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OSCILLATION CRITERIA FOR A SECOND-ORDER QUASILINEAR NEUTRAL FUNCTIONAL DYNAMIC EQUATION ON TIME SCALES

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Our aim is to establish some sufficient conditions for the oscillation of the second-order quasilinear neutral functional dynamic equation

$$\left(p(t)\left([y(t)+r(t)y(\tau(t))]^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,y(\delta(t)))=0,\quad t\in[t_0,\infty)_{\mathbb{T}},$$

on a time scale $\mathbb{T}$, where $|f(t,u)|\geq q(t)\left|u^{\beta}\right|$, r, p, and q are real-valued rd-continuous positive functions defined on $\mathbb{T}$, and γ and $\beta>0$ are ratios of odd positive integers. Our results do not require that $\gamma=\beta\geq 1$, $p^{\Delta}(t)\geq 0$,

$$\int_{t_0}^{\infty}\left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}}\Delta t=\infty,\qquad\text{and}\qquad\int_{t_0}^{\infty}\delta^{\beta}(s)q(s)[1-r(\delta(s))]^{\beta}\Delta s=\infty.$$

Some examples are considered to illustrate the main results.

1. Introduction

In this paper, we consider the quasilinear neutral functional dynamic equation

$$\left(p(t)\left([y(t)+r(t)y(\tau(t))]^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,y(\delta(t)))=0,\tag{1.1}$$

on a time scale $\mathbb{T}$. Throughout this paper, we assume the following hypotheses:

(h_1) r, p, and q are real-valued rd-continuous positive functions defined on $\mathbb{T}$ and $0\leq r(t)<1$;

(h_2) γ is a ratio of odd positive integers, $\tau\colon\mathbb{T}\to\mathbb{T}$, $\delta\colon\mathbb{T}\to\mathbb{T}$, $\tau(t)\leq t$ for all $t\in\mathbb{T}$, and

$$\lim_{t\to\infty}\delta(t)=\lim_{t\to\infty}\tau(t)=\infty;$$

(h_3) $f(t,u)\colon\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $uf(t,u)>0$ for all $u\neq 0$, and there exists a positive rd-continuous function $q(t)$ defined on $\mathbb{T}$ and such that $|f(t,u)|\geq q(t)|u^{\beta}|$, where $\beta>0$ is a ratio of odd positive integers.

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We also consider the following two cases:

$$\int_{t_0}^{\infty} \left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}} \Delta t = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} \left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}} \Delta t < \infty. \tag{1.3}$$

Since we are interested in the oscillatory and asymptotic behavior of solutions of (1.1) near infinity, we assume that $\sup \mathbb{T} = \infty$ and define a time scale interval $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}$. Throughout this paper, these assumptions are supposed to hold. Let $\tau^*(t) = \min\{\tau(t), \delta(t)\}$ and let $T_0 = \min\{\tau^*(t) : t \geq 0\}$ and $\tau^*_{-1}(t) = \sup\{s \geq 0 : \tau^*(s) \leq t\}$ for $t \geq T_0$. Clearly, if $\tau^*(t) \leq t$, then $\tau^*_{-1}(t) \geq t$ for $t \geq T_0$; $\tau^*_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau^*(t)$ when the latter exists. Throughout the paper, we use the following notation:

$$x(t) := y(t) + r(t)y(\tau(t)), \quad x^{[1]} := p(x^{\Delta})^{\gamma}, \quad x^{[2]} := (x^{[1]})^{\Delta}. \tag{1.4}$$

By a solution of (1.1), we mean a nontrivial real-valued function y that has the properties $x \in C^1_{rd}[\tau^*_{-1}(t_0), \infty)$ and $x^{[1]} \in C^1_{rd}[\tau^*_{-1}(t_0), \infty)$, where C_r is the space of rd-continuous functions. Our attention is restricted to the solutions of (1.1) that exist on some half line $[t_y, \infty)$ and satisfy the condition $\sup\{|y(t)| : t > t_1\} > 0$ for any $t_1 \geq t_y$. A solution y of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Much recent attention has been given to dynamic equations on time scales (or measure chains), and we refer the reader to the landmark paper of Hilger [6] for a comprehensive treatment of the subject. Since then, several authors have expounded on various aspects of this new theory [5]. The book [4] by Bohner and Peterson on the subject of time scales summarizes and organizes much of time-scale calculus.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [9]), i.e., the cases where $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}} = t : t = q^k$, $k \in \mathbb{N}$, $q > 1$.

Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, they can model insect populations that are continuous while in season, die out, say, in winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [4]). There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in *New Scientist* [14] discusses several possible applications. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and $f^{\sigma}(t)$ denotes $f(\sigma(t))$ for any function $f : \mathbb{T} \to \mathbb{R}$.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of second-order neutral dynamic equations on time scales; we refer the reader to [1–3, 7, 8, 11–13, 15, 16]. We note that all the results obtained in these papers are given for neutral delay dynamic equations in the case where (1.2) holds,

$$\gamma = \beta \geq 1, \quad p^{\Delta}(t) \geq 0, \quad \text{and} \quad \int_{t_0}^{\infty} \delta^{\gamma}(s)q(s)[1 - r(\delta(s))]^{\gamma} \Delta s = \infty. \tag{1.5}$$

The question now is as follows: Is it possible to find new oscillation criteria for (1.1) without (1.5)? Our interest is to give an affirmative answer to this question and to establish some oscillation criteria for (1.1) that do not require that

$$\gamma = \beta \geq 1, \quad p^{\Delta}(t) \geq 0, \quad \text{and} \quad \int_{t_0}^{\infty} \delta^{\beta}(s) q(s) [1 - r(\delta(s))]^{\beta} \Delta s = \infty. \tag{1.6}$$

The paper is organized as follows: In Sec. 2, we consider the case where (1.2) holds. In Sec. 3, we consider the case where (1.3) holds. Our results are essentially new for (1.1) even in the case where $\gamma = \beta$ and can be applied if $\gamma < 1$ and/or $\beta < 1$. Applications to equations to which previously known criteria for oscillation are not applicable are given.

2. Oscillation Criteria in the Case Where (1.2) Holds

In this section, we establish some sufficient conditions for oscillation of (1.1) in the case where (1.2) holds. In Sec. 2.1, we consider the case where $\delta(t) > t$, and the case where $\delta(t) \leq t$ is considered in Sec. 2.2. To prove the main results we need the following lemmas, which play important roles in the proofs of the main results even in the case where (1.3) holds:

Lemma 2.1. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied, Eq. (1.1) has a nonoscillatory solution* y *on* $[t_0, \infty)_{\mathbb{T}}$, *and* x *is defined as in (1.4). Then there exists* $T > t_0$ *such that* $x(t) x^{[1]}(t) > 0$ *for* $t \geq T$.

