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RESEARCH

Predefined-time adaptive fuzzy control of pure-feedback nonlinear systems under input and output quantization

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Abstract For a class of pure-feedback nonlinear systems, a novel adaptive predefined-time fuzzy control strategy is researched in this paper. Different from the existing nonlinear systems with predefined-time control, both the input and output signals are quantized in this strategy. In the control process, firstly, by utilizing the Butterworth low-pass filter technique to handle the form of pure-feedback and FLSs to approximate the unknown functions, a novel fuzzy state observer is devised to estimate the immeasurable states. Secondly, in the traditional backstepping process, the virtual control signals are usually differentiable. Due to the discontinuity of the output quantization, they are not differentiable which makes the traditional backstepping method not applicable. To handle this issue, a command filtering technique is applied in this strategy. Thirdly, by using a class of smooth functions, an intermediate auxiliary control signal and a novel adaptive predefined-time controller are constructed. Moreover, to compensate the impact of quantization errors, Lemma 9 is proved. On this basis, the proposed strategy can ensure the systems under input and output quantization are practical predefined-time stable. Lastly, an example is applied to demonstrate the feasibility of this strategy.

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1 Introduction

In the field of modern control, convergence is an important index to measure stability. To realize a faster convergence speed, the finite/fixed-time stability has raised the attention of many scholars [1–4], which makes the closed-loop system's state converge to the equilibrium point within a finite/fixed time. For the past few years, based on the good approximation capability of fuzzy logic systems (FLSs) /neural network (NNs), many significant results about adaptive intelligent finite/fixed-time control have been obtained [5–8]. Among them, for single-input single-output (SISO) nonlinear systems, considering the full-state constraints and actuator failures, Zhang et al. [5] developed an adaptive neural finite-time control approach; for multiple-input multiple-output (MIMO) nonlinear systems, the issue of the nonsingular fixed-time output feedback control is resolved in literature [6]; for stochastic nonlinear systems, an adaptive fuzzy finite-time control scheme is presented by Sui et al. [7]; for multi-agent nonlinear systems, Wu et al. [8] investigated a fixed-time fuzzy consensus control strategy. Notably, the bounds on the settling times in above results [5–8] can not be arbitrarily set. However, many engineering applications, for example autonomous vehicle rendezvous and missile guidance [9], need to ensure the desired performance of the system in preset time, which motivates the research about predefined-time control.

A predefined-time stability represents a particular case of the fixed time stability where the convergence time can be chosen a priori. It is presented by Sánchez-Torres et al. [10]. Subsequently, a sufficient condition of predefined-time stability is put forward in literature [11]. Based on the excellent features of predefined-time stability, its applications in robotics, rigid spacecraft and other fields have been researched, such as [12–15]. Noting that the nonlinearities of systems in above literatures [12–15] are known or need to satisfy the linear growth conditions. With the development of the complex systems, the nonlinearities of practical control systems are often unknown. Therefore, based on FLSs/NNs, the issue of adaptive intelligent predefined-time control has raised attention of many scholars. In particular, for the unknown strict-feedback nonlinear systems (SFNSs), the issue of an adaptive fuzzy predefined-time control is researched in literature [16]. On the basis of [16] and considering the impact of input saturation and output hysteresis, Zhang et al. [17] developed a predefined-time adaptive fuzzy controller. It is well known that the pure-feedback nonlinear systems refer to a more general class of nonlinear systems which have no affine appearance of the state variables. Furthermore, [18] presented a global adaptive NNs control algorithm for unknown pure-feedback systems to achieve zero tracking error within a predefined time. Notably, the signal quantization is not considered in the above predefined-time outcomes [16–18]. However, in networks systems, signals are required to quantize before transmission due to the limited communication capacity. With the wide application of networks systems, quantization has attracted extensive attention.

Quantization can be viewed as a mapping from a continuous set to a discrete set. In the quantized control, due to the nonlinear feature of quantization, the performance and stability of the system are affected. Thus, it is significant to ensure the system's stability while guaranteeing the relatively communication rates. In recent years, for the nonlinear systems, many significant outcomes about adaptive quantized control have been developed [19–21]. Notably, the results in [19–21] only considered the quantization in the input channel, which means that the control process still depends on continuous states/output. Actually, in practical remote control systems, control input signal and sensor information are communicated by network. Due to the limited bandwidth of the communication channel, both the control input and states/output signals should be quantized before transmission. To better meet the needs of practical engineering, by combining dynamic filtering technology with backstepping technology, the issue of the adaptive output feedback control for the nonlinear systems under input and output quantization is resolved in literature [22]. Based on [22], considering the effect of the sensor failures, an adaptive quantized output controller is established in [23]. Notably, the systems in [22,23] are linear parameterizations. In addition, their parameters are bounded as a prior knowledge and nonlinear functions need to satisfy global Lipschitz continuity condition. To remove these limitations, for a class of unknown SFNSs with output quantization, Lu et al. [24] raised an adaptive fuzzy output feedback control strategy by using FLSs to approximate the unknown nonlinearities. It is mentioning that the above literatures [22–24] only ensure asymptotic stability of SFNSs. However, in practical engineering, the pure-feedback nonlinear systems are more general than SFNSs. Moreover, the stability of the system is expected to be achieved within a predefined time.

Although the problems of predefined-time control and the output quantization of SFNSs have been solved separately, it is difficult to design an adaptive predefined-time control strategy for unknown pure-feedback nonlinear systems with input and output quantization. There are three main difficulties: firstly, the existing state observers under input and output quantization are only applied to SFNSs, in which the variables x_{j+1} are affine in the $\dot{x}_j (1 \leq j \leq m-1)$ equations. Since the pure-feedback nonlinear systems are non-affine systems, these observers are unsuitable. Thus, how to design a corresponding state observer to estimate the immeasurable states? Secondly, in the existing predefined-time control processes, the virtual control signals must be differentiable. However, when output quantization is applied to establish the virtual control signals in each recursive step, the virtual control signals are discontinuous and their derivatives cannot be calculated as often done in standard backstepping design progress. This means that the traditional backstepping progress is not applicable. Thus, how to design the control progress to obtain the controller? Thirdly, the existing adaptive predefined-time control results are often used the term "$-\frac{\varrho_i \bar{\sigma}_i^2}{\sqrt{\varrho_i^2 \bar{\sigma}_i^2 + \sigma_i^2}}$" to avoid the singularity problem. However, when the input and output signals are quantized, it is difficult to deal with the quantization errors generated in the above term. Thus, how to construct the controller to handle this difficulty and ensure the practical predefined-time stability of systems?

In this article, the above difficulties are overcome. The main contributions are shown below:

(1) A novel adaptive fuzzy predefined-time control scheme is presented in this article. Compared with the existing predefined-time control scheme [21] with input quantization, the input and output signals are both quantized in this scheme. In the backstepping process, because of the quantized output's discontinuity, a command filtering technique is applied to avoid the partial derivatives of the virtual control signals. On this basis and a class of smooth functions, an intermediate auxiliary control signal is established and a novel adaptive predefined-time controller is obtained.

(2) Different from the existing results under input and output quantization [22–24], an unknown pure-feedback nonlinear system under input and output quantization is researched in this article, which is more general than the systems in [22–24]. By utilizing the Butterworth low-pass filter technique to handle the form of pure-feedback and FLSs to approximate the unknown functions, a novel fuzzy state observer is devised to estimate the immeasurable states. Moreover, Lemma 9 is presented to compensate the quantization errors. Based on Lemma 9, the practical predefined-time stability of system is ensured, which can achieve a faster converge speed than [22–24].

The rest of this article is organized as below: Preparations and problem formulation are displayed in Sect. 2. In Sect. 3, an predefined-time adaptive fuzzy quantized control scheme is proposed. Stability analysis is shown in Sect. 4. Section 5 represents the simulation example. Conclusion is summarized in Sect. 6.

2 Preparations and problem formulation

2.1 Definitions and lemmas

Definition 1 [10]. The equilibrium point $\upsilon = 0$ of the system $\dot{\upsilon} = \hbar(\upsilon)$ is practical predefined-time stable (PPTS) if there exist the constants $\varpi > 0$ and $T^* > 0$ to make $\|\upsilon(t)\| \leq \varpi$, for all $t \geq T^*$. Where $\upsilon \in \Re^m$ and $\hbar(\upsilon) : \Re^m \to \Re^m$ represent the state variable and nonlinear function, respectively.

Lemma 1 [16]. *If there exist Lyapunov function V and constants $\gamma \in (0, 1)$; $\Gamma_3 > 0$ such that*

$$\dot{V} \leq -\frac{\pi}{\gamma T^*}\left(V^{1+\frac{\gamma}{2}} + V^{1-\frac{\gamma}{2}}\right) + \Gamma_3 \tag{1}$$

then the system $\dot{\upsilon} = \hbar(\upsilon)$ is PPTS, and V can maintain in the area $V \leq \frac{\gamma T^* \Gamma_3}{\pi}$ within a predefined time $2T^*$.

Definition 2 [25]. *Uniform Quantizer*: A uniform quantizer is represented as

$$q(\upsilon) = \begin{cases} \delta_i sgn(\upsilon), \delta_i - \frac{b}{2} < |\upsilon| \leq \delta_i + \frac{b}{2}, \\ 0, |\upsilon| \leq \delta_0, \end{cases} \tag{2}$$

where $i = 1, \ldots, n$. The parameter b stands for the quantization interval's length, $\delta_0 = \frac{b}{2}$ determines the size of the deadzone for $q(\upsilon)$, $\delta_1 = \delta_0 + \frac{b}{2}$, $\delta_{i+1} = \delta_i + b$. The following property is met:

$$|q(\upsilon) - \upsilon| \leq \tau_\upsilon \tag{3}$$

where $\tau_\upsilon \geq \frac{b}{2}$.

Lemma 2 [26]. *For $\forall \chi \geq \upsilon > 0$ and $\jmath > 0$, one obtains*

$$\upsilon(\chi - \upsilon)^\jmath \leq \frac{\jmath}{1+\jmath}(\chi^{\jmath+1} - \upsilon^{\jmath+1}). \tag{4}$$

Lemma 3 [27]. *For $\upsilon_j \in \Re$, one has*

$$\sum_{j=1}^{m} |\upsilon_j|^p \geq \begin{cases} \left(\sum_{j=1}^{m} |\upsilon_j|\right)^p, 0 < p \leq 1 \\ \frac{1}{m^{p-1}}\left(\sum_{j=1}^{m} |\upsilon_j|\right)^p, p > 1 \end{cases} \tag{5}$$

Lemma 4 [28]. *For real variables υ and $\check{\upsilon}$, and any constants $\iota \in (0, 1)$ and $\ell > 0$, we have*

$$|\upsilon|^\iota |\check{\upsilon}|^{1-\iota} \leq \iota \ell |\upsilon| + (1-\iota)\ell^{\frac{-\iota}{1-\iota}} |\check{\upsilon}|. \tag{6}$$

Lemma 5 [29]. *For real variables $\upsilon \geq 0$ and $\check{\upsilon} \geq 0$, and a constant $\prime \geq 1$, the following inequalities hold:*

$$\begin{aligned} &|\upsilon^{1/\prime} - \check{\upsilon}^{1/\prime}| \leq 2^{1-1/\prime} |\upsilon - \check{\upsilon}|^{1/\prime}; \\ &|\upsilon^\prime - \check{\upsilon}^\prime| \leq b_1 |\upsilon - \check{\upsilon}| \left(|\upsilon - \check{\upsilon}|^{\prime-1} + \check{\upsilon}^{\prime-1}\right), \end{aligned} \tag{7}$$

where $b_1 > 0$ denotes a constant.

