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ORIGINAL PAPER

Bogdanov–Takens bifurcation of an enzyme-catalyzed reaction model

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Abstract Enzyme-catalyzed reactions are frequently observed in the chemical process, and could be described by the mathematical model, such as the Gray–Scott model with Langmuir–Hinshelwood mechanism. The complex dynamical behaviors are analyzed in this work, including the existence and their stability of equilibrium points and the bifurcations of the model. By using stability theory, normal form technique and bifurcation analysis, the stability and the saddle-node bifurcation, Hopf bifurcation and Bogdanov–Takens bifurcation are explored in detail. Numerical simulations are also carried out to verify the validity of theoretical results.

Keywords Enzyme-catalyzed reaction · Stability · Hopf bifurcation · Bogdanov–Takens bifurcation

1 Introduction

In the early 20th century, scientists began to study some phenomena, such as catalytic reactions and cell division in living organisms, and found the involved changes of matters in space and time. Reaction-diffusion equations, which describe nonlinear interactions between diffusion and reaction terms, were developed to explain self-organizing phenomena in chemical and biological processes. As the result of the nonlinear interaction between diffusion and reaction terms, self-organizing patterns will emerge, forming various type of patterns, when some specific parameter conditions are satisfied. Turing initiated the research of such self-organizing patterns in 1952 [1], which is now often referred to as the Turing pattern.

Among the various autocatalytic reaction models, Gray–Scott model is one of popular models, which was proposed by Gray and Scott [2–4], when they considered the autocatalytic process in a continuous stirred tank reactor and studied the interaction between the chemical and the catalyst. They found that the system could exhibit different self-organizing phenomena, s uch as multistability, hysteresis, extinction, ignition and self sustained oscillations. The model takes the following dimensionless form

$$\begin{cases} \frac{\partial u}{\partial t} = -uv^2 + a(1-u) + D_u \frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial v}{\partial t} = uv^2 - (a+b)v + D_v \frac{\partial^2 v}{\partial x^2}, \end{cases} \tag{1}$$

where the state variables $u(x,t)$ and $v(x,t)$ respectively represent the concentrations of reactants and autocatalysts, a is the dimensionless feed rate, b is the dimensionless rate constant of the second reaction, and D_u and D_v represent the diffusion coefficients of the reactant and autocatalyst, respectively. In practice, this is also the model of chemical substances in a gel reac-

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tor, where the rate a can be relatively easily modified, while b depends on the system temperature.

The enzyme-catalyzed reaction model not only provides new ideas and methods to study self-organization phenomena in chemical reactions, but also has been widely applied in other fields such as biology, physics, and geology. Chen et al. [5] investigated a general reaction-diffusion model with non-local activators and inhibitors and applied the results to Klausmeier–Gray–Scott water-plant model and Holling-Tanner predator–prey model. In [6], Dong et al. proposed a new method for identifying the Gray–Scott model through deterministic learning and interpolation, which is based on local identification results and uses interpolation to achieve global identification. Gandhi et al. [7] considered the spatially localized structures in the Gray–Scott model and focused on the regime in diffusivities by using a combination of numerical continuation technique and weakly nonlinear theory. Saadi et al. [8] discussed a homotopy, which took a Schnakenberg-like glycolysis model to the Gray–Scott model. Several distinct codimension-two bifurcations were discovered by numerical continuation. Kuznetsov et al. [9] studied the homoclinic orbits of the second-order Gray–Scott model and gave the standard form of the Bogdanov–Takens bifurcation of the system.

The Bogdanov–Takens bifurcation is an important research topic in bifurcation theory. Many scholars have applied it to other models and obtained various research results. Yuan et al. [10] studied the bifurcations analysis in a generalist predator–prey model with stage structure. They showed saddle-node bifurcation of codimension 1 and 2, Hopf bifurcation, Bogdanov–Takens bifurcation, and bifurcations of nilpotent singularities of elliptic and focus type of codimension 3. And they also found that the nilpotent focus of codimension 4 serves as an organizing center to connect all the codimension 3 bifurcations in the two-dimensional center manifold of the system, and the bifurcations are also associated with a third order cubic Liénard system. In [11], Xiang et al. considered the Holling–Tanner model with constant-yield prey harvesting. They gave the analysis of the nilpotent cusp of codimension 4 and the Bogdanov–Takens bifurcation of codimension 4. And they also showed the situation that Hopf bifurcation of codimension 3 occurs. Jiao et al. [12] considered a delayed predator–prey system with double Allee effect in prey and presented the detailed bifurcation situation. Su et al. [13] studied a dynamic model of the initial lung infection response of the innate immune system, which has a high-dimensional bifurcation, including the Bogdanov–Takens bifurcation of codimension 3 and the Hopf bifurcation of 2 codimension.

In the heterogeneous catalysis, for the vast majority of surface catalytic reactions the Langmuir–Hinshelwood mechanism is preferred [14] to describe the kinetics of enzyme-catalyzed surface reactions. For the enzyme-catalyzed reaction model, we will introduce the Langmuir–Hinshelwood mechanism, which is similar to the Michaelis-Menten type functional reaction function. Moreover, if the concentration of substance v represented by $\frac{v}{1+mv}$, then v will reach saturation during the reaction as v is sufficiently large, i.e. $\lim_{v\to\infty} \frac{v}{1+mv} = \frac{1}{m}$. Therefore, from the perspective of controllability of chemical reactions, considering the Langmuir–Hinshelwood reaction mechanism may be more practical. So we will consider the impact on the dynamics of local system and find that the model could exhibit the stability, instability and bifurcation.

The paper is organized as follows. The enzyme-catalyzed reaction model with Langmuir–Hinshelwood mechanism is first formulated. The existence and their stability of equilibrium points are given in Sect. 2. In Sect. 3, specific bifurcations are presented, including saddle-node bifurcation, Hopf bifurcation, and Bogdanov–Takens bifurcation. Numerical simulations are given to validate theoretical results in Sect. 4. Some conclusions are drawn in Sect. 5.

2 Existence and stability of equilibria

With the Langmuir–Hinshelwood mechanism the enzyme-catalyzed reaction model takes the form as follows

$$\begin{cases} \frac{\partial u}{\partial t} = -u + \frac{u^2 v}{1+mv}, \\ \frac{\partial v}{\partial t} = b - dv - \frac{u^2 v}{1+mv}. \end{cases} \tag{2}$$

where the state variables $u(x,t)$ and $v(x,t)$ respectively represent the concentration of autocatalyst and reactant, where b, d, and m are positive constants. Let $f(u,v) = -u + \frac{u^2 v}{1+mv}$ and $g(u,v) = b - dv - \frac{u^2 v}{1+mv}$, solving the $f(u,v) = 0$ and getting that $u = 0$ or $u = \frac{mv+1}{v}$. Now, we consider these two cases respectively.

At the boundary equilibrium $E_0(u_0, v_0) = (0, \frac{b}{d})$, the Jacobian matrix is

$$J_{E_0} = \begin{bmatrix} -1 & 0 \\ 0 & -d \end{bmatrix}.$$

The eigenvalues of J_{E_0} are $\lambda_1 = -1 < 0, \lambda_2 = -d < 0$. So E_0 is a stable node of the system (2).

Then we discuss the case of positive equilibrium. From $f(u, v) = 0$, we have $u = \frac{mv+1}{v}$, substituting it into $g(u, v) = 0$, one has the following equation

$$h(v) \triangleq dv^2 - bv + mv + 1 = 0.$$

The discriminant of $h(v)$ can be obtained as

$$\Delta = (m - b)^2 - 4d.$$

According to the discriminant, when $d = \frac{(m-b)^2}{4}$, $h(v)$ has a unique solution

$$v_1 = \frac{b - m}{2d}.$$

When $d < \frac{(m-b)^2}{4}$, there are two distinct roots

$$v_2 = \frac{b - m + \sqrt{\Delta}}{2d}, \quad v_3 = \frac{b - m - \sqrt{\Delta}}{2d}.$$

In order to ensure that the obtained roots are positive, here we assume that $b - m > 0$.

Therefore, we have the following result.

Theorem 1 *Assume that $b - m > 0$, system (2) has only one boundary equilibrium $E_0\,(0, v_0) = \left(0, \frac{b}{d}\right)$, and*

(i) *if $d > \frac{(m-b)^2}{4}$, then there are no real equilibria, therefore no positive equilibria;*
(ii) *If $d = \frac{(m-b)^2}{4}$, then there is a unique positive equilibrium $E_1\,(u_1, v_1) = \left(\frac{b+m}{2}, \frac{2}{b-m}\right)$;*
(iii) *If $d < \frac{(m-b)^2}{4}$, then there are two positive equilibria $E_2\,(u_2, v_2) = \left(\frac{m\sqrt{\Delta}+bm-m^2+2d}{b-m+\sqrt{\Delta}}, \frac{b-m+\sqrt{\Delta}}{2d}\right)$ and $E_3\,(u_3, v_3) = \left(\frac{m\sqrt{\Delta}-bm+m^2-2d}{m-b+\sqrt{\Delta}}, \frac{b-m-\sqrt{\Delta}}{2d}\right)$.*

Next, we will present the stability analysis of the equilibrium points E_1, E_2 and E_3. As for E_1, we have the following statements.