Proof. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. (Note that, in the case where $y(t)$ is negative, the proof is similar because the transformation $y(t) = -z(t)$ transforms (1.1) into the same form.) Since y is a positive solution of (1.1) and $q(t) > 0$, we have [by (h_3)]

$$(x^{[1]}(t))^{\Delta} \leq -q(t) y^{\beta}(\delta(t)) < 0 \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{2.1}$$

Then $x^{[1]}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$ and is of one sign. We claim that $x^{[1]}(t) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. Assume the contrary. Then there is $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $x^{[1]}(t_2) = c < 0$ (note that $x^{[1]}(t)$ is strictly decreasing). Then it follows from (2.1) that $x^{[1]}(t) \leq c$ for $t \geq t_2$, and, therefore,

$$x^{\Delta}(t) \leq \frac{c^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)} \quad \text{for} \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.2}$$

Integrating the last inequality from t_2 to t, we find from (1.2) that

$$x(t) = x(t_2) + \int_{t_2}^{t} x^{\Delta}(s) \Delta s \leq x(t_2) + c^{\frac{1}{\gamma}} \int_{t_2}^{t} \frac{\Delta s}{p^{\frac{1}{\gamma}}(s)} \to -\infty \quad \text{as} \quad t \to \infty, \tag{2.3}$$

which implies that x is eventually negative. This contradiction completes the proof.

Lemma 2.2. *Assume that conditions (h_1)–(h_3) and (1.2) are satisfied, Eq. (1.1) has a nonoscillatory solution y on $[t_0,\infty)_{\mathbb{T}}$, and x is defined as in (1.4). Then there exists $T \geq t_0$ such that*

$$\left(p(t)\left(x^{\Delta}(t)\right)^{\gamma}\right)^{\Delta} + P(t)x^{\beta}(\delta(t)) \leq 0 \quad \textit{for} \quad t \geq T, \tag{2.4}$$

where

$$P(t) = q(t)(1 - r(\delta(t)))^{\beta}. \tag{2.5}$$

Proof. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0,\infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0,\infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ on $[t_1,\infty)_{\mathbb{T}}$. (Note that, in the case where $y(t)$ is negative, the proof is similar because the transformation $y(t) = -z(t)$ transforms (1.1) into the same form.) Since y is a positive solution of (1.1) and $q(t) > 0$, it follows from Lemma 2.1 that (note that $x^{[1]}(t) > 0$ and $p(t) > 0$)

$$x(t) > 0, \quad x^{\Delta}(t) > 0, \quad \text{and} \quad \left(x^{[1]}(t)\right)^{\Delta} < 0 \quad \text{for} \quad t \geq t_1. \tag{2.6}$$

Since $\tau(t) \leq t$ and $r(t) \geq 0$, it follows from (1.4) and (2.6) that

$$x(t) = y(t) + r(t)y(\tau(t)) \leq y(t) + r(t)x(\tau(t)) \leq y(t) + r(t)x(t) \quad \text{for} \quad t \geq t_1.$$

Thus, $y(t) \geq (1 - r(t))x(t)$ for $t \geq t_1$. Then, for $t \geq t_2$, where $t_2 > t_1$ is chosen large enough, we have

$$y(\delta(t)) \geq (1 - r(\delta(t)))x(\delta(t)). \tag{2.7}$$

Relation (2.1) and the last inequality yield inequality (2.4), which completes the proof.

2.1. Oscillation of (1.1) in the Case Where $\delta(t) > t$. In this subsection, we establish some sufficient conditions for the oscillation of (1.1) in the case where (1.2) is true and $\delta(t) > t$. We introduce the following notation:

$$Q(t) := P(t)\left(\frac{p^{1/\gamma}(t)P(t,T)}{p^{1/\gamma}(t)P(t,T) + \sigma(t) - t}\right)^{\beta}\eta^{\sigma}(t), \quad P(t,T) := \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s,$$

and

$$\eta^{\sigma}(t) := \begin{cases} 1 & \text{if} \quad \beta = \gamma, \\ c_2\left(\int_T^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s\right)^{\beta-\gamma} & \text{if} \quad \beta < \gamma, \\ c_1 & \text{if} \quad \beta > \gamma, \end{cases} \tag{2.8}$$

where $T \geq t_0$ is chosen sufficiently large and c_1 and c_2 are arbitrary positive constants. We begin with the following theorem:

Theorem 2.1. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. Let* y *be a nonoscillatory solution of (1.1) and make the Riccati substitution*

$$u(t) := \frac{x^{[1]}(t)}{x^{\gamma}(t)}, \tag{2.9}$$

where x *is defined as in (1.4). Then* $u(t) > 0$ *for* $t \geq T$ *(here* T *is as in Lemma 2.2) and*

$$u^{\Delta}(t) + Q(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} \left(u^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} \leq 0 \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}}. \tag{2.10}$$

Proof. Let y be as above and assume, without loss of generality, that there is $t_1 > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq t_1$. Then it follows from Lemma 2.1 and (1.4) that there exists $T > t_1$ such that

$$x(t) > 0, \quad x^{[1]}(t) > 0, \quad \text{and} \quad x^{[2]}(t) < 0 \quad \text{for} \quad t \geq T.$$

By the quotient rule [4] (Theorem 1.20) and the definition of $u(t)$, we have

$$u^{\Delta}(t) = \frac{x^{\gamma}(t)x^{[2]}(t) - (x^{\gamma}(t))^{\Delta}\, x^{[1]}(t)}{x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}} = \frac{x^{[2]}(t)}{\left(x^{\delta}(t)\right)^{\gamma}} \frac{\left(x^{\delta}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}} - \frac{(x^{\gamma}(t))^{\Delta}\, x^{[1]}(t)}{x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}}.$$

It follows from Lemma 2.2 that

$$u^{\Delta}(t) \leq -P(t) \frac{\left(x^{\delta}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}} - \frac{(x^{\gamma}(t))^{\Delta}\, x^{[1]}(t)}{x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.11}$$

By the Pötzsche chain rule ([4], Theorem 1.90), if $f^{\Delta}(t) > 0$ and $\gamma > 1$ (note that $f^{\sigma} \geq f$), then

$$\left(f^{\gamma}(t)\right)^{\Delta} = \gamma \int_0^1 \left[f(t) + \mu h f^{\Delta}(t)\right]^{\gamma-1} f^{\Delta}(t)dh$$

$$\geq \gamma \int_0^1 (f(t))^{\gamma-1} f^{\Delta}(t)dh = \gamma (f(t))^{\gamma-1} f^{\Delta}(t). \tag{2.12}$$

Also by the Pötzsche chain rule ([4], Theorem 1.90), if $f^{\Delta}(t) > 0$ and $0 < \gamma \leq 1$, then

$$\left(f^{\gamma}(t)\right)^{\Delta} = \gamma \int_0^1 \left[f(t) + h\mu(t) f^{\Delta}(t)\right]^{\gamma-1} dh\; f^{\Delta}(t)$$

$$\geq \gamma \int_0^1 \left(f^{\sigma}(t)\right)^{\gamma-1} dh f^{\Delta}(t) = \gamma (f^{\sigma}(t))^{\gamma-1} f^{\Delta}(t). \tag{2.13}$$

Using (2.12), (2.13), the relation $f(t) = x(t)$, and the fact that $x(t)$ is increasing and $x^{[1]}(t)$ is decreasing, for $\gamma > 1$ we get

$$\frac{\left((x(t))^{\gamma}\right)^{\Delta} x^{[1]}(t)}{(x(t))^{\gamma}\,(x^{\sigma}(t))^{\gamma}} \geq \frac{\gamma\left(x^{[1]}(t)\right)^{\sigma}\left(\left(x^{[1]}(t)\right)^{\sigma}\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}x^{\sigma}(t)\,(x^{\sigma}(t))^{\gamma}} = \gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}\left(u^{\sigma}(t)\right)^{\frac{1}{\gamma}+1}.$$

