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RESEARCH

Fixed-time prescribed performance tracking for nonlinear systems with unknown time-varying input delay

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Abstract This paper investigates the fixed-time prescribed performance tracking control problem for a class of nonlinear systems with multiple uncertainties. The considered systems involve input delay, coefficients, nonlinear functions and external disturbances which are both unknown, posing significant challenges. To overcome these challenges, a compensation system is introduced to eliminate the impact of time-varying input delay. Subsequently, new adaptive parameters are introduced into the Lyapunov–Krasovskii functional to address unknown external disturbances. By incorporating a specific funnel function to constrain the transient behavior of tracking error, along with backstepping method and bounded estimation techniques, a novel fixed-time control tracking scheme is proposed which ensures the prescribed transient performance. Ultimately, the efficacy of the proposed control methodology is substantiated through simulation examples.

Keywords Unknown time-varying delay · Fixed-time control · Prescribed performance · Saturation

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1 Introduction

Time delay refers to the phenomenon between the response and the input of the system, which is widely existed in various fields in the real world, including biology, control engineering, transportation and so on [1–3]. The study of time delay systems can help us understand the stability and control performance of the system, and provide a theoretical basis for system design. For example, in control engineering, time delay have an important impact on the stability and performance of the system [4,5]. In the field of biology, the conduction delay in the nervous system is a kind of time-delay phenomenon, and studying the time-delay system is helpful to understand the dynamic characteristics of the nervous system [6]. Therefore, the study of time-delay systems is of great significance and value for theoretical research and practical application.

In recent years, a wealth of results have been obtained for systems with constant or known time-varying delay, such as [7–13]. To be specific, the study conducted in [7] addressed the global stabilization issue of nonlinear systems featuring unknown control directions and constant parameter uncertainties within the delay domain. In the realm of feedback systems with input delay, an adaptive neural control approach was proposed in [9]. The predicator-based controller for uncertain nonlinear systems with matching conditions was developed in [11,12], which included finite integrals of past control values. References [13] examined strictly feedback nonlinear systems using the Pade approximation method and utilized fuzzy systems to approximate unknown functions, but the Pade approximation method is only applicable to situations where the input delay is very small. However, in many practical systems, specific information about system delay is unknown, and unknown delay are frequently encountered in practical engineering systems. Therefore, studying the impact of unknown delay is crucial and has received widespread attention, such as [14–18]. For example, for a class of uncertain nonlinear systems with unknown time-varying input delay and interference, the asymptotic tracking controller was designed in [16].

It is noteworthy that, in contrast to asymptotic stability, fixed-time stability offers superior control performance and is better suited for practical systems, see as [19–23]. However, to our knowledge, there are relatively few results concerning fixed-time control problems. Designing a controller for fixed-time control remains a highly challenging and difficult task when such a requirement is imposed on the system. While the fixed-time control problem for high-order nonlinear systems was addressed in [23], the presence of singularity issues during the analysis restricts the overall generality of the proof.

As is well known, the concept of prescribed performance control was initially introduced by Bechlioulis and Rovithakis [24]. Prescriptive performance control entails the convergence of tracking errors to a predefined small residual set, where the maximum fluctuation of errors is less than a predetermined constant, and the convergence time is not less than a specified duration. There are numerous intriguing results associated with prescribed performance control. For instance, for uncertain MIMO nonlinear systems, a new robust tracking controller was proposed in [25], which can guarantee output tracking with specified performance. In the realm of uncertain strict feedback nonlinear systems with arbitrary relative degrees and unknown control directions, an asymptotic tracking control approach was introduced in [26], which guarantees the specified transient behavior of the system. A hybrid control strategy was proposed in [27] to ensure asymptotic convergence and transient behavior of tracking errors.

Furthermore, in recent years, the combination of backstepping control with neural network control or fuzzy logic systems has been widely applied to mitigate the impact of uncertainties in the system, such as [8,10,23,28–33]. Thanks to the unique general approximation, adaptive capabilities, and learning abilities of radial basis function neural networks (RBFNNs), they can effectively approximate unknown continuous functions. In more specific terms, the design of adaptive control strategies utilizing RBFNNs to approximate unknown functions for nonlinear systems with unknown nonlinear functions was discussed in [8,23,29]. Moreover, a self-adaptive output feedback control scheme for nonlinear quantized system, developed based on the backstepping method and general approximation, was presented in [33].

In particular, when considering unknown input delay and fixed-time control, the design of the controller becomes more complex in the presence of uncertainties such as unknown control coefficients, unknown nonlinear functions, and external disturbances. To date, there appear to be no existing results to address these challenges. Therefore, inspired by the aforementioned discussion, this paper addresses the problem of fixed-time prescribed performance adaptive control for nonlinear systems with uncertainties including unknown input delay, unknown control coefficients and external disturbances. The contributions of this paper are highlighted as follows:

1. We introduce a novel bounded estimation mechanism in the L-K functional to design adaptive parameters and combine it with the backstepping method. This marks the first consideration of the fixed-time prescribed performance adaptive tracking control problem for nonlinear systems with various uncertainties, including unknown coefficients, delay, nonlinear functions and external disturbances. Importantly, our proposed approach not only eliminates the reliance on priori knowledge of desired signal but also effectively enhances the disturbance rejection performance of the closed-loop system, further extending existing results as shown in [20,34–40].
2. In contrast to the existing work on unknown input delay [16,23], this paper introduces a novel adaptive tracking scheme by incorporating a special funnel function and a logarithmic L–K functional. A key feature of our design is that it ensures globally prescribed transient performance, independent of initial conditions.
3. To address the impact of unknown input delays, introduce a compensation system aimed at elim-

inating the effects of these delay while demonstrating the fixed-time stability of the compensator system. Furthermore, our approach does not rely on restrictive conditions such as bounded inputs or prior information about signals, and rigorously establishes the fixed-time stability of the compensation system. Different from the majority of current results with input delay, such as [10,16,23,28,41–44].

The paper is structured as follows: Sect. 2 presents the problem formulation and some preliminary results. The controller design and the stability analysis are detailed in Sect. 3. Section 4 presents the simulation examples. Lastly, the paper wraps up with the conclusion in Sect. 5.

2 Preliminary

2.1 Problem formulation

Consider a class of nth-order nonlinear systems

$$
\begin{aligned}
&\dot{x}_i(t) = g_i x_{i+1}(t) + f_i(x,t) + d_i(x,t),\\
&\dot{x}_n(t) = g_n u(t-\tau(t)) + f_n(x,t) + d_n(x,t),\\
&y = x_1, \quad r(t) = y - y_r,
\end{aligned}
\tag{1}
$$

where the system state variable $x = [x_1, x_2, ..., x_n]^T$, $u(t-\tau(t)) \in R$ is the control input with input delay, $\tau(t)$ denotes unknown time-varying input delay, $r(t)$ is tracking error, y is the system output, y_r denotes the desired signal, the constants g_i is unknown and can take either positive or negative values for $i = 1, 2, ..., n$. $f_i(x,t)$ and $d_i(x,t)$ represent the unknown smooth nonlinear function and unknown additive disturbance for $i = 1, 2, ..., n$, respectively. In the subsequent sections of the paper, when it is not misleading, nonlinear functions $f_i(\cdot)$, $d_i(\cdot)$, etc., will be abbreviated as f_i, d_i, etc.

The control objective of this paper is to design an adaptive controller for nonlinear system (1) such that

1. All the closed-loop signals of the system are semi-globally fixed-time uniformly ultimately bounded;
2. The tracking error $r(t)$ will remain within a small bounded range of the origin in finite time;
3. The tracking error $r(t)$ can be guided to a predetermined accuracy set $\Theta_r = \{r(t) \in R \ \ ||r(t)| < \varepsilon\}$ within a specified finite time T_k, where both ε and T_k can be preassigned, ensuring the transient performance of $r(t)$.

In order to achieve the control objective for system (1), we introduce the following assumptions and definition.

Assumption 1 The nonlinear external disturbance $d_i(x,t)$ is bounded by constants $d_{im} > 0$ for $i = 1, 2, ..., n$.

Assumption 2 The desired trajectory y_r and its first derivative $\dot{y}_r$ exist and bounded.

Assumption 3 Without loss of generality, we assume that the signs of g_i, where $i = 1, 2, ..., n$, are positive throughout this article.

Assumption 4 The input time-varying delay $\tau(t)$ is bounded such that $\tau(t) < \tau_{max}$ for all $t \in R$ and slowly varying such that $\dot{\tau}(t) < \bar{\mu} < 1$, where τ_{max} and $\bar{\mu}$ are positive constants. Moreover, there exists a sufficiently accurate constant estimate $\hat{\tau} \in R$ of τ which is available. Then defined $\tilde{\tau} \triangleq \tau - \hat{\tau}$ and $\tilde{\tau}$ satisfies $|\tilde{\tau}| \leqslant \bar{\tau}$ for all $t \in R$ where $\bar{\tau}$ is a known positive constant.

Remark 1 Since the bounds on input delay are attainable in many applications, it is reasonable to assume that the maximum allowable error $\bar{\tau}$ and error estimation $\hat{\tau}$ are known in Assumption 4. Such assumptions are common in addressing the problem of unknown time-varying input delay, as in [16].

Definition 1 [45] Consider the nonlinear system

$$
\dot{x} = f(x,t), \quad f(t,0) = 0, \quad x(0) = x_0, \tag{2}
$$

where state vector $x \in R^n$ and $f : R^+ \times R^n \to R^n$ is the nonlinear function. For any initial state $x(0)$, if the solution of the system can converge to the set Ω in finite time, then set Ω is referred to as the system's fixed-time attractor, i.e. $\exists T_{max}$, such that for any $t \geqslant T(x_0)$, $x(t, x_0) \in \Omega$ where $T(x_0) \leqslant T_{max}$. In particular, when $\Omega = 0$, the system is termed fixed-time stable at the origin.

Lemma 1 [46] *For all $x_0 \in R^n$, if there exists a globally radially unbounded and positive definite C^1 function $V(x)$, for some constants α, β, p, q, $k > 0$ with $0 < pk < 1$ and $qk > 1$, such that*

$$
\dot{V}(x) \leqslant -(\alpha V^p(x) + \beta V^q(x))^k \tag{3}
$$

holds, then the system (2) is globally fixed-time stable at the origin and the settling time function $T(x_0)$ is satisfied

$$T(x_0) \leqslant T_m = \frac{\Gamma(m_p)\Gamma(m_q)}{\alpha^k \Gamma(k)(q-p)} \left(\frac{\alpha}{\beta}\right)^{m_q}, \tag{4}$$

where $m_p = \frac{1-pk}{q-p} > 0$, $m_q = \frac{qk-1}{q-p} > 0$ *and* $\Gamma(z) := \int_0^{+\infty} e^{-t} t^{z-1} dt$ *denote the gamma function* [47].

Lemma 2 [48] *For* $x > 0$, $y > 0$, $m > 0$, $n > 0$ *and* $p > 0$, *which yields*

$$|x|^m |y|^n \leqslant \frac{m}{m+n} p|x|^{m+n} + \frac{n}{m+n} p^{-\frac{m}{n}} |y|^{m+n}. \tag{5}$$

Lemma 3 [49] *For* $z_i \in R$, $0 \leqslant i \leqslant n$ *and* $0 \leqslant a \leqslant 1$, *than*

$$\left(\sum_{i=1}^n |z_i|\right)^a \leqslant \sum_{i=1}^n |z_i|^a \leqslant n^{1-a} \left(\sum_{i=1}^n |z_i|\right)^a. \tag{6}$$

Lemma 4 [49] *For* $x_i > 0$ *and* $0 \leqslant i \leqslant n$, *such that*

$$\left(\sum_{i=1}^n x_i\right)^2 \leqslant n \sum_{i=1}^n {x_i}^2. \tag{7}$$

2.2 State transformation

The presence of unknown control coefficients g_i complicates the design of the controller for system (1). To facilitate the controller design, we introduce a linear transformation on the system state x_i. Through this linear transformation, system (1) is transformed into a nonlinear system with only one unknown coefficient.