Theorem 2 *The following statements about E_1 are true.*

(i) *If $d = \frac{1}{2}$, then E_1 is a cusp of codimension two;*
(ii) *If $d < \frac{1}{2}$, then E_1 is a saddle-node with an unstable parabolic sector;*
(iii) *If $d > \frac{1}{2}$, then E_1 is a saddle-node with a stable parabolic sector.*

Proof Since the Jacobian matrix of system (2) at equilibrium E_1 is

$$J_{E_1} = \begin{pmatrix} 1 & d \\ -2 & -2d \end{pmatrix},$$

we have

$$tr\,J_{E_1} = 1 - 2d, \quad det\,J_{E_1} = 0.$$

Now we translate $E_1(u_1, v_1) = (\frac{bm-m^2+2d}{b-m}, \frac{b-m}{2d})$ to the origin with $(u, v) = (U + u_1, V + v_1)$, then system (2) becomes

$$\begin{cases} \dot{U} = a_{10}U + a_{01}V + a_{20}U^2 + a_{11}UV + a_{02}V^2 \\ \quad +a_{21}U^2V + a_{12}UV^2 + a_{03}V^3 + P\,(U, V)\,, \\ \dot{V} = b_{10}U + b_{01}V + b_{20}U^2 + b_{11}UV + b_{02}V^2 \\ \quad +b_{21}U^2V + b_{12}UV^2 + b_{03}V^3 + Q\,(U, V)\,, \end{cases} \tag{3}$$

where

$$a_{10} = 1, \quad a_{01} = d, \quad a_{20} = \frac{b - m}{bm - m^2 + 2d},$$
$$a_{11} = \frac{8d^2}{(b - m)(bm - m^2 + 2d)},$$
$$a_{02} = -\frac{2d^2m}{bm - m^2 + 2d}, \quad a_{21} = \frac{4d^2}{(bm - m^2 + 2d)^2},$$
$$a_{12} = -\frac{16d^3m}{(b - m)(bm - m^2 + 2d)^2},$$
$$a_{03} = \frac{4d^3m^2}{(bm - m^2 + 2d)^2}, \quad b_{10} = -2, \quad b_{01} = -2d,$$
$$b_{20} = -\frac{b - m}{bm - m^2 + 2d},$$
$$b_{11} = -\frac{8d^2}{(b - m)(bm - m^2 + 2d)},$$
$$b_{02} = \frac{2d^2m}{bm - m^2 + 2d},$$

$$b_{21} = -\frac{4d^2}{(bm - m^2 + 2d)^2},$$
$$b_{12} = \frac{16d^3 m}{(b-m)(bm - m^2 + 2d)^2},$$
$$b_{03} = -\frac{4d^3 m^2}{(bm - m^2 + 2d)^2},$$

and $P(U, V)$, $Q(U, V)$ are terms of at least order four in U and V.

When $d = \frac{1}{2}$, we find that both the trace and determinant of J_{E_1} are equal to zero, indicating that both of eigenvalues of J_{E_1} are also equal to zero. After a transformation to (3) by letting $(U, V) = (x, 2x + 2y)$, then we get the following form

$$\begin{cases} \frac{dx}{dt} = y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{30}x^3 + c_{21}x^2 y \\ \qquad +c_{12}xy^2 + c_{03}y^3 + P_1(x, y), \\ \frac{dy}{dt} = d_{20}x^2 + d_{11}xy + d_{02}y^2 + d_{30}x^3 + d_{21}x^2 y \\ \qquad +d_{12}xy^2 + d_{03}y^3 + Q_1(x, y), \end{cases} \tag{4}$$

where

$$c_{20} = \frac{b^2 - 4mb + 3m^2 - 4}{(b-m)(mb - m^2 + 1)}, \quad c_{11} = \frac{4}{b-m},$$
$$c_{02} = -\frac{2m}{mb - m^2 + 1},$$
$$c_{30} = -\frac{2(2m^2 b - 2m^3 + b + 3m)}{(b-m)(mb - m^2 + 1)^2},$$
$$c_{21} = \frac{2(6m^2 b - 6m^3 + b + 7m)}{(b-m)(mb - m^2 + 1)^2},$$
$$c_{12} = -\frac{12m(mb - m^2 + \frac{2}{3})}{(b-m)(mb - m^2 + 1)^2},$$
$$c_{03} = \frac{4m^2}{(mb - m^2 + 1)^2},$$
$$d_{20} = \frac{b^2 - 4mb + 3m^2 - 4}{2(b-m)(mb - m^2 + 1)},$$
$$d_{11} = \frac{2}{b-m}, \quad d_{02} = -\frac{m}{mb - m^2 + 1},$$
$$d_{30} = -\frac{2m^2 b - 2m^3 + b + 3m}{(b-m)(mb - m^2 + 1)^2},$$
$$d_{21} = \frac{(6m^2 b - 6m^3 + b + 7m)}{(b-m)(mb - m^2 + 1)^2},$$
$$d_{12} = -\frac{6m(mb - m^2 + \frac{2}{3})}{(b-m)(mb - m^2 + 1)^2},$$
$$d_{03} = \frac{2m^2}{(mb - m^2 + 1)^2},$$

and $P_1(x, y)$, $Q_1(x, y)$ are terms of at least order four in x and y.

To facilitate further calculation, we apply the change $(x, y) = (x_1, y_1 + d_{02}x_1 y_1 - c_{02}y_1^2)$ to system (4) to eliminate the y^2 term and rewrite system (4) as

$$\begin{cases} \dot{x}_1 = y_1 + e_{20}x_1^2 + e_{11}x_1 y_1 + e_{30}x_1^3 + e_{21}x_1^2 y_1 \\ \qquad +e_{12}x_1 y_1^2 + e_{03}y_1^3 + P_2(x_1, y_1), \\ \dot{y}_1 = f_{20}x_1^2 + f_{11}x_1 y_1 + f_{30}x_1^3 + f_{21}x_1^2 y_1 \\ \qquad +f_{12}x_1 y_1^2 + f_{03}y_1^3 + Q_2(x_1, y_1), \end{cases} \tag{5}$$

where

$$e_{20} = c_{20}, \quad e_{11} = d_{02} + c_{11}, \quad e_{30} = c_{30},$$
$$e_{21} = c_{21} + c_{11}d_{02},$$
$$e_{12} = c_{12} - c_{02}c_{11} + 2c_{02}d_{02}, \quad e_{03} = c_{03} - 2c_{02}^2,$$
$$f_{20} = d_{20}, \quad f_{11} = d_{11}, \quad f_{30} = d_{30},$$
$$f_{21} = d_{21} - c_{20}d_{02} + 2c_{02}d_{20},$$
$$f_{12} = d_{12} + d_{02}^2 + c_{02}d_{11} - c_{11}d_{02},$$
$$f_{03} = d_{03} - 2c_{02}d_{02},$$

and $P_2(x_1, y_1)$, $Q_2(x_1, y_1)$ are terms of at least order four in x_1 and y_1.

Further, we reduce (5) into the following form by the transformation $(x_2, y_2) = (x_1, y_1 + e_{20}x_1^2 + e_{11}x_1 y_1 + M(x_1, y_1))$. Here the $M(x_1, y_1)$ is the term of at least order three in x_1 and y_1.

$$\begin{cases} \dot{x}_2 = y_2, \\ \dot{y}_2 = g_{20}x_2^2 + g_{11}x_2 y_2 + g_{02}y_2^2 + g_{30}x_2^3 + g_{21}x_2^2 y_2 \\ \qquad +g_{12}x_2 y_2^2 + g_{03}y_2^3 + Q_2(x_2, y_2), \end{cases} \tag{6}$$

where

$$
\begin{aligned}
& g_{20}=f_{20}, \quad g_{11}=f_{11}+2e_{20}, \quad g_{02}=e_{11},\\
& g_{30}=f_{30}+e_{11}f_{20}-e_{20}f_{11},\\
& g_{21}=f_{21}-e_{11}e_{20}, \quad g_{12}=f_{12}-e_{11}^2,\\
& g_{03}=f_{03},
\end{aligned}
$$

and $Q_2(x_2, y_2)$ is the term of at least order four in x_2 and y_2.

Then to eliminate the y_2^2 term in system (6), we first apply the scaling transformation $dt=(1-g_{02}x_2)d\tau$ to it and obtain the system as

$$
\begin{cases}
\frac{dx_2}{d\tau}=(1-g_{02}x_2)y_2,\\
\frac{dy_2}{d\tau}=(1-g_{02}x_2)(g_{20}x_2^2+g_{11}x_2y_2+g_{02}y_2^2+g_{30}x_2^3\\
\qquad +g_{21}x_2^2y_2+g_{12}x_2y_2^2+g_{03}y_2^3+Q_2(x_2, y_2)).
\end{cases} \tag{7}
$$

Further transformation $(x_3, y_3)=(x_2, (1-g_{02}x_2)y_2)$ for system (7) yields the following result

$$
\begin{cases}
\frac{dx_3}{d\tau}=y_3,\\
\frac{dy_3}{d\tau}=g_{20}x_3^2+g_{11}x_3y_3+Q_3(x_3, y_3),
\end{cases} \tag{8}
$$

where

$$
g_{20}=\frac{b^2-4bm+3m^2-4}{2(b-m)(bm-m^2+1)},
$$
$$
g_{11}=\frac{2b^2-6bm+4m^2-6}{(b-m)(bm-m^2+1)},
$$

and $Q_3(x_3, y_3)$ is the term of at least order three in x_3 and y_3.

Simple calculation gives

$$
g_{20}g_{11}=\frac{(b^2-4bm+3m^2-4)(b^2-3bm+2m^2-3)}{(b-m)^2(bm-m^2+1)^2}.
$$

If $b=2m\pm\sqrt{m^2+4}$ or $b=\frac{3m}{2}\pm\frac{\sqrt{m^2+12}}{2}$, then $g_{20}g_{11}=0$. However, note that $b-m>0$, $d=\frac{1}{2}$ and $\Delta=(m-b)^2-4d=0$, then one has that $b=m+\sqrt{2}$. So it follows that $g_{20}g_{11}\neq 0$ and $E_1(u_1, v_1)$ is a cusp of codimension two.