Also for $0 < \gamma \leq 1$, we have

$$\frac{(x^{\gamma}(t))^{\Delta}\,x^{[1]}(t)}{x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}} \geq \frac{\gamma\left(x^{[1]}(t)\right)^{\sigma}\left(\left(x^{[1]}\right)^{\sigma}(t)\right)^{\frac{1}{\gamma}}}{p^{\frac{1}{\gamma}}(t)\,(x^{\sigma}(t))^{\gamma}\,x^{\sigma}(t)} = \gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}\left(u^{\sigma}(t)\right)^{1+\frac{1}{\gamma}}.$$

Thus,

$$\frac{(x^{\gamma}(t))^{\Delta}\,x^{[1]}(t)}{x^{\gamma}(t)\,(x^{\sigma}(t))^{\gamma}} \geq \gamma\frac{1}{p^{\frac{1}{\gamma}}}\left(u^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} \quad \text{for} \quad \gamma > 0.$$

Substituting in (2.11), we obtain

$$u^{\Delta}(t) \leq -P(t)\frac{\left(x^{\delta}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}} - \gamma\frac{1}{p^{\frac{1}{\gamma}}(t)}\left(u^{\sigma}\right)^{1+\frac{1}{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.14}$$

Next, consider the coefficient of P in (2.14). Since $x^{\sigma} = x + \mu x^{\Delta}$, we have

$$\frac{x^{\sigma}(t)}{x(t)} = 1 + \mu(t)\frac{x^{\Delta}}{x(t)} = 1 + \frac{\mu(t)}{p^{\frac{1}{\gamma}}(t)}\frac{\left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}}{x(t)}. \tag{2.15}$$

Since $x^{[1]}(t)$ is decreasing, we get

$$x(t) = x(T) + \int_{T}^{t}\left(x^{[1]}(s)\right)^{\frac{1}{\gamma}}\left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s > \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}\int_{T}^{t}\left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s.$$

Hence,

$$\frac{x(t)}{\left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}} \geq \int_{T}^{t}\left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}}\Delta s = P(t,T) \quad \text{for} \quad t \geq T. \tag{2.16}$$

This and (2.15) imply that

$$\frac{x^{\sigma}(t)}{x(t)} = 1 + \mu(t)\frac{x^{\Delta}(t)}{x(t)} = 1 + \frac{\mu(t)}{p^{\frac{1}{\gamma}}(t)}\frac{\left(x^{[1]}(t)\right)^{\frac{1}{\gamma}}}{x(t)} \leq \frac{p^{\frac{1}{\gamma}}(t)P(t,T) + \mu(t)}{p^{\frac{1}{\gamma}}(t)P(t,T)} \quad \text{for} \quad t \geq T.$$

Hence,

$$\frac{x(t)}{x^\sigma(t)} \geq \frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\sigma(t)-t} \quad \text{for} \quad t \geq T.$$

Thus, for $t \geq T$, we have

$$\frac{x^\delta(t)}{x^\sigma(t)} = \frac{x^\delta(t)}{x(t)}\frac{x(t)}{x^\sigma(t)} \geq \left(\frac{x^\delta(t)}{x(t)}\right)\frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\sigma(t)-t}. \tag{2.17}$$

Since $\delta(t) > t$ and $x(t)$ is increasing, we have

$$x^\delta(t) > x(t). \tag{2.18}$$

This and (2.17) guarantee that

$$\frac{\left(x^\delta(t)\right)^\beta}{(x^\sigma(t))^\gamma} \geq \left(\frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\sigma(t)-t}\right)^\beta \left(x^\sigma(t)\right)^{\beta-\gamma} \quad \text{for} \quad t \geq T. \tag{2.19}$$

Consider several cases.

Case 1: $\beta < \gamma$. Since $x^{[1]}(t)$ is positive and decreasing, it follows from Lemma 2.1 that $x^{[1]}(t) \leq x^{[1]}(t_2) = c$ for $t \geq t_2$. This implies that

$$x(\sigma(t)) \leq x(t_2) + c^{\frac{1}{\gamma}}\left(\int_{t_2}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s\right).$$

Thus,

$$x^{\beta-\gamma}(\sigma(t)) > (c_2)^\beta \left(\int_{t_2}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s\right)^{\beta-\gamma}, \tag{2.20}$$

where

$$c_2 = \left(\frac{1}{c}\right)^\beta.$$

Case 2: $\beta = \gamma$. In this case, we see that $(x^\sigma(t))^{\beta-\gamma} = 1$.

Case 3: $\beta > \gamma$. In this case, since $x^\Delta(t) > 0$, there exists $t_2 \geq t_1$ such that $x^\sigma(t) > x(t) > c > 0$. This implies that $(x^\sigma(t))^{\beta-\gamma} > c_1$, where $c_1 = c^{\beta-\gamma}$.

Combining these three cases and using the definition of $\eta^\sigma(t)$, we conclude that

$$\left(x^\sigma(t)\right)^{\beta-\gamma} \geq \eta^\sigma(t).$$

This and (2.19) yield

$$\frac{\left(x^{\delta}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}} \geq \left(\frac{p^{\frac{1}{\gamma}}(t)P(t,T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\sigma(t)-t}\right)^{\beta} \eta^{\sigma}(t) \quad \text{for} \quad t \geq T. \tag{2.21}$$

Substituting (2.21) into (2.14), we obtain inequality (2.10), which completes the proof.

Theorem 2.2 (Leighton–Wintner type)**.** *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. Furthermore, assume that*

$$\int_{t_0}^{\infty} Q(s)\Delta s = \infty. \tag{2.22}$$

Then every solution of (1.1) oscillates.

Proof. Assume the contrary and let y be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$ (where T is as in Theorem 2.1). We consider only this case because the proof for $y(t) < 0$ is similar. Let u be defined as in Theorem 2.1. Then it follows from Theorem 2.1 that $u(t) > 0$ for $t \geq T$ and the following inequality is true:

$$-u^{\Delta}(t) \geq Q(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} \left(u^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} > Q(t) \quad \text{for} \quad t \geq T. \tag{2.23}$$

It follows from the definition of $x^{[1]}(t)$ that

$$x^{\Delta}(t) = \left(\frac{x^{[1]}(t)}{p(t)}\right)^{\frac{1}{\gamma}}.$$

Integrating from T to t, we obtain

$$x(t) = x(T) + \int_{T}^{t} \left(\frac{1}{p(s)} x^{[1]}(s)\right)^{\frac{1}{\gamma}} \Delta s \quad \text{for} \quad t \geq T.$$

Taking into account that $x^{[1]}(t)$ is positive and decreasing, we get

$$x(t) \geq x(T) + \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}} \int_{T}^{t} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s \quad \text{for} \quad t \geq T.$$

Hence,

$$u(t) = \frac{x^{[1]}(t)}{x^{\gamma}(t)} \leq \left(\int_{t_0}^{t} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s\right)^{-\gamma} \quad \text{for} \quad t \in [T,\infty)_{\mathbb{T}},$$

which implies, in view of (1.2), that

$$\lim_{t\to\infty} u(t) = 0.$$

Integrating (2.23) from T to ∞ and using the fact that

$$\lim_{t\to\infty} u(t) = 0,$$

we obtain

$$u(T) \geq \int_T^\infty Q(s)\Delta s,$$

which contradicts (2.22). The proof is complete.