Define $e_1 = x_1$ and $e_i = x_i / g_i g_{i+1} \ldots g_n$, then we have

$$\begin{aligned}
\dot{e}_1(t) &= a_0 e_2(t) + \tilde{f}_1(x,t) + \tilde{d}_1(x,t), \\
\dot{e}_i(t) &= e_{i+1}(t) + \tilde{f}_i(x,t) + \tilde{d}_i(x,t), \\
\dot{e}_n(t) &= u(t - \tau(t)) + \tilde{f}_n(x,t) + \tilde{d}_n(x,t), \\
y &= e_1, \quad r(t) = y - y_r = e_1 - y_r,
\end{aligned} \tag{8}$$

where $a_0 = g_1 g_2 \ldots g_n$, $\tilde{d}_1(x,t) = d_1(x,t)$, $\tilde{d}_i(x,t) = d_i(x,t)/g_i g_{i+1} \ldots g_n$ for $i = 2, 3, \ldots, n$, and $\tilde{f}_1(x,t) = f_1(x,t)$, $\tilde{f}_i(x,t) = f_i(x,t)/g_i g_{i+1} \ldots g_n$ for $i = 2, 3, \ldots, n$. According to Assumption 3, we can easily get that there is a constant $\bar{d}_i > 0$ such that $\left|\tilde{d}_i(x,t)\right| \leqslant \bar{d}_i$.

2.3 Compensation system

To mitigate the impact of input delay in system (1), we introduce the following compensation system:

$$\begin{aligned}
\dot{\lambda}_i &= -\hat{p}_i \lambda_i - p_i \lambda_i^3 + \lambda_{i+1}, \text{ for } i = 2, \ldots, n-1, \\
\dot{\lambda}_n &= -\hat{p}_n \lambda_n - p_n \lambda_n^3 + u(t) - u(t - \hat{\tau}) \\
&\quad - sign(\lambda_n) \int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}} |\dot{u}(s)| ds,
\end{aligned} \tag{9}$$

where $\hat{p}_i$ and $p_i > 0$ are designed parameters and the initial condition is $\lambda_i(0) = 0$ for $i = 2, 3, \ldots, n$.

Remark 2 It is worth mentioning that:

1. In the existing literature as in [23,47,50,51], they have also investigated the adaptive control problem with input delay. These references introduced novel coordinate transformations to effectively compensate for the effects of input delay. However, the design approach mentioned above overlooked the impact of the compensation function on the stability of the closed-loop system, leading to certain limitations in the conclusions. In contrast, in this paper, we introduce an auxiliary system (9) to effectively eliminate the influence of time-varying input delay and the expression of the compensator system aids us in proving its fixed-time convergence property in subsequent sections.
2. In the compensation system, the initial condition is set as $\lambda_i(0) = 0$, which means that when there is no input delay in the system, λ_i remain zero, and thus it will not affect the controller design of system (1). When the input delay of the system is non-zero, we will utilize the compensation system to mitigate the impact of input delay.
3. In the compensation system (9), there are signals related to u in the auxiliary system. In subsequent sections, we will demonstrate the boundedness of the input u, thereby obtaining the fixed-time boundedness of the auxiliary system (9).

2.4 Funnel constraint function

To impose transient behavior constraints on the system tracking error, inspired by [52–54], we define the funnel function as follows:

Definition 2 For any convergence time T_k and positive error precision ε, a function taking non-negative values $\mu(t) \in \mathbb{R}$ is called a funnel constraint function if it satisfies:

1. $\mu(0) = 0$ and $\mu(t) > 0$ for all $t > 0$;
2. $\mu(t)$ and $\dot{\mu}(t)$ are bounded by positive constants μ_d and μ_{dm};
3. there exist a time $t_s \leqslant T_k$, such that $\mu(t) \geqslant \frac{1}{\varepsilon}$ holds for all $t > t_s$.

Then, the prescribed performance funnel can be defined as

$$F = \left\{(t, r) \in R^+ \times R \,|\mu(t)\,|r| < 1\right\}. \tag{10}$$

Remark 3 The tracking error evolution curve within the funnel is depicted in Fig. 1. By constructing an appropriate L-K functional, we confine the product of the tracking error $r(t)$ and the funnel constraint function $\mu(t)$ within F, thereby regulating the transient behavior of tracking error $r(t)$. Moreover, there are various options for the specific expression of funnel constraint function $\mu(t)$, such as

$$\mu(t) = \frac{t}{\varepsilon(\alpha t + (1-\alpha)T_k)}, \tag{11}$$

$$\mu(t) = \begin{cases} t^\beta T_k^\beta/\varepsilon, & t \in [0, T_k), \\ 1/\varepsilon, & t \in [T_k, \infty), \end{cases} \tag{12}$$

where $0 < \alpha < 1$, $\beta > 0$ are design constants.

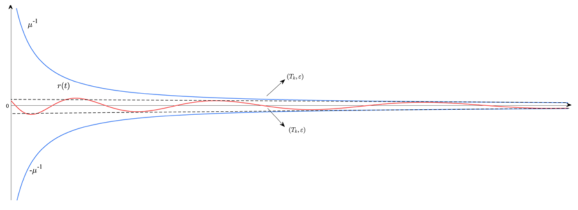

Fig. 1 The trajectory of tracking error within the funnel

2.5 RBF neural networks

Radial basis function neural networks (RBFNNs) are widely used for the control and identification of nonlinear systems with uncertainties due to their ability to effectively approximate nonlinear functions [55,56]. In this section, we will introduce the fundamental knowledge of RBFNNs to facilitate their use in the subsequent controller design process.

For any continuous unknown function $f(Z)$, there exist suitable neural network $W^{*T}S(Z)$ such that

$$f(Z) = W^{*T}S(Z) + \delta(Z), \tag{13}$$

where $S(Z) = (s_1(Z), ..., s_l(Z)) \in R^l$ denote the basis function vector, where the Gaussian function $s_i(Z)$ are defined as following:

$$s_i(Z) = exp[-(Z-\xi_i)^T(Z-\xi_i)/\sigma_i^2], \quad i = 1, 2, ..., l, \tag{14}$$

where σ_i is the width of the Gaussian functions and $\xi_i = [\xi_1, \xi_2, ..., \xi_n]^T$ denotes the center of the receptive domain. Additionally, $l > 1$ represents the number of nodes in the neural network, $Z = [Z_1, Z_2, ..., Z_q]^T \in \Omega_Z \in R$ represents the input vector, $W = (w_1, ..., w_l) \in R^l$ is the weight vector and $\delta(Z)$ represents the approximation error which satisfies $|\delta(Z)| \leqslant \varepsilon$ with $\varepsilon > 0$. The ideal weight vector $W^* = [w_1, w_2, ..., w_q]^T \in R^l$ is define as

$$W^* := arg \min_{W \in R^l}\left\{\sup_{Z \in \Omega_Z}\left|F(Z) - W^T S(Z)\right|\right\}. \tag{15}$$

3 Controller design and stability analysis

3.1 Controller design

In this section, we will utilize the backstepping method to provide the detailed procedure for designing fixed-time adaptive controller.

To begin with, by using system (8) and the compensator system (9), we introduce the following coordinate transformation:

$$z_1 = \mu(t)(e_1 - y_r),$$
$$z_i = e_i - \alpha_{i-1} + \lambda_i, \quad \text{for } i = 2, 3, ..., n, \tag{16}$$

where α_{i-1} is the virtual controller that will be designed in the subsequent steps.

Step 1: According to (8) and (9), differentiating z_1, one has

$$\begin{aligned}\dot{z}_1 &= \mu(t)(\dot{e}_1 - \dot{y}_r) + \dot{\mu}(t)(e_1 - y_r)\\ &= \mu(t)(a_0 e_2(t) + \tilde{f}_1(x,t)\\ &\quad + \tilde{d}_1(x,t) - \dot{y}_r) + \dot{\mu}(t)(e_1 - y_r).\end{aligned} \tag{17}$$

Then let the Lyapounov function be

$$V_1 = \frac{1}{2a_0}\ln\frac{1}{1-z_1^2} + \frac{\hat{\theta}_1^2}{2\beta_1} + \frac{\hat{\eta}_1^2}{2\gamma_1}, \tag{18}$$

where β_1 and γ_1 are positive design parameters, $\hat{\theta}_1 = \theta_1 - \theta_1^*$ and $\hat{\eta}_1 = \eta_1 - \eta_1^*$ are the estimation error with θ_1^* and η_1^* represent the estimate of θ_1 and η_1, respectively. The positivity definiteness and continuous differentiability of V_1 for $|z_1| < 1$ can be confirmed, establishing it as a suitable candidate for a Lyapunov function.

Further, by differentiating V_1 with respect to t, we get

$$\begin{aligned}\dot{V}_1 &= \frac{\chi z_1 \dot{z}_1}{a_0} + \frac{\hat{\theta}_1\dot{\hat{\theta}}_1}{\beta_1} + \frac{\hat{\eta}_1\dot{\hat{\eta}}_1}{\gamma_1}\\ &= \frac{\chi z_1}{a_0}(\mu(t)(a_0 e_2(t) + \tilde{f}_1(x,t)\\ &\quad + \tilde{d}_1(x,t) - \dot{y}_r) + \dot{\mu}(t)(e_1 - y_r))\\ &\quad + \frac{\hat{\theta}_1\dot{\hat{\theta}}_1}{\beta_1} + \frac{\hat{\eta}_1\dot{\hat{\eta}}_1}{\gamma_1}\\ &= \chi z_1(\mu(t)z_2(t) + F_1(Z_1) + \frac{\mu(t)\tilde{d}_1}{a_0} + \mu(t)\alpha_1)\\ &\quad + \frac{\hat{\theta}_1\dot{\hat{\theta}}_1}{\beta_1} + \frac{\hat{\eta}_1\dot{\hat{\eta}}_1}{\gamma_1},\end{aligned} \tag{19}$$

where $F_1(Z_1) = (\mu(t)(\tilde{f}_1(x,t) - \dot{y}_r - a_0\lambda_2) + \dot{\mu}(t)(e_1 - y_r))/a_0$ and $Z_1 = [x_1, x_2, ..., x_n, \mu, \dot{\mu}, y_r, \dot{y}_r, \lambda_2]^T$ and $\chi = \frac{1}{1-z_1^2}$.

Since $F_1(Z_1)$ is an unknown continuous function, we will utilize RBFNNs from Sect. 2.5 to approximate $F_1(Z_1)$. Therefore, by using RBFNNs in (13) and Young's inequality, we infer that

$$\begin{aligned}\chi z_1 F_1(Z_1) =& \chi z_1 W_1^{*T} S_1(Z_1) + \chi z_1 \delta_1(Z_1)\\ \leqslant& \frac{(\chi z_1)^2}{2a_1}\theta_1 S_1^T S_1 + \frac{a_1}{2} + |\chi z_1|\varepsilon_1,\end{aligned} \tag{20}$$

where $\theta_1 = \left\| W_1^{*T} \right\|^2$, $a_1 > 0$ is design parameter and ε_1 is the upper bound of the estimation error.

Inserting the above inequality into (19), one has

$$\begin{aligned}\dot{V}_1 \leqslant& \chi z_1\mu(t)(z_2(t) + \alpha_1) + \frac{(\chi z_1)^2}{2a_1}\theta_1 S_1^T S_1\\ &+ \frac{a_1}{2} + \frac{\hat{\theta}_1\dot{\hat{\theta}}_1}{\beta_1} + \frac{\hat{\eta}_1\dot{\hat{\eta}}_1}{\gamma_1} + |\chi z_1|\left(\frac{\mu_d\bar{d}_1}{a_0} + \varepsilon_1\right)\\ \leqslant& \chi z_1\mu(t)(z_2(t) + \alpha_1) + \frac{(\chi z_1)^2}{2a_1}\theta_1 S_1^T S_1\\ &+ \frac{a_1}{2} + \frac{\hat{\theta}_1\dot{\hat{\theta}}_1}{\beta_1} + \frac{\hat{\eta}_1\dot{\hat{\eta}}_1}{\gamma_1} + \frac{(\chi z_1)^2}{2b_1}\eta_1 + \frac{b_1}{2},\end{aligned} \tag{21}$$

where $\mu_d > 0$ is the upper bound of μ, $\eta_1 = (\frac{\mu_d\bar{d}_1}{a_0} + \varepsilon_1)^2$ and $b_1 > 0$ is design parameter.

