Nonlinear Oscillations, Vol. 12, No. 2, 2009

SOLUTION OF ONE HEAT EQUATION WITH DELAY

D. Ya. Khusainov, A. F. Ivanov, and I. V. Kovarzh

UDC 517.929

We prove the existence and uniqueness of a solution of a boundary-value problem for a heat equation with delay. For the construction of a solution, we use a special "delayed exponential function."

Introduction

In recent years, there has been increasing interest in the investigation of differential equations containing distributed parameters with delay. Equations of parabolic type with delayed argument are considered in the course of investigation of population dynamics in ecological systems with inhomogeneous external medium, manpower dynamics with regard for migration, the dynamics of generators with delayed feedback, etc. [1, 2]. It should be noted that differential equations with concentrated parameters of different types with aftereffect are fairly well studied [3–6], whereas there are not as many works devoted to studying partial differential equations with delay (see [7]).

1. Solution of a One-Dimensional Heat Equation by the Fourier Method

Consider the first boundary-value problem for the heat equation

$$u_t(x,t) = a^2 u_{xx}(x,t) + f(x,t).$$

We seek a classical solution of the first boundary-value problem, i.e., a function $u(x,t)$ defined for $0 \le x \le l$, $t \ge 0$, twice continuously differentiable with respect to x, continuously differentiable with respect to t, and satisfying the initial condition $u(x,0) = \varphi(x)$, $0 \le x \le l$, and the boundary conditions $u(0,t) = \mu_1(t)$ and $u(l,t) = \mu_2(t)$, $t \ge 0$. For the construction of a solution, one often uses the method of separation of variables (Fourier method). A solution is sought in the form of a sum, namely,

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t),$$

where $u_1(x,t)$ is a solution of the homogeneous equation with the zero boundary conditions $u_1(0,t) \equiv 0$ and $u_1(l,t) \equiv 0$, $t \ge 0$, and the nonzero initial condition

$$u_1(x,0) = \Phi(x), \quad \Phi(x) = \varphi(x) - \mu_1(0) - \frac{x}{l}[\mu_2(0) - \mu_1(0)], \quad 0 \le x \le l,$$

Shevchenko Kiev National University, Kiev, Ukraine.

Translated from Neliniini Kolyvannya, Vol. 12, No. 2, pp. 251–272, April–June, 2009. Original article submitted May 27, 2008.

 1536–0059/09/1202–0260

$u_2(x,t)$ is a solution of the inhomogeneous equation with the right-hand side $F(x,t) = f(x,t) - \dot{\mu}_1(t) - \frac{x}{l}[\dot{\mu}_2(t) - \dot{\mu}_1(t)]$, zero boundary conditions $u_2(0,t) \equiv 0$ and $u_2(l,t) \equiv 0$, $t \geq 0$, and nonzero initial condition $u_2(x,0) = 0$, $0 \leq x \leq l$, and $u_3(x,t)$ has the form

$$u_3(x,t) = \mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)], \quad 0 \leq x \leq l, \quad t \geq 0.$$

A solution $u_1(x,t)$ of the homogeneous equation is sought in the form of a product of two functions:

$$u_1(x,t) = X(x)\,T(t).$$

As a result of the separation of variables, the problem is reduced to the eigenvalue problem, and a solution of the homogeneous equation is represented in the form of a series in eigenfunctions of the Sturm–Liouville problem. The same eigenfunctions are used for the construction of a solution of the second boundary-value problem.

2. Solution of a One-Dimensional Heat Equation with Delay

Consider the first boundary-value problem for the one-dimensional heat equation with delay

$$u_t(x,t) = a_1^2 u_{xx}(x,t) + a_2^2 u_{xx}(x,t-\tau) + c_1 u(x,t) + c_2 u(x,t-\tau) + f(x,t) \tag{1}$$

defined for $0 \leq x \leq l$ and $t \geq 0$. The initial condition has the form

$$u(x,t) = \varphi(x,t), \quad 0 \leq x \leq l, \quad -\tau \leq t \leq 0, \tag{2}$$

and the boundary conditions are as follows:

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \quad t \geq -\tau. \tag{3}$$

Furthermore, we assume that the following condition of "consistency of boundary and initial conditions" is satisfied:

$$\varphi(0,t) = \mu_1(t), \quad \varphi(l,t) = \mu_2(t), \quad -\tau \leq t \leq 0.$$

We seek a solution in the form of a sum:

$$u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t),$$

where the functions $u_1(x,t)$, $u_2(x,t)$, and $u_3(x,t)$ are defined as follows:

$u_1(x,t)$ is a solution of the homogeneous equation

$$\frac{\partial u_1(x,t)}{\partial t} = a_1^2 \frac{\partial^2 u_1(x,t)}{\partial x^2} + a_2^2 \frac{\partial^2 u_1(x,t-\tau)}{\partial x^2} + c_1 u_1(x,t) + c_2 u_1(x,t-\tau)$$

with the zero boundary conditions $u_1(0,t) = 0$ and $u_1(l,t) = 0$, $t \geq -\tau$, and the nonzero initial condition

$$u_1(x,t) = \Phi(x,t),$$

$$\Phi(x,t) = \varphi(x,t) - \mu_1(t) - \frac{x}{l}[\mu_2(t) - \mu_1(t)], \quad 0 \leq x \leq l, \quad -\tau \leq t \leq 0;$$

$u_2(x,t)$ is a solution of the inhomogeneous equation

$$\frac{\partial u_2(x,t)}{\partial t} = a_1^2 \frac{\partial^2 u_2(x,t)}{\partial x^2} + a_2^2 \frac{\partial^2 u_2(x,t-\tau)}{\partial x^2} + c_1 u_2(x,t) + c_2 u_2(x,t-\tau) + F(x,t),$$

$$F(x,t) = f(x,t) - \frac{d}{dt}\left\{\mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)]\right\} + c_1\left\{\mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)]\right\}$$

$$+ c_2\left\{\mu_1(t-\tau) + \frac{x}{l}[\mu_2(t-\tau) - \mu_1(t-\tau)]\right\}, \quad 0 \leq x \leq l,$$

with the zero boundary conditions $u_2(0,t) = 0$ and $u_2(l,t) = 0$, $t \geq -\tau$, and the zero initial condition $u_2(x,t) \equiv 0$, $0 \leq x \leq l$, $-\tau \leq t \leq 0$;

$u_3(x,t) = \mu_1(t) + \dfrac{x}{l}[\mu_2(t) - \mu_1(t)]$.

2.1. Consider the homogeneous equation with delayed argument

$$\frac{\partial u_1(x,t)}{\partial t} = a_1^2 \frac{\partial^2 u_1(x,t)}{\partial x^2} + a_2^2 \frac{\partial^2 u_1(x,t-\tau)}{\partial x^2} + c_1 u_1(x,t) + c_2 u_1(x,t-\tau) \tag{4}$$

with the zero boundary conditions $u_1(0,t) = 0$ and $u_1(l,t) = 0$, $t \geq -\tau$, and the nonzero initial condition $u_1(x,t) = \Phi(x,t)$, $0 \leq x \leq l$, $-\tau \leq t \leq 0$. We seek its solution by using the Fourier method, i.e., the function $u_1(x,t)$ is sought in the form of a product:

$$u_1(x,t) = X(x)\,T(t).$$

Substituting this product into the homogeneous equation, we get

$$X(x)\,T'(t) = a_1^2 X''(x)\,T(t) + a_2^2 X''(x) T(t-\tau) + c_1 X(x)\,T(t) + c_2 X(x)\,T(t-\tau).$$

Separating the variables, we obtain

$$\frac{X''(x)}{X(x)} = \frac{T'(t) - c_1 T(t) - c_2 T(t-\tau)}{a_1^2 T(t) + a_2^2 (t-\tau)} = -\lambda^2.$$

Then the equation splits into the following two equations:

$$X''(x) + \lambda^2 X(x) = 0, \quad T'(t) + \left(\lambda^2 a_1^2 - c_1\right) T(t) + \left(\lambda^2 a_2^2 - c_2\right) T(t-\tau) = 0. \tag{5}$$