In what follows, we consider the case where

$$\int_{t_0}^\infty Q(s)\Delta s < \infty. \tag{2.24}$$

Theorem 2.3. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. Furthermore, assume that there exists a positive* rd*-continuous* Δ*-differentiable function* $\phi(t)$ *such that*

$$\lim_{t\to\infty} \sup \int_{t_0}^t \left[\phi(s)Q(s) - \frac{p(s)((\phi^\Delta(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)}\right]\Delta s = \infty. \tag{2.25}$$

Then every solution of (1.1) oscillates.

Proof. Assume the contrary and let y be a nonoscillatory solution of Eq. (1.1). Without loss of generality we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$ (where T is as in Theorem 2.1). We consider only this case because the proof for $y(t) < 0$ is similar. Let u be defined as in Theorem 2.1. Then it follows from Theorem 2.1 that $u(t) > 0$ for $t \geq T$ and inequality (2.10) is true. Using (2.10), we get

$$u^\Delta(t) \leq -Q(t) - \frac{\gamma}{p^{\frac{1}{\gamma}}(t)}\left(u^\sigma\right)^{\frac{\gamma+1}{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.26}$$

Multiplying (2.26) by $\phi(s)$ and integrating from T to t $(t \geq T)$, we obtain

$$\int_T^t \phi(s)Q(s)\Delta s \leq -\int_T^t \phi(s)u^\Delta(s)\Delta s - \int_T^t \frac{\gamma\phi(s)}{p^{\frac{1}{\gamma}}(s)}\left(u^\sigma\right)^{\frac{\gamma+1}{\gamma}}\Delta s.$$

Using integration by parts, we get

$$\int_T^t \phi(s)Q(s)\Delta s \le u(T)\phi(T) + \int_T^t \phi^\Delta(s)u^\sigma(s)\Delta s - \int_{t_1}^t \frac{\gamma\phi(s)}{p^{\frac{1}{\gamma}}(s)}(u^\sigma)^{\frac{\gamma+1}{\gamma}}\Delta s.$$

Setting $B = \phi^\Delta(s),\ A = \gamma\phi(s)p^{-1/\gamma}(s),$ and $u = u^\sigma$ and using the inequality

$$Bu - Au^{\frac{\gamma+1}{\gamma}} \le \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^\gamma},$$

we get

$$\int_T^t \phi(s)Q(s)\Delta s \le u(t_2)\phi(T) + \int_{t_1}^t \frac{p(s)(\phi^\Delta(s))^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)}\Delta s,$$

i.e.,

$$\int_{t_2}^t \left[\phi(s)Q(s) - \frac{p(s)(\phi^\Delta(s))^{\gamma+1}(s)}{(\gamma+1)^{\gamma+1}\phi^\gamma(s)}\right]\Delta s < \phi(T)u(T),$$

which contradicts condition (2.25). Then every solution of (1.1) oscillates. The proof is complete.

From Theorem 2.3, we can obtain different conditions for the oscillation of (1.1) by using different choices of $\phi(t)$. For instance, if $\phi(t) = t$, we have the following result:

Corollary 2.1. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. Furthermore, let*

$$\lim_{t\to\infty}\sup \int_{t_0}^t \left[sQ(s) - \frac{p(s)}{(\gamma+1)^{\gamma+1}s^\gamma}\right]\Delta s = \infty. \tag{2.27}$$

Then every solution of (1.1) oscillates.

Another method for choosing test functions can be developed by considering the function class $\Re$ that consists of kernels of two variables. Following [11], we say that a function H belongs to $\Re$ if H is defined for $t_0 \le s \le t$, $t, s \in [t_0, \infty)_{\mathbb{T}}$, $H(t,s) \ge 0$, $H(t,t) = 0$ for $t \ge s \ge t_0$, and, for every fixed t, $H^{\Delta_i}(t,s)$ is delta integrable with respect to the variable i, $i = 1, 2$. Important examples of H are $H(t,s) = (t-s)^m$ for $m \ge 1$ in the case where $\mathbb{T} = \mathbb{R}$ and $H(t,s) = (t-s)^{\underline{k}}$, $k \in \mathbb{N}$, $t^{\underline{k}} = t(t-1)\ldots(t-k+1)$, in the case where $\mathbb{T} = \mathbb{Z}$.

The theorem below gives new oscillation criteria for (1.1), which can be considered as an extension of a Kamenev-type oscillation criterion. The proof is similar to that in [11] (Theorem 3.3) if one uses inequality (2.10) and is thus omitted.

Theorem 2.4. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. Suppose that* $\phi(t)$ *is defined as in Theorem 2.3,* $H \in \Re$, *and, for* $t > s$, *one has*

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)Q(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}(H^{\Delta_s}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)H^{\gamma}(t,s)} \right] \Delta s = \infty. \tag{2.28}$$

Then every solution of (1.1) oscillates.

Properly choosing the functions H, one can establish a number of oscillation criteria for (1.1) on different types of time scales. For instance, if there exists a function $h(t,s) \in \Re$ such that

$$H^{\Delta_s}(t,s) := -h(t,s)H^{\frac{\gamma}{1+\gamma}}(t,s), \tag{2.29}$$

we deduce the following oscillation result from Theorem 2.4:

Corollary 2.2. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. Suppose that* $\phi(t)$ *is defined as in Theorem 2.3,* $H \in \Re$, *and, for* $t > s$, *one has*

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\phi(s)Q(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}(h(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)} \right] \Delta s = \infty.$$

Then every solution of Eq. (1.1) is oscillatory.

As a special case, by choosing $H(t,s) = (t-s)^m$ for $m \geq 1$, we deduce the following Kamenev-type oscillation criterion from Corollary 2.2:

Corollary 2.3. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. If, for* $m > 1$,

$$\lim_{t\to\infty} \sup \frac{1}{t^m} \int_{t_0}^{t} \left[(t-s)^m Q(s) - \frac{m^{\gamma+1}p(s)((t-s)^{m-1})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(t-s)^{m\gamma}} \right] \Delta s = \infty,$$

then every solution of (1.1) oscillates.

In what follows, we give an example to illustrate the results of this subsection. To obtain conditions for oscillation we use the following facts:

$$\int_{t_0}^{\infty} \frac{\Delta s}{s^{\nu}} = \infty \quad \text{if} \quad 0 \leq \nu \leq 1, \qquad \text{and} \qquad \int_{t_0}^{\infty} \frac{\Delta s}{s^{\nu}} < \infty \quad \text{if} \quad \nu > 1. \tag{2.30}$$

For more details, we refer the reader to [4] (Theorem 5.68 and Corollary 5.71).