Lemma 6 [30]. *If there is a bounded function $\tilde{\upsilon}$ such that $|\tilde{\upsilon}_j| \leq \upsilon_j^*$ with υ_j^* being a boundary. For $\mu_1 \in (0, 1)$, $\mu_2 > 1$, $q_j > 1$, one has*

$$-\sum_{j=1}^{m} \frac{\tilde{\upsilon}_j^2}{q_j} \leq -\Big(\sum_{j=1}^{m} \frac{\tilde{\upsilon}_j^2}{2q_j}\Big)^{\mu_1} - \frac{1}{m^{\mu_2-1}}\Big(\sum_{j=1}^{m} \frac{\tilde{\upsilon}_j^2}{2q_j}\Big)^{\mu_2} + C, \tag{8}$$

where $C = \sum_{j=1}^{m} \left(\frac{(\upsilon_j^*)^2}{2q_j}\right)^{\mu_2} + (1-\mu_1)\mu_1^{\frac{\mu_1}{1-\mu_1}}$.

Lemma 7 [31]. *If a matrix $\Upsilon_1 \in \Re^{m\times m}$ is stable, it yields*

$$\|e^{\Upsilon_1}\| \leq \omega_1 e^{-\omega_2 t} \tag{9}$$

where $\omega_1 = \sqrt{\lambda_{\max}(\Upsilon_2)/\lambda_{\min}(\Upsilon_2)}$ and $\omega_2 = 1/\lambda_{\max}(\Upsilon_2)$; Υ_2 is a symmetric positive definite matrix and it meets $\Upsilon_1^T\Upsilon_2 + \Upsilon_2\Upsilon_1 = -2I$.

2.2 System description

In this article, considering a pure-feedback nonlinear system under input and output quantization as follows:

$$\begin{cases} \dot{\upsilon}_i = \hbar_i(\bar{\upsilon}_i, \upsilon_{i+1}), 1 \leq i \leq m-1, \\ \dot{\upsilon}_m = \hbar_m(\bar{\upsilon}_m, q(u)), \\ y = \upsilon_1, \end{cases} \tag{10}$$

where $\bar{\upsilon}_i = [\upsilon_1, \ldots, \upsilon_i]^T \in \Re^i$, $y \in \Re$ and $u \in \Re$ represent the state vector, the output and control input of the system, respectively; $\hbar_i(\cdot) \in \Re$ is an unknown smooth function. $q(u)$ denotes the output of the quantizer (2) and takes the quantized value.

Remark 1 If system is unknown SFNSs, the adaptive fuzzy control with output quantization was researched in literature [24]. Moreover, the adaptive fuzzy predefined-time control with input quantization was investigated in literature [21]. However, system (10) studied in this paper is in pure-feedback form, which is more general than the strict-feedback form. To the best of our knowledge, so far, there are no results on predefined-time control with input and output quantization for system (10).

Control Objective: In this paper, under input and output quantization, the objective is to establish an adaptive fuzzy predefined-time quantized controller that guarantees the practical predefined-time stability of system (10).

Select the following function transformations:

$$\hbar_i(\bar{\upsilon}_i, \upsilon_{i+1}) = H_i(\bar{\upsilon}_i, \upsilon_{i+1}) + \upsilon_{i+1}, 1 \leq i \leq m-1$$
$$\hbar_m(\bar{\upsilon}_m, q(u)) = H_m(\bar{\upsilon}_m, q(u)) + q(u) \tag{11}$$

Thus, the system (10) is equivalent as follows:

$$\begin{cases} \dot{\upsilon}_i = H_i(\bar{\upsilon}_i, \upsilon_{i+1}) + \upsilon_{i+1}, 1 \leq i \leq m-1, \\ \dot{\upsilon}_m = H_m(\bar{\upsilon}_m, q(u)) + q(u), \\ y = \upsilon_1, \end{cases} \tag{12}$$

Assumption 1: Suppose that the function $\hbar_i$ meets the global Lipschitz condition, i.e., there exists a constant $\varpi_i (1 \leq i \leq m)$ such that the following inequality is met:

$$|\hbar_i(\chi_1) - \hbar_i(\chi_2)| \leq \varpi_i \|\chi_1 - \chi_2\|, \forall \chi_1, \chi_2 \in \Re^i. \tag{13}$$

2.3 Fuzzy logic systems

Lemma 8 [32]. *For a continuous function $H(\upsilon)$ on a compact set Ξ, there exists an FLS $\vartheta^T\varphi(\upsilon)$, it yields:*

$$\sup_{\upsilon\in\Xi} |H(\upsilon) - \vartheta^T\varphi(\upsilon)| \leq \epsilon, \tag{14}$$

in which $\epsilon > 0$ denotes the fuzzy minimum approximation error; $\upsilon = [\upsilon_1, \upsilon_2, ..., \upsilon_m]^T \in \Re^m$ stands for the FLS's input; $\vartheta^T = [\vartheta_1, \vartheta_2, \ldots, \vartheta_M]$ with M standing for the fuzzy rules' number; $\varphi(\upsilon) = (\varphi_1(\upsilon), \varphi_2(\upsilon), \ldots, \varphi_M(\upsilon))^T$ denotes a fuzzy basic function vector and the following $\varphi_o(\upsilon)$ is chosen:

$$\varphi_o(\upsilon) = \frac{\prod_{j=1}^{m} \mu_{A_j^o}(\upsilon_j)}{\sum_{o=1}^{M}[\prod_{j=1}^{m} \mu_{A_j^o}(\upsilon_j)]}$$

where $A_j^o(o = 1, \ldots, M, j = 1, \ldots, m)$ represents the fuzzy set and $\mu_{A_j^o}$ denotes the membership function.

3 Predefined-time adaptive fuzzy quantized control design

3.1 Fuzzy state observer design

Firstly, rewrite the system (12) as

$$\begin{cases} \dot{\upsilon}_i = H_i(\hat{\bar{\upsilon}}_i, \hat{\upsilon}_{i+1,f}) + \upsilon_{i+1} + \Delta H_i, 1 \leq i \leq m-1, \\ \dot{\upsilon}_m = H_m(\hat{\bar{\upsilon}}_m, q(u)_f) + q(u) + \Delta H_m, \\ y = \upsilon_1, \end{cases} \tag{15}$$

where $\Delta H_i = H_i(\bar{\upsilon}_i, \upsilon_{i+1}) - H_i(\hat{\bar{\upsilon}}_i, \hat{\upsilon}_{i+1,f})$; $\hat{\upsilon}_i$ denotes the estimation of υ_i; $\hat{\upsilon}_{i+1,f}, q(u)_f$ represent the signal filters defined by [34,35], as shown below:

$$\hat{\upsilon}_{i+1,f} = H_L(s)\hat{\upsilon}_{i+1} \approx \hat{\upsilon}_{i+1},$$
$$q(u)_f = H_L(s)q(u) \approx q(u)$$

with $H_L(s)$ standing for a Butterworth low-pass filter.

Assumption 2: An unknown constant $\wp_{f_M} > 0$ exists and satisfies $|\hat{\upsilon}_{i+1} - \hat{\upsilon}_{i+1,f}| \leq \wp_{f_M}$; $|q(u) - q(u)_f| \leq \wp_{f_M}$, $i = 1, \ldots, m-1$.

Utilizing the FLSs to approximate the functions $H_i(\hat{\bar{\upsilon}}_i, \hat{\upsilon}_{i+1,f})$ and $H_m(\hat{\bar{\upsilon}}_m, q(u)_f)$. According to Lemma 8 and the quantized output $q(y)$, a fuzzy state observer is devised, as shown below:

$$\begin{cases} \dot{\hat{\upsilon}}_i = \hat{\upsilon}_{i+1} + \hat{\vartheta}_i^T \varphi_i(\hat{\bar{\upsilon}}_i, \hat{\upsilon}_{i+1,f}) + k_i(q(y) - \hat{y}), \\ \dot{\hat{\upsilon}}_m = q(u) + \hat{\vartheta}_m^T \varphi_m(\hat{\bar{\upsilon}}_m, q(u)_f) + k_m(q(y) - \hat{y}), \\ \hat{y} = \hat{\upsilon}_1, \end{cases} \tag{16}$$

where k_j stands for the design parameter and it makes the matrix

$$A_c = \begin{bmatrix} -k_1 & \\ \vdots & I_{m-1} \\ -k_m & \ldots \; 0 \end{bmatrix} \tag{17}$$

a strict Hurwitz matrix. Thus, there exist a symmetric positive definite matrix G and a constant $d > 0$, A_c satisfies the following equality:

$$A_c^T G + G^T A_c = -dI. \tag{18}$$

The observation error $e = [e_1, \ldots, e_m]^T$ with $e_j = \upsilon_j - \hat{\upsilon}_j$ being defined. According to (15) – (17), one obtains

$$\begin{aligned} \dot{e} &= A_c e + \sum_{j=1}^{m} B_j \Big(\tilde{\vartheta}_j^T \varphi_j(Z_j) + \epsilon_j + \Delta H_j \Big) \\ &\quad + K(y - q(y)), \end{aligned} \tag{19}$$

where $B_j = [\underbrace{0, \ldots, 1}_{j}, \ldots, 0]^T$, $K = [k_1, \ldots, k_m]^T$, $\tilde{\vartheta}_j = \vartheta_j^* - \hat{\vartheta}_j$ and $Z_i = [\hat{\bar{\upsilon}}_i, \hat{\upsilon}_{i+1,f}] (i = 1, 2, \ldots, m-1)$, $Z_m = [\hat{\bar{\upsilon}}_m, q(u)_f]$.

Select a Lyapunov function candidate as follows:

$$V_0 = e^T G e. \tag{20}$$

According to the Assumptions 1–2, the property (3) and Young's inequality, the derivative of (20) holds

$$\begin{aligned} \dot{V}_0 &= e^T (A_c^T G + G^T A_c) e + 2e^T G \Big[\sum_{j=1}^{m} B_j \Big(\tilde{\vartheta}_j^T \varphi_j(Z_j) \\ &\quad + \epsilon_j + \Delta H_j \Big) + K(y - q(y)) \Big] \\ &\leq -(d - 4\|G\|^2 - \sum_{j=1}^{m} \varpi_j^2) \|e\|^2 + \sum_{j=1}^{m} \tilde{\vartheta}_j^T \tilde{\vartheta}_j \\ &\quad + \|K\|^2 \tau_y^2 + \sum_{j=1}^{m} \Big((\epsilon_j^*)^2 + \varpi_j^2 \wp_{f_M}^2 \Big) \\ &\leq -(d - 4\|G\|^2 - \sum_{j=1}^{m} \varpi_j^2) \|e\|^2 \\ &\quad + \sum_{j=1}^{m} \tilde{\vartheta}_j^T \tilde{\vartheta}_j + \Gamma_0, \end{aligned} \tag{21}$$

where $\Gamma_0 = \|K\|^2 \tau_y^2 + \sum_{j=1}^{m} \big((\epsilon_j^*)^2 + \varpi_j^2 \wp_{f_M}^2 \big)$.

3.2 Adaptive fuzzy predefined-time quantized control process

The quantized output variable cannot be directly utilized in the Lyapunov-based backstepping design due to its discontinuity. Thus, the control process is divided into two parts. Firstly, in *(i)*, incorporating second-order low-pass filters with a class of smooth functions, an intermediate auxiliary control signal will be devised by applying the unquantized output vector y and the state estimation $\hat{\upsilon}_j (j = 1, \ldots, m)$. Secondly, by replacing the output y in the virtual signals and intermediate auxiliary control signal with the quantized output $q(y)$, an actual adaptive predefined-time controller will be obtained in *(ii)*.

(i) In this section, by utilizing the continuous output y before quantization, an adaptive predefined-time fuzzy control scheme is presented with a class of smooth functions and FLSs.

Define the following coordinate transformation:

$$\begin{aligned} \varrho_1 &= y, \\ \varrho_{\iota+1} &= \hat{\upsilon}_{\iota+1} - \hat{\alpha}_{\iota,1}, \\ \tilde{\alpha}_{\iota,1} &= \hat{\alpha}_{\iota,1} - \alpha_\iota, \end{aligned} \tag{22}$$

where $\tilde{\alpha}_{\iota,1}$ stand for filtering errors; α_ι represent the virtual control signals and $\hat{\alpha}_{\iota,1}$ are acquired from the following second-order low-pass filters:

$$\begin{aligned} \dot{\hat{\alpha}}_{\iota,1} &= \hat{\alpha}_{\iota,2} \\ \dot{\hat{\alpha}}_{\iota,2} &= -2\theta_\iota \zeta_\iota \hat{\alpha}_{\iota,2} - \zeta_\iota^2 (\hat{\alpha}_{\iota,1} - \alpha_\iota) \end{aligned} \tag{23}$$

with $\hat{\alpha}_{\iota,1}(0) = \alpha_\iota(0)$, $\hat{\alpha}_{\iota,2}(0) = 0 (\iota = 1, \ldots, m-1)$, $\theta_\iota > 0$ and $\zeta_\iota > 0$ denoting the damping factors and the natural frequencies of the filters, respectively.