Since the boundary conditions are zero, for the first equation we obtain the zero boundary conditions

$$X(0) = 0, \quad X(l) = 0.$$

A solution is nonzero only if

$$\lambda = \lambda_n = \frac{\pi n}{l}, \quad n = 1, 2, 3, \ldots,$$

and each eigenvalue $\lambda_n = \dfrac{n\pi}{l}$ is associated with

$$X_n(x) = A_n \sin \frac{\pi n}{l} x, \quad n = 1, 2, 3, \ldots,$$

where A_n is an arbitrary constant. Substituting the obtained values $\lambda_n = \dfrac{\pi n}{l}$ into the second equation in (5), we obtain the following differential equations with delayed argument:

$$\dot{T}_n(t) = \left[c_1 - \left(\frac{\pi n}{l} a_1\right)^2\right] T_n(t) + \left[c_2 - \left(\frac{\pi n}{l} a_2\right)^2\right] T_n(t - \tau), \quad n = 1, 2, 3, \ldots . \tag{6}$$

We introduce initial conditions for each equation in (6) as follows: We expand the function $\Phi(x, t),\ 0 \le x \le l,\ -\tau \le t \le 0,$ in a series in the eigenfunctions of the first equation, i.e., we represent it in the form

$$\Phi(x, t) = \sum_{k=1}^{\infty} \Phi_n(t) \sin \frac{\pi n}{l} x, \quad \Phi_n(t) = \frac{2}{l} \int_0^l \Phi(\xi, t)\, \sin \frac{\pi n}{l} \xi \, d\xi.$$

Substituting the value of $\Phi(x, t)$ and taking the integral, we obtain the following initial conditions for each equation in (6):

$$T_n(t) = \Phi_n(t), \quad n = 1, 2, 3, \ldots, \quad -\tau \le t \le 0,$$

$$\Phi_n(t) = \frac{2}{l} \int_0^l \varphi(\xi, t) \sin \frac{\pi n}{l} \xi \, d\xi + \frac{2}{\pi n} \left[(-1)^n \mu_2(t) - \mu_1(t)\right].$$

Let us find a solution of the Cauchy problem for each equation in (6) in analytic form. First, we give several auxiliary statements. As shown in [8], a solution of the system of linear homogeneous differential equations with pure delay

$$\dot{x}(t) = b x(t - \tau), \quad t \ge 0, \quad x(t) \equiv \varphi(t), \quad -\tau \le t \le 0,$$

has the form

$$x(t) = e_\tau^{bt} \varphi(-\tau) + \int_{-\tau}^{0} e_\tau^{b(t-\tau-s)} \varphi'(s) \, ds,$$

where the function e_τ^{bt} on the segment $(k-1)\tau \le t < k\tau$ is a matrix polynomial of degree k "glued" at the nodes $t = k\tau$. This polynomial is called a "delayed" exponential function.

Consider the scalar differential equation with delay

$$\dot{x}(t) = ax(t) + bx(t-\tau), \quad t \ge 0, \quad \tau > 0, \quad x(t) \equiv \varphi(t), \quad -\tau \le t \le 0, \tag{7}$$

where $\varphi(t)$ is an arbitrary continuously differentiable function that determines the initial condition.

Definition 1. *The function*

$$e_\tau^{bt} = \begin{cases} 0, & -\infty < t < -\tau, \\ 1, & -\tau \le t < 0, \\ 1 + b\dfrac{t}{1!}, & 0 \le t < \tau, \\ 1 + b\dfrac{t}{1!} + b^2\dfrac{(t-\tau)^2}{2!}, & \tau \le t < 2\tau, \\ \dots\dots\dots & \dots\dots\dots \\ 1 + b\,\dfrac{t}{1!} + \ldots + b^k\,\dfrac{[t-(k-1)\tau]^k}{k!}, & (k-1)\tau \le t < k\tau, \\ \dots\dots\dots & \dots\dots\dots \end{cases} \tag{8}$$

is called the delayed exponential function.

It was proved in [8] that the function e_τ^{bt} is a solution of the linear homogeneous equation with pure delay

$$\dot{x}(t) = bx(t-\tau), \quad t \ge 0,$$

that satisfies the unit initial condition $x(t) \equiv 1, \ -\tau \le t \le 0.$

Let us show that a solution of the Cauchy problem for the equation with delay (7) can also be represented in the integral form with analogous function.

Lemma 1. *The function*

$$x_0(t) = e^{at}e_\tau^{b_1}, \quad b_1 = e^{-a\tau}b, \quad t \ge 0, \tag{9}$$

where $e_\tau^{b_1 t}$ is the delayed exponential function defined by (8), is a solution of Eq. (7) that satisfies the initial condition

$$x_0(t) = e^{at}, \quad -\tau \le t \le 0. \tag{10}$$

Proof. The fact that the function $x_0(t)$ satisfies condition (10) follows from the definition of e^{at} and $e_\tau^{b_1 t}$. Let us show that, for $t \ge 0$, the function $x_0(t)$ is a solution of Eq. (7). Differentiating (9), we obtain

$$\frac{d}{dt}\left(e^{at}\,e_\tau^{b_1 t}\right) = a\left(e^{at}\,e_\tau^{b_1 t}\right) + e^{at}\,b_1\,e_\tau^{b_1(t-\tau)} = a\left(e^{at}\,e_\tau^{b_1 t}\right) + e^{at}\,e^{-a\tau}\,b\,e_\tau^{b_1(t-\tau)}$$

$$= a\left(e^{at}\,e_\tau^{b_1 t}\right) + b\left(e^{a(t-\tau)}\,e_\tau^{(t-\tau)}\right).$$

Taking (9) into account, we get

$$\frac{d}{dt}\, x_0(t) = a x_0(t) + b x_0(t-\tau),$$

i.e., Lemma 1 is proved.

Theorem 1. *A solution* $x(t)$ *of Eq. (7) that satisfies the initial condition* $x(t) \equiv \varphi(t),\ \ -\tau \le t \le 0,$ *has the form*

$$x(t) = e^{a(t+\tau)}\, e_{\tau}^{b_1 t} \varphi(-\tau) + \int\limits_{-\tau}^{0} e^{a(t-s)} e_{-\tau}^{b_1(t-\tau-s)} \left[\varphi'(s) - a\varphi(s)\right]\, ds. \tag{11}$$

Proof. We seek a solution of Eq. (7) that satisfies the initial condition $x(t) \equiv \varphi(t),\ \ -\tau \le t \le 0,$ in the form

$$x(t) = x_0(t)c + \int\limits_{-\tau}^{0} x_0(t-\tau-s)\, y(s)\, ds, \tag{12}$$

where c is an unknown constant, $y(t)$ is an unknown continuously differentiable function, and $x_0(t)$ is defined by (9). According to Lemma 1, the function $x_0(t)$ is a solution of Eq. (7). Therefore, for any c and $y(t)$, expression (12) is also a solution of Eq. (7). We choose c and $y(t)$ so that the initial conditions are satisfied, i.e., $x(t) \equiv \varphi(t),\ \ -\tau \le t \le 0,$ or, with regard for (12),

$$x_0(t)\, c + \int\limits_{-\tau}^{0} x_0(t-\tau-s)\, y(s)\, ds \equiv \varphi(t) \quad \text{for} \quad -\tau \le t \le 0.$$

We set $t = -\tau$. It follows from the definition of delayed exponential function that $x_0(-\tau) = e^{-a\tau}$, $x_0(-2\tau - s) = 0$ if $-\tau < s \le 0$, and $x_0(-2\tau - s) = e^{-a\tau}$ if $s = -\tau$. Therefore, $\varphi(-\tau) = e^{-a\tau}\, c$, whence $c = e^{a\tau}\varphi(-\tau)$, and relation (12) takes the form

$$x(t) = e^{a(t+\tau)} e_{\tau}^{b_1 t} \varphi(-\tau) + \int\limits_{-\tau}^{0} e^{a(t-\tau-s)}\, e_{\tau}^{b_1(t-\tau-s)} y(s)\, ds.$$