Remark 4 In contrast to the general approach in adaptive controllers for handling unknown external disturbances and NNs error, in this paper, we consider the unknown external disturbances $\tilde{d}_1$ and NNs error ε_1 as a collective uncertainty, and then introduce the estimate η_1 to quantify this uncertainty. This approach not only eliminates the need for a priori knowledge of boundary information but also enhances the robustness of the closed-loop system.

Then, let the virtual control signal α_1 and the adaptive law be designed as

$$\begin{aligned}\alpha_1 =& -\frac{1}{2a_1}\chi r\theta_1^* S_1^T S_1 - \frac{1}{2b_1}\chi r\eta_1^*\\ &- \frac{1}{2}\chi z_1\mu - K_1\chi r - L_1\chi^3 r,\end{aligned} \tag{22}$$

$$\dot{\theta}_1^* = \frac{\beta_1}{2a_1}(\chi z_1)^2 S_1^T S_1 - m_1\theta_1^* - \frac{q_1}{\beta_1}\theta_1^{*3}, \tag{23}$$

$$\dot{\eta}_1^* = \frac{\gamma_1}{2b_1}(\chi z_1)^2 - r_1\eta_1^* - \frac{s_1}{\gamma_1}\eta_1^{*3}, \tag{24}$$

where m_1, q_1, r_1, s_1, K_1 and L_1 are positive design parameters and error $r = e_1 - y_r$.

Then substituting (22)–(24) into (21), we arrive at

$$\dot{V}_1 \leqslant -K_1\chi^2 z_1^2 - L_1\chi^4 z_1^2 + m_1\frac{\hat{\theta}_1\theta_1^*}{\beta_1} + q_1\frac{\hat{\theta}_1\theta_1^{*3}}{\beta_1^2} + r_1\frac{\hat{\eta}_1\eta_1^*}{\gamma_1} + s_1\frac{\hat{\eta}_1\eta_1^{*3}}{\gamma_1^2} + \frac{1}{2}z_2^2 + \frac{a_1}{2} + \frac{b_1}{2}. \quad (25)$$

Noting that

$$\begin{aligned} &-K_1(z_1^2\chi^2 + \ln\chi - \ln\chi) \\ &= -K_1((\chi-1)\chi + \ln\chi - \ln\chi) \\ &\leqslant -\frac{K_1}{2}\chi^2 + K_1\ln\chi - K_1\ln\chi + \frac{K_1}{2} \\ &\leqslant -K_1\ln\chi + \frac{K_1}{2}, \\ &-L_1(z_1^2\chi^4 + \ln^2\chi - \ln^2\chi) \leqslant -\frac{L_1}{4}\chi^4 \\ &+ L_1\ln^2\chi - L_1\ln^2\chi + \frac{L_1}{4} \\ &\leqslant -L_1\ln^2\chi + \frac{L_1}{4}. \end{aligned}$$

Then we can rewrite (25) as

$$\dot{V}_1 \leqslant -K_1\ln\chi - L_1\ln^2\chi + m_1\frac{\hat{\theta}_1\theta_1^*}{\beta_1} + n_1\frac{\hat{\theta}_1\theta_1^{*3}}{\beta_1^2} + r_1\frac{\hat{\eta}_1\eta_1^*}{\gamma_1} + s_1\frac{\hat{\eta}_1\eta_1^{*3}}{\gamma_1^2} + \frac{1}{2}z_2^2 + \bar{\sigma}_1, \quad (26)$$

where $\bar{\sigma}_1 = \frac{K_1}{2} + \frac{L_1}{4} + \frac{a_1}{2} + \frac{b_1}{2}$.

Step 2: With the help of (8) and (9), differentiating z_2, we deduce that

$$\begin{aligned} \dot{z}_2 &= \dot{e}_2 - \dot{\alpha}_1 + \dot{\lambda}_2 \\ &= e_3(t) + \tilde{f}_2(x,t) + \tilde{d}_2(x,t) - \dot{\alpha}_1 + \lambda_3 \\ &\quad - \hat{p}_2\lambda_2 - p_2\lambda_2^3. \end{aligned} \quad (27)$$

Define the Lyapounov function as follow

$$V_2 = V_1 + \frac{z_2^2}{2} + \frac{\hat{\theta}_2^2}{2\beta_2} + \frac{\hat{\eta}_2^2}{2\gamma_2}, \quad (28)$$

where the design parameters $\beta_2 > 0$ and $\gamma_2 > 0$, and the estimation errors are defined as $\hat{\theta}_2 = \theta_2 - \theta_2^*$ and $\hat{\eta}_2 = \eta_2 - \eta_2^*$ with θ_2^* and η_2^* being the estimates of θ_2 and η_2, respectively.

Thus, $\dot{V}_2(t)$ can be derived from (27) and (28) that

$$\begin{aligned} \dot{V}_2 &= z_2\dot{z}_2 + \frac{\hat{\theta}_2\dot{\hat{\theta}}_2}{\beta_2} + \frac{\hat{\eta}_2\dot{\hat{\eta}}_2}{\gamma_2} + \dot{V}_1 \\ &= z_2(e_3(t) + \tilde{f}_2(x,t) + \tilde{d}_2(x,t) \\ &\quad - \dot{\alpha}_1 + \lambda_3 - \hat{p}_2\lambda_2 - p_2\lambda_2^3) + \frac{\hat{\theta}_2\dot{\hat{\theta}}_2}{\beta_2} + \frac{\hat{\eta}_2\dot{\hat{\eta}}_2}{\gamma_2} + \dot{V}_1 \\ &= z_2(z_3(t) + F_2(Z_2) + \tilde{d}_2 + \alpha_2) + \frac{\hat{\theta}_2\dot{\hat{\theta}}_2}{\beta_2} \\ &\quad + \frac{\hat{\eta}_2\dot{\hat{\eta}}_2}{\gamma_2} + \dot{V}_1, \end{aligned} \quad (29)$$

where $F_2(Z_2) = \tilde{f}_2(x,t) - \dot{\alpha}_1 - \hat{p}_2\lambda_2 - p_2\lambda_2^3$ and $Z_2 = [x_1, x_2, ..., x_n, \theta_1^*, \eta_1^*, \lambda_2]^T$.

Similar to the calculation of (20), one can get

$$\begin{aligned} z_2F_2(Z_2) =& z_2W_2^{*T}S_2(Z_2) + z_2\delta_2(Z_2) \\ &\leqslant \frac{z_2^2}{2a_2}\theta_2S_2^TS_2 + \frac{a_2}{2} + |z_2|\,\varepsilon_2, \end{aligned} \quad (30)$$

where $\theta_2 = \left\|W_2^{*T}\right\|^2$, $a_2 > 0$ is design parameter and ε_2 is the upper bound of identify error.

Substituting (30) into (29), we can get

$$\dot{V}_2 \leqslant z_2(z_3(t) + \alpha_2) + \frac{z_2^2}{2a_2}\theta_2S_2^TS_2 + \frac{a_2}{2} + \frac{\hat{\theta}_2\dot{\hat{\theta}}_2}{\beta_2} + \frac{\hat{\eta}_2\dot{\hat{\eta}}_2}{\gamma_2} + \dot{V}_1 + \frac{z_2^2}{2b_2}\eta_2 + \frac{b_2}{2}, \quad (31)$$

where $\eta_2 = (\bar{d}_2 + \varepsilon_2)^2$ and $b_2 > 0$ is design parameter.

Then, the virtual control signal α_2 and the adaptive law are designed as below

$$\begin{aligned} \alpha_2 =& -\frac{1}{2a_2}z_2\theta_2^*S_2^TS_2 - \frac{1}{2b_2}z_2\eta_2^* \\ &- (K_2+1)z_2 - L_2z_2^3, \end{aligned} \quad (32)$$

$$\dot{\theta}_2^* = \frac{\beta_2}{2a_2}z_2^2S_2^TS_2 - m_2\theta_2^* - \frac{q_2}{\beta_2}\theta_2^{*3}, \quad (33)$$

$$\dot{\eta}_2^* = \frac{\gamma_2}{2b_2}z_2^2 - r_2\eta_2^* - \frac{s_2}{\gamma_2}\eta_2^{*3}, \quad (34)$$

where m_2, q_2, r_2, s_2, K_2 and L_2 are positive design parameters.

Recalling the inequality in Lemma 2, one has

$$|z_2|^{\frac{3}{2}} = (z_2^2)^{\frac{3}{4}} \times 1^{\frac{1}{4}}$$
$$\leqslant \frac{\frac{3}{4}}{1} \times \frac{4}{3}(z_2^2)^{\frac{3}{4}+\frac{1}{4}} + \frac{1}{4} \times \left(\frac{4}{3}\right)^{-\frac{3}{4}/\frac{1}{4}} \times 1^{\frac{3}{4}+\frac{1}{4}}$$
$$= z_2^2 + \frac{1}{4} \times \left(\frac{4}{3}\right)^{-3} = z_2^2 + \sigma_2, \tag{35}$$

where $\sigma_2 = \frac{1}{4} \times (\frac{4}{3})^{-3}$. Taking (32)–(35) into (31), yields

$$\dot{V}_2 \leqslant -\left(K_2 + \frac{1}{2}\right) z_2^2 - L_2 z_2^4 + m_2 \frac{\hat{\theta}_2 \theta_2^*}{\beta_2} + q_2 \frac{\hat{\theta}_2 \theta_2^{*3}}{\beta_2^2}$$
$$+ r_2 \frac{\hat{\eta}_2 \eta_2^*}{\gamma_2} + s_2 \frac{\hat{\eta}_2 \eta_2^{*3}}{\gamma_2^2}$$
$$+ \frac{1}{2} z_3^2 + K_2 |z_2|^{\frac{3}{2}} - K_2 |z_2|^{\frac{3}{2}} + \frac{a_2}{2} + \frac{b_2}{2} + \dot{V}_1$$
$$\leqslant -K_1 \ln \chi - L_1 \ln^2 \chi - K_2 |z_2|^{\frac{3}{2}} - L_2 z_2^4$$
$$+ \sum_{i=1}^{2} m_i \frac{\hat{\theta}_i \theta_i^*}{\beta_i} + \sum_{i=1}^{2} q_i \frac{\hat{\theta}_i \theta_i^{*3}}{\beta_i^2} + \sum_{i=1}^{2} r_i \frac{\hat{\eta}_i \eta_i^*}{\gamma_i}$$
$$+ \sum_{i=1}^{2} s_i \frac{\hat{\eta}_i \eta_i^{*3}}{\gamma_i^2} + \sum_{i=1}^{2} \bar{\sigma}_i + \frac{1}{2} z_3^2, \tag{36}$$

where $\bar{\sigma}_2 = K_2 \sigma_2 + \frac{a_2}{2} + \frac{b_2}{2}$.

Step i ($3 \leqslant i \leqslant n-1$): Based on (8) and (9), differentiating z_i, we have

$$\dot{z}_i = \dot{e}_i - \dot{\alpha}_{i-1} + \dot{\lambda}_i$$
$$= e_{i+1}(t) + \tilde{f}_i(x,t) + \tilde{d}_i(x,t) - \dot{\alpha}_{i-1}$$
$$+ \lambda_{i+1} - \hat{p}_i \lambda_i - p_i \lambda_i^3.$$

Then consider the following Lyapounov function defined as

$$V_i = V_{i-1} + \frac{z_i^2}{2} + \frac{\hat{\theta}_i^{\,2}}{2\beta_i} + \frac{\hat{\eta}_i^{\,2}}{2\gamma_i},$$

where β_i and γ_i are the positive design parameters, and $\hat{\theta}_i = \theta_i - \theta_i^*$ and $\hat{\eta}_i = \eta_i - \eta_i^*$ are the estimation errors, θ_i^* and η_i^* represent the estimates of unknown constants θ_i and η_i, respectively.