For the filters (23), define a vector $\tilde{\alpha}_\iota = [\tilde{\alpha}_{\iota,1}, \tilde{\alpha}_{\iota,2}]^T$ and $\tilde{\alpha}_{\iota,2} = \hat{\alpha}_{\iota,2}$. According to (23), the derivative of $\tilde{\alpha}_\iota$ is acquired as

$$\dot{\tilde{\alpha}}_\iota = Q_\iota \tilde{\alpha}_\iota + \bar{\Xi}\Pi_\iota \tag{24}$$

where $\bar{\Xi} = [-1, 0]^T$ and $\Pi_\iota = \dot{\alpha}_\iota$. Because $\theta_\iota > 0$, $\zeta_\iota > 0$,

$$Q_\iota = \begin{bmatrix} 0 & 1 \\ -\zeta_\iota^2 & -2\theta_\iota\zeta_\iota \end{bmatrix}$$

is a Hurwitz matrix. Thus, for arbitrary matrix $E_\iota > 0$, there is a matrix $R_\iota > 0$ such that $Q_\iota^T R_\iota + R_\iota Q_\iota = -E_\iota$.

Consider a class of smooth functions $sg_j(\cdot)$ as follows:

$$sg_j(\varrho_j) = \begin{cases} \frac{\varrho_j}{|\varrho_j|}, & |\varrho_j| \geq \sigma_j \\ \frac{\varrho_j}{(\sigma_j^2 - \varrho_j^2)^2 + |\varrho_j|}, & |\varrho_j| < \sigma_j \end{cases} \tag{25}$$

and a class of switched functions

$$\psi_j(\varrho_j) = \begin{cases} 1, & |\varrho_j| \geq \sigma_j \\ 0, & |\varrho_j| < \sigma_j \end{cases} \tag{26}$$

where $\sigma_j > 0$ and $j = 1, \ldots, m$ denote the designed parameters. From (25) and (26), the following properties hold:

$$sg_j(\varrho_j)\psi_j(\varrho_j) = \begin{cases} \frac{\varrho_j}{|\varrho_j|}, & |\varrho_j| \geq \sigma_j \\ 0, & |\varrho_j| < \sigma_j \end{cases} \tag{27}$$

and

$$[\psi_j(\varrho_j)]^{\Im} = \psi_j(\varrho_j), \tag{28}$$

where $\Im = 1, 2, \ldots$ stands for positive integer.

Step 1. According to (15) and (22), one obtains

$$\begin{aligned} \dot{\varrho}_1 &= e_2 + \hat{\vartheta}_1^T \varphi_1(Z_1) + \epsilon_1 + \tilde{\vartheta}_1^T \varphi_1(Z_1) \\ &\quad + \varrho_2 + \tilde{\alpha}_{1,1} + \alpha_1 + \Delta H_1. \end{aligned} \tag{29}$$

A Lyapunov function candidate is chosen as follows:

$$V_1 = \frac{1}{2}(|\varrho_1| - \sigma_1)^2 \psi_1(\varrho_1) \tag{30}$$

and its derivative is calculated as

$$\begin{aligned} \dot{V}_1 &= (|\varrho_1| - \sigma_1) sg_1(\varrho_1)\psi_1 \big[e_2 + \hat{\vartheta}_1^T \varphi_1(Z_1) + \epsilon_1 + \varrho_2 \\ &\quad + \tilde{\alpha}_{1,1} + \alpha_1 + \Delta H_1\big] \\ &\quad + \tilde{\vartheta}_1^T (|\varrho_1| - \sigma_1) sg_1(\varrho_1)\psi_1 \varphi_1(Z_1). \end{aligned} \tag{31}$$

By utilizing Young's inequality, one has

$$\begin{aligned} &(|\varrho_1| - \sigma_1) sg_1(\varrho_1)\psi_1(e_2 + \tilde{\alpha}_{1,1} + \epsilon_1 + \Delta H_1) \\ &\leq \frac{1+\varpi_1^2}{2}\|e\|^2 + \frac{\|\tilde{\alpha}_1\|^2}{2} + \frac{\epsilon_1^{*2}}{2} + \frac{\varpi_1^2 \wp_{fM}^2}{2} \\ &\quad + \frac{5}{2}(|\varrho_1| - \sigma_1)^2 \psi_1. \end{aligned} \tag{32}$$

Substituting (32) into (31), it yields

$$\begin{aligned} \dot{V}_1 &\leq \frac{1+\varpi_1^2}{2}\|e\|^2 + \frac{\|\tilde{\alpha}_1\|^2}{2} + \frac{\epsilon_1^{*2}}{2} + \frac{\varpi_1^2 \wp_{fM}^2}{2} \\ &\quad + (|\varrho_1| - \sigma_1) sg_1(\varrho_1)\psi_1 \big[\varrho_2 + \alpha_1 + \hat{\vartheta}_1^T \varphi_1(Z_1) \\ &\quad + \frac{5}{2}(|\varrho_1| - \sigma_1) sg_1(\varrho_1)\big] \\ &\quad + \tilde{\vartheta}_1^T (|\varrho_1| - \sigma_1) sg_1(\varrho_1)\psi_1 \varphi_1(Z_1). \end{aligned} \tag{33}$$

Define a virtual control signal α_1 as follows:

$$\begin{aligned} \alpha_1 &= -\frac{\beta_1}{2^{1+\frac{\gamma}{2}}}(|\varrho_1| - \sigma_1)^{1+\gamma} sg_1(\varrho_1) \\ &\quad - \frac{\beta_2}{2^{1-\frac{\gamma}{2}}}(|\varrho_1| - \sigma_1)^{1-\gamma} sg_1(\varrho_1) \\ &\quad - \hat{\vartheta}_1^T \varphi_1(Z_1) - \frac{11}{4}(|\varrho_1| - \sigma_1) sg_1(\varrho_1) \\ &\quad - (\sigma_2 + 1) sg_1(\varrho_1) \end{aligned} \tag{34}$$

with $0 < \beta_1 < 1, 0 < \beta_2 < 1$ being the designed constants and $\beta_1 = \frac{\pi}{h\gamma T^*}$, $\beta_2 = \frac{\pi}{\gamma T^*}$, $h = \min\{(2/m)^{\gamma/2}, (m-1)^{-\gamma/2}\}$.

According to (34), (33) further becomes as follows:

$$\begin{aligned} \dot{V}_1 &\leq \frac{1+\varpi_1^2}{2}\|e\|^2 + \frac{\|\tilde{\alpha}_1\|^2}{2} + \frac{\epsilon_1^{*2}}{2} + \frac{\varpi_1^2 \wp_{fM}^2}{2} \\ &\quad + \tilde{\vartheta}_1^T (|\varrho_1| - \sigma_1) sg_1(\varrho_1)\psi_1 \varphi_1(Z_1) \\ &\quad - \beta_1 \Big[\frac{(|\varrho_1| - \sigma_1)^2}{2}\psi_1\Big]^{1+\frac{\gamma}{2}} \\ &\quad - \beta_2 \Big[\frac{(|\varrho_1| - \sigma_1)^2}{2}\psi_1\Big]^{1-\frac{\gamma}{2}} - \frac{1}{4}(|\varrho_1| - \sigma_1)^2 \psi_1 \\ &\quad + (|\varrho_1| - \sigma_1)\psi_1(|\varrho_2| - \sigma_2 - 1). \end{aligned} \tag{35}$$

Step $\iota(2 \leq \iota \leq m-1)$. From the coordinate transformation (22) and the observer (16), we have

$$\begin{aligned} \dot{\varrho}_\iota &= \varrho_{\iota+1} + \tilde{\alpha}_{\iota,1} + \alpha_\iota + \hat{\vartheta}_\iota^T \varphi_\iota(Z_\iota) + k_\iota(q(y) - y) \\ &\quad + k_\iota e_1 - \hat{\alpha}_{\iota-1,2} \\ &= \varrho_{\iota+1} + \tilde{\alpha}_{\iota,1} + \alpha_\iota + \hat{\vartheta}_\iota^T \varphi_\iota(Z_\iota) + k_\iota(q(y) - y) \\ &\quad + k_\iota e_1 - \hat{\alpha}_{\iota-1,2} + \tilde{\vartheta}_\iota^T \varphi_\iota(Z_\iota) - \tilde{\vartheta}_\iota^T \varphi_\iota(Z_\iota). \end{aligned} \tag{36}$$

The following Lyapunov function is considered

$$V_\iota = V_{\iota-1} + \frac{1}{2}(|\varrho_\iota| - \sigma_\iota)^2 \psi_\iota. \tag{37}$$

From (36), its derivative is obtained as:

$$\begin{aligned} \dot{V}_\iota &= \dot{V}_{\iota-1} + (|\varrho_\iota| - \sigma_\iota) sg_\iota(\varrho_\iota)\psi_\iota \Big[\varrho_{\iota+1} + \alpha_\iota + \hat{\vartheta}_\iota^T \varphi_\iota(Z_\iota) \\ &\quad + k_\iota(q(y) - y) + \tilde{\alpha}_{\iota,1} + k_\iota e_1 - \hat{\alpha}_{\iota-1,2} - \tilde{\vartheta}_\iota^T \varphi_\iota(Z_\iota)\Big] \end{aligned}$$

$$+\tilde{\vartheta}_\iota^T(|\varrho_\iota|-\sigma_\iota)sg_\iota(\varrho_\iota)\psi_\iota\varphi_\iota(Z_\iota). \quad (38)$$

By applying Young's inequality, the following inequality can be obtained:

$$(|\varrho_\iota|-\sigma_\iota)sg_\iota(\varrho_\iota)\psi_\iota\Big[k_\iota e_1+\tilde{\alpha}_{\iota,1}+k_\iota(q(y)-y)-\tilde{\vartheta}_\iota^T\varphi_\iota(Z_\iota)\Big] \le \frac{k_\iota^2\|e\|^2}{2}+\frac{\|\tilde{\alpha}_\iota\|^2}{2}+\frac{k_\iota^2\tau_y^2}{2}+\frac{\tilde{\vartheta}_\iota^T\tilde{\vartheta}_\iota}{2}+2(|\varrho_\iota|-\sigma_\iota)^2\psi_\iota. \quad (39)$$

Bringing (39) into (38), it yields

$$\begin{aligned}\dot{V}_\iota \le{}& \dot{V}_{\iota-1}+\frac{k_\iota^2\|e\|^2}{2}+\frac{\|\tilde{\alpha}_\iota\|^2}{2}+\frac{k_\iota^2\tau_y^2}{2}+\frac{\tilde{\vartheta}_\iota^T\tilde{\vartheta}_\iota}{2}\\ &+\tilde{\vartheta}_\iota^T(|\varrho_\iota|-\sigma_\iota)sg_\iota(\varrho_\iota)\psi_\iota\varphi_\iota(Z_\iota)\\ &+(|\varrho_\iota|-\sigma_\iota)sg_\iota(\varrho_\iota)\psi_\iota\Big[\varrho_{\iota+1}+\alpha_\iota+\hat{\vartheta}_\iota^T\varphi_\iota(Z_\iota)\\ &+2(|\varrho_\iota|-\sigma_\iota)sg_\iota(\varrho_\iota)-\hat{\alpha}_{\iota-1,2}\Big]. \end{aligned} \quad (40)$$