On the segment $-\tau \le t \le 0,$ we divide the integral into two integrals. As a result, we obtain

$$\varphi(t) = e^{a(t+\tau)} \varphi(-\tau) + \int\limits_{-\tau}^{t} e^{a(t-\tau-s)}\, e_{\tau}^{b_1(t-\tau-s)}\, y(s)\, ds + \int\limits_{t}^{0} e^{a(t-\tau-s)}\, e_{\tau}^{b_1(t-\tau-s)}\, y(s)\, ds.$$

In the first integral, we have $-\tau \le s \le t$. Therefore, $-\tau \le t - \tau - s \le t,$ and the delayed exponential function is equal to

$$e_{\tau}^{b_1(t-\tau-s)} \equiv 1, \quad -\tau \le s \le t.$$

In the second integral, we have $t \le s \le 0$. Therefore, $t-\tau \le t-\tau-s \le -\tau$, and the delayed exponential function is equal to

$$e_\tau^{b_1(t-\tau-s)} = 0 \quad \text{if} \quad t < s \le 0, \quad \text{and} \quad e_\tau^{b_1(t-\tau-s)} = 1 \quad \text{if} \quad s = t.$$

Thus, on the segment $-\tau \le t \le 0$, we get

$$e^{a(t+\tau)}\varphi(-\tau) + \int_{-\tau}^{t} e^{a(t-\tau-s)}\, y(s)\, ds = \varphi(t). \tag{13}$$

Differentiating relation (13), we obtain

$$ae^{a(t+\tau)}\varphi(-\tau) + a\int_{-\tau}^{t} e^{a(t-\tau-s)}\, y(s)\, ds + e^{-a\tau}\, y(t) = \varphi'(t). \tag{14}$$

Solving the system of equations (13), (14), we get

$$y(t) = e^{a\tau}\left[\varphi'(t) - a\varphi(t)\right].$$

Substituting this expression into (12), we obtain the statement of Theorem 1.

Remark 1. Relation (11) can be rewritten in the form

$$x(t) = e^{a\tau}\left\{x_0(t)\varphi(-\tau) + \int_{-\tau}^{0} x_0(t-\tau-s)\left[\varphi'(s) - a\varphi(s)\right]\, ds\right\}.$$

Remark 2. If $a = 0$, i.e., Eq. (7) is an equation with pure delay, then we obtain the results presented in [7]:

$$x(t) = e_\tau^{bt}\varphi(-\tau) + \int_{-\tau}^{0} e_\tau^{b(t-\tau-s)}\varphi'(s)\, ds.$$

We now return to the differential equations (6) with corresponding initial conditions

$$\dot{T}_n(t) = \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right] T_n(t) + \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2\right] T_n(t-\tau),$$

$$T_n(t) = \Phi_n(t), \quad -\tau \le t \le 0, \quad n = 1, 2, 3, \ldots .$$

Denote

$$D_n = \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2\right] e^{-\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\tau}.$$

It follows from (11) that solutions of the Cauchy problem for each equation in (6) have the form

$$T_n(t) = e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](t+\tau)} e_\tau^{D_n t}\Phi_n(-\tau)$$

$$+\int_{-\tau}^{0} e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](t-s)} e_\tau^{D_n(t-\tau-s)}\left[\Phi_n'(s)-\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right]\Phi_n(s)\right]ds. \quad (15)$$

This yields the following solution of the first boundary-value problem for Eq. (4):

$$u_1(x,t) = \sum_{n=1}^{\infty}\sin\frac{\pi n}{l}x\left\{e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](t+\tau)} e_\tau^{D_n t}\Phi_n(-\tau)\right.$$

$$\left.+\int_{-\tau}^{0} e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](t-s)} e_\tau^{D_n(t-\tau-s)}\left[\Phi_n'(s)-\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right]\Phi_n(s)\right]ds\right\},$$

$$\Phi_n(t) = \frac{2}{l}\int_0^l \varphi(\xi,t)\sin\frac{\pi n}{l}\xi\,d\xi + \frac{2}{\pi n}\left[(-1)^n\mu_2(t)-\mu_1(t)\right]. \quad (16)$$

2.2. Consider the inhomogeneous equation

$$\frac{\partial u_2(x,t)}{\partial t} = a_1^2\frac{\partial^2 u_2(x,t)}{\partial x^2} + a_2^2\frac{\partial^2 u_2(x,t-\tau)}{\partial x^2} + c_1u_2(x,t) + c_2u_2(x,t-\tau) + F(x,t) \quad (17)$$

with the zero boundary conditions $u_2(0,t) = 0$ and $u_2(l,t) = 0,\ t \ge -\tau$, and the zero initial condition $u_2(x,t) \equiv 0,\ 0 \le x \le l,\ -\tau \le t \le 0$. We seek a solution in the form of a Fourier series in the eigenfunctions $\sin\frac{\pi n}{l}x,\ n = 1,2,\ldots$:

$$u_2(x,t) = \sum_{n=1}^{\infty} u_{2n}(t)\sin\frac{\pi n}{l}x, \quad n = 1,2,\ldots,$$

where t is regarded as a parameter. We also represent the function $F(x,t)$ in the form of a series:

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t)\sin\frac{\pi n}{l}x, \quad F_n(t) = \frac{2}{l}\int_0^l F(s,t)\sin\frac{\pi n}{l}\xi\,d\xi.$$

Since

$$F(x,t) = f(x,t) - \frac{d}{dt}\left\{\mu_1(t)+\frac{x}{l}[\mu_2(t)-\mu_1(t)]\right\} + c_1\left\{\mu_1(t)+\frac{x}{l}[\mu_2(t)-\mu_1(t)]\right\}$$

$$+c_2\left\{\mu_1(t-\tau)+\frac{x}{l}[\mu_2(t-\tau)-\mu_1(t-\tau)]\right\},$$

taking the integrals we obtain

$$F_n(t) = \frac{2}{l}\int_0^l f(\xi,t)\sin\frac{\pi n}{l}\xi\,d\xi + \frac{2}{\pi n}\frac{d}{dt}\left[(-1)^n\mu_2(t) - \mu_1(t)\right]$$

$$-\frac{2}{\pi n}c_1\left[(-1)^n\mu_2(t) - \mu_1(t)\right] - \frac{2}{\pi n}c_2\left[(-1)^n\mu_2(t-\tau) - \mu_1(t-\tau)\right], \quad t \geq 0.$$

Then each function $u_{2n}(t),\ n = 1, 2, \dots,$ is a solution of the corresponding equation

$$\dot{u}_{2n}(t) = \left[c_1 - \left(\frac{\pi n}{l}a_1\right)^2\right]u_{2n}(t) + \left[c_2 - \left(\frac{\pi n}{l}a_2\right)^2\right]u_{2n}(t-\tau) + F_n(t) \tag{18}$$

with zero initial condition $u_{2n}(t) \equiv 0,\ -\tau \leq t \leq 0.$

We again present several auxiliary results. Consider the Cauchy problem for the inhomogeneous equation with delay

$$\dot{x}(t) = ax(t) + bx(t-\tau) + f(t), \quad t \geq 0, \quad \tau > 0, \tag{19}$$

with zero initial condition.

Theorem 2. *A solution $\overline{x}(t)$ of the inhomogeneous equation (19) that satisfies the zero initial conditions has the form*

$$\overline{x}(t) = \int_0^t e^{a(t-s)}\, e_\tau^{b_1(t-\tau-s)}\, f(s)\,ds, \quad t \geq 0, \quad b_1 = e^{-a\tau}b. \tag{20}$$

Proof. Since $x_0(t)$ is a solution of the homogeneous equation (7), using the method of variation of constants and taking into account the form of the function $x_0(t)$ we seek a solution $\overline{x}(t)$ of the inhomogeneous equation (19) in the form

$$\overline{x}(t) = \int_0^t e^{a(t-\tau-s)}\, e_\tau^{b_1(t-\tau-s)}\, c(s)\,ds, \tag{21}$$

where $c(s),\ 0 \leq s \leq t,$ is an unknown function.