Next, for the situation $d\neq\frac{1}{2}$, we apply the transformation $(U, V)=(-dx-\frac{y}{2}, x+y)$ to system (3) and obtain the following system

$$
\begin{cases}
\dot{\bar{x}}=\bar{c}_{20}\bar{x}^2+\bar{c}_{11}\bar{x}\bar{y}+\bar{c}_{02}\bar{y}^2+P_4(\bar{x}, \bar{y}),\\
\dot{\bar{y}}=\bar{d}_{01}\bar{y}+\bar{d}_{20}\bar{x}^2+\bar{d}_{11}\bar{x}\bar{y}+\bar{d}_{02}\bar{y}^2+Q_4(\bar{x}, \bar{y}),
\end{cases} \tag{9}
$$

where

$$
\begin{aligned}
\bar{c}_{20}&=-\frac{d(b^3-3bm^2+(3m^2-8d)b-m^3)}{4(2d-1)(bm-m^2+2d)},\\
\bar{c}_{11}&=-\frac{-m^3+3bm^2+(-3b^2-8d^2+4d)m+b(b^2-8d^2-4d)}{4(2d-1)(bm-m^2+2d)},\\
\bar{c}_{02}&=-\frac{32d^3m-b^3+3b^2m+16d^2b-3bm^2-16d^2m+m^3}{16d(2d-1)(bm-m^2+2d)},\\
\bar{d}_{01}&=1-2d,\\
\bar{d}_{20}&=-\frac{d(b^3-3b^2m+(3m^2-8d)b-m^3)(d-1)}{4(2d-1)(bm-m^2+2d)},\\
\bar{d}_{11}&=-\frac{((-8b-8m)d^2+(4m-4b)d+(b-m)^3)(d-1)}{4(2d-1)(bm-m^2+2d)},\\
\bar{d}_{02}&=-\frac{(-m^3+3b^2m+(-32d^3-3b^2+16d^2)m+b^3-16d^2b)(d-1)}{16d(d-\frac{1}{2})(bm-m^2+2d)},
\end{aligned}
$$

and $P_4(\bar{x}, \bar{y}))$ and $Q_4(\bar{x}, \bar{y}))$ are the terms of at least order three in $\bar{x}$ and $\bar{y}$.

After introduction of a new time variable $\tau=(1-2d)t$ to system (9), the we arrive at the following form

$$
\begin{cases}
\frac{\mathrm{d}\bar{x}}{\mathrm{d}\tau}=\bar{e}_{20}\bar{x}^2+\bar{e}_{11}\bar{x}\bar{y}+\bar{e}_{02}\bar{y}^2+P_5(\bar{x}, \bar{y})),\\
\frac{\mathrm{d}\bar{y}}{\mathrm{d}\tau}=\bar{y}+\bar{f}_{20}\bar{x}^2+\bar{f}_{11}\bar{x}\bar{y}+\bar{f}_{02}\bar{y}^2+Q_5(\bar{x}, \bar{y})),
\end{cases} \tag{10}
$$

where $\bar{e}_{ij}=\frac{\bar{c}_{ij}}{1-2d}$, $\bar{f}_{ij}=\frac{\bar{d}_{ij}}{1-2d}(i+j\leq 2, 0\leq i, j\leq 2)$, $P_5(\bar{x}, \bar{y}))=\frac{P_4(\bar{x}, \bar{y}))}{1-2d}$ and $Q_5(\bar{x}, \bar{y}))=\frac{Q_4(\bar{x}, \bar{y}))}{1-2d}$.

By calculation, it can be obtained that $\bar{e}_{20}=\frac{d(b^3-3b^2m+(3m^2-8d)b-m^3)}{4(1-2d)^2(bm-m^2+2d)}\neq 0$. From Theorem 7.1 in [15], the origin is always a saddle-node. When $d<\frac{1}{2}$, E_1 is a saddle-node with an unstable parabolic sector, and it is a saddle-node with a stable parabolic sector when $d>\frac{1}{2}$.

The proof is completed. □

If $\Delta>0$, then $h(u)$ has two solutions $E_2(u_2, v_2)$ and $E_3(u_3, v_3)$. Now, we intend to give their stability results.

Theorem 3 *E_2 is a saddle and E_3 is*

(i) *a source when $tr\,J_{E_3} > 0$, i.e. $d > \frac{(b-m+1)(b-m-1)}{(b-m)^2}$;*
(ii) *a center or a fine focus when $tr\,J_{E_3} = 0$, i.e. $d = \frac{(b-m+1)(b-m-1)}{(b-m)^2}$;*
(iii) *a sink when $tr\,J_{E_3} < 0$, i.e. $d < \frac{(b-m+1)(b-m-1)}{(b-m)^2}$.*

Proof The Jacobian matrix at $E_2(u_2, v_2)$ and $E_3(u_3, v_3)$ are

$$J_{E_{2,3}} = \begin{pmatrix} 1 & -2 \\ \frac{1}{v_{2,3}^2} & -\frac{dv_{2,3}^2+1}{v_{2,3}^2} \end{pmatrix}.$$

The determinants of J_{E_2} and J_{E_3} are, respectively.

$$\det J_{E_2} = -\frac{2d\sqrt{\Delta}(b-m+\sqrt{\Delta})}{(b-m+\sqrt{\Delta})^2} < 0,$$

and

$$\det J_{E_3} = \frac{2d\sqrt{\Delta}(b-m-\sqrt{\Delta})}{(b-m-\sqrt{\Delta})^2} > 0.$$

So E_2 is a saddle and the stability of E_3 is up to the sign of the trace

$$tr\,J_{E_3} = \frac{2(d-1)(b-m)\sqrt{\Delta}(2-2d)(b^2+m^2)+4m(d-1)b-4d}{(b-m-\sqrt{\Delta})^2}.$$

It follows that E_3 is a source when $d > \frac{(b-m+1)(b-m-1)}{(b-m)^2}$; it is a center or a fine focus when $d = \frac{(b-m+1)(b-m-1)}{(b-m)^2}$ and is a sink when $d < \frac{(b-m+1)(b-m-1)}{(b-m)^2}$. □

3 Bifurcation analysis

3.1 Saddle-node bifurcation

After checking the existence of equilibrium points, we find that the number of equilibria changes with respect to the parameter d. Specifically, when $d = \frac{(m-b)^2}{4}$, the saddle-node bifurcation may occur. In the following discussion, we will investigate the saddle-node bifurcation around E_1.

Theorem 4 *When bifurcation parameter $d \equiv d_{SN} = \frac{(m-b)^2}{4}$, system (2) will experience the saddle-node bifurcation around E_1.*

Proof By applying Sotomayor's theorem [16], we need to verify the transversality condition around $d \equiv d_{SN}$. The Jacobian matrix at E_1 is given by

$$J_{E_1} = \begin{pmatrix} 1 & d \\ -2 & -2d \end{pmatrix}$$

Let M and N represent the eigenvectors of the eigenvalue λ_1 of J_{E_1} and $J_{E_1}^T$, respectively. To be specific, we have

$$M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} -d \\ 1 \end{pmatrix}, \qquad N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Moreover, we have

$$F_d(E_1, d_{SN}) = \begin{pmatrix} 0 \\ \frac{2}{m-b} \end{pmatrix},$$

and

$$\begin{aligned} &D^2F(E_1, d_{SN})(M, M) \\ &= \begin{pmatrix} \frac{\partial^2 f}{\partial u^2}M_1^2 + 2\frac{\partial^2 f}{\partial u \partial v}M_1M_2 + \frac{\partial^2 f}{\partial v^2}M_2^2 \\ \frac{\partial^2 g}{\partial u^2}M_1^2 + 2\frac{\partial^2 f}{\partial u \partial v}M_1M_2 + \frac{\partial^2 f}{\partial v^2}M_2^2 \end{pmatrix}_{(E_1, d_{SN})} \\ &= \begin{pmatrix} -\frac{(b-m)^3}{4} \\ \frac{(b-m)^3}{4} \end{pmatrix}. \end{aligned}$$

Hence, M and N satisfy the transversality conditions, which are given by

$$N^T F_d(E_1, d_{SN}) = -\frac{2}{b-m} \neq 0,$$

and

$$N^T[D^2F(E_1, d_{SN})(M, M)] = \frac{(b-m)^3}{4} \neq 0.$$

Therefore, the system (2) exhibits the saddle-node bifurcation around E_1. □

3.2 Hopf bifurcation

From Theorem 3, we find that when $d = \frac{(b-m+1)(b-m-1)}{(b-m)^2}$, E_3 becomes a center or a fine focus, indicating that the Hopf bifurcation may happen. Next, we will explore whether or not the Hopf bifurcation around E_3 exists.

According to the Hopf theorem [17], we need to verify the transversal condition

$$\left.\frac{\mathrm{d}trJ_{E_2}}{\mathrm{d}d}\right|_{d=c} = (4((-b^4+4b^3m+(1-6m^2)b^2+(4m^3-2m) \times b-m^4+m^2-1)\sqrt{\frac{(b^2-2bm+m^2-2)^2}{(b-m)^2}} + (b^4-4b^3m+(6m^2-3)b^2 + (6m-4m^3)b+m^4-3m^2+3)(b-m))) /(\sqrt{\frac{(b^2-2bm+m^2-2)^2}{(b-m)^2}}(b-m - \sqrt{\frac{(b^2-2bm+m^2-2)^2}{(b-m)^2}})^3(b-m)) \neq 0.$$

Hence the Hopf bifurcation happens at E_3 in system (2).

Then we want to give the direction of the Hopf bifurcation. Translating $E_3(u_3, v_3)$ into $(0, 0)$ with $(\hat{u}, \hat{v}) = (u-u_3, v-v_3)$, system (2) becomes

$$\begin{cases}\dot{\tilde{u}} = \tilde{a}_{10}\tilde{u}+\tilde{a}_{01}\tilde{v}+\tilde{a}_{20}\tilde{u}^2+\tilde{a}_{11}\tilde{u}\tilde{v}+\tilde{a}_{02}\tilde{v}^2 \\ \quad +\tilde{P}(\tilde{u},\tilde{v}), \\ \dot{\tilde{v}} = \tilde{b}_{10}\tilde{u}+\tilde{b}_{01}\tilde{v}+\tilde{b}_{20}\tilde{u}^2+\tilde{b}_{11}\tilde{u}\tilde{v}+\tilde{b}_{02}\tilde{v}^2 \\ \quad +\tilde{Q}(\tilde{u},\tilde{v}),\end{cases} \tag{11}$$

where

$$\tilde{a}_{10}=1, \quad \tilde{a}_{01}=\frac{1}{v_3^2}, \quad \tilde{a}_{20}=\frac{v_3}{mv_3+1},$$
$$\tilde{a}_{11}=\frac{2}{v_3(mv_3+1)},$$
$$\tilde{a}_{02}=-\frac{m}{v_3^2(mv_3+1)}, \quad \tilde{b}_{10}=-2,$$
$$\tilde{b}_{01}=\frac{-d^*v_3^2-1}{v_3^2}, \quad \tilde{b}_{20}=-\frac{v_3}{mv_3+1},$$
$$\tilde{b}_{11}=-\frac{2}{v_3(mv_3+1)}, \quad \tilde{b}_{02}=\frac{m}{v_3^2(mv_3+1)},$$

and $\tilde{P}(\tilde{u},\tilde{v})$, $\tilde{Q}(\tilde{u},\tilde{v})$ are the terms of at least order three in $\tilde{u}$ and $\tilde{v}$.