Example 2.1. Consider the following second-order neutral dynamic equation:

$$\left[y(t) + \frac{1}{2}y(\tau(t)) \right]^{\Delta\Delta} + \frac{\lambda\,(\sigma(t)-1)}{t^3} y(\delta(t)) = 0 \quad \text{for} \quad t \in [2,\infty)_{\mathbb{T}}, \tag{2.31}$$

where $\mathbb{T}$ is a time scale such that

$$\int_1^\infty \frac{\sigma(s)}{s^3}\Delta s < \infty.$$

Here, $\gamma = 1$, $\tau(t) < t$, $\delta(t) > t$, $\tau(t), \delta(t) \in \mathbb{T}$,

$$\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty,$$

$r(t) = 1/2$, $p(t) = 1$, $f(t,u) = q(t)u$,

$$q(t) = \frac{\lambda(\sigma(t)-1)}{t^3},$$

and $\lambda > 0$ is a constant. Now take an arbitrary $T \geq 2$. Since $p(t) = 1$, we have $P(t,T) = P(t,T) = t - T$. This gives

$$Q(t) := P(t)\frac{P(t,T)}{P(t,T)+\sigma(t)-t} = \frac{\lambda(\sigma(t)-1)}{2t^3}\frac{t-T}{t-T+\sigma(t)-t} = \frac{\lambda(\sigma(t)-1)}{t^3}\frac{t-T}{\sigma(t)-T}.$$

It is easy to see that assumptions (h_1)–(h_3) hold and also (2.24) is satisfied because

$$\int_{t_0}^\infty Q(s)\Delta s = \frac{\lambda}{2}\int_{t_0}^\infty \frac{(\sigma(s)-1)}{s^3}\frac{s-T}{\sigma(s)-T}\Delta s \leq \frac{\lambda}{2}\int_2^\infty \frac{\sigma(s)}{s^3} - \frac{1}{s^3}\Delta s < \infty.$$

To apply Corollary 2.1, it remains to discuss condition (2.27). Note that

$$\lim_{t\to\infty} \sup \int_{t_0}^t \left[sQ(s) - \frac{r(s)}{(\gamma+1)^{\gamma+1}s^\gamma}\right]\Delta s$$

$$= \lim_{t\to\infty} \sup \int_2^t \left(\frac{\lambda s(\sigma(s)-1)}{2s^3}\frac{s-T}{\sigma(s)-T} - \frac{1}{4s}\right)\Delta s$$

$$> \lim_{t\to\infty} \sup \int_t^t \left(\frac{\lambda s^2}{2s^3} - \frac{T}{2s^2(s-1)} - \frac{1}{4s}\right)\Delta s = \infty,$$

provided that $\lambda > 1/2$. Hence, by Corollary 2.1, every solution of (2.31) oscillates if $\lambda > 1/2$.

2.2. Oscillation Criteria in the Case Where $\delta(t) \leq t$. In this subsection, we establish some sufficient conditions for the oscillation of (1.1) in the case where $\delta(t) \leq t$. We use the following notation:

$$A(t) := P(t)\alpha^\beta(t)\eta^\sigma(t),$$

where $\eta^{\sigma}(t)$ is defined as in (2.8),

$$\alpha(t) := \frac{p^{\frac{1}{\gamma}}(t)P(\delta(t),T)}{p^{\frac{1}{\gamma}}(t)P(t,T)+\mu(t)}, \qquad \text{and} \qquad P(u,v) := \int_v^u \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s.$$

Theorem 2.5. *Assume that conditions (h_1)–(h_3) and (1.2) are satisfied. Let y be a nonoscillatory solution of (1.1) and make the Riccati substitution*

$$w(t) := \frac{x^{[1]}(t)}{x^{\gamma}(t)}, \tag{2.32}$$

where x is defined as in (1.4). Then $w(t) > 0$ for $t \geq T$ (here, T is as in Lemma 2.2) and

$$w^{\Delta}(t) + A(t) + \gamma \frac{1}{p^{\frac{1}{\gamma}}(t)} \left(w^{\sigma}\right)^{1+\frac{1}{\gamma}}(t) \leq 0 \quad \textit{for} \quad t \in [T,\infty)_{\mathbb{T}}. \tag{2.33}$$

Proof. Let y be as above and assume, without loss of generality, that there is $t_1 > t_0$ such that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq t_1$. From the definition of w, by the quotient rule [4] (Theorem 1.20) and as in the proof of Theorem 2.1, we get

$$w^{\Delta}(t) \leq -P(t)\frac{\left(x^{\delta}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}} - \gamma \frac{1}{p^{\frac{1}{\gamma}}(t)} \left(w^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} \quad \text{for} \quad t \geq T. \tag{2.34}$$

Now consider the coefficient of $P(t)$ in (2.34). Since $x^{[1]}(t) = p\left(x^{\Delta}\right)^{\gamma}(t)$ is decreasing for $t \geq T,$ we have

$$x^{\sigma}(t) - x(\delta(t)) = \int_{\delta(t)}^{\sigma(t)} \frac{x^{[1]}(s)}{p^{\frac{1}{\gamma}}(s)}\Delta s \leq x^{[1]}(\delta(t)) \int_{\delta(t)}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s,$$

and this implies that

$$\frac{x^{\sigma}(t)}{x(\delta(t))} \leq 1 + \frac{x^{[1]}(\delta(t))}{x(\delta(t))} \int_{\delta(t)}^{\sigma(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s. \tag{2.35}$$

On the other hand, we have

$$x(\delta(t)) > x(\delta(t)) - x(T) = \int_T^{\delta(t)} \frac{x^{[1]}(s)}{p^{\frac{1}{\gamma}}(s)}\Delta s \geq (x^{[1]})(\delta(t)) \int_T^{\delta(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s,$$

which leads to

$$\frac{x^{[1]}(\delta(t))}{x(\delta(t))} < \left(\int_T^{\delta(t)} \frac{1}{p^{\frac{1}{\gamma}}(s)}\Delta s\right)^{-1}.$$

This and (2.35) imply that

$$\frac{x^{\sigma}(t)}{x(\delta(t))} < 1 + \frac{\int_{\delta(t)}^{\sigma(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}{\int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s} = \frac{\int_{T}^{\sigma(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}{\int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}$$

$$= \frac{\int_{T}^{t} p^{-\frac{1}{\gamma}}(s)\Delta s + \int_{t}^{\sigma(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}{\int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s}$$

$$= \frac{\int_{T}^{t} p^{-\frac{1}{\gamma}}(s)\Delta s + \mu(t) p^{-\frac{1}{\gamma}}(t)}{\int_{T}^{\delta(t)} p^{-\frac{1}{\gamma}}(s)\Delta s} = \frac{1}{\alpha(t)} \quad \text{for} \quad t \geq T,$$

where we have used the fact that

$$\int_{t}^{\sigma(t)} f(s)\Delta s = \mu(t) f(t).$$

Hence,

$$x(\delta(t)) \geq \alpha(t) x^{\sigma}(t) \quad \text{for} \quad t \geq T. \tag{2.36}$$

This implies that

$$\frac{\left(x^{\delta}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}} \geq (\alpha(t))^{\beta} \left(x^{\sigma}(t)\right)^{\beta-\gamma} \quad \text{for} \quad t \geq T.$$

As in the proof of Theorem 2.1, since $(x^{\sigma}(t))^{\beta-\gamma} \geq \eta^{\sigma}(t)$, we have

$$\frac{\left(x^{\delta}(t)\right)^{\beta}}{(x^{\sigma}(t))^{\gamma}} \geq (\alpha(t))^{\beta} \eta^{\sigma}(t) \quad \text{for} \quad t \geq T. \tag{2.37}$$

Substituting (2.37) into (2.34), we obtain the desired inequality (2.33). This completes the proof.