Differentiating (21), we get

$$\frac{d}{dt}\overline{x}(t) = e^{a(t-\tau-s)}\, e_\tau^{b_1(t-\tau-s)}\, c(s)\Big|_{s=t} + \int_0^t \left[ae^{a(t-\tau-s)}\, e_\tau^{b_1(t-\tau-s)} + e^{a(t-\tau-s)}\, b_1\, e_\tau^{b_1(t-2\tau-s)}\right]c(s)ds.$$

Substituting (21) and the expression obtained for the derivative into Eq. (19), we write

$$e^{-a\tau}c(t)+\int\limits_0^t\left[ae^{a(t-\tau-s)}\,e_\tau^{b_1(t-\tau-s)}+b\,e^{a(t-2\tau-s)}\,e_\tau^{b_1(t-2\tau-s)}\right]\,c(s)\,ds$$

$$=a\left[\int\limits_0^t e^{a(t-\tau-s)}\,e_\tau^{b_1(t-\tau-s)}\,c(s)\,ds\right]+b\left[\int\limits_0^{t-\tau} e^{a(t-2\tau-s)}\,e_\tau^{b_1(t-2\tau-s)}\,c(s)\,ds\right]+f(t),$$

whence

$$e^{-a\tau}c(t)+\int\limits_{t-\tau}^t be^{a(t-2\tau-s)}\,e_\tau^{b_1(t-2\tau-s)}\,c(s)\,ds=f(t).$$

Since $e_\tau^{b_1(t-2\tau-s)}=0$ for $t-\tau<s\le t$ and $e_\tau^{b_1(t-2\tau-s)}=1$ for $s=t-\tau$, we have $e^{-a\tau}c(t)=f(t)$ and $c(t)=e^{a\tau}f(t)$. This yields relation (20).

Using the result obtained, we write the solution of the Cauchy problem for Eqs. (18) with zero initial condition in the form

$$u_{2n}(t)=\int\limits_0^t e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_\tau^{D_n(t-\tau-s)}\,F_n(s)\,ds,\quad D_n=\left[c_2-\left(\frac{\pi n}{l}\,a_2\right)^2\right]\,e^{-\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right]\tau}.$$

A solution of the inhomogeneous heat equation with delayed argument (17) with zero boundary and initial conditions has the form

$$u_2(x,t)=\sum_{n=1}^{\infty}\left[\int\limits_0^t e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}\,e_\tau^{D_n(t-\tau-s)}F_n(s)\,ds\right]\,\sin\frac{\pi n}{l}\,x,$$

$$F_n(t)=\frac{2}{l}\int\limits_0^l f(\xi,t)\sin\frac{\pi n}{l}\,\xi\,d\xi+\frac{2}{\pi n}\,\frac{d}{dt}\,[(-1)^n\mu_2(t)-\mu_1(t)]$$

$$-\frac{2}{\pi n}\,c_1\,[(-1)^n\mu_2(t)-\mu_1(t)]-\frac{2}{\pi n}\,c_2\,[(-1)^n\mu_2(t-\tau)-\mu_1(t-\tau)]\,,\quad t\ge 0.$$

Combining all relations obtained, we write a solution of the boundary-value problem for the heat equation with delay in the form

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\Bigg\{e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t+\tau)}\,e_{\tau}^{D_n t}\Phi_n(-\tau)+\int\limits_{-\tau}^{0}e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}\,e_{\tau}^{D_n(t-\tau-s)}$$

$$\times\left[\Phi_n'(s)-\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right]\Phi_n(s)\right]ds\Bigg\}$$

$$+\sum_{n=1}^{\infty}\left[\int\limits_{0}^{t}e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}\,e_{\tau}^{D_n(t-\tau-s)}F_n(s)\,ds\right]\sin\frac{\pi n}{l}\,x+\mu_1(t)+\frac{x}{l}\left[\mu_2(t)-\mu_1(t)\right], \tag{22}$$

where

$$F_n(t)=\frac{2}{l}\int\limits_{0}^{l}f(\xi,t)\sin\frac{\pi n}{l}\,\xi\,d\xi+\frac{2}{\pi n}\,\frac{d}{dt}\left[(-1)^n\mu_2(t)-\mu_1(t)\right]$$

$$-\frac{2}{\pi n}c_1\left[(-1)^n\mu_2(t)-\mu_1(t)\right]-\frac{2}{\pi n}c_2\left[(-1)^n\mu_2(t-\tau)-\mu_1(t-\tau)\right], \tag{23}$$

$$\Phi_n(t)=\frac{2}{l}\int\limits_{0}^{l}\varphi(\xi,t)\sin\frac{\pi n}{l}\,\xi\,d\xi+\frac{2}{\pi n}\left[(-1)^n\mu_2(t)-\mu_1(t)\right],$$

$$D_n=\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right]e^{-\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right]\tau}.$$

A solution of the differential equation with delay (17) is represented in the form of a formal Fourier series.

We now formulate the following theorem on the convergence of solutions of the boundary-value problem (1)–(3):

Theorem 3. *Let the functions $F(x,t)$ and $\Phi(x,t)$ be such that the Fourier coefficients $F_n(t)$, $\Phi_n(t)$, and $\Phi_n'(t)$ satisfy the relations*

$$\lim_{n\to\infty}n^{2(k-2)}\left[\Phi_n'(s)+\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right]\Phi_n(s)\right]e^{-\left(\frac{\pi n}{l}\,a_1\right)^2(t^*-s)}=0,$$

$$\lim_{n\to\infty}n^{2(k-1)}\left|F_n(s)\right|e^{-\left(\frac{\pi n}{l}\,a_1\right)^2(t^*+\tau)}=0,\quad -\tau\le s\le 0,\quad (k-1)\tau\le t^*<k\tau.$$

Then the function $u(x,t)$, represented in the form of series (22), has the continuous derivative with respect to t and the continuous second derivative with respect to x and is a solution of Eq. (1) that satisfies the initial condition (2) and boundary condition (3). Furthermore, the series can be differentiated term by term twice with respect to x and once with respect to t, and the series obtained converge absolutely and uniformly for $0\le x\le l$, $-\tau\le t$.

Proof. We rewrite series (22) as a sum of three series, namely,

$$u(x,t) = S_1(x,t) + S_2(x,t) + S_3(x,t) + \mu_1(t) + \frac{x}{l}[\mu_2(t) - \mu_1(t)],$$

where

$$S_1(x,t) = \sum_{n=1}^{\infty} A_n(t) \sin\frac{\pi n}{l}\,x, \quad S_2(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{\pi n}{l}\,x, \quad S_3(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin\frac{\pi n}{l}\,x,$$

$$A_n(t) = e^{\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right](t+\tau)}\, e_\tau^{D_n t} \Phi_n(-\tau), \quad C_n(t) = \int_0^t e^{\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}\, e_\tau^{D_n(t-\tau-s)} F_n(s)\, ds,$$

$$B_n(t) = \int_{-\tau}^{0} e^{\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}\, e_\tau^{D_n(t-\tau-s)} \left[\Phi_n'(s) - \left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right]\, \Phi_n(s)\right] ds.$$

1. Consider the coefficients $A_n(t)$, $n = 1, 2, \ldots$, of the first series $S_1(x,t)$. For an arbitrary fixed time t^*: $(k-1)\tau \le t^* < k\tau$, we obtain

$$A_n(t^*) = e^{\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right](t^*+\tau)}\, e_\tau^{D_n t^*}\, \Phi_n(-\tau)$$

$$= e^{\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right](t^*+\tau)} \left\{1 + D_n \frac{t^*}{1!} + D_n^2 \frac{(t^*-\tau)^2}{2!} + \ldots + D_n^k \frac{[t^* - (k-1)\tau]^k}{k!}\right\} \Phi_n(-\tau).$$