Differentiating $V_i(t)$, yields that

$$\dot{V}_i = z_i \dot{z}_i + \frac{\hat{\theta}_i \dot{\hat{\theta}}_i}{\beta_i} + \frac{\hat{\eta}_i \dot{\hat{\eta}}_i}{\gamma_i} + \dot{V}_{i-1}$$
$$= z_i (e_{i+1}(t) + \tilde{f}_i(x,t) + \tilde{d}_i(x,t) - \dot{\alpha}_{i-1}$$
$$+ \lambda_{i+1} - \hat{p}_i \lambda_i - p_i \lambda_i^3) + \frac{\hat{\theta}_i \dot{\hat{\theta}}_i}{\beta_i} + \frac{\hat{\eta}_i \dot{\hat{\eta}}_i}{\gamma_i} + \dot{V}_{i-1}$$
$$= z_i (z_{i+1}(t) + F_i(Z_i) + \tilde{d}_i + \alpha_i) + \frac{\hat{\theta}_i \dot{\hat{\theta}}_i}{\beta_i}$$
$$+ \frac{\hat{\eta}_i \dot{\hat{\eta}}_i}{\gamma_i} + \dot{V}_{i-1}, \tag{37}$$

where $F_i(Z_i) = \tilde{f}_i(x,t) - \dot{\alpha}_{i-1} - \hat{p}_i \lambda_i - p_i \lambda_i^3$ and $Z_i = [x_1, ..., x_n, \theta_1^*, ..., \theta_i^*, \eta_1^*, ..., \eta_i^*, \lambda_2, ..., \lambda_i]^T$.

Similar as in (20), we immediately get

$$z_i F_i(Z_i) = z_i W_i^{*T} S_i(Z_i) + z_i \delta_i(Z_i)$$
$$\leqslant \frac{z_i^2}{2a_i} \theta_i S_i^T S_i + \frac{a_i}{2} + |z_i| \varepsilon_i, \tag{38}$$

where $\theta_i = \|W_i^{*T}\|^2$, the design parameter $a_i > 0$ and $\varepsilon_i > 0$ is a constant.

Inserting (38) into (37), we can get

$$\dot{V}_i \leqslant z_i (z_{i+1}(t) + \alpha_i) + \frac{z_i^2}{2a_i} \theta_i S_i^T S_i + \frac{a_i}{2} + \frac{\hat{\theta}_i \dot{\hat{\theta}}_i}{\beta_i}$$
$$+ \frac{\hat{\eta}_i \dot{\hat{\eta}}_i}{\gamma_i} + \frac{z_i^2}{2b_i} \eta_i + \frac{b_i}{2} + \dot{V}_{i-1}, \tag{39}$$

where $\eta_i = (\bar{d}_i + \varepsilon_i)^2$ and b_i is positive design parameter.

Next, we design the virtual controller α_i and the adaptive law as follows

$$\alpha_i = -\frac{1}{2a_i} z_i \theta_i^* S_i^T S_i - \frac{1}{2b_i} z_i \eta_i^*$$
$$- (K_i + 1) z_i - L_i z_i^3, \tag{40}$$
$$\dot{\theta}_i^* = \frac{\beta_i}{2a_i} z_i^2 S_i^T S_i - m_i \theta_i^* - \frac{q_i}{\beta_i} \theta_i^{*3}, \tag{41}$$
$$\dot{\eta}_i^* = \frac{\gamma_i}{2b_i} z_i^2 - r_i \eta_i^* - \frac{s_i}{\gamma_i} \eta_i^{*3}, \tag{42}$$

where $m_i > 0$, $q_i > 0$, $r_i > 0$, $s_i > 0$, $K_i > 0$ and $L_i > 0$ are design parameters.

Furthermore, holds (40)–(42) on the hand, one can rewrite (39) as

$$\dot{V}_i \leqslant -\left(K_i + \frac{1}{2}\right) z_i^2 - L_i z_i^4 + m_i \frac{\hat{\theta}_i \theta_i^*}{\beta_i} + q_i \frac{\hat{\theta}_i \theta_i^{*3}}{\beta_i^2}$$
$$+ r_i \frac{\hat{\eta}_i \eta_i^*}{\gamma_i} + s_i \frac{\hat{\eta}_i \eta_i^{*3}}{\gamma_i^2}$$

$$\begin{aligned}
&+\frac{1}{2}z_{i+1}^2+K_i|z_i|^{\frac{3}{2}}-K_i|z_i|^{\frac{3}{2}}+\frac{a_i}{2}+\frac{b_i}{2}+\dot{V}_{i-1}\\
&\leqslant -K_1\ln\chi-L_1\ln^2\chi-\sum_{j=2}^{i}K_j\left|z_j\right|^{\frac{3}{2}}\\
&-\sum_{j=2}^{i}L_jz_j^4+\sum_{j=1}^{i}m_j\frac{\hat{\theta}_j\theta_j^*}{\beta_j}\\
&+\sum_{j=1}^{i}q_j\frac{\hat{\theta}_j\theta_j^{*3}}{\beta_j^2}+\sum_{j=1}^{i}r_j\frac{\hat{\eta}_j\eta_j^*}{\gamma_j}\\
&+\sum_{j=1}^{i}s_j\frac{\hat{\eta}_j\eta_j^{*3}}{\gamma_j^2}+\sum_{j=1}^{i}\bar{\sigma}_j+\frac{1}{2}z_{i+1}^2, \qquad (43)
\end{aligned}$$

where $\bar{\sigma}_i=K_i\sigma_i+\frac{a_i}{2}+\frac{b_i}{2}$.

Step n: Noting the definition on (8) and (9), differentiating z_n, we deduce that

$$\begin{aligned}
\dot{z}_n&=\dot{e}_n-\dot{\alpha}_{n-1}+\dot{\lambda}_n\\
&=u(t-\tau(t))+\tilde{f}_n(x,t)+\tilde{d}_n(x,t)\\
&\quad-\dot{\alpha}_{n-1}-\hat{p}_n\lambda_n-p_n\lambda_n^3\\
&\quad+u(t)-u(t-\hat{\tau})-sign(\lambda_n)\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds.
\end{aligned}$$

Consider the Lyapounov function candidate as

$$V_n=V_{n-1}+\frac{z_n^2}{2}+\frac{\hat{\theta_n}^2}{2\beta_n}+\frac{\hat{\eta_n}^2}{2\gamma_n}, \qquad (44)$$

where the design parameters $\beta_n>0$ and $\gamma_n>0$, and θ_n^*, η_n^* denote the estimation of unknown constants θ_n and η_n, respectively, and $\hat{\eta_n}=\eta_n-\eta_n^*$, $\hat{\theta_n}=\theta_n-\theta_n^*$ mean the estimation errors.

Furthermore, the time derivative of $V_n(t)$ can be calculated as follows:

$$\begin{aligned}
\dot{V}_n&=z_n\dot{z}_n+\frac{\hat{\theta}_n\dot{\hat{\theta}}_n}{\beta_n}+\frac{\hat{\eta}_n\dot{\hat{\eta}}_n}{\gamma_n}+\dot{V}_{n-1}\\
&=z_n(u(t-\tau(t))+\tilde{f}_n(x,t)+\tilde{d}_n(x,t)\\
&\quad-\dot{\alpha}_{n-1}-\hat{p}_n\lambda_n-p_n\lambda_n^3+u(t)\\
&\quad-u(t-\hat{\tau})-sign(\lambda_n)\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds)\\
&\quad+\frac{\hat{\theta}_n\dot{\hat{\theta}}_n}{\beta_n}+\frac{\hat{\eta}_n\dot{\hat{\eta}}_n}{\gamma_n}+\dot{V}_{n-1}\\
&=z_n(u(t)+u(t-\tau(t))-u(t-\hat{\tau})-sign(z_n)\\
&\quad\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds+F_n(Z_n)+\tilde{d}_n)\\
&\quad+\frac{\hat{\theta}_n\dot{\hat{\theta}}_n}{\beta_n}+\frac{\hat{\eta}_n\dot{\hat{\eta}}_n}{\gamma_n}+\dot{V}_{n-1}, \qquad (45)
\end{aligned}$$

where $F_n(Z_n)=\tilde{f}_n(x,t)-\dot{\alpha}_{n-1}-\hat{p}_n\lambda_n-p_n\lambda_n^3+(sign(z_n)-sign(\lambda_n))\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds$ and $Z_n=[x_1,...,x_n,\theta_1^*,...,\theta_n^*,\eta_1^*,...,\eta_n^*,\lambda_2,...,\lambda_n]^T$.

Following the similar calculation process as in (20), we infer that

$$\begin{aligned}
z_nF_n(Z_n)=&z_nW_n^{*T}S_n(Z_n)+z_n\delta_n(Z_n)\\
&\leqslant\frac{z_n^2}{2a_n}\theta_nS_n^TS_n+\frac{a_n}{2}+|z_n|\,\varepsilon_n, \qquad (46)
\end{aligned}$$

where the design parameters a_n and ε_n are positive constants, and $\vartheta_n=\left\|W_n^{*T}\right\|^2$. Next, we estimate the first few terms in (45), and by utilizing Assumption 4, we can derive that

$$\begin{aligned}
&z_n(u(t)+u(t-\tau(t))-u(t-\hat{\tau})-sign(z_n)\\
&\quad\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds)\\
&\leqslant z_n\left(\int_{t-\hat{\tau}}^{t-\tau(t)}\dot{u}(s)ds-sign(z_n\right)\\
&\quad\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds+u(t))\\
&\leqslant z_nu(t)+|z_n|\left|\int_{t-\hat{\tau}}^{t-\tau(t)}\dot{u}(s)ds\right|\\
&\quad-|z_n|\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds\\
&\leqslant z_nu(t), \qquad (47)
\end{aligned}$$

where $sign(\cdot)=\begin{cases}1,\cdot>0,\\0,\cdot=0,\\-1,\cdot<0.\end{cases}$ Substituting (46)–(47) into (45), yields

$$\begin{aligned}
\dot{V}_n\leqslant& z_nu(t)+\frac{z_n^2}{2a_n}\theta_nS_n^TS_n+\frac{a_n}{2}+\frac{\hat{\theta}_n\dot{\hat{\theta}}_n}{\beta_n}+\frac{\hat{\eta}_n\dot{\hat{\eta}}_n}{\gamma_n}\\
&+\dot{V}_{n-1}+\frac{z_n^2}{2b_n}\eta_n+\frac{b_n}{2}, \qquad (48)
\end{aligned}$$

where $\eta_n = (\bar{d}_n + \varepsilon_n)^2$ and $b_n > 0$ is design parameter.

Then, the actual controller u and the adaptive law are selected as

$$u = -\frac{1}{2a_n} z_n \theta_n^* S_n^T S_n - \frac{1}{2b_n} z_n \eta_n^* - \left(K_n + \frac{1}{2}\right) z_n - L_n z_n^3, \tag{49}$$

$$\dot{\theta}_n^* = \frac{\beta_n}{2a_n} z_n^2 S_n^T S_n - m_n \theta_n^* - \frac{q_n}{\beta_n} \theta_n^{*3}, \tag{50}$$

$$\dot{\eta}_n^* = \frac{\gamma_n}{2b_n} z_n^2 - r_n \eta_n^* - \frac{s_n}{\gamma_n} \eta_n^{*3}, \tag{51}$$

where $m_n > 0$, $q_n > 0$, $r_n > 0$, $s_n > 0$, $K_n > 0$ and $L_n > 0$ are design parameters.

In view of (49)–(51), we arrive at

$$\begin{aligned}
\dot{V}_n \leqslant & -\left(K_n + \frac{1}{2}\right) z_n^2 - L_n z_n^4 + m_n \frac{\hat{\theta}_n \theta_n^*}{\beta_n} + q_n \frac{\hat{\theta}_n \theta_n^{*3}}{\beta_n^2} \\
& + r_n \frac{\hat{\eta}_n \eta_n^*}{\gamma_n} + s_n \frac{\hat{\eta}_n \eta_n^{*3}}{\gamma_n^2} \\
& + K_n |z_n|^{\frac{3}{2}} - K_n |z_n|^{\frac{3}{2}} + \frac{a_n}{2} + \frac{b_n}{2} + \dot{V}_{n-1} \\
\leqslant & -K_1 \ln \chi - L_1 \ln^2 \chi - \sum_{j=2}^{n} K_j |z_j|^{\frac{3}{2}} - \sum_{j=2}^{n} L_j z_j^4 \\
& + \sum_{j=1}^{n} m_j \frac{\hat{\theta}_j \theta_j^*}{\beta_j} + \sum_{j=1}^{n} q_j \frac{\hat{\theta}_j \theta_j^{*3}}{\beta_j^2} + \sum_{j=1}^{n} r_j \frac{\hat{\eta}_j \eta_j^*}{\gamma_j} \\
& + \sum_{j=1}^{n} s_j \frac{\hat{\eta}_j \eta_j^{*3}}{\gamma_j^2} + \sum_{j=1}^{n} \bar{\sigma}_j,
\end{aligned} \tag{52}$$

where $\bar{\sigma}_n = K_n \sigma_n + \frac{a_n}{2} + \frac{b_n}{2}$.

