varepsilon) + R_1(y_1(\cdot,\varepsilon),\varepsilon)$$

$$-\ell\int_a^b K(\cdot,\tau)\left(A_1(\tau)y_1^{(1)}(\tau,\varepsilon) + R(y_1(\tau,\varepsilon),\tau,\varepsilon)\right)d\tau\Bigg] = 0. \tag{22}$$

Condition (22) is satisfied if condition (17) is satisfied. Under the same condition, the operator equation (21) is solvable:

$$c_1 = -B_0^{-}\mathcal{P}_{N(Q^*)}\Bigg[\ell_1 y_1^{(1)}(\cdot,\varepsilon) + R_1(y_1(\cdot,\varepsilon),\varepsilon)$$

$$-\ell\int_a^b K(\cdot,\tau)\left(A_1(\tau)y_1^{(1)}(\tau,\varepsilon) + R(y_1(\tau,\varepsilon),\tau,\varepsilon)\right)d\tau\Bigg]. \tag{23}$$

We represent the second approximation $y_2^{(1)}(t,\varepsilon)$ of $y(t,\varepsilon)$ in the form

$$y_2(t,\varepsilon) = U(t)\mathcal{P}_{N(Q)}c_1 + y_2^{(1)}(t,\varepsilon).$$

We assume that the third approximation $y_3^{(1)}(t,\varepsilon)$ of $y(t,\varepsilon)$ is a particular solution of the boundary-value problem

$$\dot{y}_3 = A(t)y_3 + \varepsilon\left\{\varphi_0(t,c_0) + A_1(t)\left[U(t)\mathcal{P}_{N(Q)}c_2 + y_2^{(1)}(t,\varepsilon)\right] + R(y_2(t,\varepsilon),t,\varepsilon)\right\},$$

$$\ell y_3 = \varepsilon\left\{J_0(x_0(\cdot,c_0)) + \ell_1\left[U(\cdot)\mathcal{P}_{N(Q)}c_2 + y_2^{(1)}(\cdot,\varepsilon)\right] + R_1(y_2(\cdot,\varepsilon),\varepsilon)\right\},$$

which has the form

$$y_3^{(1)}(t,\varepsilon) = \varepsilon\left(G\left[\varphi_0(t,c_0) + A_1(t)\left[U(t)\mathcal{P}_{N(Q)}c_2 + y_2^{(1)}(t,\varepsilon)\right] + R(y_2(t,\varepsilon),t,\varepsilon)\right]\right)(t)$$
$$+ \varepsilon U(t)Q^-\left\{J_0(x_0(\cdot,c_0)) + \ell_1\left[U(\cdot)\mathcal{P}_{N(Q)}c_2 + y_2^{(1)}(\cdot,\varepsilon)\right] + R_1(y_2(\cdot,\varepsilon),\varepsilon)\right\}.$$

Using the condition for the solvability of this problem, we get

$$\mathcal{P}_{N(Q^*)}\varepsilon\Bigg[J_0(x_0(\cdot,c_0)) + \ell_1\left[U(\cdot)\mathcal{P}_{N(Q)}c_2 + y_2^{(1)}(\cdot,\varepsilon)\right] + R_1(y_2(\cdot,\varepsilon),\varepsilon)$$
$$-\ell\int_a^b K(\cdot,\tau)\left(\varphi_0(t,c_0) + A_1(t)\left[U(t)\mathcal{P}_{N(Q)}c_2 + y_2^{(1)}(t,\varepsilon)\right] + R(y_2(\tau,\varepsilon),\tau,\varepsilon)\right)d\tau\Bigg] = 0, \quad \varepsilon \neq 0.$$

Taking into account that the element c_0 satisfies Eq. (8), we obtain the system

$$B_0c_2 = -\mathcal{P}_{N(Q^*)}\Bigg[\ell_1 y_2^{(1)}(\cdot,\varepsilon) + R_1(y_2(\cdot,\varepsilon),\varepsilon)$$
$$-\ell\int_a^b K(\cdot,\tau)\left(A_1(\tau)y_2^{(1)}(\tau,\varepsilon) + R(y_2(\tau,\varepsilon),\tau,\varepsilon)\right)d\tau\Bigg], \qquad (24)$$

where the operator B_0 has the form (16). We determine the first approximation c_2 of $c(\varepsilon)$.