Substituting the value of D_n, we get

$$A_n(t^*) = e^{\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right](t^*+\tau)} \Bigg\{1 + \left[c_2 - \left(\frac{\pi n}{l}\,a_2\right)^2\right]\, e^{-\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right]\tau} \frac{t^*}{1!}$$

$$+ \left[\left[c_2 - \left(\frac{\pi n}{l}\,a_2\right)^2\right]\, e^{-\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right]\tau}\right]^2 \frac{(t^*-\tau)^2}{2!}$$

$$+ \ldots + \left[\left[c_2 - \left(\frac{\pi n}{l}\,a_2\right)^2\right]\, e^{-\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right]\tau}\right]^k \frac{[t^* - (k-1)\tau]^k}{k!}\Bigg\} \Phi_n(-\tau),$$

or

$$A_n(t^*) = \Bigg\{e^{\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right](t^*+\tau)} + \left[c_2 - \left(\frac{\pi n}{l}\,a_2\right)^2\right]\, e^{\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right]t^*} \frac{t^*}{1!}$$

$$+ \left[c_2 - \left(\frac{\pi n}{l}\,a_2\right)^2\right]^2\, e^{\left[c_1 - \left(\frac{\pi n}{l}\,a_1\right)^2\right](t^*-\tau)} \frac{(t^*-\tau)^2}{2!}$$

$$+\ldots+\left[c_2-\left(\frac{\pi n}{l}\,a_2\right)^2\right]^k e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right][t^*-(k-1)\tau]}\,\frac{[t^*-(k-1)\tau]^k}{k!}\right\}\Phi_n(-\tau).$$

By assumption, we have $(k-1)\tau\le t^*<k\tau$. Therefore, if

$$c_1-\left(\frac{\pi n}{l}\,a_1\right)^2<0 \quad\text{and}\quad c_2-\left(\frac{\pi n}{l}\right)^2<-1$$

or if the "stronger" inequality

$$n>\frac{l}{\pi}\,\max\left\{\frac{\sqrt{|c_1|}}{|a_1|},\ \frac{\sqrt{|1+c_2|}}{|a_2|}\right\}$$

is satisfied, then

$$|A_n(t^*)|\le|\Phi_n(-\tau)|e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right][t^*-(k-1)\tau]}\left|\left\{1+\left[c_2-\left(\frac{\pi n}{l}\,a_2\right)^2\right]\frac{t^*}{1!}\right.\right.$$

$$\left.\left.+\left[c_2-\left(\frac{\pi n}{l}\,a_2\right)^2\right]^2\frac{(t^*-\tau)^2}{2!}+\ldots+\left[c_2-\left(\frac{\pi n}{l}\,a_2\right)^2\right]^k\frac{[t^*-(k-1)\tau]^k}{k!}\right\}\right|$$

$$\le|\Phi_n(-\tau)|e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right][t^*-(k-1)\tau]}\left|\left[c_2-\left(\frac{\pi n}{l}\,a_2\right)^2\right]^k\right|$$

$$\times\left|1+\frac{t^*}{1!}+\frac{(t^*-\tau)^2}{2!}+\ldots+\frac{[t^*-(k-1)\tau]^k}{k!}\right|,$$

and there exists a continuous function $N_1(t^*)$ for which

$$|A_n(t^*)|\le N_1(t^*)\left(\frac{\pi n}{l}\,a_2\right)^{2k}e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right][t^*-(k-1)\tau]}\,|\Phi_n(-\tau)|\,.$$

Thus, if, at time t^*, $(k-1)\tau\le t^*<k\tau$, one has

$$\lim_{n\to\infty}n^{2k}e^{-\left(\frac{\pi n}{l}\,a_1\right)^2[t^*-(k-1)\tau]}\,\Phi_n(-\tau)=0,$$

then the series

$$S_1(x,t^*)=\sum_{n=1}^{\infty}A_n(t^*)\sin\frac{\pi n}{l}\,x$$

converges uniformly and absolutely.

2. Consider the coefficients $B_n(t)$, $n=1,2,\ldots,$ of the second series $S_2(x,t)$. For a fixed time t^*, $(k-1)\tau\le t^*<k\tau$, we perform the change of variables $t^*-\tau-s=\xi$ and divide the integral into a sum of two integrals:

$$B_n(t^*) = \int\limits_{t^*-\tau}^{(k-1)\tau} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi+\tau)}\, e_\tau^{D_n \xi} \left[\Phi_n'(t-\tau-\xi) - \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\, \Phi_n(t-\tau-\xi)\right] d\xi$$

$$+ \int\limits_{(k-1)\tau}^{t^*} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi+\tau)}\, e_\tau^{D_n \xi} \left[\Phi_n'(t-\tau-\xi) - \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\, \Phi_n(t-\tau-\xi)\right] d\xi.$$

Using the representation of the delayed exponential function $e_\tau^{D_n \xi}$ on each segment, we obtain

$$B_n(t^*) = \int\limits_{t^*-\tau}^{(k-1)\tau} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi+\tau)} \left[\Phi_n'(t^*-\tau-\xi) - \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\, \Phi_n(t^*-\tau-\xi)\right]$$

$$\times \left\{1 + D_n \frac{\xi}{1!} + D_n^2 \frac{(\xi-\tau)^2}{2!} + \ldots + D_n^{k-1} \frac{[\xi-(k-2)\tau]^{k-1}}{(k-1)!}\right\} d\xi$$

$$+ \int\limits_{(k-1)\tau}^{t^*} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi+\tau)} \left[\Phi_n'(t^*-\tau-\xi) - \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\, \Phi_n(t^*-\tau-\xi)\right]$$

$$\times \left\{1 + D_n \frac{\xi}{1!} + D_n^2 \frac{(\xi-\tau)^2}{2!} + \ldots + D_n^{k} \frac{[\xi-(k-1)\tau]^{k}}{k!}\right\} d\xi.$$

Substituting the value of $D_n,$ we get

$$B_n(t^*) = \int\limits_{t^*-\tau}^{(k-1)\tau} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi+\tau)} \left[\Phi_n'(t^*-\tau-\xi) - \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\, \Phi_n(t^*-\tau-\xi)\right]$$

$$\times \left\{1 + \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2\right]\, e^{-\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\tau}\, \frac{\xi}{1!}\right.$$

$$+ \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2\right]^2\, e^{-2\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\tau}\, \frac{(\xi-\tau)^2}{2!}$$

$$\left. + \ldots + \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2\right]^{k-1}\, e^{-(k-1)\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\tau}\, \frac{[\xi-(k-2)\tau]^{k-1}}{(k-1)!}\right\} d\xi$$

$$+ \int\limits_{(k-1)\tau}^{t^*} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi+\tau)} \left[\Phi_n'(t^*-\tau-\xi) - \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\, \Phi_n(t^*-\tau-\xi)\right]$$

$$\times \left\{ 1 + \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2 \right] e^{-\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\tau} \frac{\xi}{1!} \right.$$

$$+ \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2 \right]^2 e^{-2\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\tau} \frac{(\xi-\tau)^2}{2!}$$

$$\left. + \ldots + \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2 \right]^k e^{-k\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\tau} \frac{[\xi-(k-1)\tau]^k}{k!} \right\} d\xi,$$

or

$$B_n(t^*) = \int\limits_{t^*-\tau}^{(k-1)\tau} \left[\Phi_n'(t^*-\tau-\xi) - \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2 \right] \Phi_n(t^*-\tau-\xi) \right]$$

$$\times \left\{ e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi+\tau)} + \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2 \right] e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\xi} \frac{\xi}{1!} \right.$$

$$+ \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2 \right]^2 e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi-\tau)} \frac{(\xi-\tau)^2}{2!}$$

$$\left. + \ldots + \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2 \right]^{k-1} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right][\xi-(k-2)\tau]} \frac{[\xi-(k-2)\tau]^{k-1}}{(k-1)!} \right\} d\xi$$

$$+ \int\limits_{(k-1)\tau}^{t^*} \left[\Phi_n'(t^*-\tau-\xi) - \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2 \right] \Phi_n(t^*-\tau-\xi) \right]$$

$$\times \left\{ e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi+\tau)} + \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2 \right] e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]\xi} \frac{\xi}{1!} \right.$$

$$+ \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2 \right]^2 e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](\xi-\tau)} \frac{(\xi-\tau)^2}{2!}$$

$$\left. + \ldots + \left[c_2 - \left(\frac{\pi n}{l}\, a_2\right)^2 \right]^{k} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right][\xi-(k-1)\tau]} \frac{[\xi-(k-1)\tau]^{k}}{k!} \right\} d\xi.$$