3.2 Stability analysis

Now we are in the position to the proof of Theorem 1.

Theorem 1 *Suppose that Assumptions 1–4 hold. For systems (1) with unknown time-varying input delay and disturbance, an adaptive controller (49) coupled with the virtual controller (22), (32), (40) and the adaptive laws (23), (24), (33), (34), (41), (42), (50) and (51) exist. By appropriately choosing the function $\mu(t)$ to meet Definition 2, the following assertions hold for any bounded initial states:*

1. *All the closed-loop signals of the system are semi-globally fixed-time uniformly ultimately bounded;*
2. *The tracking error $r(t)$ will remain within a small bounded range of the origin in finite time if suitable design parameters are chosen;*
3. *The tracking error $r(t)$ can be guided to a predetermined accuracy set $\Theta_r = \{r(t) \in R \ \ ||r(t)| < \varepsilon\}$ within a specified finite time T_k, where both ε and T_k can be pre-set, ensuring the transient performance of $r(t)$.*

Proof First, let us prove the first two assertions. Notice that for any $1 \leqslant i \leqslant n$, one has

$$\hat{\theta}_i \theta_i^* \leqslant \frac{1}{2}\theta_i^2 - \frac{1}{2}\hat{\theta}_i^2, \quad \hat{\eta}_i \eta_i^* \leqslant \frac{1}{2}\eta_i^2 - \frac{1}{2}\hat{\eta}_i^2.$$

Then, by differentiating $V_n(t)$ with respect to t, we can derive that

$$\begin{aligned}
\dot{V}_n \leqslant & -K_1 \ln \chi - L_1 \ln^2 \chi - \sum_{j=2}^{n} K_j |z_j|^{\frac{3}{2}} \\
& - \sum_{j=2}^{n} L_j z_j^4 + \sum_{j=1}^{n} m_j \frac{\hat{\theta}_j \theta_j^*}{\beta_j} \\
& + \sum_{j=1}^{n} q_j \frac{\hat{\theta}_j \theta_j^{*3}}{\beta_j^2} + \sum_{j=1}^{n} r_j \frac{\hat{\eta}_j \eta_j^*}{\gamma_j} \\
& + \sum_{j=1}^{n} s_j \frac{\hat{\eta}_j \eta_j^{*3}}{\gamma_j^2} + \sum_{j=1}^{n} \bar{\sigma}_j \\
\leqslant & -K_1 \ln \chi - L_1 \ln^2 \chi - \sum_{j=2}^{n} K_j |z_j|^{\frac{3}{2}} - \sum_{j=2}^{n} L_j z_j^4 \\
& - \sum_{j=1}^{n} \frac{m_j}{2\beta_j} \hat{\theta}_j^2 - \sum_{j=1}^{n} \frac{q_j}{2\gamma_j} \hat{\eta}_j^2 + \sum_{j=1}^{n} r_j \frac{\hat{\theta}_j \theta_j^{*3}}{\beta_j^2} \\
& + \sum_{j=1}^{n} s_j \frac{\hat{\eta}_j \eta_j^{*3}}{\gamma_j^2} + \sum_{j=1}^{n} \bar{\sigma}_j \\
& + \sum_{j=1}^{n} \frac{m_j}{2\beta_j} \theta_j^2 + \sum_{j=1}^{n} \frac{q_j}{2\gamma_j} \eta_j^2 \\
\leqslant & -K_1 \ln \chi - L_1 \ln^2 \chi - \sum_{j=2}^{n} K_j (z_j^2)^{\frac{3}{4}} - \sum_{j=2}^{n} L_j (z_j^2)^2 \\
& - \left(\sum_{j=1}^{n} \frac{m_j}{2\beta_j} \hat{\theta}_j^2\right)^{\frac{3}{4}} + \left(\sum_{j=1}^{n} \frac{m_j}{2\beta_j} \hat{\theta}_j^2\right)^{\frac{3}{4}}
\end{aligned}$$

$$
\begin{aligned}
&-\left(\sum_{j=1}^{n}\frac{q_j}{2\gamma_j}\hat{\eta}_j^2\right)^{\frac{3}{4}}+\left(\sum_{j=1}^{n}\frac{q_j}{2\gamma_j}\hat{\eta}_j^2\right)^{\frac{3}{4}}\\
&-\sum_{j=1}^{n}\frac{m_j}{2\beta_j}\hat{\theta}_j^2-\sum_{j=1}^{n}\frac{q_j}{2\gamma_j}\hat{\eta}_j^2+\sum_{j=1}^{n}r_j\frac{\hat{\theta}_j\theta_j^{*3}}{\beta_j^2}\\
&+\sum_{j=1}^{n}s_j\frac{\hat{\eta}_j\eta_j^{*3}}{\gamma_j^2}+\sum_{j=1}^{n}\bar{\sigma}_j\\
&+\sum_{j=1}^{n}\frac{m_j}{2\beta_j}\theta_j^2+\sum_{j=1}^{n}\frac{q_j}{2\gamma_j}\eta_j^2. \qquad (53)
\end{aligned}
$$

In addition, with the help of Lemma 2, one gets

$$
\left(\sum_{j=1}^{n}\frac{m_j}{2\beta_j}\hat{\theta}_j^2\right)^{\frac{3}{4}} \leqslant \sum_{j=1}^{n}\frac{m_j}{2\beta_j}\hat{\theta}_j^2+\frac{1}{4}\times\left(\frac{3}{4}\right)^3, \qquad (54)
$$

$$
\left(\sum_{j=1}^{n}\frac{q_j}{2\gamma_j}\hat{\eta}_j^2\right)^{\frac{3}{4}} \leqslant \sum_{j=1}^{n}\frac{q_j}{2\gamma_j}\hat{\eta}_j^2+\frac{1}{4}\times\left(\frac{3}{4}\right)^3, \qquad (55)
$$

$$
-K_1\ln\chi \leqslant -K_1\ln^{\frac{3}{4}}\chi+\frac{K_1}{4}\times\left(\frac{3}{4}\right)^3. \qquad (56)
$$

Moreover, utilizing Young's inequality, we can deduce that

$$
\begin{aligned}
\hat{\theta}_j\theta_j^{*3} &\leqslant -\hat{\theta}_j^4+3\hat{\theta}_j^3\theta_j-3\hat{\theta}_j^2\theta_j^2+\hat{\theta}_j\theta_j^3\\
&\leqslant -\hat{\theta}_j^4-3\hat{\theta}_j^2\theta_j^2+\frac{9w^{\frac{4}{3}}}{4}\hat{\theta}_j^4\\
&\quad+\frac{3}{4w^4}\theta_j^4+3\hat{\theta}_j^2\theta_j^2+\frac{1}{12}\theta_j^4\\
&\leqslant -\left(1-\frac{9w^{\frac{4}{3}}}{4}\right)\hat{\theta}_j^4+\left(\frac{3}{4w^4}+\frac{1}{12}\right)\theta_j^4, \qquad (57)
\end{aligned}
$$

$$
\hat{\eta}_j\eta_j^{*3} \leqslant -\left(1-\frac{9v^{\frac{4}{3}}}{4}\right)\hat{\eta}_j^4+\left(\frac{3}{4v^4}+\frac{1}{12}\right)\eta_j^4, \qquad (58)
$$

where w_j and v_j are positive design parameters.

Inserting (54)–(58) into (53), yields

$$
\begin{aligned}
\dot{V}_n &\leqslant -K_1\ln^{\frac{3}{4}}\chi-L_1\ln^2\chi-\sum_{j=2}^{n}K_j(z_j^2)^{\frac{3}{4}}\\
&\quad-\sum_{j=2}^{n}L_j(z_j^2)^2-\left(\sum_{j=1}^{n}m_j\frac{\hat{\theta}_j^2}{2\beta_j}\right)^{\frac{3}{4}}\\
&\quad-\left(\sum_{j=1}^{n}q_j\frac{\hat{\eta}_j^2}{2\beta_j}\right)^{\frac{3}{4}}-\sum_{j=1}^{n}r_j\left(1-\frac{9w^{\frac{4}{3}}}{4}\right)\frac{\hat{\theta}_j^4}{\beta_j^2}\\
&\quad-\sum_{j=1}^{n}s_j\left(1-\frac{9v^{\frac{4}{3}}}{4}\right)\frac{\hat{\eta}_j^4}{\gamma_j^2}+\sum_{j=1}^{n}\bar{\sigma}_j\\
&\quad+\sum_{j=1}^{n}r_j\left(\frac{3}{4w^4}+\frac{1}{12}\right)\frac{\theta_j^4}{\beta_j^2}\\
&\quad+\sum_{j=1}^{n}s_j\left(\frac{3}{4v^4}+\frac{1}{12}\right)\frac{\eta_j^4}{\gamma_j^2}+\sum_{j=1}^{n}\frac{m_j}{2\beta_j}\theta_j^2\\
&\quad+\sum_{j=1}^{n}\frac{q_j}{2\gamma_j}\eta_j^2+\frac{1}{2}\times\left(\frac{3}{4}\right)^3+\frac{K_1}{4}\times\left(\frac{3}{4}\right)^3\\
&\leqslant -H_1\left(\frac{\ln\chi}{2a_0}\right)^{\frac{3}{4}}-H_1\sum_{j=2}^{n}\left(\frac{z_j^2}{2}\right)^{\frac{3}{4}}-H_1\left(\sum_{j=1}^{n}\frac{\hat{\theta}_j^2}{2\beta_j}\right)^{\frac{3}{4}}\\
&\quad-H_1\left(\sum_{j=1}^{n}\frac{\hat{\eta}_j^2}{2\beta_j}\right)^{\frac{3}{4}}-H_2\left(\frac{\ln\chi}{2a_0}\right)^2-H_2\sum_{j=2}^{n}\left(\frac{z_j^2}{2}\right)^2\\
&\quad-H_2\sum_{j=1}^{n}\left(\frac{\hat{\theta}_j^2}{\beta_j}\right)^2-H_2\sum_{j=1}^{n}\left(\frac{\hat{\eta}_j^2}{\gamma_j}\right)^2+\mathcal{F}, \qquad (59)
\end{aligned}
$$

where $H_1 := \min_{1\leqslant j\leqslant n}\{(2a_0)^{\frac{3}{4}}K_1, 2^{\frac{3}{4}}K_j, m_j^{\frac{3}{4}}, q_j^{\frac{3}{4}}\}$, $H_2 := \min_{1\leqslant j\leqslant n}\{4a_0^2L_1, 4L_j, r_j(1-\frac{9w^{\frac{4}{3}}}{4}), s_j(1-\frac{9v^{\frac{4}{3}}}{4})\}$ and $\mathcal{F} = \sum_{j=1}^{n}\bar{\sigma}_j+\sum_{j=1}^{n}(\frac{3}{4w^4}+\frac{1}{12})\frac{\theta_j^4}{\beta_j^2}+\sum_{j=1}^{n}(\frac{3}{4v^4}+\frac{1}{12})\frac{\eta_j^4}{\gamma_j^2}+\sum_{j=1}^{n}\frac{1}{2\beta_j}\theta_j^2+\sum_{j=1}^{n}\frac{1}{2\gamma_j}\eta_j^2+\frac{1}{2}\times(\frac{3}{4})^3+\frac{K_1}{4}\times(\frac{3}{4})^3$.