The following virtual control signals α_ι are devised

$$\begin{aligned}\alpha_\iota ={}& -\frac{\beta_1}{2^{1+\frac{\gamma}{2}}}(|\varrho_\iota|-\sigma_\iota)^{1+\gamma}sg_\iota(\varrho_\iota)\\ &-\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}(|\varrho_\iota|-\sigma_\iota)^{1-\gamma}sg_\iota(\varrho_\iota)+\hat{\alpha}_{\iota-1,2}\\ &-\frac{13}{4}(|\varrho_\iota|-\sigma_\iota)sg_\iota(\varrho_\iota)-\hat{\vartheta}_\iota^T\varphi_\iota(Z_\iota)\\ &-(\sigma_{\iota+1}+1)sg_\iota(\varrho_\iota). \end{aligned} \quad (41)$$

Bringing (41) into (40), (40) is further computed as follows:

$$\begin{aligned}\dot{V}_\iota \le{}& \frac{1}{2}(1+\sum_{l=2}^{\iota}k_l^2+\varpi_1^2)\|e\|^2+\sum_{l=1}^{\iota}\frac{\|\tilde{\alpha}_l\|^2}{2}+\frac{\epsilon_1^{*2}}{2}\\ &+\sum_{l=2}^{\iota}\frac{\tilde{\vartheta}_l^T\tilde{\vartheta}_l}{2}+\frac{\varpi_1^2\wp_{f_M}^2}{2}+\sum_{l=2}^{\iota}\frac{k_l^2\tau_y^2}{2}\\ &+\sum_{l=1}^{\iota}\tilde{\vartheta}_l^T(|\varrho_l|-\sigma_l)sg_l(\varrho_l)\psi_l\varphi_l(Z_l)\\ &-\beta_1\sum_{l=1}^{\iota}\Big[\frac{(|\varrho_l|-\sigma_l)^2}{2}\psi_l\Big]^{1+\frac{\gamma}{2}}+\nabla_\iota\\ &-\beta_2\sum_{l=1}^{\iota}\Big[\frac{(|\varrho_l|-\sigma_l)^2}{2}\psi_l\Big]^{1-\frac{\gamma}{2}}\\ &-\frac{1}{4}(|\varrho_\iota|-\sigma_\iota)^2\psi_\iota+(|\varrho_\iota|-\sigma_\iota)\psi_\iota(|\varrho_{\iota+1}|\\ &-\sigma_{\iota+1}-1) \end{aligned} \quad (42)$$

where

$$\begin{aligned}\nabla_\iota ={}& -\frac{1}{4}(|\varrho_{\iota-1}|-\sigma_{\iota-1})^2\psi_{\iota-1}-(|\varrho_\iota|-\sigma_\iota)^2\psi_\iota\\ &+(|\varrho_{\iota-1}|-\sigma_{\iota-1})\psi_{\iota-1}(|\varrho_\iota|-\sigma_\iota-1). \end{aligned} \quad (43)$$

For the term ∇_ι, we have the following analysis: If $|\varrho_\iota| \le \sigma_\iota+1$, obviously, there is $(|\varrho_{\iota-1}|-\sigma_{\iota-1})\psi_{\iota-1}(|\varrho_\iota|-\sigma_\iota-1) \le 0$, then $\nabla_\iota \le 0$; if $|\varrho_\iota| > \sigma_\iota+1$, by using Young's inequality, we have $(|\varrho_{\iota-1}|-\sigma_{\iota-1})\psi_{\iota-1}(|\varrho_\iota|-\sigma_\iota-1) \le \frac{1}{4}(|\varrho_{\iota-1}|-\sigma_{\iota-1})^2\psi_{\iota-1}+(|\varrho_\iota|-\sigma_\iota-1)^2$. Because of $|\varrho_\iota| > \sigma_\iota+1$, $(|\varrho_\iota|-\sigma_\iota-1)^2 \le (|\varrho_\iota|-\sigma_\iota)^2\psi_\iota$. Thus, $\nabla_\iota \le 0$ always holds.

Step m. According to the observer (16) and the coordinate transformation (22), we have

$$\begin{aligned}\dot{\varrho}_m ={}& q(u)+\hat{\vartheta}_m^T\varphi_m(Z_m)+k_m(q(y)-y)+k_me_1\\ &-\hat{\alpha}_{m-1,2}+\tilde{\vartheta}_m^T\varphi_m(Z_m)-\tilde{\vartheta}_m^T\varphi_m(Z_m). \end{aligned} \quad (44)$$

Choosing a Lyapunov function candidate as below

$$V_m = V_{m-1}+\frac{1}{2}(|\varrho_m|-\sigma_m)^2\psi_m. \quad (45)$$

and its derivative as

$$\begin{aligned}\dot{V}_m ={}& \dot{V}_{m-1}+(|\varrho_m|-\sigma_m)sg_m(\varrho_m)\psi_m(q(u)-v)\\ &+\tilde{\vartheta}_m^T(|\varrho_m|-\sigma_m)sg_m(\varrho_m)\psi_m\varphi_m(Z_m)\\ &+(|\varrho_m|-\sigma_m)sg_m(\varrho_m)\psi_m\Big[v+k_m(q(y)-y)+k_me_1\\ &+\hat{\vartheta}_m^T\varphi_m(Z_m)-\hat{\alpha}_{m-1,2}-\tilde{\vartheta}_m^T\varphi_m(Z_m)\Big]. \end{aligned} \quad (46)$$

Similar to *Step* ι, by utilizing the Young's inequality, the derivative of V_m is further computed as

$$\begin{aligned}\dot{V}_m \le{}& \dot{V}_{m-1}+\frac{k_m^2\|e\|^2}{2}+\frac{k_m^2\tau_y^2}{2}+\frac{\tilde{\vartheta}_m^T\tilde{\vartheta}_m}{2}\\ &+(|\varrho_m|-\sigma_m)sg_m(\varrho_m)\psi_m(q(u)-v)\\ &+(|\varrho_m|-\sigma_m)sg_m(\varrho_m)\psi_m\Big[v+\hat{\vartheta}_m^T\varphi_m(Z_m)\\ &-\hat{\alpha}_{m-1,2}+\frac{3}{2}(|\varrho_m|-\sigma_m)sg_m(\varrho_m)\Big]\\ &+\tilde{\vartheta}_m^T(|\varrho_m|-\sigma_m)sg_m(\varrho_m)\psi_m\varphi_m(Z_m). \end{aligned} \quad (47)$$

Design an intermediate auxiliary control signal v, as shown below

$$\begin{aligned}v ={}& -\frac{\beta_1}{2^{1+\frac{\gamma}{2}}}(|\varrho_m|-\sigma_m)^{1+\gamma}sg_m(\varrho_m)\\ &-\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}(|\varrho_m|-\sigma_m)^{1-\gamma}sg_m(\varrho_m)+\hat{\alpha}_{m-1,2}\\ &-\hat{\vartheta}_m^T\varphi_m(Z_m)-(\frac{5}{2}+1)(|\varrho_m|-\sigma_m)sg_m(\varrho_m). \end{aligned} \quad (48)$$

Substituting (48) into (47), $\dot{V}_m$ is further computed as

$$\dot{V}_m \le \frac{1}{2}(1+\sum_{l=2}^{m}k_l^2+\varpi_1^2)\|e\|^2+\sum_{l=1}^{m-1}\frac{\|\tilde{\alpha}_l\|^2}{2}$$

$$+\frac{\epsilon_1^{*2}}{2}+\sum_{l=2}^{m}\frac{\tilde{\vartheta}_l^T\tilde{\vartheta}_l}{2}+\sum_{l=2}^{m}\frac{k_l^2\tau_y^2}{2}+\frac{\varpi_1^2\wp_{f_M}^2}{2}$$
$$+\sum_{l=1}^{m}\vartheta_l^T(|\varrho_l|-\sigma_l)sg_l(\varrho_l)\psi_l\varphi_l(Z_l)$$
$$+(|\varrho_m|-\sigma_m)sg_m(\varrho_m)\psi_m(q(u)-v)$$
$$-\beta_1\sum_{l=1}^{m}\Big[\frac{(|\varrho_l|-\sigma_l)^2}{2}\psi_l\Big]^{1+\frac{\gamma}{2}}$$
$$-\beta_2\sum_{l=1}^{m}\Big[\frac{(|\varrho_l|-\sigma_l)^2}{2}\psi_l\Big]^{1-\frac{\gamma}{2}}$$
$$-(|\varrho_m|-\sigma_m)^2\psi_m. \quad (49)$$

(ii) In this section, the quantized output $q(y)$ displaces the continuous output y in above section. Thus, an actual adaptive predefined-time quantized controller is got as follows:

$$u=-\frac{\beta_1}{2^{1+\frac{\gamma}{2}}}(|\varrho_m^q|-\sigma_m)^{1+\gamma}sg_m(\varrho_m^q)$$
$$-\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}(|\varrho_m^q|-\sigma_m)^{1-\gamma}sg_m(\varrho_m^q)+\hat{\alpha}_{m-1,2}^q$$
$$-\hat{\vartheta}_m^T\varphi_m(Z_m)-\frac{7}{2}(|\varrho_m^q|-\sigma_m)sg_m(\varrho_m^q), \quad (50)$$

where $\varrho_1^q = q(y), \varrho_{\iota+1}^q = \hat{\upsilon}_{\iota+1} - \hat{\alpha}_{\iota,1}^q$ and $\hat{\alpha}_{\iota,1}^q$ is obtained by the following filter:

$$\dot{\hat{\alpha}}_{\iota,1}^q=\hat{\alpha}_{\iota,2}^q;$$
$$\dot{\hat{\alpha}}_{\iota,2}^q=-2\theta_\iota\zeta_\iota\hat{\alpha}_{\iota,2}^q-\zeta_\iota^2(\hat{\alpha}_{\iota,1}^q-\alpha_\iota^q). \quad (51)$$

Moreover, the virtual control signals α_ι^q, parameter adaptive laws $\dot{\hat{\vartheta}}_j$ are devised:

$$\alpha_1^q=-\frac{\beta_1}{2^{1+\frac{\gamma}{2}}}(|\varrho_1^q|-\sigma_1)^{1+\gamma}sg_1(\varrho_1^q)$$
$$-\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}(|\varrho_1^q|-\sigma_1)^{1-\gamma}sg_1(\varrho_1^q)-\hat{\vartheta}_1^T\varphi_1(Z_1)$$
$$-(\sigma_2+1)sg_1(\varrho_1^q)-\frac{11}{4}(|\varrho_1^q|-\sigma_1)sg_1(\varrho_1^q); \quad (52)$$

$$\alpha_\iota^q=-\frac{\beta_1}{2^{1+\frac{\gamma}{2}}}(|\varrho_\iota^q|-\sigma_\iota)^{1+\gamma}sg_\iota(\varrho_\iota^q)$$
$$-\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}(|\varrho_\iota^q|-\sigma_\iota)^{1-\gamma}sg_\iota(\varrho_\iota^q)-\hat{\vartheta}_\iota^T\varphi_\iota(Z_\iota)$$
$$+\hat{\alpha}_{\iota-1,2}^q-\frac{13}{4}(|\varrho_\iota^q|-\sigma_\iota)sg_\iota(\varrho_\iota^q)$$
$$-(\sigma_{\iota+1}+1)sg_\iota(\varrho_\iota^q); \quad (53)$$

$$\dot{\hat{\vartheta}}_1=\lambda_1(|\varrho_1^q|-\sigma_1)sg_1(\varrho_1^q)\psi_1(\varrho_1^q)\varphi_1(Z_1)$$
$$-\beta_1\hat{\vartheta}_1^{1+\gamma}-(\beta_2+3\lambda_1)\hat{\vartheta}_1,\ \hat{\vartheta}_1(t_0)\geq 0; \quad (54)$$
$$\dot{\hat{\vartheta}}_l=\lambda_l(|\varrho_l^q|-\sigma_l)sg_l(\varrho_l^q)\psi_l(\varrho_l^q)\varphi_l(Z_l)$$
$$-\beta_1\hat{\vartheta}_l^{1+\gamma}-(\beta_2+4\lambda_l)\hat{\vartheta}_l,\ \hat{\vartheta}_l(t_0)\geq 0, \quad (55)$$

where $\lambda_j > 0$ denote the designed constants and functions $sg_j(\varrho_j^q)$, $\psi_j(\varrho_j^q)$ are as:

$$sg_j(\varrho_j^q)=\begin{cases}\frac{\varrho_j^q}{|\varrho_j^q|}, & |\varrho_j^q|\geq\sigma_j\\ \frac{\varrho_j^q}{(\sigma_j^2-(\varrho_j^q)^2)^2+|\varrho_j^q|}, & |\varrho_j^q|<\sigma_j\end{cases} \quad (56)$$

$$\psi_j(\varrho_j^q)=\begin{cases}1, & |\varrho_j^q|\geq\sigma_j\\ 0, & |\varrho_j^q|<\sigma_j\end{cases} \quad (57)$$

and $j = 1, 2, \ldots, m, l = 2, \ldots, m$.