As in the previous case, since $t^* \geq (k-1)\tau,$ the following inequality holds for sufficiently large n:

$$n > \frac{l}{\pi} \max \left\{ \frac{\sqrt{|c_1|}}{|a_1|}, \frac{\sqrt{|1+c_2|}}{|a_2|} \right\}.$$

It follows from properties of definite integrals and the second mean-value theorem that there exist s_1 and s_2, $t^* - \tau \le s_1 \le (k-1)\tau$, $(k-1)\tau \le s_2 \le t^*$, such that

$$|B_n(t^*)|$$

$$\le \left| \left[c_2 - \left(\frac{\pi n}{l} a_2 \right)^2 \right]^{k-1} \right| \left| (k\tau - t^*) \left[\Phi_n'(t^* - \tau - s_1) - \left[c_1 - \left(\frac{\pi n}{l} a_1 \right)^2 \right] \Phi_n(t^* - \tau - s_1) \right] \right.$$

$$\left. \times\, e^{-\left[\left(\frac{\pi n}{l} a_1 \right)^2 - c_1 \right] [s_1 - (k-2)\tau]} \left\{ 1 + \frac{s_1}{1!} + \frac{(s_1 - \tau)^2}{2!} + \ldots + \frac{[s_1 - (k-2)\tau]^{k-1}}{(k-1)!} \right\} \right|$$

$$+ \left| \left[c_2 - \left(\frac{\pi n}{l} a_2 \right)^2 \right]^{k} \right| \left| [t^* - (k-1)\tau] \left[\Phi_n'(t^* - \tau - s_2) - \left[c_1 - \left(\frac{\pi n}{l} a_1 \right)^2 \right] \Phi_n(t^* - \tau - s_2) \right] \right.$$

$$\left. \times\, e^{-\left[\left(\frac{\pi n}{l} a_1 \right)^2 - c_1 \right] [s_2 - (k-1)\tau]} \left\{ 1 + \frac{s_2}{1!} + \frac{(s_2 - \tau)^2}{2!} + \ldots + \frac{[s_2 - (k-1)\tau]^{k}}{k!} \right\} \right|,$$

and one can find continuous functions $N_2^1(t^*, s)$ and $N_2^2(t^*, s)$ bounded for $(k-1)\tau \le t^* < k\tau$, $-\tau \le t \le 0$, and such that

$$|B_n(t^*)| \le \left| \left[c_2 - \left(\frac{\pi n}{l} a_2 \right)^2 \right]^{k-1} \right| \left| \left[\Phi_n'(t^* - \tau - s_1) - \left[c_1 - \left(\frac{\pi n}{l} a_1 \right)^2 \right] \Phi_n(t^* - \tau - s_1) \right] \right.$$

$$\left. \times\, e^{-\left[\left(\frac{\pi n}{l} a_1 \right)^2 - c_1 \right] [s_1 - (k-2)\tau]} \right| \left| N_2^1(t^*, s_1) \right|$$

$$+ \left| \left[c_2 - \left(\frac{\pi n}{l} a_2 \right)^2 \right]^{k} \right| \left| \left[\Phi_n'(t^* - \tau - s_2) - \left[c_1 - \left(\frac{\pi n}{l} a_1 \right)^2 \right] \Phi_n(t^* - \tau - s_2) \right] \right.$$

$$\left. \times\, e^{-\left[\left(\frac{\pi n}{l} a_1 \right)^2 - c_1 \right] [s_2 - (k-1)\tau]} \right| \left| N_2^2(t^*, s_2) \right|.$$

Assume that the functions $\Phi_n'(s)$ and $\Phi_n(s)$ increase “not too rapidly” on the segment $-\tau \le s \le 0$, i.e., for the time t^*, $(k-1)\tau \le t^* < k\tau$, and an arbitrary s, $-\tau \le s \le 0$, the following relation is true:

$$\lim_{n \to \infty} n^{2(k-2)} \left[\Phi_n'(s) + \left[c_1 - \left(\frac{\pi n}{l} a_1 \right)^2 \right] \Phi_n(s) \right] e^{-\left(\frac{\pi n}{l} a_1 \right)^2 (t^* - s)} = 0.$$

Then

$$\lim_{n \to \infty} B_n(t^*) = 0,$$

and the series $S_2(x, t)$ also converges uniformly and absolutely.

3. Consider the coefficients $C_n(t^*),\ n=1,2,\dots,$ of the third series $S_3(x,t)$.

For an arbitrary fixed time $t^*,\ (k-1)\tau \le t^* < k\tau,$ we perform the change of variables $t-\tau-\xi=s$ and represent the integral in the form of a sum of integrals in which the delayed exponential function has the same structure:

$$C_n(t^*) = \int_{-\tau}^{t^*-\tau} e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](\xi+\tau)} e_\tau^{D_n\xi} F_n(t^*-\tau-\xi)\,d\xi$$

$$= \int_{-\tau}^{0} e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](\xi+\tau)} F_n(t^*-\tau-\xi)\,d\xi$$

$$+ \int_{0}^{\tau} e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](\xi+\tau)} \left[1+D_n\frac{\xi}{1!}\right] F_n(t^*-\tau-\xi)\,d\xi$$

$$+\ldots+ \int_{(k-2)\tau}^{t^*-\tau} e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](\xi+\tau)} \left[1+D_n\frac{\xi}{1!}+D_n^2\frac{(\xi-\tau)^2}{2!}\right.$$

$$\left. +\ldots+D_n^{k-1}\frac{(\xi-(k-2)\tau)^{k-1}}{(k-1)!}\right] F_n(t^*-\tau-\xi)\,d\xi.$$

Substituting the value of D_n and using the mean-value theorem, we show that there exist times $-\tau\le s_1\le 0$, $0\le s_2\le\tau,\dots,(k-2)\tau\le s_k\le t^*-\tau$ such that

$$C_n(t^*) \le \tau e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](\tau+s_1)} F_n(t^*-\tau-s_1)$$

$$+\tau e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](\tau+s_1)} \left[1+\left[c_2-\left(\frac{\pi n}{l}a_2\right)^2\right] e^{-\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right]\tau} \frac{s_2}{1!}\right] F_n(t^*-\tau-s_2)$$

$$+\ldots+\tau e^{\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right](\tau+s_k)} \left[1+\left[c_2-\left(\frac{\pi n}{l}a_2\right)^2\right] e^{-\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right]\tau} \frac{s_k}{1!}\right.$$

$$\left.+\ldots+\tau\left[c_2-\left(\frac{\pi n}{l}a_2\right)^2\right]^{k-1} e^{-(k-1)\tau\left[c_1-\left(\frac{\pi n}{l}a_1\right)^2\right]} \frac{[s_k-(k-2)\tau]^{k-1}}{(k-1)!}\right] F_n(t^*-\tau-s_k),$$

and the following inequality hold for sufficiently large n:

$$n > \frac{l}{\pi}\max\left\{\frac{\sqrt{|c_1|}}{|a_1|},\ \frac{\sqrt{|1+c_2|}}{|a_2|}\right\}.$$

Therefore, one can find a function $N_3(t^*, s)$ continuous and bounded for $-\tau \le s \le t^*$ and such that

$$|C_n(t^*)| \le \tau \max_{-\tau \le s \le t^*} |F_n(s)|\, N_3(t^*, s) \left(\frac{\pi n}{l}\, a_2\right)^{2(k-1)} e^{-\left[\left(\frac{\pi n}{l}\, a_1\right)^2 - c_1\right](t^*+\tau)}.$$

Assume that the functions $F_n(s)$ increase "not too rapidly" on the segment $-\tau \le s \le t^*$, i.e., the following condition is satisfied:

$$\lim_{n\to\infty} n^{2(k-1)}\, |F_n(s)|\, e^{-\left(\frac{\pi n}{l}\, a_1\right)^2 (t^*+\tau)} = 0.$$

Then

$$\lim_{n\to\infty} C_n(t^*) = 0,$$

and the series also converges absolutely and uniformly.