A criterion for the solvability of the operator system (24) has the form

$$\mathcal{P}_{N(B_0^*)}\mathcal{P}_{N(Q^*)}\Bigg[\ell_1 y_1^{(2)}(\cdot,\varepsilon) + R_1(y_2(\cdot,\varepsilon),\varepsilon)$$
$$-\ell\int_a^b K(\cdot,\tau)\left(A_1(\tau)y_2^{(1)}(\tau,\varepsilon) + R(y_2(\tau,\varepsilon),\tau,\varepsilon)\right)d\tau\Bigg] = 0. \qquad (25)$$

Thus, if $\mathcal{P}_{N(B_0^*)}\mathcal{P}_{N(Q^*)} = 0,$ then solvability conditions of the type (25) for the corresponding operator systems are satisfied at each step of the iterative procedure. Continuing this process, we obtain the following iterative procedure for the determination of a solution $y(t,\cdot) \in C[0;\varepsilon_0],$ $y(t,0) = 0,$ of the boundary-value problem (11), (12):

$$c_k = -B_0^- \mathcal{P}_{N(Q^*)}\Bigg[\ell_1 y_k^{(1)}(\cdot,\varepsilon) + R_1(y_k(\cdot,\varepsilon),\varepsilon) - \ell\int_a^b K(\cdot,\tau)\Big(A_1(\tau)y_k^{(1)}(\tau,\varepsilon) + R(y_k(\tau,\varepsilon),\tau,\varepsilon)\Big)\,d\tau\Bigg], \qquad (26)$$

$$y_{k+1}^{(1)}(t,\varepsilon) = \varepsilon\Big(G\Big[\varphi_0(t,c_0) + A_1(t)[U(t)\mathcal{P}_{N(Q)}c_k + y_k^{(1)}(t,\varepsilon)] + R(y_k(t,\varepsilon),t,\varepsilon)\Big]\Big)(t)$$

$$+\, \varepsilon U(t)Q^-\Big\{J_0(x_0(\cdot,c_0)) + \ell_1\Big[U(\cdot)\mathcal{P}_{N(Q)}c_k + y_k^{(1)}(\cdot,\varepsilon)\Big] + R_1(y_k(\cdot,\varepsilon),\varepsilon)\Big\},$$

$$y_{k+1}(t,\varepsilon) = U(t)\mathcal{P}_{N(Q)}c_k + y_{k+1}^{(1)}(t,\varepsilon), \quad k = 0,1,2,\ldots, \qquad (27)$$

$$y_0(t,\varepsilon) = y_0^{(1)}(t,\varepsilon) = 0.$$

Thus, we have proved the following theorem:

Theorem 2 (sufficient condition)**.** *Suppose that, under condition (6), the boundary-value problem (3), (4) has a family of solutions of the form (5), and the operator* B_0 *satisfies the following conditions:*

(i) B_0 *is a generalized inverse operator;*

(ii) $\mathcal{P}_{N(B_0^*)}\mathcal{P}_{N(Q^*)} = 0.$

Then, for any element $c = c_0 \in \mathbf{B}_1$ *that satisfies the equation for generating constants (7), the boundary-value problem (11), (12) has at least one solution* $y(t,\cdot) \in C[0;\varepsilon_0],$ $y(t,0) = 0.$ *This solution can be determined by using the iterative procedure (26), (27), which converges on* $[0;\varepsilon_0].$ *The boundary-value problem (1), (2) has at least one solution that turns into the generating solution* $x_0(t,c_0)$ *for* $\varepsilon = 0.$ *This solution* $x(t,\cdot) \in C[0;\varepsilon_*]$ *can be determined by using the convergent iterative procedure (26), (27) and the relation* $x_k(t,\varepsilon) = x_0(t,c_0) + y_k(t,\varepsilon),$ $k = 0,1,2,\ldots$.