By using Lemmas 3 and 4, we conclude that

$$
\begin{aligned}
\dot{V}_n &\leqslant -H_1\left(\left(\frac{\ln\chi}{2a_0}\right)^{\frac{3}{4}}+\left(\sum_{j=1}^{n}\frac{z_j^2}{2}\right)^{\frac{3}{4}}\right.\\
&\quad\left.+\left(\sum_{j=1}^{n}\frac{\hat{\theta}_j^2}{2\beta_j}\right)^{\frac{3}{4}}+\left(\sum_{j=1}^{n}\frac{\hat{\eta}_j^2}{2\beta_j}\right)^{\frac{3}{4}}\right)\\
&\quad-\frac{H_2}{n}\left(\left(\frac{\ln\chi}{2a_0}\right)^2+\left(\sum_{j=1}^{n}\frac{z_j^2}{2}\right)^2+\left(\sum_{j=1}^{n}\frac{\hat{\theta}_j^2}{\beta_j}\right)^2\right.\\
&\quad\left.+\left(\sum_{j=1}^{n}\frac{\hat{\eta}_j^2}{\gamma_j}\right)^2\right)+\mathcal{F}
\end{aligned}
$$

$$\leqslant -H_1\left(\frac{\ln\chi}{2a_0}+\sum_{j=1}^{n}\frac{z_j^2}{2}+\sum_{j=1}^{n}\frac{\hat{\theta}_j^2}{2\beta_j}+\sum_{j=1}^{n}\frac{\hat{\eta}_j^2}{2\beta_j}\right)^{\frac{3}{4}}-\frac{H_2}{n}\left(\frac{\ln\chi}{2a_0}+\sum_{j=1}^{n}\frac{z_j^2}{2}+\sum_{j=1}^{n}\frac{\hat{\theta}_j^2}{\beta_j}+\sum_{j=1}^{n}\frac{\hat{\eta}_j^2}{\gamma_j}\right)^2+\mathcal{F}$$
$$\leqslant -H_1V_n-(1-\varepsilon_1)\frac{H_2}{n}V_n-\varepsilon_1\frac{H_2}{n}V_n+\mathcal{F}, \tag{60}$$

where $0<\varepsilon_1<1$. Therefore, according to inequality (60), for $V_n\leqslant n\mathcal{F}/H_2$, we have the boundedness of V_n. Furthermore, if $V_n>n\mathcal{F}/H_2$, one can obtain

$$\dot{V}_n\leqslant -\sigma V_n^{\frac{3}{4}}-\nu V_n^2,$$

where $\sigma=H_1$ and $\nu=(1-\varepsilon_1)\frac{H_2}{n}$. Thus, according to Lemma 1, V_n is bounded, and the settling time can be set as

$$T\leqslant T_m=\frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\frac{4}{5}\sigma}\left(\frac{\sigma}{\nu}\right)^{\frac{4}{5}}. \tag{61}$$

Review the expression for V_n, we have the boundedness of z_i, $\hat{\theta}_i$ and $\hat{\eta}_i$ for $i=1,2,...,n$. And owing to the initial value $z_1(0)=\mu(0)r(0)=0<1$, which means that for all $t\geqslant 0$, $z_1(t)$ strictly within the set $\Omega_{z_1}=\{z_1(t)\in R\,||z_1(t)|<1\}$. As $r(0)$ is bounded and $\mu(t)>0$ for $t>0$, $r(t)=z_1(t)/\mu(t)$ is well defined and bounded. Hence, χ and the virtual controller α_i are bounded. Moreover, the estimated parameters $\theta_i^*=\theta_i-\hat{\theta}_i$ and $\eta_i^*=\eta_i-\hat{\eta}_i$ are bounded due to the boundedness of $\theta_i,\hat{\theta}_i,\eta_i,\hat{\eta}_i$ for $i=1,2,...,n$. Besides, since u is made up of bounded signals, we get the boundedness of u.

Next, we will prove the fixed-time boundedness of the compensator system λ_i. To begin with, consider the following Lyapounov function as

$$V_\lambda=\sum_{i=2}^{n}\frac{\lambda_i^2}{2}. \tag{62}$$

Recalling the definition in (9), we can infer that

$$\dot{V}_\lambda=\sum_{i=2}^{n}\lambda_i\dot{\lambda}_i$$
$$\leqslant -\left(\hat{p}_2-\frac{1}{2}\right)\lambda_2^2-\sum_{i=3}^{n-1}(\hat{p}_i-1)\lambda_i^2-\left(\hat{p}_n-\frac{1}{2}\right)\lambda_n^2-\sum_{i=2}^{n}p_i\lambda_i^4+\lambda_n(u(t)-u(t-\hat{\tau})-sign(\lambda_n)\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds). \tag{63}$$

Furthermore, based on the boundedness of u, we have the following fact

$$\lambda_n(u(t)-u(t-\hat{\tau})-sign(\lambda_n)\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds)$$
$$\leqslant\lambda_n(u(t)-u(t-\hat{\tau}))-|\lambda_n|\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}}|\dot{u}(s)|ds$$
$$\leqslant\lambda_n h_u\leqslant\frac{1}{4\epsilon_2}\lambda_n^2+\epsilon_2h_u^2, \tag{64}$$

where $h_u>0$ is a constant and ϵ_2 is a design parameter. Thus, putting (64) into (63), yields

$$\dot{V}_\lambda\leqslant -\left(\hat{p}_2-\frac{1}{2}\right)\lambda_2^2-\sum_{i=3}^{n-1}(\hat{p}_i-1)\lambda_i^2-\left(\hat{p}_n-\frac{1}{2}-\frac{1}{4\epsilon_2}\right)\lambda_n^2-\sum_{i=2}^{n}p_i\lambda_i^4+\epsilon_2h_u^2$$
$$\leqslant -P_1\left(\sum_{i=2}^{n}\lambda_i^2\right)^{\frac{3}{4}}-\frac{(1-\epsilon_1)P_2}{n}\left(\sum_{i=2}^{n}\lambda_i^2\right)^2-\frac{\epsilon_1P_2}{n}\left(\sum_{i=2}^{n}\lambda_i^2\right)^2+\zeta$$
$$\leqslant -\sigma_1\left(\sum_{i=2}^{n}\lambda_i^2\right)^{\frac{3}{4}}-\nu_1\left(\sum_{i=2}^{n}\lambda_i^2\right)^2, \tag{65}$$

where $\sigma_1=P_1=\min\limits_{3\leqslant i\leqslant n-1}\left\{\hat{p}_2-\frac{1}{2},\hat{p}_i-1,\hat{p}_n-\frac{1}{2}-\frac{1}{4\epsilon_2}\right\}$, $P_2=\min\limits_{2\leqslant i\leqslant n}\{p_i\}$, $\zeta=\epsilon_2h_u^2+\frac{nP_1}{4}\times(\frac{3}{4})^3$ and

$\nu_1 = \frac{(1-\epsilon_1)P_2}{n}$. Therefore, we immediately observe the fixed-time boundedness of compensation system, and the fixed time can be given as

$$T \leqslant T_m = \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\frac{4}{5}\sigma_1}\left(\frac{\sigma_1}{\nu_1}\right)^{\frac{4}{5}}. \tag{66}$$

Therefore, based on coordinate transformation (16), we immediately get the boundedness of e_i and x_i for $i = 1, 2, ..., n$. Then we get the conclusion that all the closed-loop signals of the system are semi-globally fixed-time uniformly ultimately bounded.

Next, we proceed to prove the third assertion. Building upon the previous analysis where we have established that $z_1 \in \Omega_{z_1}$, we now recall the definition and properties of $\mu(t)$ to obtain $|r(t)| < \frac{1}{\mu(t)} \leqslant \varepsilon$ for all $t > t_s$. This implies that the tracking error $r(t)$ can be guided to a pre-specified accuracy region $(-\varepsilon, \varepsilon)$ within the specified finite time $t_s \leqslant T_k$.

Remark 5 Due to the presence of unknown time delay in the system input, these delays cannot be explicitly expressed, meaning that direct introduction of approximations or upper bound of unknown delay to eliminate its impact is not feasible. Given the unknown nature of the delay, neither coordinate transformations nor delay compensation within the controller can be employed. Therefore, in this paper, compensation for the unknown delay is achieved by introducing an additional compensator system using an input u containing approximation and approximation error. Furthermore, the integral term ensures the boundedness of the compensator system in fixed-time. Particularly, the compensator system described in this paper can also be applied to scenarios with known input delay by replacing $u(t-\hat{\tau})$ and $sign(\lambda_n)\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}} |\dot{u}(s)|ds$ in the compensator system with $u(t-\tau(t))$, thus directly compensating for the effects of known input time delay.

Remark 6 The existing methods for nonlinear systems with input delay as in [28,41–44,57] were not applicable to address unknown delay. This paper, however, has obtained fixed-time control results through compensation systems and the construction of a novel L-K functional. In comparison to asymptotic control, fixed-time control offers rapid convergence, high precision, and independence from initial conditions. Additionally, the transient behavior constraint method designed here is applicable to more complex systems, as well as scenarios where reference signal information is not provided in advance or where the desired trajectories are generated by online planners and measurement devices. Typical applications include missile attitude control systems, spacecraft control [58], intelligent driving systems, and so on.

Remark 7 In the control method proposed in this paper, while ensuring globally specified transient performance, it eliminates most of the initial condition-dependent restrictions in most prescribed performance bound-based results and the requirement for any high-order ($n \geqslant 2$) derivatives of y_r, relaxing the conditions in related results such as [34–37].

3.3 Corollary

According to the proof process of the previous Theorem 1, even when saturation exists in the input delay of a nonlinear system, we can still draw conclusions regarding fixed-time control. Therefore, in this section, we introduce input saturation to the system (1) with unknown input delay, and consider the following system:

$$\begin{aligned}
\dot{e}_1(t) &= a_0 e_2(t) + \tilde{f}_1(x,t) + \tilde{d}_1(x,t), \\
\dot{e}_i(t) &= e_{i+1}(t) + \tilde{f}_i(x,t) + \tilde{d}_i(x,t), \\
\dot{e}_n(t) &= S(u(t-\tau(t))) + \tilde{f}_n(x,t) + \tilde{d}_n(x,t), \\
y &= e_1,
\end{aligned} \tag{67}$$

where the definition of system variables is the same as in (8), and let the system saturation be

$$S(u-\tau(t)) = \begin{cases} \bar{u}, u(t-\tau(t)) > \bar{u}, \\ u(t-\tau(t)), -\tilde{u} \leqslant u(t-\tau(t)) \leqslant \bar{u}, \\ -\tilde{u}, u(t-\tau(t)) < -\tilde{u}, \end{cases} \tag{68}$$

where $\bar{u}$ and $\tilde{u}$ are positive known constants.

Assumption 5 [29,59] For given system input with saturation and time-varying delay, as well as the nonlinear system (67), there are feasible actual input control thresholds and the controller u, allowing the system output y to track the desired trajectory y_r.

First of all, owing to the presence of unknown time delay and input saturation in the system (67), the compensator system (9) will be modified as follows:

$$\dot{\lambda}_i = -\hat{p}_i\lambda_i - p_i\lambda_i^3 + \lambda_{i+1}, \text{ for } i = 2, ..., n-1$$
$$\dot{\lambda}_n = -\hat{p}_n\lambda_n - p_n\lambda_n^3 + u(t) - sign(w_n)(\bar{u} + \tilde{u}), \tag{69}$$

where $p_i > 0$ are designed parameters and the initial condition is $\lambda_i(0) = 0$.

Then, the coordinate transformations can be given as

$$w_1 = \mu(t)(e_1 - y_r),$$
$$w_i = e_i - \alpha_{i-1} + \lambda_i, \quad \text{for } i = 2, 3, ..., n, \tag{70}$$

where α_{i-1} is the virtual controller.

Similar to the design process in Sects. 3.1 and 3.2, we can also apply the backstepping method to obtain the controller design for the i-th step ($1 \leqslant i < n$). Therefore, for the sake of brevity and readability, we will not delve into the specific design process here. However, in the design of the n-th step, due to the consideration of input saturation in the system, unlike the previous design methods, we will focus on introducing the design process for step n.

Step n: Based on (67) and (69), differentiating w_n, one has

$$\begin{aligned}\dot{w}_n &= \dot{e}_n - \dot{\alpha}_{n-1} + \dot{\lambda}_n \\ &= S(u(t-\tau(t))) + \tilde{f}_n(x,t) + \tilde{d}_n(x,t) \\ &\quad - \dot{\alpha}_{n-1} - \hat{p}_n\lambda_n - p_n\lambda_n^3 + u(t) \\ &\quad - sign(w_n)(\bar{u} + \tilde{u}).\end{aligned}$$

Let the Lyapounov function be

$$V_n = V_{n-1} + \frac{w_n^2}{2} + \frac{\hat{\vartheta}_n^2}{2\varsigma_n} + \frac{\hat{\mu}_n^2}{2\xi_n}, \tag{71}$$

where $\varsigma_n > 0$ and $\xi_n > 0$ are the design parameters, $\hat{\vartheta}_n = \vartheta_n - \vartheta_n^*$, $\hat{\mu}_n = \mu_n - \mu_n^*$ denote the estimation errors and ϑ_n^*, μ_n^* represent the estimation of ϑ_n and μ_n, respectively.