Remark 2 In this backstepping process, the time derivative of α_j^q is replaced with the variable $\alpha_{j,2}^q$ by filters (51). Thus, the issues of output quantization and "explosion of complexity" are resolved. Although dynamic surface technology also usually is applied to handle the above issues ([22]–[23]), the differential of the first-order filter generated by the quantized output is discontinuous. Moreover, for higher-order filters, smoother outputs require more complicated computations. To balance computational complexity with filter performance, a second-order command filter is applied. In addition, a class of functions sg_j, ψ_j are applied to devise a novel adaptive fuzzy predefined-time controller.

4 Stability analysis

To compensate the impact of the quantization errors, the following lemma is given.

Lemma 9 *Define the quantization errors as follows:*

$$\Upsilon_{\varrho_j}=\varrho_j-\varrho_j^q,\ \Upsilon_{\alpha_\iota}=\alpha_\iota-\alpha_\iota^q,\ \Upsilon_{\hat{\alpha}_{\iota,1}}=\hat{\alpha}_{\iota,1}-\hat{\alpha}_{\iota,1}^q,$$
$$\Upsilon_{\hat{\alpha}_{\iota,2}}=\hat{\alpha}_{\iota,2}-\hat{\alpha}_{\iota,2}^q,\ \Upsilon_v=u-v, \quad (58)$$

where $j = 1, \ldots, m$, $\iota = 1, \ldots, m-1$. There exist the positive constants $\aleph_{\varrho_j}, \aleph_{\alpha_\iota}, \aleph_{\hat{\alpha}_\iota}, \aleph_v$, respectively, such that $|\Upsilon_{\varrho_j}| \leq \aleph_{\varrho_j}, |\Upsilon_{\alpha_\iota}| \leq \aleph_{\alpha_\iota}, \|\Upsilon_{\hat{\alpha}_\iota}\| \leq \aleph_{\hat{\alpha}_\iota}, |\Upsilon_v| \leq \aleph_v$ with $\hat{\alpha}_\iota = [\hat{\alpha}_{\iota,1}, \hat{\alpha}_{\iota,2}]^T$.

Proof (1) When $j = \iota = 1$, based on the characteristic of uniform quantizer (3), it yields

$$|\varrho_1-\varrho_1^q|=|y-q(y)|\leq\tau_y\triangleq\aleph_{\varrho_1}. \quad (59)$$

From (34) and (52), the following formula is obtained:

$$\begin{aligned}\Upsilon_{\alpha_1} &= \alpha_1 - \alpha_1^q \\ &= -\frac{\beta_1}{2^{1+\frac{\gamma}{2}}}\big[(|\varrho_1| - \sigma_1)^{1+\gamma} sg_1(\varrho_1) \\ &\quad -(|\varrho_1^q| - \sigma_1)^{1+\gamma} sg_1(\varrho_1^q)\big] \\ &\quad -\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}\big[(|\varrho_1| - \sigma_1)^{1-\gamma} sg_1(\varrho_1) \\ &\quad -(|\varrho_1^q| - \sigma_1)^{1-\gamma} sg_1(\varrho_1^q)\big] \\ &\quad -\frac{11}{4}\Big[(|\varrho_1| - \sigma_1) sg_1(\varrho_1) \\ &\quad -(|\varrho_1^q| - \sigma_1) sg_1(\varrho_1^q)\Big] \\ &\quad -(\sigma_2 + 1)\big[sg_1(\varrho_1) - sg_1(\varrho_1^q)\big]. \end{aligned} \tag{60}$$

When $|\varrho_1| \geq \sigma_1, |\varrho_1^q| \geq \sigma_1$, according to Lemma 5, define a term as:

$$\begin{aligned}\Theta &= \frac{\beta_1}{2^{1+\frac{\gamma}{2}}}\Big|(|\varrho_1| - \sigma_1)^{1+\gamma} sg_1(\varrho_1) \\ &\quad -(|\varrho_1^q| - \sigma_1)^{1+\gamma} sg_1(\varrho_1^q)\Big| \\ &\quad +\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}\Big|(|\varrho_1| - \sigma_1)^{1-\gamma} sg_1(\varrho_1) \\ &\quad -(|\varrho_1^q| - \sigma_1)^{1-\gamma} sg_1(\varrho_1^q)\Big| \end{aligned} \tag{61}$$

and it yields

$$\Theta \leq \frac{\beta_1 b_1}{2^{1+\frac{\gamma}{2}}}\aleph_{\varrho 1}(\aleph_{\varrho_1}^{\gamma} + \sigma_1^{\gamma}) + \beta_2 \cdot 2^{\frac{3\gamma}{2}-1}\aleph_{\varrho_1}^{1-\gamma}. \tag{62}$$

When $|\varrho_1| \geq \sigma_1, |\varrho_1^q| < \sigma_1$, we have

$$\begin{aligned}\Theta &\leq \frac{\beta_1}{2^{1+\frac{\gamma}{2}}}\Big|(|\varrho_1| - \sigma_1)^{1+\gamma}\Big| + \frac{\beta_1}{2^{1+\frac{\gamma}{2}}}\Big|(|\varrho_1^q| - \sigma_1)^{1+\gamma}\Big| \\ &\quad +\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}\Big|(|\varrho_1| - \sigma_1)^{1-\gamma}\Big| + \frac{\beta_2}{2^{1-\frac{\gamma}{2}}}\Big|(|\varrho_1^q| - \sigma_1)^{1-\gamma}\Big| \\ &\leq \frac{\beta_1}{2^{1+\frac{\gamma}{2}}}\Big||\varrho_1| - |\varrho_1^q|\Big|^{1+\gamma} + \frac{\beta_1\sigma_1^{1+\gamma}}{2^{1+\frac{\gamma}{2}}} \\ &\quad +\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}\Big||\varrho_1| - |\varrho_1^q|\Big|^{1-\gamma} + \frac{\beta_2\sigma_1^{1-\gamma}}{2^{1-\frac{\gamma}{2}}} \\ &\leq \frac{\beta_1}{2^{1+\frac{\gamma}{2}}}\aleph_{\varrho_1}^{1+\gamma} + \frac{\beta_2}{2^{1-\frac{\gamma}{2}}}\aleph_{\varrho_1}^{1-\gamma} + \frac{\beta_1\sigma_1^{1+\gamma}}{2^{1+\frac{\gamma}{2}}} + \frac{\beta_2\sigma_1^{1-\gamma}}{2^{1-\frac{\gamma}{2}}}. \end{aligned} \tag{63}$$

Thus, when $|\varrho_1| < \sigma_1, |\varrho_1^q| \geq \sigma_1$, the above inequalities also hold.

When $|\varrho_1| < \sigma_1, |\varrho_1^q| < \sigma_1$, similar to (63), we have

$$\Theta \leq \frac{\beta_1\sigma_1^{1+\gamma}}{2^{\frac{\gamma}{2}}} + \beta_2\sigma_1^{1-\gamma}2^{\frac{\gamma}{2}}. \tag{64}$$

According to (62) – (64), we have

$$\begin{aligned}|\Upsilon_{\alpha_1}| &\leq \frac{\beta_1 b_1}{2^{1+\frac{\gamma}{2}}}\aleph_{\varrho 1}(\aleph_{\varrho_1}^{\gamma} + \sigma_1^{\gamma}) + \beta_2 \cdot 2^{\frac{3\gamma}{2}-1}\aleph_{\varrho_1}^{1-\gamma} \\ &\quad +\frac{\beta_1\sigma_1^{1+\gamma}}{2^{\frac{\gamma}{2}}} + \beta_2\sigma_1^{1-\gamma}2^{\frac{\gamma}{2}} + \frac{11}{4}\aleph_{\varrho_1} \\ &\quad +2(\sigma_2 + 1) \triangleq \aleph_{\alpha_1}. \end{aligned} \tag{65}$$

From (23) and (51), one obtains

$$\dot{\Upsilon}_{\hat{\alpha}_1} = Q_1\Upsilon_{\hat{\alpha}_1} + \bar{B}_1\Upsilon_{\alpha_1}, \tag{66}$$

where $\bar{B}_1 = [0, \zeta_1^2]$.

By reducing the equation, we obtain

$$\Upsilon_{\hat{\alpha}_1}(t) = e^{Q_1 t}\Upsilon_{\hat{\alpha}_1}(0) + \int_0^t e^{Q_1(t-\nu)}\bar{B}\Upsilon_{\alpha_1}(\nu)d\nu. \tag{67}$$

Because Q_1 is invertible, the following inequality is met

$$\begin{aligned}\|\Upsilon_{\hat{\alpha}_1}(t)\| &\leq \|e^{Q_1 t}\| \cdot \|\Upsilon_{\hat{\alpha}_1}(0)\| \\ &\quad +\aleph_{\alpha_1}\|\bar{B}\| \cdot \|Q_1^{-1}(I - e^{Q_1 t})\|. \end{aligned} \tag{68}$$

Due to $\hat{\alpha}_{1,1}(0) = \alpha_1(0)$, $\hat{\alpha}_{1,2}(0) = 0$, thus $\|\Upsilon_{\hat{\alpha}_1}(0)\| = |\Upsilon_{\hat{\alpha}_1}(0)|$. From Lemma 7, the above inequality goes further

$$\begin{aligned}\|\Upsilon_{\hat{\alpha}_1}(t)\| &\leq \omega_1 e^{-\omega_2 t}|\Upsilon_{\hat{\alpha}_1}(0))| \\ &\quad +\aleph_{\alpha_1}\|\bar{B}\| \cdot \|Q_1^{-1}\|(1 + \omega_1) \triangleq \aleph_{\hat{\alpha}_1}. \end{aligned} \tag{69}$$

Therefore, $|\Upsilon_{\hat{\alpha}_{1,1}}| \leq \aleph_{\hat{\alpha}_1}$ and $|\Upsilon_{\hat{\alpha}_{1,2}}| \leq \aleph_{\hat{\alpha}_1}$ are established.

(2) From the definition of ϱ_2, it yields

$$|\varrho_2 - \varrho_2^q| = |\hat{\upsilon}_2 - \hat{\alpha}_{1,1} - \hat{\upsilon}_2 + \hat{\alpha}_{1,1}^q| \leq \aleph_{\hat{\alpha}_1} \triangleq \aleph_{\varrho_2}. \tag{70}$$

Similarly the procedure in (1), according to α_2 (41), α_2^q (53), the second-order low-pass filters (23) and (51), there exist the positive constants $\aleph_{\alpha_2}$ and $\aleph_{\hat{\alpha}_2}$ such that

$$|\Upsilon_{\alpha_2}| \leq \aleph_{\alpha_2}, \|\Upsilon_{\hat{\alpha}_2}\| \leq \aleph_{\hat{\alpha}_2}. \tag{71}$$

(3) According to the recursive method, $\Upsilon_{\varrho_j}, j = 3, \ldots, m$; $\Upsilon_{\alpha_\iota}, \Upsilon_{\hat{\alpha}_{\iota,1}}, \Upsilon_{\hat{\alpha}_{\iota,2}}, \iota = 3, \ldots, m-1$ and Υ_v are bounded, as shown below:

$$|\Upsilon_{\varrho_j}| \leq \aleph_{\varrho_j}, |\Upsilon_{\alpha_\iota}| \leq \aleph_{\alpha_\iota}, \|\Upsilon_{\hat{\alpha}_\iota}\| \leq \aleph_{\hat{\alpha}_\iota}, |\Upsilon_v| \leq \aleph_v. \tag{72}$$

The proof of Lemma 9 is accomplished.