Theorem 2.6 (Leighton–Wintner type). *Assume that conditions (h_1)–(h_3) and (1.2) are satisfied. Furthermore, assume that*

$$\int_{t_0}^{\infty} A(s)\Delta s = \infty. \tag{2.38}$$

Then every solution of (1.1) oscillates.

Proof. Assume the contrary and let y be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(\tau(t))) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$ (where T is as in Theorem 2.5). We consider only this case because the proof in the case where $y(t) < 0$ is similar. Let w be defined as in Theorem 2.2. Then it follows from Theorem 2.5 that $w(t) > 0$ for $t \geq T$ and inequality (2.33) is true. Using (2.33), we get

$$-w^{\Delta}(t) \geq A(t) + \frac{\gamma}{p^{\frac{1}{\gamma}}(t)} \left(w^{\sigma}(t)\right)^{1+\frac{1}{\gamma}} > Q(t) \quad \text{for} \quad t \geq T. \tag{2.39}$$

It follows from the definition of $x^{[1]}(t)$ that

$$x^{\Delta}(t) = \left(\frac{x^{[1]}(t)}{p(t)}\right)^{\frac{1}{\gamma}}.$$

Integrating from T to t, we obtain

$$x(t) = x(T) + \int_T^t \left(\frac{1}{p(s)} x^{[1]}(s)\right)^{\frac{1}{\gamma}} \Delta s \quad \text{for} \quad t \geq T.$$

Taking into account that $x^{[1]}(t)$ is positive and decreasing, we get

$$x(t) \geq x(T) + \left(x^{[1]}(t)\right)^{\frac{1}{\gamma}} \int_T^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s \quad \text{for} \quad t \geq T.$$

Hence,

$$w(t) = \frac{x^{[1]}(t)}{x^{\gamma}(t)} \leq \left(\int_{t_0}^t \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s\right)^{-\gamma} \quad \text{for} \quad t \in [T, \infty)_{\mathbb{T}},$$

which, in view of (1.2), implies that

$$\lim_{t \to \infty} w(t) = 0.$$

Integrating (2.39) from T to ∞ and using the fact that

$$\lim_{t \to \infty} w(t) = 0,$$

we obtain

$$w(T) \geq \int_T^{\infty} A(s) \Delta s,$$

which contradicts (2.38). The proof is complete.

In what follows, we consider the case where

$$\int_{t_0}^{\infty} A(s)\Delta s < \infty \tag{2.40}$$

and proceed as in the proof of Theorem 2.3 [using inequality (2.33)] to get the following results:

Theorem 2.7. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. Furthermore, assume that there exists a positive* rd*-continuous* Δ*-differentiable function* $\phi(t)$ *such that*

$$\lim_{t\to\infty}\sup \int_{t_0}^{t} \left[\phi(s)A(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right]\Delta s = \infty. \tag{2.41}$$

Then every solution of (1.1) oscillates.

Theorem 2.8. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. Suppose that* $\phi(t)$ *is defined as in Theorem 2.3,* $H \in \Re$*, and, for* $t > s$*, one has*

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)}\int_{t_0}^{t} \left[H(t,s)\phi(s)A(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}(H^{\Delta_s}(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)H^{\gamma}(t,s)}\right]\Delta s = \infty. \tag{2.42}$$

Then every solution of (1.1) oscillates.

Properly choosing the functions H, one can establish a number of oscillation criteria for (1.1) on different types of time scales. For instance, if there exists a function $h(t,s) \in \Re$ such that (2.29) holds, then Theorem 2.8 yields the following oscillation result:

Corollary 2.4. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. Suppose that* $\phi(t)$ *is defined as in Theorem 2.3,* $H \in \Re$*, and, for* $t > s$*, one has*

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_0)}\int_{t_0}^{t} \left[H(t,s)\phi(s)A(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}(h(t,s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right]\Delta s = \infty. \tag{2.43}$$

Then every solution of Eq. (1.1) oscillates.

As a special case, by choosing $H(t,s) = (t-s)^m$ for $m \geq 1$, we deduce the following Kamenev-type oscillation criterion from Corollary 2.2:

Corollary 2.5. *Assume that conditions* (h_1)–(h_3) *and (1.2) are satisfied. If, for* $m > 1$*, one has*

$$\lim_{t\to\infty}\sup \frac{1}{t^m}\int_{t_0}^{t} \left[(t-s)^m A(s) - \frac{m^{\gamma+1}p(s)((t-s)^{m-1})^{\gamma+1}}{(\gamma+1)^{\gamma+1}(t-s)^{m\gamma}}\right]\Delta s = \infty, \tag{2.44}$$

then every solution of (1.1) oscillates.

In what follows, we give an example to illustrate the results. To obtain conditions for oscillation we use relations (2.30).

Example 2.2. Assume that $\mathbb{T} = \mathbb{R}$ and consider the second-order neutral dynamic equation

$$\left(\frac{1}{t^2}\left(\left(y(t) + \frac{\delta^{-1}(t) - 1}{\delta^{-1}(t)} y(\tau(t))\right)'\right)^{\gamma}\right)' + \frac{\lambda}{t} y^{\gamma}(\delta(t)) = 0, \quad t \in [1, \infty)_{\mathbb{R}}, \tag{2.45}$$

where $\gamma > 0$ and is a ratio of odd positive integers, $\tau(t), \delta(t) \in \mathbb{T}$,

$$\lim_{t\to\infty} \delta(t) = \lim_{t\to\infty} \tau(t) = \infty,$$

$\tau(t) \le t$, $\delta(t) \le t$, and we assume that $\delta^{-1}(t)$ (the inverse of the function $\delta(t)$) exists. Here,

$$\gamma = \beta > 0, \quad p(t) = \frac{1}{t^2}, \quad r(t) = \frac{\delta^{-1}(t) - 1}{\delta^{-1}(t)} = 1 - \frac{1}{\delta^{-1}(t)}, \quad \text{and} \quad q(t) = \frac{\lambda}{t}, \quad \lambda > 0.$$

This implies (in view of the fact that $\alpha(t) = 1$ and $\eta^{\sigma}(t) = 1$) that

$$A(t) = P(t) = q(t)(1 - r(\delta(t))^{\gamma} = \frac{\lambda}{t^{\gamma+1}}.$$

We use Theorem 2.7. It is easy to see that conditions (h_1)–(h_3) and (1.2) are satisfied because

$$\int_{t_0}^{\infty} \left(\frac{1}{p(t)}\right)^{\frac{1}{\gamma}} \Delta t = \int_{t_0}^{\infty} t^{\frac{2}{\gamma}} dt = \infty.$$

Relation (2.40) is also satisfied because

$$\int_{t_0}^{\infty} A(s)\Delta s = \lambda \int_{t_0}^{\infty} \frac{1}{s^{\gamma+1}} ds < \infty.$$