4. Relationship between Necessary and Sufficient Conditions

A relationship between necessary and sufficient conditions for the existence of solutions of a weakly nonlinear boundary-value problem in a Banach space in the critical case is established by the following statement:

Corollary 1. *Suppose that a functional* $F(c)$ *has the Fréchet derivative* $F^{(1)}(c)$ *for a certain element* c_0 *of the Banach space* $\mathbf{B}_1$ *that satisfies the operator equation for generating constants (7). If* $F^{(1)}(c)$ *has an inverse, then the boundary-value problem (1), (2) has a unique solution for every* c_0.

Proof. We represent the Fréchet derivative of the functional $F(c)$ as follows:

$$F^{(1)}(c)[h] = \mathcal{P}_{N(Q^*)}\Bigg[J^{(1)}(v,\varepsilon)|_{v=x_0(\cdot,c),\varepsilon=0}[x_0^{(1)}(t,c)[h]]$$

$$-\,\ell\int\limits_a^b K(\cdot,\tau)Z^{(1)}(v,\tau,\varepsilon)|_{v=x_0,\varepsilon=0}[x_0^{(1)}(\tau,c)[h]]\,d\tau\Bigg].$$

This representation follows from the theorem on a superposition of differential mappings in a Banach space [13, p. 131]. Let us determine the Fréchet derivative $x_0^{(1)}(t,c)[h]$.

Since $x_0(t,c) = U(t)\mathcal{P}_{N(Q)}c + U(t)Q^{-}\alpha + (Gf)(t)$, we have

$$x_0^{(1)}(t,c)[h] = \frac{\partial x_0(t,c+\lambda h)}{\partial\lambda}\bigg|_{\lambda=0}$$

$$= \frac{\partial}{\partial\lambda}\left[U(t)\mathcal{P}_{N(Q)}c + \lambda U(t)\mathcal{P}_{N(Q)}h + U(t)Q^{-}\alpha + (Gf)(t)\right]\bigg|_{\lambda=0}$$

$$= \frac{\partial}{\partial\lambda}[U(t)\mathcal{P}_{N(Q)}c]\bigg|_{\lambda=0} + \frac{\partial}{\partial\lambda}\,[\lambda U(t)\mathcal{P}_{N(Q)}h]|_{\lambda=0} + \frac{\partial}{\partial\lambda}[U(t)Q^{-}\alpha]\bigg|_{\lambda=0} + \frac{\partial}{\partial\lambda}[(Gf)(t)]\bigg|_{\lambda=0}$$

$$= U(t)\mathcal{P}_{N(Q)}h,$$

$$Z^{(1)}(v,\tau,\varepsilon)\Big|_{v=x_0,\varepsilon=0} = A_1(t), \quad J^{(1)}(v,\varepsilon)\Big|_{v=x_0(\cdot,c)} = \ell_1.$$

Thus,

$$F^{(1)}(c)[h] = \mathcal{P}_{N(Q^*)}\left[\ell_1 U(\cdot)\mathcal{P}_{N(Q)}[h] - \ell\int\limits_a^b K(\cdot,\tau)A_1(\tau)U(\tau)\mathcal{P}_{N(Q)}d\tau[h]\right] = B_0[h].$$

The operator B_0 is invertible by virtue of the invertibility of the operator $F^{(1)}(c)$. Consequently, an equation of the form (15) has a unique solution, and, hence, the boundary-value problem (1), (2) has a unique solution.

Thus, the condition of the invertibility of the operator $B_0 = F^{(1)}(c_0)$ interrelates the necessary and sufficient conditions for the existence of solutions of a weakly nonlinear boundary-value problem in a Banach space in the critical case.

Remark 1. In the case of finite-dimensional spaces $\mathbf{B}_1 = \mathbb{R}^n$ and $\mathbf{B}_2 = \mathbb{R}^m$, the condition of the invertibility of the operator $F^{(1)}(c)$ is equivalent to the condition that the root c_0 of the equation for generating constants is simple.

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