Remark 3 Lemma 9 proves the boundness of the quantization errors. It is significant tool to ensure the practical predefined-time stability of system (10).

Theorem 1 *For arbitrary initial conditions satisfying $V(0) \le \Lambda$ where $\Lambda > 0$ stands for a constant, under Assumptions 1 and 2, consider the pure-feedback nonlinear system with input and output quantization (10). If the observer (16), the second-order low-pass filter (51), the virtual control signals (52)–(53), the parameter adaptive laws (54)–(55) and the actual quantized controller (48) are adopted, the closed-loop system (12) is PPTS. Furthermore, within a predefined time $2T^*$, the variable ϱ_1 can converge to a neighborhood around the origin.*

Proof Choose the following Lyapunov function:

$$V = V_0 + V_m + \sum_{\iota=1}^{m-1} \tilde{\alpha}_\iota^T R_\iota \tilde{\alpha}_\iota + \sum_{j=1}^{m} \frac{\tilde{\vartheta}_j^2}{2\lambda_j}. \quad (73)$$

From (21), (23)–(24), (49) and (54)–(55), $\dot{V}$ is computed as

$$\begin{aligned}\dot{V} \le& -\Big[d - 4\|G\|^2 - \sum_{j=1}^{m} \varpi_j^2 - \frac{1}{2}(1 + \sum_{l=2}^{m} k_l^2 + \varpi_1^2)\Big]\|e\|^2 + \Gamma_1 \\ &+ \underbrace{\sum_{\iota=1}^{m-1} \frac{\|\tilde{\alpha}_\iota\|^2}{2} + \sum_{\iota=1}^{m-1}[-\tilde{\alpha}_\iota^T E_\iota \tilde{\alpha}_\iota + 2\tilde{\alpha}_\iota^T R_\iota \bar{\Xi}\Pi_\iota]}_{\Delta_1} - \beta_1 \sum_{l=1}^{m}\Big[\frac{(|\varrho_l| - \sigma_l)^2}{2}\psi_l\Big]^{1+\frac{\gamma}{2}} - \beta_2 \sum_{l=1}^{m}\Big[\frac{(|\varrho_l| - \sigma_l)^2}{2}\psi_l\Big]^{1-\frac{\gamma}{2}} \\ &+ \underbrace{\sum_{j=1}^{m} \rho_j \tilde{\vartheta}_j^T \vartheta_j + \beta_1 \sum_{j=1}^{m} \frac{\tilde{\vartheta}_j^T \hat{\vartheta}_j^{1+\gamma}}{\lambda_j} + \beta_2 \sum_{j=1}^{m} \frac{\tilde{\vartheta}_j^T \hat{\vartheta}_j}{\lambda_j} + \sum_{l=2}^{m} 4\tilde{\vartheta}_l^T \hat{\vartheta}_l + 3\tilde{\vartheta}_1^T \hat{\vartheta}_1}_{\Delta_2} - (|\varrho_m| - \sigma_m)^2\psi_m \\ &+ (|\varrho_m| - \sigma_m) sg_m(\varrho_m)\psi_m\Big[(q(u) - u) + (u - v)\Big] \end{aligned} \quad (74)$$

where $\Gamma_1 = \Gamma_0 + \frac{\epsilon_1^{*2}}{2} + \sum_{l=1}^{m} \frac{k_l^2 \tau_y^2}{2} + \sum_{l=1}^{m} \frac{\aleph_{\varrho_l}^2}{2} + \frac{\varpi_1^2 \wp_{f_M}^2}{2}$ and $\rho_1 = 3/2, \rho_2 = 2, \ldots, \rho_m = 2$.

According to the definition of v (48), u (50), the property (3) and Lemma 9, the following inequality is obtained:

$$\begin{aligned}&(|\varrho_m| - \sigma_m) sg_m(\varrho_m)\psi_m\Big[(q(u) - u) + (u - v)\Big] \\ &\le (|\varrho_m| - \sigma_m)^2\psi_m + \frac{\tau_u^2}{2} + \frac{\aleph_v^2}{2}. \end{aligned} \quad (75)$$

For the term Δ_1, define a set Ω_ι as follows: $\Omega_\iota = \Big\{2e^T G e + \sum_{j=1}^{\iota+1}(|\varrho_j| - \sigma_j)^2\psi_j + \sum_{j=1}^{\iota} 2\tilde{\alpha}_j^T R_j \tilde{\alpha}_j + \sum_{j=1}^{\iota+1} \frac{\tilde{\vartheta}_j^T \tilde{\vartheta}_j}{\lambda_j} \le 2\Lambda\Big\}$ with $\iota = 1, 2, \ldots, m-1$. Since $\Omega_\iota \in \Re^{m+3\iota+2}$ is a compact set. Thus, from $\Pi_\iota = \dot{\alpha}_\iota$, $|\Pi_\iota| \le \bar{\Pi}_\iota$ is obtained on Ω_ι with $\bar{\Pi}_\iota > 0$ representing a constant. Therefore, it yields

$$\begin{aligned}2\tilde{\alpha}_\iota^T R_\iota \bar{\Xi}\Pi_\iota &\le \|\tilde{\alpha}_\iota\|^2\|\Pi_\iota\|^2 + \|R_\iota\|^2 \\ &\le \|\tilde{\alpha}_\iota\|^2\bar{\Pi}_\iota^2 + \|R_\iota\|^2. \end{aligned} \quad (76)$$

Substituting the above inequality to Δ_1, we have

$$\Delta_1 \le -\sum_{\iota=1}^{m-1}\Big[\lambda_{\min}(E_\iota) - \frac{1}{2} - \bar{\Pi}_\iota^2\Big]\|\tilde{\alpha}_\iota\|^2 + \sum_{\iota=1}^{m-1}\|R_\iota\|^2. \quad (77)$$

According to Lemma 2, the following inequalities are obtained:

$$\begin{aligned}\tilde{\vartheta}_j^T \hat{\vartheta}_j^{1+\gamma} &\le \frac{1+\gamma}{2+\gamma}\Big[(\vartheta_j^*)^{2+\gamma} - \tilde{\vartheta}_j^{2+\gamma}\Big]; \\ \tilde{\vartheta}_j^T \hat{\vartheta}_j &\le \frac{(\vartheta_j^*)^2}{2} - \frac{\tilde{\vartheta}_j^2}{2}. \end{aligned} \quad (78)$$

From Lemma 4 with $\iota = \frac{2-\gamma}{2}$, $\ell = \frac{2}{2-\gamma}$, $\upsilon = 1$, $\check{\upsilon} = \sum_{j=1}^{m} \frac{\tilde{\vartheta}_j^2}{2\lambda_j}$, the following inequality holds:

$$\Big(\sum_{j=1}^{m} \frac{\tilde{\vartheta}_j^2}{2\lambda_j}\Big)^{1-\frac{\gamma}{2}} \le \sum_{j=1}^{m} \frac{\tilde{\vartheta}_j^2}{2\lambda_j} + \frac{\gamma}{2}\Big(\frac{2-\gamma}{2}\Big)^{\frac{2-\gamma}{\gamma}}. \quad (79)$$

Substituting the above inequality to the term Δ_2, we have

$$\Delta_2 \le -\sum_{j=1}^{m} \frac{1+\gamma}{2+\gamma}\frac{\beta_1}{\lambda_j}\tilde{\vartheta}_j^{2+\gamma} - \beta_2\Big(\sum_{j=1}^{m} \frac{\tilde{\vartheta}_j^2}{2\lambda_j}\Big)^{1-\frac{\gamma}{2}} + \Gamma_2.$$

(80)

where $\Gamma_2 = \frac{\beta_2\gamma}{2}(\frac{2-\gamma}{2})^{\frac{2-\gamma}{\gamma}} + \sum_{j=1}^{m}\frac{1+\gamma}{2+\gamma}\frac{\beta_1}{\lambda_j}(\vartheta_j^*)^{2+\gamma} + \sum_{j=1}^{m}\frac{\beta_2(\vartheta_j^*)^2}{2\lambda_j} + \sum_{j=1}^{m}\rho_j(\vartheta_j^*)^2$.

Bringing (75), (77) and (80) into (74), $\dot{V}$ is further calculated as

$$\begin{aligned}\dot{V} \leq & -\Big[d - 4\|G\|^2 - \sum_{j=1}^{m}\varpi_j^2 \\ & -\frac{1}{2}(1+\sum_{l=2}^{m}k_l^2+\varpi_1^2)\Big]\|e\|^2 \\ & +\frac{\tau_u^2}{2}+\frac{\aleph_v^2}{2}+\Gamma_1+\Gamma_2+\sum_{\iota=1}^{m-1}\|R_\iota\|^2 \\ & -\beta_1\sum_{l=1}^{m}\Big[\frac{(|\varrho_l|-\sigma_l)^2}{2}\psi_l\Big]^{1+\frac{\gamma}{2}} \\ & -\beta_2\sum_{l=1}^{m}\Big[\frac{(|\varrho_l|-\sigma_l)^2}{2}\psi_l\Big]^{1-\frac{\gamma}{2}} \\ & -\sum_{j=1}^{m}\frac{1+\gamma}{2+\gamma}\frac{\beta_1}{\lambda_j}\tilde{\vartheta}_j^{2+\gamma} - \beta_2(\sum_{j=1}^{m}\frac{\tilde{\vartheta}_j^2}{2\lambda_j})^{1-\frac{\gamma}{2}} \\ & -\sum_{\iota=1}^{m-1}\Big[\lambda_{\min}(E_\iota)-\frac{1}{2}-\bar{\Pi}_\iota^2\Big]\|\tilde{\alpha}_\iota\|^2. \end{aligned} \tag{81}$$

According to (81), obviously, when $t \to \infty$, the system's error variables can be stabilized in a residual set. Therefore, there exist the positive constants e^* and $\tilde{\alpha}_\iota^*$ such that $\|e\| \leq e^*$ and $\|\tilde{\alpha}_\iota\| \leq \tilde{\alpha}_\iota^*$ hold.

Define a constant $D = d - 4\|G\|^2 - \sum_{j=1}^{m}\varpi_j^2 - \frac{1}{2}(1+\sum_{l=2}^{m}k_l^2+\varpi_1^2)$, which meets $D \geq 2\lambda_{\max}(G)$, according to Lemma 6, we have

$$\begin{aligned}-D\|e\|^2 &\leq -(e^TGe)^{1+\frac{\gamma}{2}} - (e^TGe)^{1-\frac{\gamma}{2}} + \Im_1 \\ &\leq -\beta_1(e^TGe)^{1+\frac{\gamma}{2}} - \beta_2(e^TGe)^{1-\frac{\gamma}{2}} + \Im_1,\end{aligned} \tag{82}$$

where $\Im_1 = (e^*/\lambda_{\max}(G))^{1+\frac{\gamma}{2}} + \frac{\gamma}{2}(1-\frac{\gamma}{2})^{\frac{2-\gamma}{\gamma}}$. Similarly, define a constant $\Lambda_\iota = \lambda_{\min}(E_\iota) - \frac{1}{2} - \bar{\Pi}_\iota^2$, which meets $\Lambda_\iota \geq 2\lambda_{\max}(R_\iota)$, we obtain

$$\begin{aligned}&-\sum_{\iota=1}^{m-1}\Lambda_\iota\tilde{\alpha}_\iota^T R_\iota\tilde{\alpha}_\iota \\ &\leq -\Big(\sum_{\iota=1}^{m-1}\tilde{\alpha}_\iota^T R_\iota\tilde{\alpha}_\iota\Big)^{1-\frac{\gamma}{2}} + \Im_2 \\ &\quad -(m-1)^{-\frac{\gamma}{2}}\Big(\sum_{\iota=1}^{m-1}\tilde{\alpha}_\iota^T R_\iota\tilde{\alpha}_\iota\Big)^{1+\frac{\gamma}{2}} \\ &\leq -\beta_1(m-1)^{-\frac{\gamma}{2}}\Big(\sum_{\iota=1}^{m-1}\tilde{\alpha}_\iota^T R_\iota\tilde{\alpha}_\iota\Big)^{1+\frac{\gamma}{2}} + \Im_2 \\ &\quad -\beta_2\Big(\sum_{\iota=1}^{m-1}\tilde{\alpha}_\iota^T R_\iota\tilde{\alpha}_\iota\Big)^{1-\frac{\gamma}{2}},\end{aligned} \tag{83}$$

where $\Im_2 = \sum_{\iota=1}^{m-1}[(\tilde{\alpha}_\iota^*)^2/\lambda_{\max}(R_\iota)]^{1+\frac{\gamma}{2}} + \frac{\gamma}{2}(1-\frac{\gamma}{2})^{\frac{2-\gamma}{\gamma}}$.