Thus, we have shown that, for the series $S_1(x,t)$, $S_2(x,t)$, and $S_3(x,t)$ to converge absolutely and uniformly, it is only necessary that the Fourier coefficients $F_n(t)$, $-\tau \le s \le t^*$, $\Phi_n(t)$, and $\Phi_n'(t)$, $-\tau \le t \le 0$, increase "not too rapidly" with respect to the index n. The convergence of the derivatives of the function $u(x,t)$ follows from the properties of differentiability of the delayed exponential function.

The representation of a solution of the boundary-value problem (1)–(3) in the form (22), (23) is not always convenient, e.g., for the estimation of the influence of initial, boundary, and external actions. It is necessary to separate these factors into different terms.

We rewrite (22) as follows:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\frac{\pi n}{l}\, x e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](t+\tau)} e_\tau^{D_n t}$$

$$\times \left[\frac{2}{l}\int_0^l \varphi(\xi, -\tau) \sin\frac{\pi n}{l}\, \xi\, d\xi + \frac{2}{\pi n}\left[(-1)^n \mu_2(-\tau) - \mu_1(-\tau)\right]\right]$$

$$+ \sum_{n=1}^{\infty} \sin\frac{\pi n}{l}\, x \left\{ \int_{-\tau}^{0} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](t-s)} e_\tau^{D_n(t-\tau-s)} \right.$$

$$\left. \times \left[\frac{2}{l}\int_0^l \varphi_s'(\xi, s) \sin\frac{\pi n}{l}\, \xi\, d\xi + \frac{2}{\pi n}\left[(-1)^n \dot{\mu}_2(s) - \dot{\mu}_1(s)\right]\right] ds \right\}$$

$$- \sum_{n=1}^{\infty} \sin\frac{\pi n}{l}\, x \int_{-\tau}^{0} e^{\left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right](t-s)} e_\tau^{D_n(t-\tau-s)} \left[c_1 - \left(\frac{\pi n}{l}\, a_1\right)^2\right]$$

$$\times \left[\frac{2}{l}\int_0^l \varphi(\xi, s) \sin\frac{\pi n}{l}\, \xi\, d\xi + \frac{2}{\pi n}\left[(-1)^n \mu_2(s) - \mu_1(s)\right]\right] ds$$

$$+\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\int_0^t e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_\tau^{D_n(t-\tau-s)}\frac{2}{l}\int_0^l f(\xi,s)\sin\frac{\pi n}{l}\,\xi\,d\xi\,ds$$

$$+\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\int_0^t e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_\tau^{D_n(t-\tau-s)}\frac{2}{\pi n}\left[(-1)^n\dot{\mu}_2(s)-\dot{\mu}_1(s)\right]\,ds$$

$$-\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\int_0^t e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_\tau^{D_n(t-\tau-s)}\frac{2}{\pi n}\,c_1\left[(-1)^n\dot{\mu}_2(s)-\dot{\mu}_1(s)\right]\,ds$$

$$-\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\int_0^t e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_\tau^{D_n(t-\tau-s)}$$

$$\times\frac{2}{\pi n}c_2\left[(-1)^n\dot{\mu}_2(s-\tau)-\dot{\mu}_1(s-\tau)\right]\,ds+\mu_1(t)+\frac{x}{l}[\mu_2(t)-\mu_1(t)]. \qquad (24)$$

We separate the initial and boundary actions as follows:

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,xe^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t+\tau)}e_\tau^{D_n t}\left[\frac{2}{l}\int_0^l\varphi(\xi,-\tau)\sin\frac{\pi n}{l}\,\xi\,d\xi\right]$$

$$+\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\left\{\int_{-\tau}^0 e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_\tau^{D_n(t-\tau-s)}\left[\frac{2}{l}\int_0^l\varphi'_s(\xi,s)\sin\frac{\pi n}{l}\,\xi\,d\xi\right]ds\right\}$$

$$-\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\int_{-\tau}^0 e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_\tau^{D_n(t-\tau-s)}$$

$$\times\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right]\left[\frac{2}{l}\int_0^l\varphi(\xi,s)\sin\frac{\pi n}{l}\,\xi\,d\xi\right]ds$$

$$+\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\int_0^t e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_\tau^{D_n(t-\tau-s)}\frac{2}{l}\int_0^l f(\xi,s)\sin\frac{\pi n}{l}\,\xi\,d\xi\,ds$$

$$+\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,xe^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t+\tau)}e_\tau^{D_n t}\left[\frac{2}{\pi n}\left[(-1)^n\mu_2(-\tau)-\mu_1(-\tau)\right]\right]$$

$$+\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\left\{\int_{-\tau}^{t} e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)} e_{\tau}^{D_n(t-\tau-s)}\left[\frac{2}{\pi n}\left[(-1)^n\dot{\mu}_2(s)-\dot{\mu}_1(s)\right]\right]ds\right\}$$

$$+\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\int_{-\tau}^{0} e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)} e_{\tau}^{D_n(t-\tau-s)}\left(\frac{\pi n}{l}\,a_1\right)^2\left[\frac{2}{\pi n}\left[(-1)^n\mu_2(s)-\mu_1(s)\right]\right]ds$$

$$-\,c_1\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\int_{-\tau}^{t} e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)} e_{\tau}^{D_n(t-\tau-s)}\,\frac{2}{\pi n}\left[(-1)^n\mu_2(s)-\mu_1(s)\right]ds$$

$$-\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\int_{0}^{t} e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)} e_{\tau}^{D_n(t-\tau-s)}\,\frac{2}{\pi n}c_2\left[(-1)^n\mu_2(s-\tau)-\mu_1(s-\tau)\right]$$

$$+\,\mu_1(t)+\frac{x}{l}[\mu_2(t)-\mu_1(t)].$$

We take the integral

$$\int_{-\tau}^{t} e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)} e_{\tau}^{D_n(t-\tau-s)}\left[(-1)^n\dot{\mu}_2(s)-\dot{\mu}_1(s)\right]ds$$

$$=\,e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)} e_{\tau}^{D_n(t-\tau-s)}\left[(-1)^n\mu_2(s)-\mu_1(s)\right]\Big|_{-\tau}^{t}$$

$$+\int_{-\tau}^{t}\left[(-1)^n\mu_2(s)-\mu_1(s)\right]\left[\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right]e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)} e_{\tau}^{D_n(t-\tau-s)}\right.$$

$$\left.+\left[c_2-\left(\frac{\pi n}{l}\,a_2\right)^2\right]e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s-\tau)} e_{\tau}^{D_n(t-2\tau-s)}\right]ds$$

$$=\left[(-1)^n\mu_2(t)-\mu_1(t)\right]-e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t+\tau)} e_{\tau}^{D_n t}\left[(-1)^n\mu_2(-\tau)-\mu_1(-\tau)\right]$$

$$+\int_{-\tau}^{t}\left[(-1)^n\mu_2(s)-\mu_1(s)\right]\left[\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right]e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)} e_{\tau}^{D_n(t-\tau-s)}\right.$$

$$\left.+\left[c_2-\left(\frac{\pi n}{l}\,a_2\right)^2\right]e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s-\tau)} e_{\tau}^{D_n(t-2\tau-s)}\right]ds.$$