We continue to change the system (11) by the transformation

$$\tilde{x}=-\tilde{u}, \quad \tilde{y}=\frac{1}{\sqrt{S}}(\tilde{a}_{10}\tilde{u}+\tilde{a}_{01}\tilde{v}), \quad d\tau=\sqrt{S}dt,$$

where $S = \tilde{a}_{10}\tilde{b}_{01}-\tilde{a}_{01}\tilde{b}_{10} = \frac{-d^*v_3^2+1}{v_3^2}$. Then we obtain the following form

$$\begin{cases}\dot{\tilde{x}} = -\tilde{y}+\tilde{f}(\tilde{x},\tilde{y}), \\ \dot{\tilde{y}} = \tilde{x}+\tilde{g}(\tilde{x},\tilde{y}).\end{cases} \tag{12}$$

where

$$\begin{cases}\tilde{f}(\tilde{x},\tilde{y}) = \tilde{c}_{20}\tilde{x}^2+\tilde{c}_{11}\tilde{x}\tilde{y}+\tilde{c}_{02}\tilde{y}^2+\tilde{P}_1(\tilde{x},\tilde{y}), \\ \tilde{g}(\tilde{x},\tilde{y}) = \tilde{d}_{20}\tilde{x}^2+\tilde{d}_{11}\tilde{x}\tilde{y}+\tilde{d}_{02}\tilde{y}^2+\tilde{Q}_1(\tilde{x},\tilde{y}),\end{cases}$$

and

$$\tilde{c}_{20} = -\frac{\tilde{a}_{02}\tilde{a}_{10}^2}{\sqrt{-\tilde{a}_{01}\tilde{b}_{10}+\tilde{a}_{10}\tilde{b}_{01}}\tilde{a}_{01}^2} + \frac{\tilde{a}_{11}\tilde{a}_{10}}{\sqrt{-\tilde{a}_{01}\tilde{b}_{10}+\tilde{a}_{10}\tilde{b}_{01}}\tilde{a}_{01}} - \frac{\tilde{a}_{20}}{\sqrt{-\tilde{a}_{01}\tilde{b}_{10}+\tilde{a}_{10}\tilde{b}_{01}}},$$
$$\tilde{c}_{11} = -\frac{2\tilde{a}_{02}\tilde{a}_{10}}{\tilde{a}_{01}^2}+\frac{\tilde{a}_{11}}{\tilde{a}_{01}},$$
$$\tilde{c}_{02} = \frac{\tilde{a}_{02}\tilde{b}_{10}}{\sqrt{-\tilde{a}_{01}\tilde{b}_{10}+\tilde{a}_{10}\tilde{b}_{01}}\tilde{a}_{01}} - \frac{\tilde{a}_{02}\tilde{a}_{10}\tilde{b}_{01}}{\sqrt{-\tilde{a}_{01}\tilde{b}_{10}+\tilde{a}_{10}\tilde{b}_{01}}\tilde{a}_{01}^2},$$
$$\tilde{d}_{20} = \frac{\tilde{b}_{02}\tilde{a}_{10}^2}{(-\tilde{a}_{01}\tilde{b}_{10}+\tilde{a}_{10}\tilde{b}_{01})\tilde{a}_{10}} - \frac{\tilde{b}_{11}\tilde{a}_{10}}{S} + \frac{\tilde{a}_{01}\tilde{b}_{20}}{S} + \frac{\tilde{a}_{02}\tilde{a}_{10}^2}{S\tilde{a}_{01}^2} - \frac{\tilde{a}_{11}\tilde{a}_{10}}{S\tilde{a}_{01}} + \frac{\tilde{a}_{20}}{S},$$
$$\tilde{d}_{11} = \frac{2\tilde{b}_{02}\tilde{a}_{10}}{S\tilde{a}_{01}} - \frac{\tilde{b}_{11}}{\sqrt{S}} + \frac{2\tilde{a}_{02}\tilde{a}_{10}}{\sqrt{S}\tilde{a}_{01}^2} - \frac{\tilde{a}_{11}}{\sqrt{S}\tilde{a}_{01}},$$
$$\tilde{d}_{02} = -\frac{\tilde{b}_{02}\tilde{b}_{10}}{S} + \frac{\tilde{b}_{02}\tilde{a}_{10}\tilde{b}_{01}}{S\tilde{a}_{01}} - \frac{\tilde{a}_{02}\tilde{b}_{10}}{S\tilde{a}_{01}} + \frac{\tilde{a}_{02}\tilde{a}_{10}\tilde{b}_{01}}{S\tilde{a}_{01}^2},$$

and $\tilde{P}_1(\tilde{x},\tilde{y})$, $\tilde{Q}_1(\tilde{x},\tilde{y})$ are the terms of at least order three in $\tilde{x}$ and $\tilde{y}$.

We can get the first-order Lyapunov number of the system by using MATLAB, which is

$$\begin{aligned}\sigma &= \frac{1}{16}[\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} + \tilde{f}_{\tilde{x}\tilde{y}\tilde{y}} + \tilde{g}_{\tilde{x}\tilde{x}\tilde{y}} + \tilde{g}_{\tilde{y}\tilde{y}\tilde{y}}] \\ &+ \frac{1}{16}[\tilde{f}_{\tilde{x}\tilde{y}}(\tilde{f}_{\tilde{x}\tilde{x}} + \tilde{f}_{\tilde{y}\tilde{y}}) - \tilde{g}_{\tilde{x}\tilde{y}}(\tilde{g}_{\tilde{x}\tilde{x}} + \tilde{g}_{\tilde{y}\tilde{y}}) \\ &- \tilde{f}_{\tilde{x}\tilde{x}}\tilde{g}_{\tilde{x}\tilde{x}} + \tilde{f}_{\tilde{y}\tilde{y}}\tilde{g}_{\tilde{y}\tilde{y}}] \\ &= -\frac{1}{8S^{\frac{3}{2}}a_{01}^4}[-2b_{20}(a_{20}\frac{b_{11}}{2})a_{01}^5] \\ &+ [(2a_{11}b_{20} + 2a_{20}b_{11} + 2b_{02}b_{20} + b_{11}^2)a_{10} \\ &- a_{11}b_{20} - 2a_{20}^2 - a_{20}b_{11}]a_{01}^4 + \ldots\end{aligned}$$

And if $\sigma < 0$, then the Hopf bifurcation is supercritical; if $\sigma > 0$, then it's subcritical [18].

Theorem 5 *When $\Delta = (b-m)^2 - 4d > 0$ and $d = \frac{(b-m+1)(b-m-1)}{(b-m)^2} > 0$, system (2) experiences the Hopf bifurcation around E_3.*

Remark 1 1. When $m = 0$, i.e. the system (2) does not have the Langmuir–Hinshelwood mechanism, the first Lyapunov coefficient is $l_1' = -\frac{6v_3(v_3^2+\frac{1}{2})\pi}{(1-dv_3^2)^{\frac{3}{2}}} < 0$. Here, $1 - dv_3^2 > 0$ and $d = \frac{b^2-1}{b^2} > 0$ needs to be satisfied. Therefore, when $m = 0$, $b^2 - 4d > 0$, $1 - dv_3^2 > 0$, and $d = \frac{b^2-1}{b^2} > 0$, it has a supercritical Hopf bifurcation occurs around E_3.

2. In Ref. [19], the phase portraits and bifurcation diagrams were present for the Gray–Scott model, that is, the model (2) with $m = 0$, but no result about the Hopf bifurcation or the Bogdanov–Takens bifurcation.

3.3 Bogdanov–Takens bifurcation

From Theorems 1 and 2, the system has a positive equilibrium point $E_1(u_1, v_1)$ when $\Delta = (m-b)^2 - 4d = 0$ and $b > m$. Furthermore, when $m = m_1 \equiv b - \sqrt{2}$ and $d \equiv d_1 = \frac{1}{2}$, we find that $tr\,J_{E_1} = det\,J_{E_1} = 0$ and E_1 is a cusp of codimension 2. Therefore, the Bogdanov–Takens bifurcation will appear around E_1 in the system. Now select m and b as the bifurcation parameters to give the detailed analysis.