Substituting (82)–(83) into (81), according to Lemma 3, the following inequality is obtained:

$$\begin{aligned}\dot{V} \leq & -\beta_1(e^TGe)^{1+\frac{\gamma}{2}} - \beta_1(m-1)^{-\frac{\gamma}{2}}\Big(\sum_{\iota=1}^{m-1}\tilde{\alpha}_\iota^T R_\iota\tilde{\alpha}_\iota\Big)^{1+\frac{\gamma}{2}} \\ & -\beta_1(m)^{-\frac{\gamma}{2}}\Big[\sum_{l=1}^{m}\frac{(|\varrho_l|-\sigma_l)^2}{2}\psi_l\Big]^{1+\frac{\gamma}{2}} \\ & -\frac{1+\gamma}{2+\gamma}\frac{\beta_1 2^{1+\frac{\gamma}{2}}}{m^{\frac{\gamma}{2}}}\Big(\sum_{j=1}^{m}\frac{\tilde{\vartheta}_j^2}{2\lambda_j}\Big)^{1+\frac{\gamma}{2}} \\ & -\beta_2(e^TGe)^{1-\frac{\gamma}{2}} - \beta_2\Big(\sum_{\iota=1}^{m-1}\tilde{\alpha}_\iota^T R_\iota\tilde{\alpha}_\iota\Big)^{1-\frac{\gamma}{2}} \\ & -\beta_2(\sum_{j=1}^{m}\frac{\tilde{\vartheta}_j^2}{2\lambda_j})^{1-\frac{\gamma}{2}} \\ & -\beta_2\sum_{l=1}^{m}\Big[\frac{(|\varrho_l|-\sigma_l)^2}{2}\psi_l\Big]^{1-\frac{\gamma}{2}} + \Gamma_3 \\ \leq & -\beta_1 h V^{1+\frac{\gamma}{2}} - \beta_2 V^{1-\frac{\gamma}{2}} + \Gamma_3 \\ \leq & -\frac{\pi}{\gamma T^*}\Big(V^{1+\frac{\gamma}{2}} + V^{1-\frac{\gamma}{2}}\Big) + \Gamma_3\end{aligned} \tag{84}$$

where $\Gamma_3 = \Gamma_1 + \Gamma_2 + \frac{\tau_u^2}{2} + \frac{\aleph_v^2}{2} + \sum_{\iota=1}^{m-1}\|R_\iota\|^2 + \Im_1 + \Im_2$.

From the definition of V and Lemma 1, within the predefined time $2T^*$, the variable ϱ_1 can converge to

$$|\varrho_1| \leq \sqrt{\frac{2\gamma\Gamma_3 T^*}{\pi}} + \sigma_1. \tag{85}$$

Thus, the proof of Theorem 1 is completed.

Remark 4 Due to output quantization $q(y)$ is discontinuous, it cannot be applied to the Lyapunov-based stability analysis of system, but unquantized output y can be applied to stability analysis. However, only the output quantization $q(y)$ is utilized to construct the controller u (50). Therefore, how to establish the relationship between quantized signals and unquantized signals is the main challenge in stability analysis. More specifically, a major difficulty in stability analysis is how to compensate for the effect from the term

$(|\varrho_m| - \sigma_m)sg_m(\varrho_m)\psi_m(u - \upsilon)$ in (75). With the help of Lemma 9, the above term can be compensated by $(|\varrho_m| - \sigma_m)^2\psi_m/2$; $\aleph_\upsilon^2/2$, which are irrelevant to the output quantization $q(y)$. Thus, since the relationship between quantized signals and unquantized signals in Lemma 9, quantized output $q(y)$ can be applied to control design and unquantized output y can be applied to stability analysis.

Remark 5 From (84) and (85), it can be acquired that increasing the parameters λ_j and decreasing the parameters $\gamma, T^*, \sigma_j, k_j, b_u, b_y$ contribute to improve the converge performance. However, too smaller T^*, σ_j may make the amplitude of the control input u too large, resulting in more control energy consumption. Thus, we need to a trade-off between the converge performance and control energy consumption.

5 Simulation example

Example 1 The following pure-feedback nonlinear system is considered:

$$\begin{aligned} \dot{\upsilon}_1 &= \hbar_1(\upsilon_1, \upsilon_2), \\ \dot{\upsilon}_2 &= \hbar_2(\bar{\upsilon}_2, q(u)), \\ y &= \upsilon_1. \end{aligned} \tag{86}$$

where $\bar{\upsilon}_2 = [\upsilon_1, \upsilon_2]^T$, y denote a state variable and the output of system, respectively; $\hbar_1(\upsilon_1, \upsilon_2) = \sin \upsilon_1\upsilon_2 + 2\upsilon_2 + \frac{\upsilon_1^2}{1+\upsilon_1^2}\upsilon_2^3$ and $\hbar_2(\bar{\upsilon}_2, q(u)) = \upsilon_1\upsilon_2 + q(u) + \frac{q(u)^3}{7}$ stand for the nonlinear functions. $q(u)$ indicates the output of the quantizer (2) and takes the quantized value.

Firstly, five fuzzy sets are defined over the interval $[-2, 2]$ with the partition points are $-2; -1; 0; 1; 2$. The following membership functions are chosen:

$$\begin{aligned} \mu_{A_1^o}(\hat{\upsilon}_1) &= \exp^{-\frac{1}{2}(\hat{\upsilon}_1-3+o)^2}, \\ \mu_{A_2^o}(\hat{\upsilon}_2) &= \exp^{-\frac{1}{2}(\hat{\upsilon}_2-3+o)^2}, \\ \mu_{A_3^o}(\hat{\upsilon}_{2,f}) &= \exp^{-\frac{1}{2}(\hat{\upsilon}_{2,f}-3+o)^2}, \\ \mu_{A_4^o}(q(u)_f) &= \exp^{-\frac{1}{2}(q(u)_f-3+o)^2}, o = 1, 2, 3, 4, 5. \end{aligned} \tag{87}$$

The fuzzy basis functions are

$$\varphi_1^o(\hat{\upsilon}_1, \hat{\upsilon}_{2,f}) = \frac{\mu_{A_1^o}(\hat{\upsilon}_1)\mu_{A_3^o}(\hat{\upsilon}_{2,f})}{\sum_{o=1}^5[\mu_{A_1^o}(\hat{\upsilon}_1)\mu_{A_3^o}(\hat{\upsilon}_{2,f})]},$$

$$\varphi_2^o(\hat{\bar{\upsilon}}_2, q(u)_f) = \frac{\mu_{A_1^o}(\hat{\upsilon}_1)\mu_{A_2^o}(\hat{\upsilon}_2)\mu_{A_4^o}(q(u)_f)}{\sum_{o=1}^5[\mu_{A_1^o}(\hat{\upsilon}_1)\mu_{A_2^o}(\hat{\upsilon}_2)\mu_{A_4^o}(q(u)_f)]} \tag{88}$$

Thus, $\varphi_1 = [\varphi_1^1, \varphi_1^2, \cdots, \varphi_1^5]^T$ and $\varphi_2 = [\varphi_2^1, \varphi_2^2, \cdots, \varphi_2^5]^T$.

From Theorem 1, we select the virtual controller, actual controller and parameter adaptive laws as follows:

$$\begin{aligned} \alpha_1^q &= -\frac{\beta_1}{2^{1+\frac{\gamma}{2}}}(|\varrho_1^q| - \sigma_1)^{1+\gamma}sg_1(\varrho_1^q) \\ &\quad -\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}(|\varrho_1^q| - \sigma_1)^{1-\gamma}sg_1(\varrho_1^q) - \hat{\vartheta}_1^T\varphi_1(Z_1) \\ &\quad -(\sigma_2 + 1)sg_1(\varrho_1^q) - \frac{11}{4}(|\varrho_1^q| - \sigma_1)sg_1(\varrho_1^q); \\ u &= -\frac{\beta_1}{2^{1+\frac{\gamma}{2}}}(|\varrho_2^q| - \sigma_2)^{1+\gamma}sg_2(\varrho_2^q) \\ &\quad -\frac{\beta_2}{2^{1-\frac{\gamma}{2}}}(|\varrho_2^q| - \sigma_2)^{1-\gamma}sg_2(\varrho_2^q) + \hat{\alpha}_{1,2}^q \\ &\quad -\hat{\vartheta}_2^T\varphi_2(Z_2) - \frac{7}{2}(|\varrho_2^q| - \sigma_2)sg_2(\varrho_2^q); \\ \dot{\hat{\vartheta}}_1 &= \lambda_1(|\varrho_1^q| - \sigma_1)sg_1(\varrho_1^q)\psi_1(\varrho_1^q)\varphi_1(Z_1) \\ &\quad -\beta_1\hat{\vartheta}_1^{1+\gamma} - (\beta_2 + 3\lambda_1)\hat{\vartheta}_1; \\ \dot{\hat{\vartheta}}_2 &= \lambda_2(|\varrho_2^q| - \sigma_2)sg_2(\varrho_2^q)\psi_2(\varrho_2^q)\varphi_2(Z_2) \\ &\quad -\beta_1\hat{\vartheta}_2^{1+\gamma} - (\beta_2 + 4\lambda_2)\hat{\vartheta}_2, \end{aligned}$$

where $\hat{\alpha}_{1,2}^q$ is generated from (51).

In this simulation, the initial condition is selected as $[\upsilon_1, \upsilon_2, \hat{\upsilon}_1, \hat{\upsilon}_2] = [0.4, 0.2, 0.5, 0.3]^T$; $\hat{\alpha}_{1,1}^q = \hat{\alpha}_{1,2}^q = 0$; $\hat{\vartheta}_1 = [\underbrace{0.3, 0.3, 0.3, 0.3, 0, 3}_{25}]^T$ and $\hat{\vartheta}_2 = [\underbrace{0.2, 0.2, 0.2, 0.2, 0.2}_{45}]^T$.

Moreover, the parameters of the quantizer (3) are selected $b_y = 0.02$ and $b_u = 0.08$, respectively; the parameters of the observer (16) are: $k_1 = 6, k_2 = 8$. In addition, the other parameters in control process are $\theta_1 = 0.707, \zeta_1 = 18, \sigma_1 = 1.1, \sigma_2 = 0.2, \lambda_1 = 0.15, \lambda_2 = 0.25, \gamma = 0.81$. In this simulation, we choose the predefined time $T^* = 4s$.