Substitution the obtained expressions, we get

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,xe^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t+\tau)}e_{\tau}^{D_n t}\left[\frac{2}{l}\int\limits_0^l\varphi(\xi,-\tau)\sin\frac{\pi n}{l}\,\xi\,d\xi\right]$$

$$+\sum_{n=1}^{\infty}\left\{\int\limits_{-\tau}^{0}e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_{\tau}^{D_n(t-\tau-s)}\left[\frac{2}{l}\int\limits_0^l\varphi_s'(\xi,s)\sin\frac{\pi n}{l}\,\xi\,d\xi\right]ds\right\}\sin\frac{\pi n}{l}\,x$$

$$-\sum_{n=1}^{\infty}\left\{\int\limits_{-\tau}^{0}e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_{\tau}^{D_n(t-\tau-s)}\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right]\right.$$

$$\left.\times\left[\frac{2}{l}\int\limits_0^l\varphi(\xi,s)\sin\frac{\pi n}{l}\,\xi\,d\xi\right]ds\right\}\sin\frac{\pi n}{l}\,x$$

$$+\sum_{n=1}^{\infty}\left\{\int\limits_{0}^{t}e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_{\tau}^{D_n(t-\tau-s)}\left[\frac{2}{l}\int\limits_0^l f(\xi,s)\sin\frac{\pi n}{l}\,\xi\,d\xi\right]ds\right\}\sin\frac{\pi n}{l}\,x$$

$$+\left(a_1^2-a_2^2\right)\sum_{n=1}^{\infty}\left\{\int\limits_{0}^{t}e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_{\tau}^{D_n(t-\tau-s)}\left[\frac{2\pi n}{l^2}\left[(-1)^n\mu_2(s)-\mu_1(s)\right]\right]ds\right\}\sin\frac{\pi n}{l}\,x$$

$$-c_2\sum_{n=1}^{\infty}\left\{\int\limits_{t-\tau}^{t}e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-\tau-s)}e_{\tau}^{D_n(t-2\tau-s)}\left[\frac{2}{\pi n}\left[(-1)^n\mu_2(s)-\mu_1(s)\right]\right]ds\right\}\sin\frac{\pi n}{l}\,x$$

$$+2\left\{\mu_1(t)+\frac{x}{l}\left[\mu_2(t)-\mu_1(t)\right]\right\}.$$

Let

$$\widetilde{S}_1[\varphi]=\sum_{n=1}^{\infty}\sin\frac{\pi n}{l}\,x\left\{e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t+\tau)}e_{\tau}^{D_n t}\left[\frac{2}{l}\int\limits_0^l\varphi(\xi,-\tau)\sin\frac{\pi n}{l}\,\xi\,d\xi\right]\right.$$

$$+\int\limits_{-\tau}^{0}e^{\left[c_1-\left(\frac{\pi n}{l}\,a_1\right)^2\right](t-s)}e_{\tau}^{D_n(t-\tau-s)}$$

$$\times \left[\frac{2}{l} \int\limits_0^l \varphi'_s(\xi, s) \sin \frac{\pi n}{l} \xi \, d\xi - \left[c_1 - \left(\frac{\pi n}{l} a_1 \right)^2 \right] \frac{2}{l} \int\limits_0^l \varphi(\xi, s) \sin \frac{\pi n}{l} \xi \, d\xi \right] ds \Bigg\} \tag{25}$$

be the sum that depends on the initial conditions, let

$$\widetilde{S}_2[f] = \sum_{n=1}^{\infty} \left\{ \int\limits_0^t e^{\left[c_1 - \left(\frac{\pi n}{l} a_1\right)^2\right](t-s)} e_\tau^{D_n(t-\tau-s)} \left[\frac{2}{l} \int\limits_0^l f(\xi, s) \sin \frac{\pi n}{l} \xi \, d\xi \right] ds \right\} \sin \frac{\pi n}{l} x \tag{26}$$

be the sum that depends on external actions, and let

$$\widetilde{S}_3[\mu_1, \mu_2] = \sum_{n=1}^{\infty} \Bigg\{ \left(a_1^2 - a_2^2\right) \frac{2\pi n}{l^2} \int\limits_0^t e^{\left[c_1 - \left(\frac{\pi n}{l} a_1\right)^2\right](t-s)} e_\tau^{D_n(t-\tau-s)} \left[(-1)^n \mu_2(s) - \mu_1(s)\right] ds$$

$$- \frac{2c_2}{\pi n} \int\limits_{t-\tau}^t e^{\left[c_1 - \left(\frac{\pi n}{l} a_1\right)^2\right](t-\tau-s)} e_\tau^{D_n(t-2\tau-s)} \left[(-1)^n \mu_2(s) - \mu_1(s)\right] ds \Bigg\} \sin \frac{\pi n}{l} x$$

$$+ 2 \left\{ \mu_1(t) + \frac{x}{l} \left[\mu_2(t) - \mu_1(t)\right] \right\} \tag{27}$$

be the sum that depends on the boundary conditions.

Then a solution of the boundary-value problem can be represented in the form

$$u(x, t) = \widetilde{S}_1[\varphi] + \widetilde{S}_2[f] + \widetilde{S}_3[\mu_1, \mu_2].$$

The following theorem is true:

Theorem 4. *Let the functions* $\varphi(x,t)$, $f(x,t)$, $\mu_1(t)$, *and* $\mu_2(t)$ *be such that, on the segment* $-\tau \le t \le t^*$, $(k-1)\tau \le t^* < k\tau$, *their Fourier coefficients*

$$\varphi_n(t) = \frac{2}{l} \int\limits_0^l \varphi(s, t) \sin \frac{\pi n}{l} s \, ds, \quad \dot{\varphi}_n(t) = \frac{2}{l} \int\limits_0^l \dot{\varphi}_t(s, t) \sin \frac{\pi n}{l} s \, ds, \quad -\tau \le t \le 0,$$

$$f_n(t) = \frac{2}{l} \int\limits_0^l f(s, t) \sin \frac{\pi n}{l} s \, ds$$

and

$$\mu_n(t) = \frac{2\pi n}{l^2}(a_1^2 - a_2^2) \int\limits_0^t e^{\left[c_1 - \left(\frac{\pi n}{l} a_1\right)^2\right](t-s)} e_\tau^{D_n(t-\tau-s)} \left[(-1)^n \mu_2(s) - \mu_1(s)\right] ds$$

$$- \frac{2c_2}{\pi n} \int\limits_{t-\tau}^t e^{\left[c_1 - \left(\frac{\pi n}{l} a_1\right)^2\right](t-\tau-s)} e_\tau^{D_n(t-2\tau-s)} \left[(-1)^n \mu_2(s) - \mu_1(s)\right] ds, \quad n = 1, 2, 3, \ldots,$$

satisfy the conditions

$$\lim_{n\to\infty} n^{2(k-1)} |\varphi_n(s)| e^{-\left(\frac{\pi n}{l} a_1\right)^2 (t^*+\tau)} = 0, \quad -\tau \le s \le 0, \quad (k-1)\tau \le t^* < k\tau,$$

$$\lim_{n\to\infty} n^{2(k-1)} |f_n(t^*)| e^{-\left(\frac{\pi n}{l} a_1\right)^2 (t^*+\tau)} = 0, \quad \lim_{n\to\infty} n^{2(k-1)} |\mu_n(t^*)| e^{-\left(\frac{\pi n}{l} a_1\right)^2 (t^*+\tau)} = 0,$$

$$(k-1)\tau \le t^* < k\tau.$$

Then, for $0 \le t \le t^*$, *a solution of the first boundary-value problem (1)–(3) has the form*

$$u(x,t) = \widetilde{S}_1[\varphi] + \widetilde{S}_2[f] + \widetilde{S}_3[\mu_1, \mu_2],$$

where $\widetilde{S}_1[\varphi]$, $\widetilde{S}_2[f]$, *and* $\widetilde{S}_3[\mu_1, \mu_2]$ *are defined by (25)–(27).*

Proof. The proof is analogous to the proof of Theorem 3.

REFERENCES

1. A. D. Bazykin, *Mathematical Biophysics of Interacting Populations* [in Russian], Nauka, Moscow (1985).
2. A. B. Zhukov, "Space and time variability of the process of forest growth," *Dokl. Akad. Nauk,* **239**, No. 1, 245–248 (1978).
3. L. É. Él'sgol'ts and S. B. Norkin, *Introduction to the Theory of Differential Equations with Deviating Argument* [in Russian], Nauka, Moscow (1971).
4. J. Hale, *Theory of Functional Differential Equations,* Springer, New York (1977).
5. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatulina, *Introduction to the Theory of Functional Differential Equations* [in Russian], Nauka, Moscow (1991).
6. B. P. Tkach, "On one partial differential equation with delayed argument," *Differents. Uravn.,* **3**, No. 10, 1796–1801 (1967).
7. A. A. Boichuk, V. F. Zhuravlev, and A. M. Samoilenko, *Generalized Inverse Operators and Noetherian Boundary-Value Problems* [in Russian], Institute of Mathematics, Ukrainian National Academy of Sciences, Kiev (1985).
8. D. Ya. Khusainov and G. V. Shuklin, "On relative controllability in systems with pure delay," *Prikl. Mekh.,* **41**, No. 2, 118–130 (2005).