Finally, we discuss (2.41). Choosing $\phi(t) = t^{\gamma}$, we note that

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\phi(s)A(s) - \frac{p(s)((\phi^{\Delta}(s))^{\gamma+1}}{(\gamma+1)^{\gamma+1}\phi^{\gamma}(s)}\right] \Delta s$$

$$= \limsup_{t\to\infty} \int_{t_0}^{t} \left[s^{\gamma}\frac{\lambda}{s^{\gamma+1}} - \frac{\gamma^{\gamma+1}(s^{\gamma-1})^{\gamma+1}}{(\gamma+1)^{\gamma+1}\,(s^{\gamma})^{\gamma}\,s^2}\right] ds$$

$$= \limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{\lambda}{s} - \frac{\gamma^{\gamma+1}}{(\gamma+1)^{\gamma+1}s^3}\right] ds = \infty,$$

provided that $\lambda > 0$. Then, by Theorem 2.7, every solution of (2.45) oscillates if $\lambda > 0$. Note that none of the results established in [1–3, 7, 8, 11–13, 15, 16] can be applied to (2.45) because

$$p^{\Delta}(t) = p'(t) = -\frac{2}{t^3} < 0.$$

3. Oscillation Criteria in the Case Where (1.3) Holds

In this section, we consider the case where $\delta(t) \leq \tau(t) \leq t$ and relation (1.3) is true and establish some sufficient conditions for the oscillation of (1.1). We use the following notation:

$$g(t) := q(t)(1 - r(t))^{\beta}, \quad \pi(t) := \int_t^{\infty} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s.$$

Remark 3.1. It follows from the proof of Lemma 2.1 that if (1.2) holds, then the case $x(t)x^{[1]}(t) < 0$ is disregarded and $x(t)x^{[1]}(t) > 0$ for $t \geq T$. Therefore, if (1.2) does not hold, i.e., relation (1.3) holds, we see that if y is a nonoscillatory solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$ and x is defined as in (1.4), then $x^{[1]}(t)$ is of one sign and there exists $T > t_0$ (where $T \geq t_0$ is chosen sufficiently large) such that

$$x(t)x^{[1]}(t) > 0 \quad \text{for} \quad t \geq T \tag{3.1}$$

or

$$x(t)x^{[1]}(t) < 0 \quad \text{for} \quad t \geq T. \tag{3.2}$$

To prove the main results of this section in the case where (1.3) holds we need the following lemma:

Lemma 3.1. *Assume that conditions (h_1)–(h_3) and (1.3) are satisfied, $\tau^{\Delta}(t) \geq 0$, and $r^{\Delta}(t) \geq 0$. Suppose that (1.1) has a nonoscillatory solution y on $[t_0, \infty)_{\mathbb{T}}$ and x is defined as in (1.4) so that (3.2) holds. Then there exists $T \geq t_0$ such that*

$$(p(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + g(t)x^{\beta}(t) \leq 0 \quad \textit{for} \quad t \geq T. \tag{3.3}$$

Proof. Assume that $y(t)$ is a positive solution of (1.1) on $[t_0, \infty)_{\mathbb{T}}$. Pick $t_1 \in [t_0, \infty)_{\mathbb{T}}$ so that $t_1 > t_0$ and so that $y(t) > 0$, $y(\tau(t)) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$. (Note that, in the case where $y(t)$ is negative, the proof is similar because the transformation $y(t) = -z(t)$ transforms (1.1) into the same form.) Since y is a positive solution of (1.1) and $q(t) > 0$, we have

$$(x^{[1]}(t))^{\Delta} \leq -q(t)y^{\beta}(\delta(t)) < 0 \quad \text{for} \quad t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.4}$$

Then $x^{[1]}(t)$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$ and is of one sign. Since y is a positive solution of (1.1), $q(t) > 0$, and (3.2) holds, we conclude that (note that $x^{[1]}(t) < 0$ and $p(t) > 0$)

$$x(t) > 0, \quad x^{\Delta}(t) < 0, \quad \text{and} \quad \left(x^{[1]}(t)\right)^{\Delta} < 0 \quad \text{for} \quad t \geq t_1. \tag{3.5}$$

Since $x(t)$ is decreasing, we may assume, without loss of generality, that $y(t)$ is also decreasing. If this is not the case, i.e., $y(t)$ and $y(\tau)$ are increasing for $t \geq t_1$, then $x(t)$ is also increasing for $t \geq t_1$ because

$$x^{\Delta}(t) = y^{\Delta}(t) + r^{\Delta}(t)y(\tau(t)) + r^{\sigma}(y(\tau(t))^{\Delta} > y^{\Delta}(t) > 0$$

(note that $r(t) \geq 0$ and $r^{\Delta}(t) \geq 0$), which contradicts the fact that $x^{\Delta}(t) < 0$ for $t \geq t_1$. This and relations (1.4) and (2.6) (note that $x(t) > y(t)$) imply that

$$x(t) = y(t) + r(t)y(\tau(t)) \leq y(\tau(t)) + r(t)x(\tau(t)) \leq y(\tau(t))[1 + r(t)] \quad \text{for} \quad t \geq t_1.$$

Thus,

$$y(\tau(t)) \geq \frac{x(t)}{1 + r(t)} \quad \text{for} \quad t \geq t_1.$$

Since $0 \leq r(t) < 1$, we have $1 \geq 1 - r^2(t)$, which implies that $1/(1 + r(t)) \geq (1 - r(t))$. Therefore,

$$y(\tau(t)) \geq x(t)(1 - r(t)) \quad \text{for} \quad t \geq t_1.$$

Since $\delta(t) \leq \tau(t)$ for $t \geq t_2$, where $t_2 > t_1$ is chosen large enough (note that $y(t)$ is decreasing), we have

$$y(\delta(t)) \geq (1 - r(t))x(t) \quad \text{for} \quad t \geq t_2. \tag{3.6}$$

Relation (3.4) and the last inequality yield inequality (3.3), which completes the proof.

Theorem 3.1. *Assume that conditions* (h_1)–(h_3) *and (1.3) are satisfied,* $\tau^{\Delta}(t) \geq 0$, *and* $r^{\Delta}(t) \geq 0$. *Furthermore, assume that (2.38) holds and there exists* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that*

$$\int_T^{\infty} \left(\frac{1}{p(s)} \int_T^s g(u)\pi^{\beta}(u)\Delta u \right)^{\frac{1}{\gamma}} \Delta s = \infty. \tag{3.7}$$

Then every solution of (1.1) oscillates.

Proof. Assume the contrary and let y be a nonoscillatory solution of Eq. (1.1). Without loss of generality, we may assume that $y(t) > 0$, $y(\tau(t)) > 0$, and $y(\delta(t)) > 0$ for $t \geq T$ (where T is chosen large enough so that the conclusions of Lemmas 2.2 and 3.1 hold). We consider only this case because the proof in the case where $y(t) < 0$ is similar. According to Remark 3.1, there are two possible cases: (3.1) and (3.2). First, we consider (3.1). In this case, we proceed as in the proof of Theorem 2.6 and define $u(t)$ as in (2.9) to get a contradiction with (2.38). Now consider (3.2). We proceed as in the proof of Lemma 3.1 to get inequality (3.3), where $x(t)$ satisfies (3.5) for $t \geq T$. Since $x^{[1]}(t) < 0$, it follows from (3.5) for $s \geq t \geq T$ that $-x^{[1]}(s) \geq -x^{[1]}(t)$, or

$$p(s)(-x^{\Delta}(s))^{\gamma} \geq p(t)(-x^{\Delta}(t))^{\gamma},$$

and, hence,

$$-x^{\Delta}(s) \geq \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \left(p(t)(-x^{\Delta}(t))^{\gamma}\right)^{\frac{1}{\gamma}}.$$