Further, by differentiating $V_n(t)$ with respect to t, we can derive

$$\begin{aligned}\dot{V}_n &= w_n\dot{w}_n + \frac{\hat{\vartheta}_n\dot{\hat{\vartheta}}_n}{\varsigma_n} \\ &\quad + \frac{\hat{\mu}_n\dot{\hat{\mu}}_n}{\xi_n} + \dot{V}_{n-1} \\ &= w_n(S(u(t-\tau(t))) + \tilde{f}_n(x,t) + \tilde{d}_n(x,t) - \dot{\alpha}_{n-1} \\ &\quad - \hat{p}_n\lambda_n - p_n\lambda_n^3 \\ &\quad + u(t) - sign(w_n)(\bar{u} + \tilde{u})) + \frac{\hat{\vartheta}_n\dot{\hat{\vartheta}}_n}{\varsigma_n} \\ &\quad + \frac{\hat{\mu}_n\dot{\hat{\mu}}_n}{\xi_n} + \dot{V}_{n-1} \\ &= w_n(u(t) + S(u(t-\tau(t))) \\ &\quad - sign(w_n)(\bar{u} + \tilde{u}) + F_n(Z_n) \\ &\quad + \tilde{d}_n) + \frac{\hat{\vartheta}_n\dot{\hat{\vartheta}}_n}{\varsigma_n} + \frac{\hat{\mu}_n\dot{\hat{\mu}}_n}{\xi_n} + \dot{V}_{n-1},\end{aligned} \tag{72}$$

where $F_n(Z_n) = \tilde{f}_n(x,t) - \dot{\alpha}_{n-1} - \hat{p}_n\lambda_n - p_n\lambda_n^3$ and $Z_n = [x_1, ..., x_n, \vartheta_1^*, ..., \vartheta_n^*, \mu_1^*, ..., \mu_n^*, \lambda_2, ..., \lambda_n]^T$.

Then, similar as in (46), one obtains

$$\begin{aligned}w_n F_n(Z_n) =& w_n W_n^{*T} S_n(Z_n) + w_n\delta_n(Z_n) \\ \leqslant& \frac{w_n^2}{2a_n}\vartheta_n S_n^T S_n + \frac{a_n}{2} + |w_n|\,\varepsilon_n,\end{aligned} \tag{73}$$

where $\theta_n = \left\| W_n^{*T} \right\|^2$, the design parameter $a_n > 0$ and $\varepsilon_n > 0$ is constant. Next, we estimate the first few terms in (72), then by using Assumption 4, we can deduce that

$$\begin{aligned}&w_n(u(t) + S(u(t-\tau(t))) - sign(w_n)(\bar{u} + \tilde{u})) \\ &\leqslant w_n u(t).\end{aligned} \tag{74}$$

Remark 8 It is worth noting that we do not employ the method for handling unknown time delay as described in Sect. 3.1 here. This is because, in the presence of input saturation, it is challenging to construct integral terms associated with unknown time delay and requires a case-by-case discussion. However, fortunately, when input saturation exists, we can estimate the input with unknown time delay by utilizing the upper and lower bounds of the saturation existing in the system input.

Substituting the above inequalities into (72), yields

$$\begin{aligned}\dot{V}_n \leqslant& w_n u(t) + \frac{w_n^2}{2a_n}\vartheta_n S_n^T S_n + \frac{a_n}{2} + \frac{\hat{\vartheta}_n\dot{\hat{\vartheta}}_n}{\varsigma_n} \\ &+ \frac{\hat{\mu}_n\dot{\hat{\mu}}_n}{\xi_n} + |w_n|\,(\bar{d}_n + \varepsilon_n) + \dot{V}_{n-1}\end{aligned}$$

$$\leqslant w_n u(t) + \frac{w_n^2}{2a_n}\vartheta_n S_n^T S_n + \frac{a_n}{2} + \frac{\hat{\vartheta}_n \dot{\hat{\vartheta}}_n}{\varsigma_n} + \frac{\hat{\mu}_n \dot{\hat{\mu}}_n}{\xi_n} + \frac{w_n^2}{2b_n}\mu_n + \frac{b_n}{2} + \dot{V}_{n-1}, \quad (75)$$

where $\mu_n = (\bar{d}_n + \varepsilon_n)^2$ and $b_n > 0$ is design parameter.

Given the actual controller u and the adaptive law be designed as

$$u = -\frac{w_n}{2a_n}\vartheta_n^* S_n^T S_n - \frac{w_n}{2b_n}\mu_n^* - \left(K_n + \frac{1}{2}\right) w_n - L_n w_n^3, \quad (76)$$

$$\dot{\vartheta}_n^* = \frac{\varsigma_n w_n^2}{2a_n} S_n^T S_n - m_n \vartheta_n^* - \frac{q_n}{\varsigma_n}\vartheta_n^{*3}, \quad (77)$$

$$\dot{\mu}_n^* = \frac{\xi_n}{2b_n} w_n^2 - r_n \mu_n^* - \frac{s_n}{\xi_n}\mu_n^{*3}, \quad (78)$$

where the design parameters m_n, q_n, r_n, s_n, K_n and L_n are positive.

Together with (43) and (76)–(78), yields

$$\begin{aligned}
\dot{V}_n \leqslant & -\left(K_n + \frac{1}{2}\right) w_n^2 - L_n w_n^4 + m_n \frac{\hat{\vartheta}_n \vartheta_n^*}{\varsigma_n} + q_n \frac{\hat{\vartheta}_n \vartheta_n^{*3}}{\varsigma_n^2} \\
& + r_n \frac{\hat{\mu}_i \mu_n^*}{\xi_i} + s_n \frac{\hat{\mu}_i \mu_i^{*3}}{\xi_i^2} \\
& + K_n w_n^{\frac{3}{2}} - K_n w_n^{\frac{3}{2}} + \frac{a_n}{2} + \frac{b_n}{2} + \dot{V}_{n-1} \\
\leqslant & -K_1 \ln \chi - L_1 \ln^2 \chi - \sum_{j=2}^{n} K_j w_j^{\frac{3}{2}} \\
& - \sum_{i=2}^{n} L_j w_j^4 + \sum_{j=1}^{n} m_j \frac{\hat{\vartheta}_j \vartheta_j^*}{\varsigma_j} \\
& + \sum_{j=1}^{n} q_j \frac{\hat{\vartheta}_j \vartheta_j^{*3}}{\varsigma_j^2} + \sum_{j=1}^{n} r_j \frac{\hat{\mu}_j \mu_j^*}{\xi_j} \\
& + \sum_{j=1}^{n} s_j \frac{\hat{\mu}_j \mu_j^{*3}}{\xi_j^2} + \sum_{j=1}^{n} \bar{\sigma}_j, \quad (79)
\end{aligned}$$

where $\bar{\sigma}_n = K_n \sigma_n + \frac{a_n}{2} + \frac{b_n}{2}$.

In accordance with the above design and the proof of Theorem 1, we immediately obtain the following theorem:

Theorem 2 *Suppose that Assumptions 1–5 hold. For systems (67) with disturbance, unknown time-varying input delay and saturation, an adaptive controller (76) coupled with the virtual controller and the adaptive laws (77) and (78) exist, ensuring that all the closed-loop signals of the system are semi-globally uniformly ultimately bounded in fixed time. By choosing appropriate function $\mu(t)$ to satisfy Definition 2, for any bounded initial states, the following facts hold:*

1. *All the closed-loop signals of the system are semi-globally fixed-time uniformly ultimately bounded;*
2. *The tracking error $r(t)$ will remain within a small bounded range of the origin in finite time if suitable design parameters are chosen;*
3. *The tracking error $r(t)$ can be guided to a predetermined accuracy set $\Theta_r = \{r(t) \in R \,||r(t)| < \varepsilon\}$ within a specified finite time T_k, where both ε and T_k can be pre-set, ensuring the transient performance of $r(t)$.*

Proof Please refer to Theorem 1 for detail.

4 Numerical example

In this section, we will present the following simulation examples to demonstrate the performance of the designed controller. Furthermore, we will conduct comparisons to highlight the innovativeness and contribution of our research results.

4.1 Example 1

Consider the nonlinear system with input delay in [60] as follows

$$\begin{aligned}
\dot{x}_1(t) &= g_1 x_2(t) + f_1(x,t) + d_1(x,t), \\
\dot{x}_2(t) &= g_2 x_3(t) + f_2(x,t) + d_2(x,t), \\
\dot{x}_3(t) &= g_3 u(t - \tau(t)) + f_3(x,t) + d_3(x,t), \\
y &= x_1, \quad (80)
\end{aligned}$$

where $g_1 = 2$, $g_2 = g_3 = 1$, unknown function, disturbance are defined as $f_1 = sin(x_1 x_3)$, $f_2 = x_1^2 x_3 e^{x_2}$, $f_3 = x_1 x_2 e^{x_3}$, $d_1 = x_3$, $d_2 = 0$, $d_3 = x_3 sin(x_1 x_2)$, respectively. To facilitate a direct comparison with the literature [60] under identical conditions, we will simulate based on the design procedures outlined in [60]. Then the initial conditions are selected as $[x_1(0), x_2(0), x_3(0)]^T = [0.01, 0, 0]^T$, and the desired trajectory $y_r(t)$ is chosen as $y_r =$

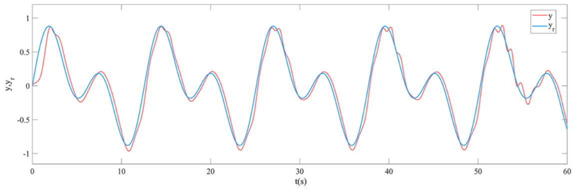

Fig. 2 The trajectory of y and y_r in [60] when $\tau = 0.02$

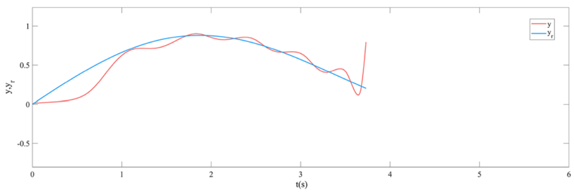

Fig. 3 The trajectory of y and y_r in [60] when $\tau = 0.03$

$0.5sint + 0.5sin(0.5t)$. The performance of the system output, based on the parameters and controller design from [60], is illustrated in Figs. 2, 3.

It is evident from Figs. 2, 3 that as the input delay increases, the controller described in the aforementioned literature may exhibit growth and oscillations. Consequently, under large input delay ($\tau \geqslant 0.03$), the state of the nonlinear system may experience an unbounded scenario. Next, we will use the scheme proposed in this paper for numerical simulation. Firstly, the compensation system is given by

$$\begin{aligned}
\dot{\lambda}_2 &= -\hat{p}_2\lambda_2 - p_2\lambda_2^3 + \lambda_3, \\
\dot{\lambda}_3 &= -\hat{p}_3\lambda_3 - p_3\lambda_3^3 + u(t) - u(t-\hat{\tau}) \\
&\quad - sign(\lambda_n)\int_{t-\hat{\tau}}^{t-\hat{\tau}+\bar{\tau}} |\dot{u}(s)|ds. \qquad (81)
\end{aligned}$$

The design parameters are set as $m_1 = 10, m_2 = m_3 = 20$, $q_1 = 4$, $q_2 = q_3 = 16$, $r_1 = r_2 = 4$, $r_3 = 4$, $s_1 = 40$, $s_2 = 20$, $s_3 = 10$, $a_1 = 1$, $a_2 = a_3 = 5$, $b_1 = b_2 = 1$, $b_3 = 0.5$, $\beta_1 = 2$, $\beta_2 = \beta_3 = 8$, $K_1 = K_2 = 0.5$, $K_3 = 9$, $L_1 = 0.2$, $L_2 = 5$, $L_3 = 3$, $\gamma_1 = 20$, $\gamma_2 = 10$, $\gamma_3 = 5$, $p_2 = 2$, $\hat{p}_2 = 5$, $p_3 = 1.5$, $\hat{p}_3 = 2$, $\hat{\tau} = \bar{\tau} = 0.04$, $T_k = 4$ seconds, $\varepsilon = 0.18$. And the initial condition are select as $[\hat{\theta}_1(0), \hat{\theta}_2(0), \hat{\theta}_3(0)]^T = [\hat{\eta}_1(0), \hat{\eta}_2(0), \hat{\eta}_3(0)]^T = [0, 0, 0]^T$. Then, the results of the numerical simulation are shown in the pictures below.