Theorem 6 *When bifurcation parameters $m = m_1$ and $d = d_1$, the Bogdanov–Takens bifurcation occurs in the small neighborhood of E_1 in system (2).*

Proof Replacing m and d with $m_1 + \varepsilon_1$ and $d_1 + \varepsilon_2$ respectively in system (2), one has

$$\begin{cases}\dot{u} = -u + \frac{u^2 v}{1+(m_1+\varepsilon_1)v}, \\ \dot{v} = b - (d_1+\varepsilon_2)v - \frac{u^2 v}{1+(m_1+\varepsilon_1)v}.\end{cases} \tag{13}$$

Using the transformation $(x, y) = (u - u_1, v - v_1)$, we expand system (13) at the origin and obtain

$$\begin{cases}\dot{x} = a_{00}(\varepsilon) + a_{10}(\varepsilon)x + a_{01}(\varepsilon)y + a_{20}(\varepsilon)x^2 \\ \quad +a_{11}(\varepsilon)xy + a_{02}(\varepsilon)y^2 + M_0(x, y, \varepsilon), \\ \dot{y} = b_{00}(\varepsilon) + b_{10}(\varepsilon)x + b_{01}(\varepsilon)y + b_{20}(\varepsilon)x^2 \\ \quad +b_{11}(\varepsilon)xy + b_{02}(\varepsilon)y^2 + N_0(x, y, \varepsilon),\end{cases} \tag{14}$$

where

$$a_{00}(\varepsilon) = -\frac{(m_1v_1+1)\varepsilon_1}{1+(m_1+\varepsilon_1)v_1}, \quad b_{00}(\varepsilon) = b - (d_1\varepsilon_2)v_1 - \frac{(m_1v_1+1)^2}{v_1(1+(m_1+\varepsilon_1)v_1)}, \quad a_{10}(\varepsilon) = \frac{1+(m_1-\varepsilon_1)v_1}{1+(m_1+\varepsilon_1)v_1},$$

$$a_{01}(\varepsilon) = \frac{(m_1v_1+1)^2}{v_1^2(1+(m_1+\varepsilon_1)v_1)^2}, \quad a_{20}(\varepsilon) = \frac{v_1}{1+(m_1+\varepsilon_1)v_1}, \quad a_{11}(\varepsilon) = \frac{2m_1v_1+2}{v_1(1+(m_1+\varepsilon_1)v_1)^2},$$

$$a_{02}(\varepsilon) = -\frac{(m_1v_1+1)^2(m_1+\varepsilon_1)}{v_1^2(1+(m_1+\varepsilon_1)v_1)^3}, \quad b_{10}(\varepsilon) = \frac{-2m_1v_1-2}{1+(m_1+\varepsilon_1)v_1},$$

$$b_{01}(\varepsilon) = \frac{-1-(m_1+\varepsilon_1)^2(d_1+\varepsilon_2)v_1^4 - 2(m_1+\varepsilon_1)(d_1+\varepsilon_2)v_1^3 + (-m_1^2-d_1-\varepsilon_2)v_1^2 - 2m_1v_1}{v_1^2(1+(m_1+\varepsilon_1)v_1)^2},$$

$$b_{20}(\varepsilon) = -\frac{v_1}{1+(m_1+\varepsilon_1)v_1}, \quad b_{11}(\varepsilon) = \frac{-2m_1v_1-2}{v_1(1+(m_1+\varepsilon_1)v_1)^2}, \quad b_{02}(\varepsilon) = \frac{(m_1v_1+1)^2(m_1+\varepsilon_1)}{v_1^2(1+(m_1+\varepsilon_1)v_1)^3},$$

and $M_0(x, y, \varepsilon)$, $N_0(x, y, \varepsilon)$ are the terms of at least order three in x and y.

In order to obtain the universal unfolding of the system, it is necessary to eliminate the y^2-term in system (14). To this end, by the transformation $(x, y) = (x_1 + \frac{a_{02}(\varepsilon)}{a_{01}(\varepsilon)} x_1 y_1, y_1 + \frac{b_{02}(\varepsilon)}{a_{01}(\varepsilon)} x_1 y_1)$, we have

$$\begin{cases} \dot{x_1} = c_{00}(\varepsilon) + c_{10}(\varepsilon)x_1 + c_{01}(\varepsilon)y_1 + c_{20}(\varepsilon)x_1^2 \\ \quad +c_{11}(\varepsilon)x_1 y_1 + M_1(x_1, y_1, \varepsilon), \\ \dot{y_1} = d_{00}(\varepsilon) + d_{10}(\varepsilon)x_1 + d_{01}(\varepsilon)y_1 + d_{20}(\varepsilon)x_1^2 \\ \quad +d_{11}(\varepsilon)x_1 y_1 + N_1(x_1, y_1, \varepsilon), \end{cases} \tag{15}$$

where

$$\begin{aligned} c_{00}(\varepsilon) &= a_{00}(\varepsilon), \quad c_{10}(\varepsilon) = a_{10}(\varepsilon) - \frac{a_{02}(\varepsilon)b_{00}(\varepsilon)}{a_{01}(\varepsilon)}, \\ c_{01}(\varepsilon) &= a_{01}(\varepsilon) - \frac{a_{00}(\varepsilon)a_{02}(\varepsilon)}{a_{01}(\varepsilon)}, \\ c_{20}(\varepsilon) &= a_{20}(\varepsilon) - \frac{a_{02}(\varepsilon)b_{10}(\varepsilon)}{a_{01}(\varepsilon)}, \\ c_{11}(\varepsilon) &= a_{11}(\varepsilon) + b_{02}(\varepsilon) - \frac{a_{02}(\varepsilon)b_{01}(\varepsilon)}{a_{01}(\varepsilon)}, \\ d_{00}(\varepsilon) &= b_{00}(\varepsilon), \quad d_{10}(\varepsilon) = b_{10}(\varepsilon) - \frac{b_{00}(\varepsilon)b_{02}(\varepsilon)}{a_{01}(\varepsilon)}, \\ d_{01}(\varepsilon) &= b_{10}(\varepsilon) - \frac{a_{00}(\varepsilon)b_{02}(\varepsilon)}{a_{01}(\varepsilon)}, \\ d_{20}(\varepsilon) &= b_{20}(\varepsilon) - \frac{b_{10}(\varepsilon)b_{02}(\varepsilon)}{a_{01}(\varepsilon)}, \\ d_{11}(\varepsilon) &= b_{11}(\varepsilon) - \frac{a_{10}(\varepsilon)b_{02}(\varepsilon)}{a_{01}(\varepsilon)} + \frac{a_{02}(\varepsilon)b_{10}(\varepsilon)}{a_{01}(\varepsilon)}, \end{aligned}$$

and $M_1(x_1, y_1, \varepsilon)$, $N_1(x_1, y_1, \varepsilon)$ are the terms of at least order three in x_1 and y_1.

Through the C^∞ change of variables $(x_2, y_2) = (x_1, c_{00}+c_{10}x_1+c_{01}y_1+c_{20}x_1^2+c_{11}x_1y_1+...)$, system (15) becomes

$$\begin{cases} \dot{x}_2 = y_2, \\ \dot{y}_2 = e_{00}(\varepsilon) + e_{10}(\varepsilon)x_2 + e_{01}(\varepsilon)y_2 + e_{20}(\varepsilon)x_2^2 \\ \quad +e_{11}(\varepsilon)x_2 y_2 + e_{02}(\varepsilon)y_2^2 + N_2(x_2, y_2, \varepsilon), \end{cases} \tag{16}$$

where

$$\begin{aligned} e_{00}(\varepsilon) =& c_{01}(\varepsilon)d_{00}(\varepsilon) - c_{00}(\varepsilon)d_{01}(\varepsilon), \\ e_{01}(\varepsilon) =& c_{10}(\varepsilon) + d_{01}(\varepsilon) - \frac{c_{11}(\varepsilon)c_{00}(\varepsilon)}{c_{01}(\varepsilon)}, \\ e_{10}(\varepsilon) =& - c_{00}(\varepsilon)d_{11}(\varepsilon) - c_{10}(\varepsilon)d_{01}(\varepsilon) \\ & + d_{00}(\varepsilon)c_{11}(\varepsilon) + c_{01}(\varepsilon)d_{10}(\varepsilon), \\ e_{01}(\varepsilon) =& c_{10}(\varepsilon) + d_{01}(\varepsilon) - \frac{c_{11}(\varepsilon)c_{00}(\varepsilon)}{c_{01}(\varepsilon)}, \\ e_{11}(\varepsilon) =& d_{11}(\varepsilon) - \frac{c_{11}(\varepsilon)c_{10}(\varepsilon)}{c_{01}(\varepsilon)} + \frac{c_{11}^2(\varepsilon)c_{00}(\varepsilon)}{c_{01}^2(\varepsilon)} \\ & + 2c_{20}(\varepsilon), \\ e_{20}(\varepsilon) =& - d_{11}(\varepsilon)c_{10}(\varepsilon) - d_{01}(\varepsilon)c_{20}(\varepsilon) \\ & + c_{11}(\varepsilon)d_{10}(\varepsilon) + c_{01}(\varepsilon)d_{20}(\varepsilon), \\ e_{02}(\varepsilon) =& \frac{c_{11}(\varepsilon)}{c_{01}(\varepsilon)}, \end{aligned}$$

and $N_2(x_2, y_2, \varepsilon)$ is the term of at least order three in x_2 and y_2.

Next, introduce a new time variable $dt = (1 - e_{02}(\varepsilon)x_2)d\tau$ and we still denote t as τ

$$\begin{cases} \dot{x}_2 = (1 - e_{02}(\varepsilon)x_2)y_2, \\ \dot{y}_2 = [e_{00}(\varepsilon) + e_{10}(\varepsilon)x_2 + e_{01}(\varepsilon)y_2 + e_{20}(\varepsilon)x_2^2 \\ \quad +e_{11}(\varepsilon)x_2 y_2 + e_{02}(\varepsilon)y_2^2 + N_2(x_2, y_2, \varepsilon)] \\ \quad \times(1 - e_{02}(\varepsilon)x_2). \end{cases} \tag{17}$$

Again by the transformation $(x_3, y_3) = (x_2, y_2(1 - e_{02}(\varepsilon)x_2))$, one has

$$\begin{cases} \dot{x}_3 = y_3, \\ \dot{y}_3 = f_{00}(\varepsilon) + f_{10}(\varepsilon)x_3 + f_{01}(\varepsilon)y_3 + f_{20}(\varepsilon)x_3^2 \\ \quad +f_{11}(\varepsilon)x_3 y_3 + N_3(x_3, y_3, \varepsilon), \end{cases} \tag{18}$$

where

$$\begin{aligned} f_{00}(\varepsilon) &= e_{00}(\varepsilon), \quad f_{10}(\varepsilon) = e_{10}(\varepsilon) - 2e_{00}(\varepsilon)e_{02}(\varepsilon), \\ f_{01}(\varepsilon) &= e_{01}(\varepsilon), \\ f_{20}(\varepsilon) &= e_{20}(\varepsilon) - 2e_{02}(\varepsilon)e_{10}(\varepsilon) + e_{00}(\varepsilon)e_{02}^2(\varepsilon), \\ f_{11}(\varepsilon) &= e_{11}(\varepsilon) - e_{01}(\varepsilon)e_{02}(\varepsilon), \end{aligned}$$

and $N_3(x_3, y_3, \varepsilon)$ is the term of at least order three in x_3 and y_3.