Integrating from $t(\geq T)$ to $u \quad (\geq t)$ and letting $u \to \infty,$ we get

$$x(t) > -x(\infty) + x(t) \geq \left(p(t)(-x^{\Delta}(t))^{\gamma}\right)^{\frac{1}{\gamma}} \int_{t}^{\infty} \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s = p^{\frac{1}{\gamma}}(t)(-x^{\Delta}(t)\pi(t) \quad \text{for} \quad t \geq T.$$

Since $p^{\frac{1}{\gamma}}(t)(-x^{\Delta}(t)$ is decreasing, this yields

$$x(t) \geq p^{\frac{1}{\gamma}}(T)(-x^{\Delta}(T)\pi(t) = c\pi(t) \quad \text{for} \quad t \geq T, \tag{3.8}$$

where $c = p^{\frac{1}{\gamma}}(T)(-x^{\Delta}(T) > 0.$ Using (3.8) in (3.3), we get

$$(p(t)\left(x^{\Delta}(t)\right)^{\gamma})^{\Delta} + g(t)c^{\beta}\pi^{\beta}(t) \leq 0 \quad \text{for} \quad t \geq T.$$

Integrating the last inequality from T to $t,$ we obtain

$$-p(t)\left(x^{\Delta}(t)\right)^{\gamma} \geq -p(T)\left(x^{\Delta}(T)\right)^{\gamma} + c^{\beta} \int_{T}^{t} g(s)\pi^{\beta}(s)\Delta s \geq c^{\beta} \int_{T}^{t} g(s)\pi^{\beta}(s)\Delta s,$$

or

$$-x^{\Delta}(t) \geq c^{\frac{\beta}{\gamma}} \left(\frac{1}{p(t)} \int_{T}^{t} g(s)\pi^{\beta}(s)\Delta s\right)^{\frac{1}{\gamma}}.$$

Integrating from T to $t,$ we get

$$\infty > x(t_1) > x(t_1) - x(t) \geq c^{\frac{\beta}{\gamma}} \int_{T}^{t} \left(\frac{1}{p(s)} \int_{T}^{s} g(u)\pi^{\beta}(u)\Delta u\right)^{\frac{1}{\gamma}} \Delta s,$$

which contradicts (3.7). This completes the proof.

Remark 3.2. Note the difference between inequality (2.4) in the case where (1.2) holds and inequality (3.3) in the case where (1.3) holds.

Example 3.1. Assume that $\mathbb{T} = \mathbb{R}$ and consider the neutral equation

$$\left(t^2\left(y(t) + (1 - \frac{1}{t})y(\lambda t)\right)'\right)' + \frac{\kappa t^2}{\alpha(t)} y(\frac{\lambda}{2}t) = 0, \quad t \in [1, \infty)_{\mathbb{R}}, \tag{3.9}$$

where

$$\tau(t) = \lambda t > \delta(t) = \frac{\lambda}{2}t \qquad \text{and} \qquad \alpha(t) = \frac{P(\delta(t), T)}{P(t, T)} > 0 \qquad \text{for any} \quad T \geq 1.$$

Here, $\gamma = \beta = 1,\ 0 < \lambda < 1,$

$$p(t) = t^2, \quad r(t) = \left(1 - \frac{1}{t}\right), \quad \text{and} \quad q(t) = \kappa t^2,\ \kappa > 0.$$

This yields (in view of the fact that $\eta^\sigma(t) = 1$)

$$A(t) = P(t)\alpha(t) = \alpha(t)q(t)(1 - r(\delta(t))) = \frac{2\kappa t}{\lambda}, \quad g(t) = 2\kappa t,$$

and

$$\pi(t) := \int_t^\infty \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s = \int_t^\infty \frac{1}{s^2} ds = \frac{1}{t}.$$

It is easy to see that assumptions (h_1)–(h_3) and (1.3) hold because

$$\int_1^\infty \left(\frac{1}{p(s)}\right)^{\frac{1}{\gamma}} \Delta s = \int_1^\infty \frac{1}{s^2} ds \le \infty. \tag{3.10}$$

To apply Theorem 3.1, it remains to discuss (2.38) and (3.7). First, we discuss (2.38). It is clear that (2.38) is satisfied because

$$\int_{t_0}^\infty A(s)\Delta s = \int_{t_0}^\infty A(s) ds = \int_1^\infty \frac{2\kappa s}{\lambda} ds = \infty.$$

It remains to discuss condition (3.7). Note that

$$\int_T^\infty \left(\frac{1}{p(s)} \int_T^s g(u)\pi^\beta(u)\Delta u\right)^{\frac{1}{\gamma}} \Delta s = 2\kappa \int_1^\infty \left(\frac{1}{s^2} \int_1^s s\frac{1}{s}\Delta u\right) ds = \kappa \int_1^\infty \left(\frac{1}{s^2}(s-1)\right) ds = \infty.$$

Then, by Theorem 3.1, every solution of (3.9) oscillates. Note that none of the results established in [1, 2, 3, 7, 8, 11–13, 15, 16] can be applied to (3.9) because (1.2) does not hold [see (3.10)].

Remark 3.3. In Theorem 3.1, we have used condition (2.38) to get a contradiction if (3.1) holds. We can also use conditions (2.41)–(2.44) to get a contradiction. In the case where (3.2) holds, we proceed as in the proof of Theorem 3.1 to get a contradiction with (3.7). Thus, the following results can similarly be stated (there are, however, no new principles involved):

Theorem 3.2. *Assume that conditions (h_1)–(h_3) and (1.3) are satisfied, $\tau^\Delta(t) \ge 0$, and $r^\Delta(t) \ge 0$. Furthermore, assume that (2.41) holds and there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that (3.7) holds. Then every solution of (1.1) oscillates.*

Theorem 3.3. *Assume that conditions (h_1)–(h_3) and (1.3) are satisfied, $\tau^\Delta(t) \ge 0$, and $r^\Delta(t) \ge 0$. Furthermore, assume that (2.42) holds and there exists $T \in [t_0, \infty)_{\mathbb{T}}$ such that (3.7) holds. Then every solution of (1.1) oscillates.*

Theorem 3.4. *Assume that conditions* (h_1)–(h_3) *and (1.3) are satisfied,* $\tau^{\Delta}(t) \geq 0$, *and* $r^{\Delta}(t) \geq 0$. *Furthermore, assume that (2.43) holds and there exists* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that (3.7) holds. Then every solution of (1.1) oscillates.*

Theorem 3.5. *Assume that conditions* (h_1)–(h_3) *and (1.3) are satisfied,* $\tau^{\Delta}(t) \geq 0$, *and* $r^{\Delta}(t) \geq 0$. *Furthermore, assume that (2.44) holds and there exists* $T \in [t_0, \infty)_{\mathbb{T}}$ *such that (3.7) holds. Then every solution of (1.1) oscillates.*

Remark 3.4. Note that the results in Theorems 3.1–3.5 are valid only if $\delta(t) \leq \tau(t) \leq t$. Hence, it would be interesting to consider the case where this condition is not satisfied and find new oscillation criteria in the case where (1.3) holds. It would also be interesting to find new conditions different from condition (3.7).

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