It can be observed from Figs. 4 and 5 that our designed adaptive tracking controllers, whether the input delay is constant or time-varying, are capable of effectively tracking the desired trajectory y_r with minimal error, demonstrating excellent tracking performance. Furthermore, we can observe from Fig. 6 that when we set $T_k = 4$ and $\varepsilon = 0.18$, the system error $r(t)$ can satisfy the predetermined transient performance criteria .

The operational trajectories of the compensation system λ_2, λ_3, the adaptive laws $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ and $\hat{\eta}_1$, $\hat{\eta}_2$, $\hat{\eta}_3$ and system state x_2, x_3, input u are depicted in Figs. 7, 8, 9 and 10, demonstrating that all signals of the closed-loop system are bounded.

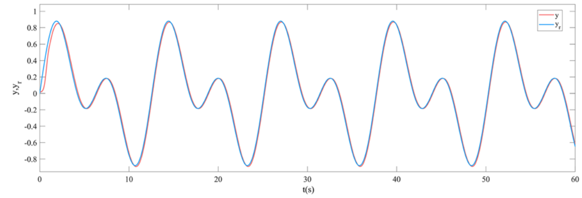

Fig. 4 Output y and desired trajectory y_r when $\tau = 0.04$

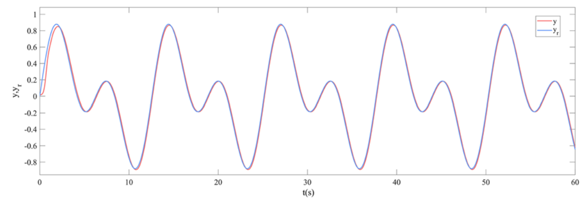

Fig. 5 Output y and desired trajectory y_r when $\tau = 0.03 + 0.02sint$

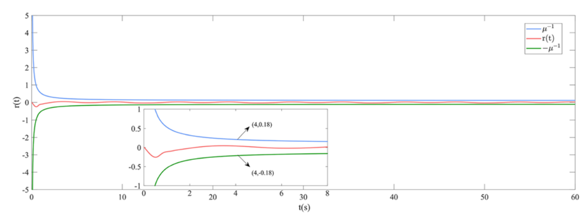

Fig. 6 The trajectory of tracking error $r(t)$

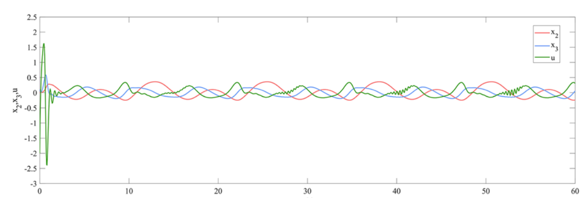

Fig. 7 The trajectories of state x_2, x_3 and input u

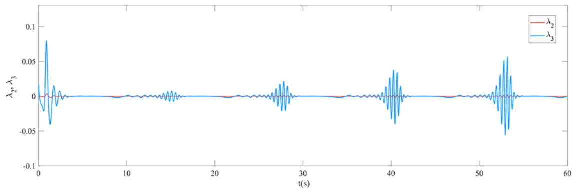

Fig. 8 The trajectory of compensation system λ_2 and λ_3

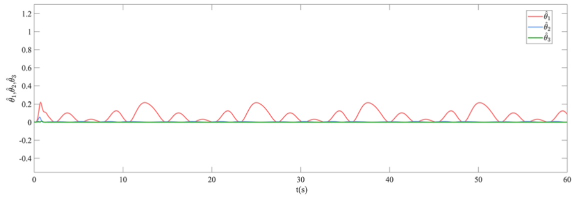

Fig. 9 The trajectory of adaptive parameters $\hat{\theta}_1$, $\hat{\theta}_2$ and $\hat{\theta}_3$

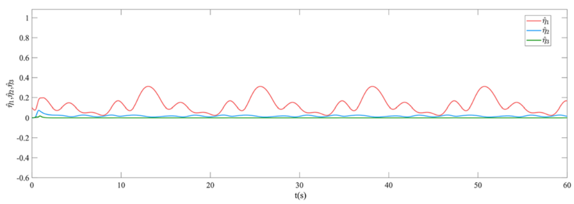

Fig. 10 The trajectory of adaptive parameters $\hat{\eta}_1$, $\hat{\eta}_2$ and $\hat{\eta}_3$

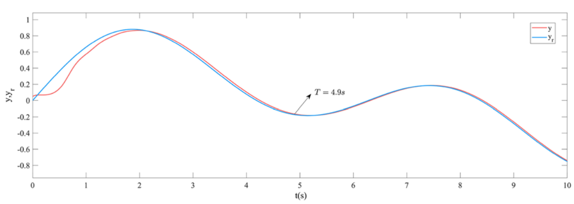

Fig. 11 The trajectory of $[x_1(0), x_2(0), , x_3(0)] = [0.05, 0.1, 0.1]$

Then, setting $w^{\frac{4}{3}} = v^{\frac{4}{3}} = \frac{2}{9}$, $\varepsilon_1 = \frac{1}{4}$, we have $\sigma = 2^{-\frac{1}{4}}$ and $\nu = 1.25$. By reviewing (61), we can calculate the setting time T_m as

$$T_m = \frac{\Gamma\left(\frac{1}{5}\right)\Gamma\left(\frac{4}{5}\right)}{\frac{4}{5}\sigma}\left(\frac{\sigma}{\nu}\right)^{\frac{4}{5}} \approx 4.9(s).$$

To demonstrate that our designed controller can achieve fixed-time tracking control, we now present the tracking trajectories for initial values $[x_1(0), x_2(0), x_3(0)]$ set at $[0.05, 0.1, 0.1]$, $[0.5, 0.6, 0.2]$, and $[0.7, 0.2, 0.3]$ in Figs. 11, 12 and 13. The time required for achieving bounded tracking will not exceed the settling time T_m .

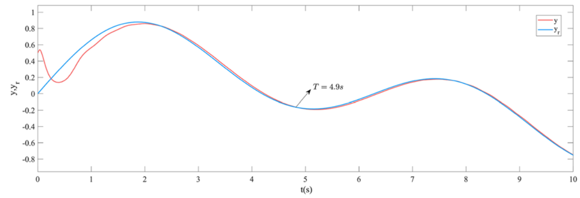

Fig. 12 The trajectory of $[x_1(0), x_2(0)] = [0.5, 0.6, 0.2]$

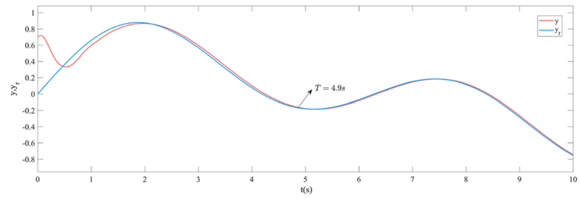

Fig. 13 The trajectory of $[x_1(0), x_2(0), x_3(0)] = [0.7, 0.2, 0.3]$

It is evident from the above three figures that as the initial value increases, the time T required for achieving bounded tracking also increases, yet all remain below 4.9 s, not exceeding the settling time T_m. This indicates that our controller can achieve the goal of fixed-time tracking control without depending on the initial value.

4.2 Example 2

Considering the following nonlinear system with input saturation

$$\begin{aligned}
\dot{x}_1(t) &= 0.5x_2(t) + 0.4sint, \\
\dot{x}_2(t) &= 0.75u(t - \tau(t)) + 1.2x_1x_2 + 0.5sin(x_2), \\
y &= x_1, \qquad (82)
\end{aligned}$$

with compensation system

$$\dot{\lambda}_2 = -\hat{p}_2\lambda_2 - p_2\lambda_2^3 + u(t) - sign(w_n)(\bar{u} + \tilde{u}). \qquad (83)$$

The design parameters are set as $m_1 = 2$, $m_2 = 2$, $q_1 = 12$, $q_2 = 4$, $r_1 = 2$, $r_2 = 4$, $s_1 = 20$, $s_2 = 40$, $a_1 = 25/3$, $a_2 = 2.5$, $b_1 = 1$, $b_2 = 2$, $\beta_1 = \beta_2 = 2$, $K_1 = 4$, $K_2 = 3$, $L_1 = 10$, $L_2 = 5$, $\gamma_1 = 10$, $\gamma_2 = 20$, $p_2 = \hat{p}_2 = 2$, $T_k = 3$ seconds, $\varepsilon = 0.1$. Let the initial condition are select as $[x_1(0), x_2(0)]^T = [0.05, 0.1]^T$, $[\hat{\theta}_1(0), \hat{\theta}_2(0)]^T = [0, 0.1]^T$ and $[\hat{\eta}_1(0), \hat{\eta}_2(0)]^T = [0.1, 0.05]^T$, $y_r = 0.5sint + 0.25sin(0.5t)$ and the saturation threshold set to $\bar{u} = 4.5$ and $\tilde{u} = 2.5$. The

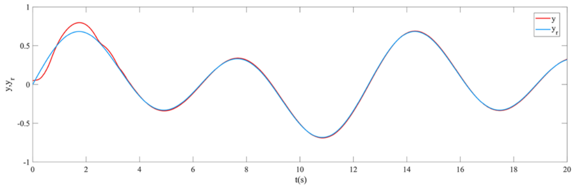

Fig. 14 Output y and desired trajectory y_r

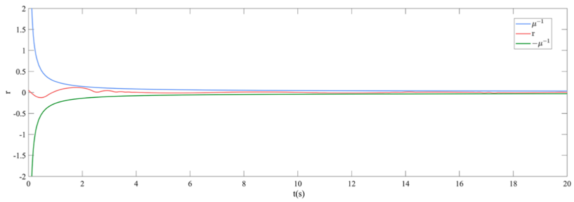

Fig. 15 The trajectory of tracking error $r(t)$

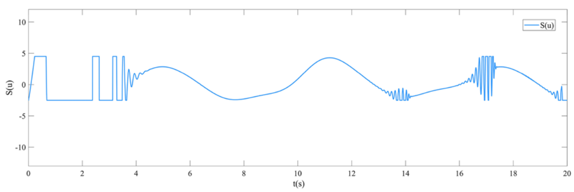

Fig. 16 The trajectory of system input u with saturation

numerical simulation results are shown in the following picture.

From Figs. 14, 15 and 16, it can be observed that our adaptive controller remains effective in the presence of unknown input delay and input saturation in nonlinear systems, as the output y can closely track the desired trajectory y_r. And the tracking error $r(t)$ satisfies the specified transient performance.

Therefore, the numerical simulation experiments above validate the effectiveness of the controller we have designed.

5 Conclusion

The study has investigated the fixed-time tracking control problem for uncertain nonlinear systems with unknown input delay, nonlinear function and external disturbances. Initially, the impact of unknown input delay have been mitigated by introducing a compensation system, and a specific funnel function has been constructed to constrain the transient performance of tracking error. Subsequently, new adaptive parameters have been introduced into the Lyapunov-Krasovskii functional to address unknown disturbances, along with a novel bounded estimation method and radial basis function neural network to handle system uncertainties. An adaptive controller is designed using backstepping method to demonstrate the fixed-time boundedness of all signals in the closed-loop system and ensure the transient behavior of tracking error. Finally, the effectiveness of the proposed method has been validated through numerical simulations.

Author contributions H.Z.W performed the conceptualization, methodology, formal analysis and editing of the manuscript.

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Data availability The authors confirm that all the data supporting the findings of this study are available in the Numerical Example section of the paper.

Declarations

Conflict of interest The authors have no Conflict of interest to declare that are relevant to the content of this article.

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