Next, there will be some classification discussions about $f_{20}(\varepsilon)$.

(a) If $f_{20}(\varepsilon) < 0$, then the transformation is $(x_4, y_4) = (x_3, \frac{y_3}{\sqrt{-f_{20}(\varepsilon)}})$ and $\tau = \sqrt{-f_{20}(\varepsilon)}t$, which will lead to

$$\begin{cases} \dot{x}_4 = y_4, \\ \dot{y}_4 = g_{00}(\varepsilon) + g_{10}(\varepsilon)x_4 + g_{01}(\varepsilon)y_4 - x_4^2 \\ \qquad + g_{11}(\varepsilon)x_4y_4 + N_4(x_4, y_4, \varepsilon), \end{cases} \tag{19}$$

where

$$g_{00}(\varepsilon) = -\frac{f_{00}(\varepsilon)}{f_{20}(\varepsilon)}, \quad g_{10}(\varepsilon) = -\frac{f_{10}(\varepsilon)}{f_{20}(\varepsilon)},$$
$$g_{01}(\varepsilon) = -\frac{f_{01}(\varepsilon)}{\sqrt{-f_{20}(\varepsilon)}}, \quad g_{11}(\varepsilon) = -\frac{f_{11}(\varepsilon)}{\sqrt{-f_{20}(\varepsilon)}},$$

and $N_4(x_4, y_4, \varepsilon)$ is the term of at least order three in x_4 and y_4.

Let $(x_5, y_5) = (x_4 - \frac{g_{10}(\varepsilon)}{2}, y_4)$, then system (19) becomes

$$\begin{cases} \dot{x}_5 = y_5, \\ \dot{y}_5 = h_{00}(\varepsilon) + h_{01}(\varepsilon)y_5 - x_5^2 + h_{11}(\varepsilon)x_5y_5 \\ \qquad + N_5(x_5, y_5, \varepsilon), \end{cases} \tag{20}$$

where

$$h_{00}(\varepsilon) = g_{00}(\varepsilon) + \frac{g_{10}^2(\varepsilon)}{4},$$
$$h_{01}(\varepsilon) = g_{01}(\varepsilon) + \frac{g_{10}(\varepsilon)g_{11}(\varepsilon)}{2}, \quad h_{11}(\varepsilon) = g_{11}(\varepsilon),$$

and $N_5(x_5, y_5, \varepsilon)$ is the term of at least order three in x_5 and y_5.

Here we suppose $f_{11}(\varepsilon) \neq 0$, then $h_{11}(\varepsilon) = g_{11}(\varepsilon) = \frac{f_{11}(\varepsilon)}{\sqrt{-f_{20}(\varepsilon)}} \neq 0$. Further, we apply the transformation $(x_6, y_6) = (h_{11}^2(\varepsilon)x_5, -h_{11}^3(\varepsilon)y_5)$ and $\tau = -\frac{1}{h_{11}(\varepsilon)}t$ to system (20) and obtain

$$\begin{cases} \dot{x}_6 = y_6, \\ \dot{y}_6 = i_1(\varepsilon) + i_2(\varepsilon)y_6 - x_6^2 + x_6y_6 + N_6(x_6, y_6, \varepsilon), \end{cases} \tag{21}$$

where

$$i_1(\varepsilon) = -h_{00}(\varepsilon)h_{11}^4(\varepsilon), \qquad i_2(\varepsilon) = -h_{01}(\varepsilon)h_{11}(\varepsilon),$$

and $N_6(x_6, y_6, \varepsilon)$ is the term of at least order three in x_6 and y_6.

(b) If $f_{20}(\varepsilon) > 0$, then the transformation is $(x_7, y_7) = (x_3, \frac{y_3}{\sqrt{f_{20}(\varepsilon)}})$, and $\tau = \sqrt{f_{20}(\varepsilon)}t$, which will give

$$\begin{cases} \dot{x}_7 = y_7, \\ \dot{y}_7 = \bar{g}_{00}(\varepsilon) + \bar{g}_{10}(\varepsilon)x_7 + \bar{g}_{01}(\varepsilon)y_7 - x_7^2 \\ \qquad + \bar{g}_{11}(\varepsilon)x_7y_7 + N_7(x_7, y_7, \varepsilon), \end{cases} \tag{22}$$

where

$$\bar{g}_{00} = -\frac{f_{00}(\varepsilon)}{f_{20}(\varepsilon)}, \quad \bar{g}_{10} = -\frac{f_{10}(\varepsilon)}{f_{20}(\varepsilon)},$$
$$\bar{g}_{01} = -\frac{f_{01}(\varepsilon)}{\sqrt{f_{20}(\varepsilon)}}, \quad \bar{g}_{11} = -\frac{f_{11}(\varepsilon)}{\sqrt{f_{20}(\varepsilon)}},$$

and $N_7(x_7, y_7, \varepsilon)$ is the term of at least order three in x_7 and y_7.

Furthermore, by the transformation $(x_8, y_8) = (x_7 + \frac{\bar{g}_{10}(\varepsilon)}{2}, y_7)$, one gets

$$\begin{cases} \dot{x}_8 = y_8, \\ \dot{y}_8 = \bar{h}_{00}(\varepsilon) + \bar{h}_{01}(\varepsilon)y_8 + x_8^2 + \bar{h}_{11}(\varepsilon)x_8y_8 \\ \qquad + N_8(x_8, y_8, \varepsilon), \end{cases} \tag{23}$$

where

$$\bar{h}_{00}(\varepsilon) = \bar{g}_{00}(\varepsilon) - \frac{\bar{g}_{10}^2(\varepsilon)}{4},$$
$$\bar{h}_{01}(\varepsilon) = \bar{g}_{01}(\varepsilon) - \frac{\bar{g}_{10}(\varepsilon)\bar{g}_{11}(\varepsilon)}{2}, \quad \bar{h}_{11}(\varepsilon) = \bar{g}_{11}(\varepsilon),$$

and $N_8(x_8, y_8, \varepsilon)$ is the term of at least order three in x_8 and y_8.

Similarly, supposing $f_{11}(\varepsilon) \neq 0$, then $\bar{h}_{11}(\varepsilon) = \bar{g}_{11}(\varepsilon) = \frac{f_{11}(\varepsilon)}{\sqrt{f_{20}(\varepsilon)}} \neq 0$. Applying transformation $(x_9, y_9) = (\bar{h}_{11}^2(\varepsilon)x_8, \bar{h}_{11}^3(\varepsilon)y_8)$ and $\tau = \frac{1}{\bar{h}_{11}(\varepsilon)}t$ to (23), one obtains

$$\begin{cases} \dot{x}_9 = y_9, \\ \dot{y}_9 = \bar{i}_1(\varepsilon) + \bar{i}_2(\varepsilon)y_9 + x^2 + xy + N_9(x_9, y_9, \varepsilon), \end{cases} \tag{24}$$

where

$$\bar{i}_1(\varepsilon) = \bar{h}_{00}(\varepsilon)\bar{h}_{11}^4(\varepsilon), \qquad \bar{i}_2(\varepsilon) = \bar{h}_{01}(\varepsilon)\bar{h}_{11}(\varepsilon),$$

and $N_9(x_9, y_9, \varepsilon)$ is the term of at least order three in x_9 and y_9.

In order to simplify the discussions, we still denote $\bar{i}_1(\varepsilon)$ and $\bar{i}_2(\varepsilon)$ as $i_1(\varepsilon)$ and $i_2(\varepsilon)$. Here with the help of the MATLAB, we have

$$\left|\frac{\partial(i_1(\varepsilon), i_2(\varepsilon))}{\partial(\varepsilon_1, \varepsilon_2)}\right|_{\varepsilon_1=\varepsilon_2=0} \neq 0.$$

So $i_1(\varepsilon)$ and $i_2(\varepsilon)$ are dependent. Then we could give the local representations of the bifurcation curves up to second-order approximation in the following ("+"for $f_{20}(\varepsilon) > 0$, "-"for $f_{20}(\varepsilon) < 0$):

(1) The saddle-node bifurcation curve $SN = \{(\varepsilon_1, \varepsilon_2) : g_1(\varepsilon_1, \varepsilon_2) = 0, g_2(\varepsilon_1, \varepsilon_2) \neq 0\}$;

(2) The Hopf bifurcation curve $H = \{(\varepsilon_1, \varepsilon_2) : g_2(\varepsilon_1, \varepsilon_2) = \pm\sqrt{-g_1(\varepsilon_1, \varepsilon_2)}, g_1(\varepsilon_1, \varepsilon_2) < 0\}$;

(3) The homoclinic bifurcation curve $HL = \{(\varepsilon_1, \varepsilon_2) : g_2(\varepsilon_1, \varepsilon_2) = \pm\frac{5}{7}\sqrt{-g_1(\varepsilon_1, \varepsilon_2)}, g_1(\varepsilon_1, \varepsilon_2) < 0\}$. □

4 Numerical simulation

In this section, we would like to demonstrate the complex dynamical behaviors through numerical simulation. Effectiveness of the theoretical analysis presented above could be confirmed through the phase portraits by using MATLAB. Here, m, b and d are parameters of system (2).