Thus, the results of this simulation can be acquired in the Fig.s 1–5. Specifically, Fig. 1 exhibits the trajectories of y with and without quantization, and also displays the trajectories of υ_1 and its estimation $\hat{\upsilon}_1$. The trajectories of the state υ_2 and its estimation $\hat{\upsilon}_2$ are shown in Fig. 2. Figure 3 exhibits the curves of

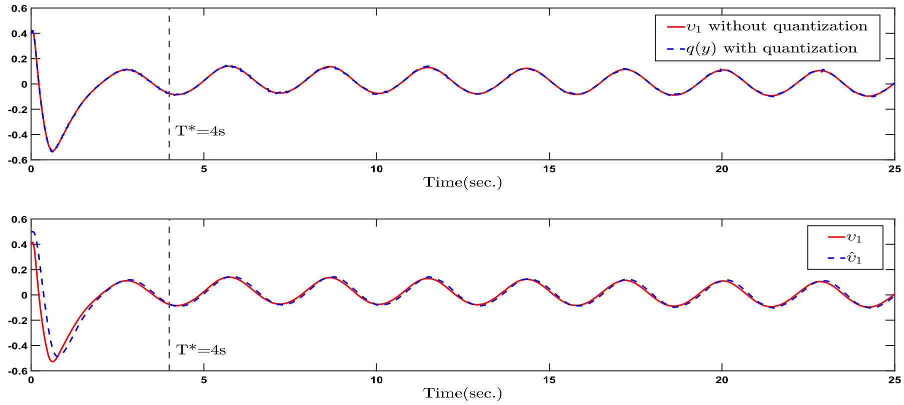

Fig. 1 The quantized output $q(y)$, state υ_1 and its estimation $\hat{\upsilon}_1$ in Example

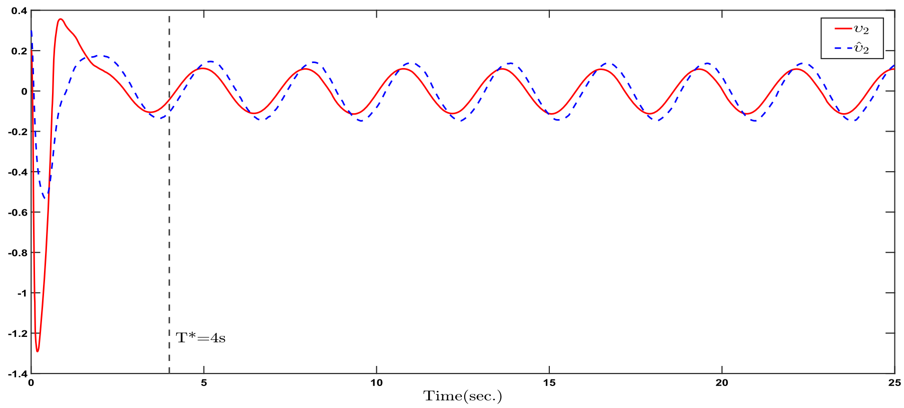

Fig. 2 The state υ_2 and its estimation $\hat{\upsilon}_2$ in Example

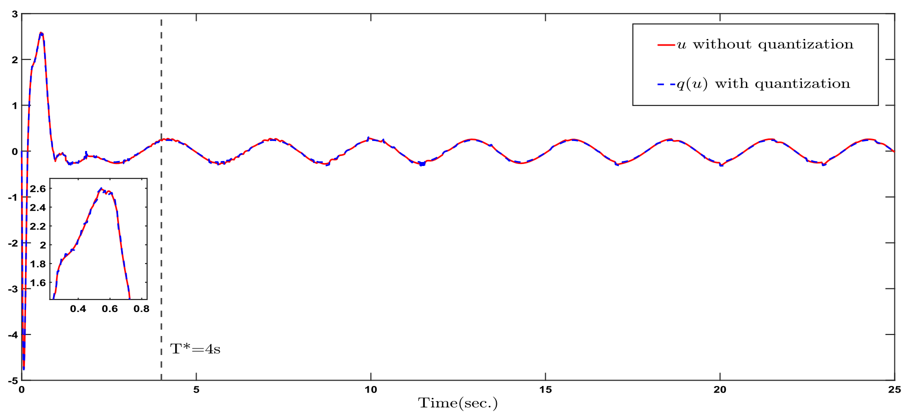

Fig. 3 The controller u with quantization and without quantization in Example

the control input signal u with and without quantization. The curves of the norm of the adaptive parameters $\hat{\vartheta}_1$, $\hat{\vartheta}_2$ and the variable ϱ_1^q are shown in Fig. 4 and Fig. 5, respectively. From Fig.s 1–5, the proposed control strategy can ensure that the system (86) under input and output quantization is practical predefined-time stable. Moreover, the variable ϱ_1^q can converge to a small domain within a predefined time $4s$.

To further indicate the superiority of input and output quantization, a comparison is presented with the existing finite-time control method in [33] under the same initial conditions. From Fig. 6, both the method in [33] and the proposed scheme can make the state variable converge to a relatively small domain. However, more energy consumption in the method in [33] is required than the proposed scheme. Thus, the input and output quantization before transmission can reduce

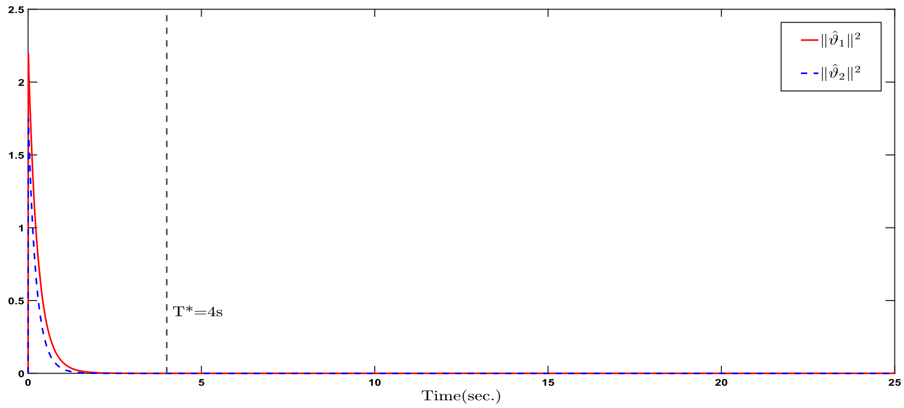

Fig. 4 The norm of the adaptive parameters $\hat{\vartheta}_1$ and $\hat{\vartheta}_2$ in Example

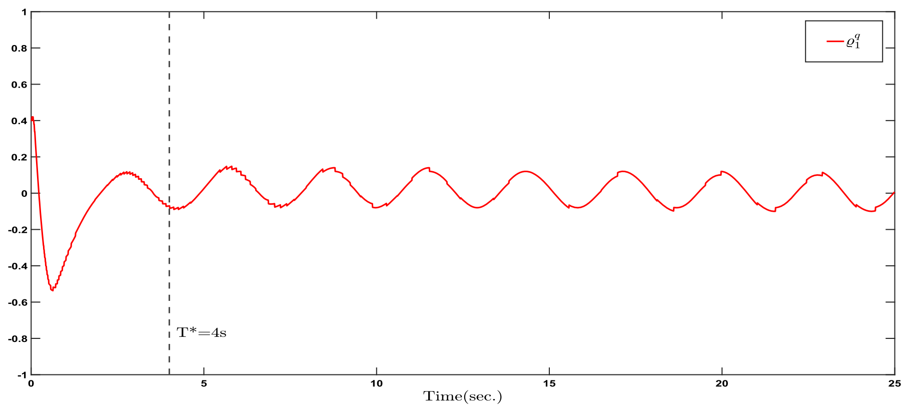

Fig. 5 The variable ϱ_1^q in Example

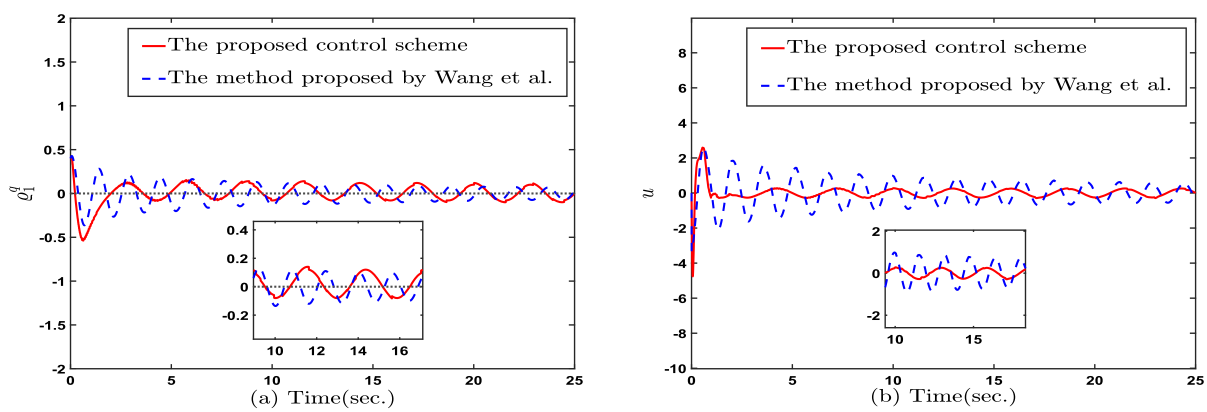

Fig. 6 Performance comparison between our proposed method and the method in [33]. (**a**) Variable ϱ_1^q; (**b**) Control input u

the energy consumption. Furthermore, to confirm the validity of the proposed predefined-time scheme, the system performance is verified when the predefined time is $T^* = 4s$, $T^* = 7s$, respectively. From Figs. 7 (a), it can be acquired that the variable ϱ_1^q can converge to a smaller region by a smaller predefined time. However, from Fig. 7 (b), with the decrease of predefined time, the control input increases. Thus, in practical application, we need to a trade-off between control consumption and settling time.

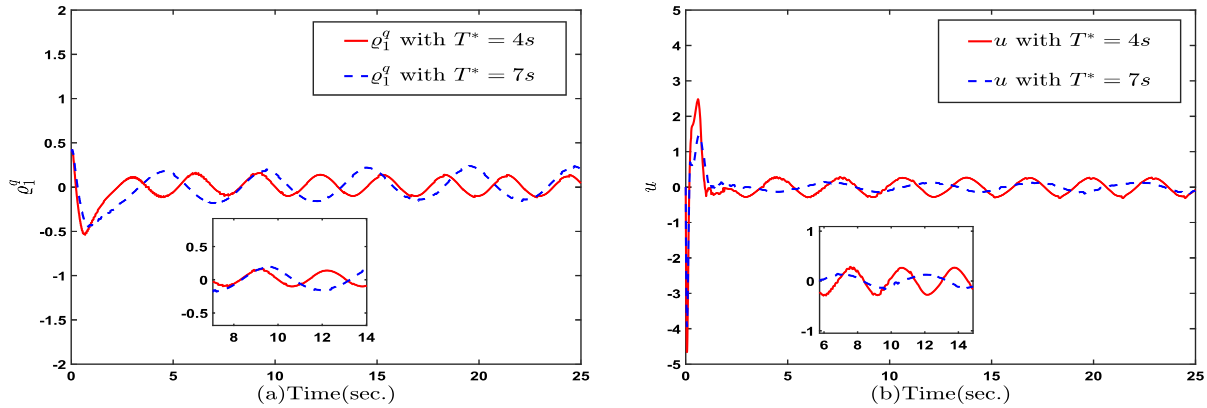

Fig. 7 Systems Performance within $T^* = 4s$, $T^* = 7s$, respectively. (**a**) Variable ϱ_1^q; (**b**) Control input u

6 Conclusion

For a class of pure-feedback nonlinear systems with input and output quantization, the issue of adaptive predefined-time control has been resolved. First of all, by using the Butterworth filter to transform the systems to the form of strict-feedback, a class of nonlinear functions have been constructed. On this basis and by utilizing the FLSs to approximate them, a new fuzzy state observer has been built. Secondly, in the backstepping control process, the command filtering technique has been applied to avoid the partial derivatives of virtual control signals. Furthermore, by applying a class of smooth functions, an intermediate auxiliary control signal has been devised. On this basis, an actual predefined-time controller has been obtained. Thirdly, Lemma 9 has been proved the boundness of quantization errors. Based on Lemma 9, the designed control scheme has guaranteed the practical predefined-time stability of the systems. Finally, the feasibility of this scheme has been proved by an example.

In this proposed scheme, only the issues of control energy consumption in signal transmission and converge time are researched, but the predefined accuracy is ignored. However, many actual systems have requirements of convergence time and control accuracy. Thus, for a class of pure-feedback nonlinear systems with input and output quantization, how to design a control scheme to make stabilization error converge to predefined accuracy within a predefined time is our future work.

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Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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