Example 1 Figure 1 shows the dynamical behavior of system (2) with given parameters $m = 0.3$ and $b = 0.3 + \sqrt{2}$. In this case, $d_{SN} = \frac{(m-b)^2}{4} = 0.5$. As shown in Fig. 1a, when $d = 0.6 > d_{SN}$, the system only has one boundary equilibrium $E_0 = (0, 1.0071)$. In Fig. 1b, when $d = 0.5 = d_{SN}$, the system has a boundary equilibrium $E_0 = (0, 3.4284)$ and a positive equilibrium $E_1 = (1.0071, 1.4142)$, which is a cusp. Saddle-node bifurcation may occur around E_1. In Fig. 1c, when $d < d_{SN}$, the system has a boundary equilibrium point $E_0 = (0, 4.2855)$ and two positive equilibrium $E_2 = (0.6909, 2.5583)$ and $E_3 = (1.3233, 0.9772)$.

Example 2 Figure 2 shows the dynamical behavior of system (2) with given parameter $m = 0.3$. In Fig. 2a, taking $b = 0.3 + \sqrt{2.4}$ and $d = 0.6$, the system has a unique positive equilibrium $E_1 = (1.0746, 1.2910)$, which is a saddle node with a stable parabolic sector. In Fig. 2b, taking $b = 0.3 + \sqrt{1.6}$ and $d = 0.4$, the system has a unique positive equilibrium point $E_1 = (0.9325, 1.5811)$, which is a saddle node with an unstable parabolic sector. As shown in Fig. 1b, with $b = 0.3 + \sqrt{2}$ and $d = 0.5$, the positive equilibrium point $E_1 = (1.0071, 1.4142)$ is a cusp of codimension 2. Therefore, the Bogdanov–Takens bifurcation may occur around E_1.

Example 3 Figure 3 shows the phase portrait of system (2) with parameters $m = 0.09$ and $b = 1.39$. At this point, $d_H = \frac{(b-m+1)(b-m-1)}{(b-m)^2} = 0.69/1.69$. In Fig. 3a, where $d = d_H$, the system has a boundary equilibrium and two positive equilibrium points $E_2 = (0.6208, 1.8841)$ and $E_3 = (0.8592, 1.3000)$. E_2 is a saddle point, while E_3 is a fine focus. In Fig. 3b, where $d = 0.2 < d_H$, the system has a boundary equilibrium and two positive equilibrium points $E_2 = (0.2683, 5.6085)$ and $E_3 = (1.2117, 0.8915)$. E_2 is a saddle point, while E_3 is a sink. In Fig. 3c, where $d = 0.4 > d_H$, the point $E_3(0.89, 1.25)$ is a source. So the system may undergo the Hopf bifurcation near E_3.

Example 4 Take parameters $m = 0.3, b = 0.3 + \sqrt{2}, d = 0.5$, then the Bogdanov–Takens bifurcation occurs around E_1. The bifurcation thresholds are $m_1 = m$ and $d_1 = d$. Then we have

$$\left|\frac{\partial(i_1(\varepsilon), i_2(\varepsilon))}{\partial(\varepsilon_1, \varepsilon_2)}\right|_{\varepsilon_1=\varepsilon_2=0} = -16.5324 \neq 0.$$

Therefore the rank of matrix $\left|\frac{\partial(i_1(\varepsilon), i_2(\varepsilon))}{\partial(\varepsilon_1, \varepsilon_2)}\right|_{\varepsilon_1=\varepsilon_2=0}$ is 2. For sufficiently small ε_i $(i = 1, 2)$, the local representation of the bifurcation curves could be approximated as follows:

$$\begin{aligned} SN = \{(\varepsilon_1, \varepsilon_2) :& -17.3481\varepsilon_1^2 - 75.8515\varepsilon_1\varepsilon_2 \\ &+ 4.7011\varepsilon_1 - 82.9553\varepsilon_2^2 + 9.9123\varepsilon_2 = 0, \\ &\varepsilon_2 \neq 0\}, \\ H = \{(\varepsilon_1, \varepsilon_2) :& -14.4343\varepsilon 1^2 - 53.4060\varepsilon_1\varepsilon_2 \\ &+ 4.3203\varepsilon_1 - 36.1388\varepsilon_2^2 + 8.1175\varepsilon_2 = 0, \varepsilon_2 < 0\}, \\ HL = \{(\varepsilon_1, \varepsilon_2) :& -5.9373\varepsilon_1^2 - 16.2542\varepsilon_1\varepsilon_2 \\ &+ 2.0177\varepsilon_1 + 4.4923\varepsilon_2^2 + 3.2624\varepsilon_2 = 0, \varepsilon_2 < 0\}. \end{aligned}$$

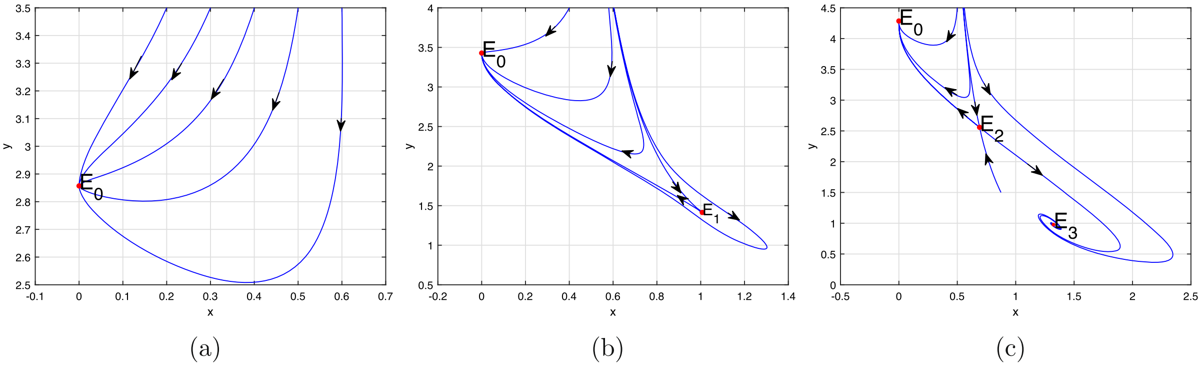

Fig. 1 Dynamics of system (2) with parameters $m = 0.3$ and $b = 0.3 + \sqrt{2}$. **a** $d = 0.6$; **b** $d = 0.5$; **c** $d = 0.4$

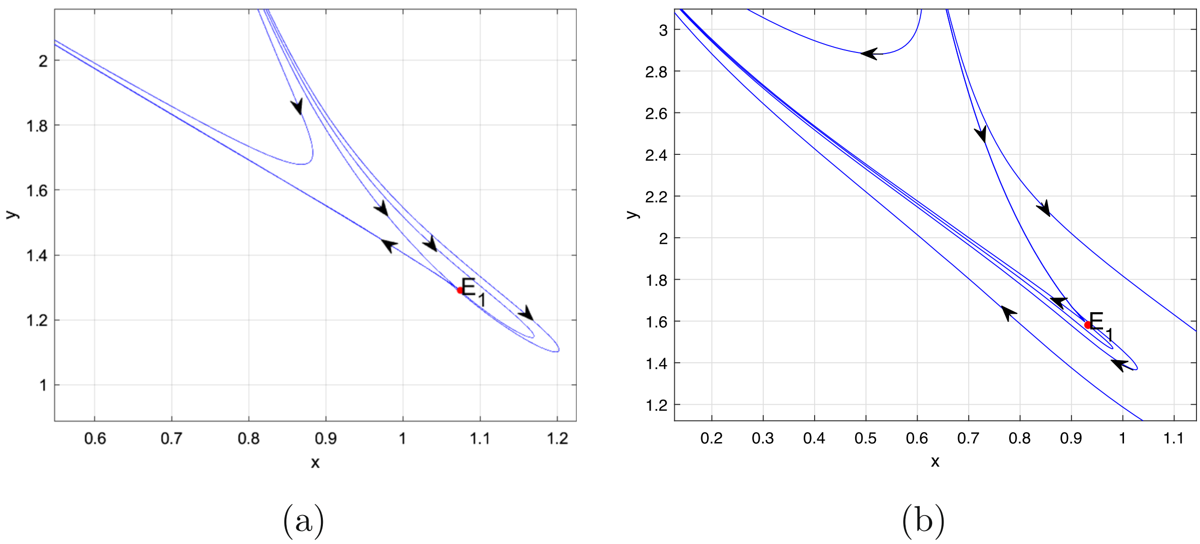

Fig. 2 Dynamics of system (2) with parameters $m = 0.3$. **a** $b = 0.3 + \sqrt{2.4}$, $d = 0.6$; **b** $b = 0.3 + \sqrt{1.6}$, $d = 0.4$

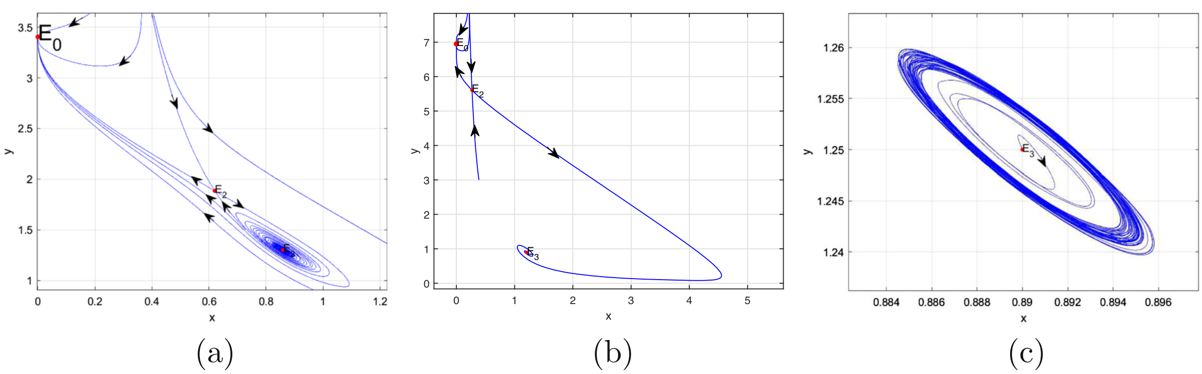

Fig. 3 Phase portraits

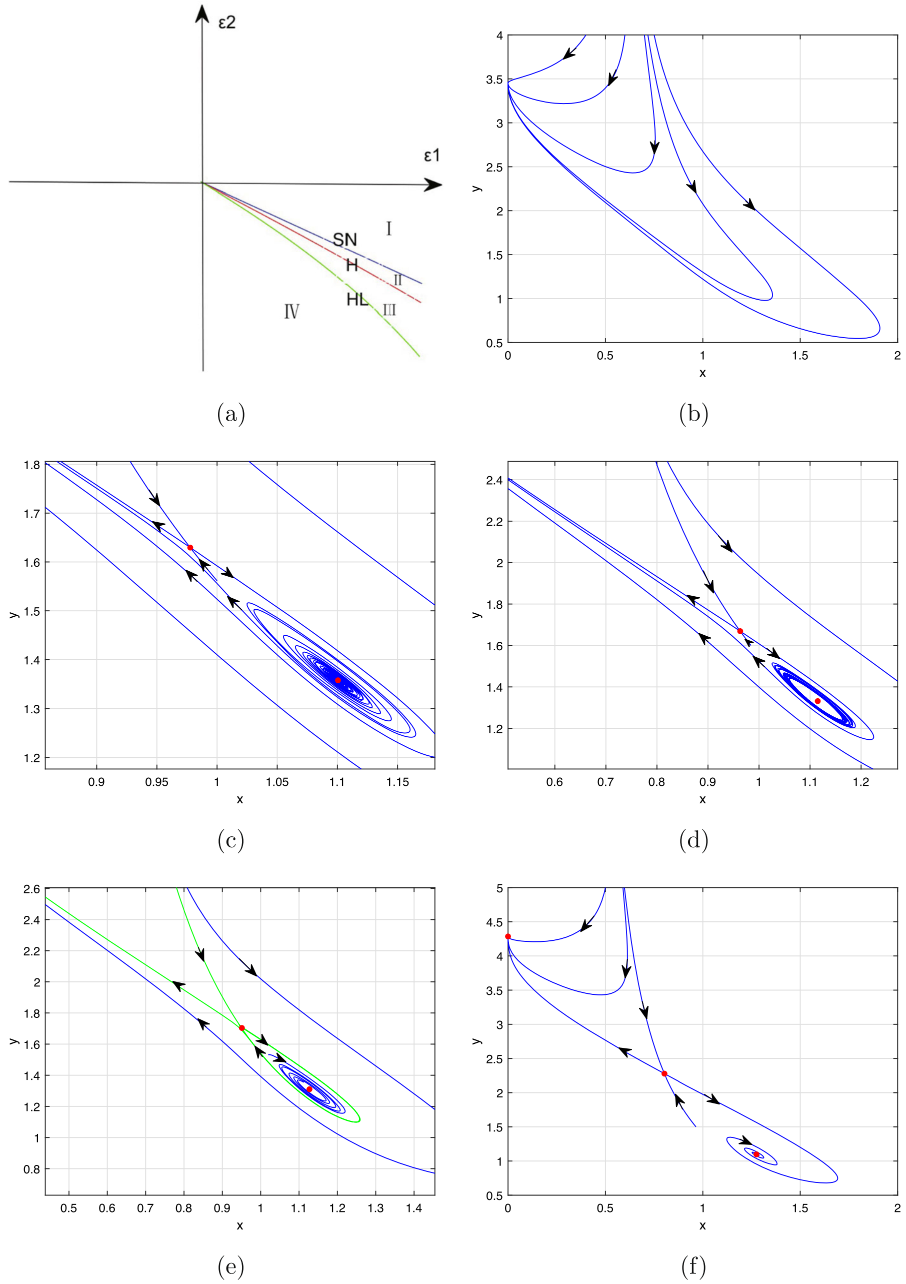

Fig. 4 **a** The subcritical Bogdanov–Takens bifurcation diagram of system (13); **b** When $(\varepsilon_1, \varepsilon_2) = (0.1, -0.0033)$, the system has no positive equilibrium point in region I; **c** When $(\varepsilon_1, \varepsilon_2) = (0.064, -0.048)$, the system has an unstable focus in region II; **d** When $(\varepsilon_1, \varepsilon_2) = (0.064, -0.05)$, the system has an unstable limit cycle in region III; **e** When $(\varepsilon_1, \varepsilon_2) = (0.064, -0.05280294849)$, the system has an unstable homoclinic orbit on curve HL; **f** When $(\varepsilon_1, \varepsilon_2) = (0.064, -0.1)$, the system has a stable focus in region IV

Figure 4 shows the subcritical Bogdanov–Takens bifurcation diagram and phase portraits of system (13). The conclusions are as follows.

(a) The bifurcation curves SN, H and HL divide the $(\varepsilon_1, \varepsilon_2)$-plane into four regions, which rotate clockwise around the critical parameter values of the Bogdanov–Takens bifurcation $(\varepsilon_1, \varepsilon_2) = (0, 0)$, as shown in Fig. 4a.

(b) In Fig. 4b, when the parameters are in region I, the system has no positive equilibrium point.

(c) When the parameter is on the curve SN, the system has a positive equilibrium point, which is a saddle node.

(d) When the parameter crosses the curve SN and enters region II, passing through the saddle-node bifurcation, the system has two positive equilibrium points, one unstable focus and the other a saddle point (See Fig. 4c).

(e) When the parameter is on the curve H, there are two positive equilibrium points, one unstable fine focus and the other a saddle point.

(f) When the parameter crosses the curve H and enters region III, passing through the subcritical Hopf bifurcation, an unstable limit cycle appears, with the focus being stable (See Fig. 4d).

(g) When the parameter crosses region III and is on the curve HL, passing through the homoclinic bifurcation, an unstable homoclinic orbit containing a stable focus appears (See Fig. 4e).

(h) When the parameter crosses the curve HL and enters region IV, the homoclinic orbit breaks and a stable focus and a saddle point appear (See Fig. 4f).

5 Conclusion

An enzyme-catalyzed reaction model is formulated in this work. Existence and their stability of equilibrium points, and the bifurcations, including the saddle-node bifurcation, the Hopf bifurcation and the Bogdanov–Takens bifurcation of codimension 2, in the system are presented. By using the stability theory, the existence and stability of equilibrium points of system (2) are provided. Moreover, the detailed bifurcation behaviors of the model are discussed through bifurcation theory, which includes the Sotomayor's theorem, Hopf analysis and perturbed theory. Numerical simulations are used to validate the results obtained. Specifically, compared to the original temporal Gray–Scott model, the system (2) exhibits richer dynamic behaviors. Effects of the diffusion on the model will be still interesting and could be further explored.

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Data Availibility All data generated or analysed during this study are included in this article.

Declarations

Conflict of interest The authors declare that they have no financial interests.

References

1. Turing, A.M.: The chemical basis of morphogenesis. Philos. Trans. R. Soc. B **237**, 1934–1990 (1952)
2. Gray, P., Scott, S.K.: Autocatalytic reactions in the isothermal, continuous stirred tank reactor: isolas and other forms of multistability. Chem. Eng. Sci. **38**, 29–43 (1983)
3. Gray, P., Scott, S.K.: Autocatalytic reactions in the isothermal, continuous stirred tank reactor: oscillations and instabilities in the system A+2B⟶3B; B⟶C. Chem. Eng. Sci. **39**, 1087–1097 (1984)
4. Gray, P., Scott, S.K.: Sustained oscillations and other exotic patterns of behavior in isothermal reactions. J. Phys. Chem. **89**, 22–32 (1985)
5. Chen, S., Shi, J., Chen, G.: Spatial pattern formation in activator-inhibitor models with nonlocal dispersal. Discrete Contin. Dyn. Syst. Ser. B **26**, 1843–1866 (2021)
6. Dong, X., Wang, C.: Identification of the Gray-Scott model via deterministic learning. Int. J. Bifurc. Chaos **31**, 2150051 (2021)
7. Gandhi, P., Zelnik, Y., Knobloch, E.: Spatially localized structures in the Gray–Scott model. Philos. Trans. R. Soc. A **376**(2018), 20170375 (2018)
8. Saadi, F., Champneys, A.: Unified framework for localized patterns in reaction–diffusion systems, the Gray–Scott and Gierer–Meinhardt cases. Philos. Trans. R. Soc. A **379**, 20200277 (2021)
9. Kuznetsov, Y., Meijer, H., Al-Hdaibat, B., Govaerts, W.: Accurate approximation of homoclinic solutions in Gray–Scott kinetic model. Int. J. Bifurc. Chaos **25**, 1550125 (2015)
10. Yuan, P., Zhu, H.: The nilpotent bifurcations in a model for generalist predatory mite and pest leafhopper with stage structure. J. Differ. Equ. **321**, 99–129 (2022)
11. Xiang, C., Lu, M., Huang, J.: Degenerate Bogdanov–Takens bifurcation of codimension 4 in Holling–Tanner model with harvesting. J. Differ. Equ. **314**, 370–417 (2022)
12. Jiao, J., Chen, C.: Bogdanov–Takens bifurcation analysis of a delayed predator-prey system with double Allee effect. Nonlinear Dyn. **104**, 1697–1707 (2021)
13. Su, J., Lu, M., Huang, J.: Bifurcations in a dynamical model of the innate immune system response to initial pulmonary infection. Qual. Theory Dyn. Syst. **21**, 41 (2022)
14. Baxter, R., Hu, P.: Insight into why the Langmuir-Hinshelwood mechanism is generally preferred. J. Chem. Phys. **116**, 4379–4381 (2002)
15. Zhang, Z., Ding, T., Huang, W., Dong, Z.: Qualitative Theory of Differential Equations. Science Press, Beijing (1992)
16. Sotomayor, J.: Generic bifurcations of dynamical system. In: Dynamical systems. Academic Press, pp. 561–582 (1973)

17. Perko, L.: Differential Equations and Dynamical Systems, 3rd edn. Texts in Applied Mathematics, Springer, New York (2001)
18. Wiggins, S., Golubitsky, M.: Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer, New York (2003)
19. Chen, T., Li, S., Llibre, J.: Phase portraits and bifurcation diagram of the Gray–Scott model. J. Math. Anal. Appl. **496**, 124840 (2021)

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