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ORIGINAL PAPER

Solitons, breathers and rational solutions for the (2 + 1)-dimensional Konopelchenko–Dubrovsky equation

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Abstract This paper explores the Konopelchenko–Dubrovsky (KD) equation and employs Hirota's bilinear method and Kadomtsev–Petviashvili (KP) hierarchy reduction technique to construct solitons, line breathers, rational solutions, and algebraic solitons within the system. These solutions are represented using $N \times N$ determinants. When the determinant size N is odd, periodic background solutions are generated, while even N values yield solutions on constant backgrounds. By utilizing asymptotic analysis, the paper elucidates explicit expressions for asymptotic algebraic solitons localized in a straight line for the algebraic soliton solutions. The dynamics of the obtained solutions are further examined and illustrated through plots.

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1 Introduction

In the 1960s, Zabusky and Kruskal utilized the finite difference method to investigate the numerical solutions of the Korteweg-de Vries (KdV) equation and discovered that solitons possess the characteristic of elastic collisions property, where their shapes and velocities remain unchanged after collisions [1]. In addition to the KdV equation, solitons can be found in various other $(1+1)$-dimensional continuous and discrete equations, such as the modified KdV equation, nonlinear Schrödinger equation, Burgers equation, sine-Gordon equation, and the nonlinear Schrödinger equations with non local characteristics [2–6]. These equations have found extensive applications in modeling nonlinear phenomena across diverse fields including mathematics, physics, optics, fluid dynamics, chemistry, biology, and more. Consequently, the investigation of solitons has emerged as a prominent and highly regarded research area [7–11]. With the advancement of soliton theory, researchers have discovered various types of soliton solutions, including line solitons, algebraic solitons, breathers, rogue waves, lumps, and peakon solutions, among others.

The significant contributions of soliton solutions to the understanding of nonlinear phenomena have led to extensive research on the mathematical structures and potential algebraic properties of soliton equations. The study of solutions for nonlinear soliton equations continues to be a thriving and ongoing subject of research. These investigations aim to deepen our knowledge of the rich dynamics and behaviors exhibited by solitons, thereby advancing our understanding of nonlinear systems and their applications.

Scholars have given various methods to derive soliton solutions, including inverse scattering transform [12], Darboux transform [13–15], Hirota bilinear method [16–23], Painlevé analysis [24,25], and Riemann-Hilbert method [26]. Another effective method is the KP hierarchy reduction technique [27] based on Hirota's bilinear method.

The Kyoto school has developed a technique that has proven successful in deriving soliton solutions for numerous integrable equations. This method has been widely applied and has yielded significant results in obtaining soliton solutions for a variety of integrable equations [28,29]. Ohta and Yang have shown that this method is also applicable for obtaining rogue wave solutions of equations such as the nonlocal nonlinear Schrödinger equation (NLSE) [30], Davey-Stewartson (DS) equation [31], and Ablowitz-Ladik equation [32]. Furthermore, this method can be used to derive (semi-) rational solutions of the third-type DS equation [33]. In 2018, Feng [34] further improved the KP hierarchy reduction method and obtained general soliton solutions for the NLSE with zero and nonzero boundary conditions using this method. Several soliton solutions for nonlocal integrable equations have also been constructed using similar approaches [35]. Compared to other methods for deriving soliton solutions and (semi-) rational solutions, the expressions obtained through the KP hierarchy method are more general and concise.

The aim of this paper is to make a study on the $(2+1)$-dimensional KD equation [36] in the stratified shear flow, internal and shallow-water waves, plasmas and other fields, with the help of KP hierarchy reduction technique and symbolic computation

$$\begin{aligned} &U_t - U_{xxx} - 6\beta U U_x + \frac{3}{2}\alpha^2 U^2 U_x - 3V_y \\ &\quad + 3\alpha U_x V = 0, \\ &U_y = V_x, \end{aligned} \tag{1.1}$$

where α and β are arbitrary constants, U and V are the analytic functions of the variables x, y and t, the subscripts denote the partial differential derivatives. Eq. (1.1) is a highly significant nonlinear partial differential equation, which has been attributed to the renowned Russian mathematicians Konopelchenko and Dubrovsky. It serves as a powerful mathematical tool for analyzing and understanding a wide range of nonlinear wave phenomena, encompassing sound waves, water waves, and light waves. Its applications extend to the exploration of various wave-related phenomena, including wave propagation, interference, and scattering. In the domain of solid mechanics, Eq. (1.1) assumes a crucial role in characterizing the vibrations and waves exhibited by elastic bodies. Specifically, it facilitates the comprehensive description of the interplay between stress and strain within elastic materials, as well as the propagation behavior of elastic waves. In the field of optics, Eq. (1.1) proves instrumental in studying the intricate nonlinear propagation dynamics of light. It enables the analysis of phenomena such as self-focusing, self-phase modulation, and optical solitons, which bear significant implications for essential applications in optical fiber communications and lasers. Eq. (1.1) can be reduced to the Gardner [37–40], KP [41–44] and modified KP (mKP) [45,46] in ocean dynamics, fluid mechanics and plasma physics. For example, (1) when $V = 0$, Eq. (1.1) are reduced to the Gardner equation, which is used to analyze the solitary waves, interfacial waves in the stratified shear flow and internal waves in a stratified ocean; (2) when $\alpha = 0$, Eq. (1.1) are reduced to the KP equation to model the shallow-water waves, propagation of weakly nonlinear dispersive long waves, etc; (3) when $\beta = 0$, Eq. (1.1) can be reduced to the mKP equation that can describe the evolution of solitary waves and the propagation of ionic acoustic and electromagnetic waves.

KD equation (1.1) have attracted a lot of attention because of its practical physical significance. In [47], abundant new exact non-travelling wave solutions of KD equation were presented based on improved tanh function method. Wang [48] propose the further improved F-expansion method to get Jacobi elliptic function solutions, soliton-like solutions, trigonometric function solutions of KD equation. Multiple lump solutions of KD equation are obtained by the Hirota bilinear method [49]. Via the Sato theory and Hirota method, Yuan and Tian [50] present the soliton solutions in terms of the Gram determinant which can yield the bright, depression and kink solitons. Breather waves [51], Wronskian solutions [52] and Painlevé property [53] of KD equation were studied. Recently, Grinevich and Santini [54] conducted research and demonstrated that rogue wave solutions or Akhmediev breathers can be formed in the presence of non-zero backgrounds, represented by elliptic functions. They also investigated the breathers and rogue wave solutions in the context of the Hirota equation and the nonlinear Schrödinger equation with elliptic function backgrounds [55]. The construction of soliton solutions on periodic function backgrounds presents a more challenging task. In this paper, the aim is to construct (semi-) rational solutions, such as solitons, linear breathers and algebraic soliton solutions, of KD equation on the zero background and periodic background. Compared with asymptotic analysis of exponential-type solitons, asymptotic behavior of algebraic soliton solutions is much more difficult to be conducted. We analyze the asymptotic behaviors of algebraic solitons of KD equation.

The organization of this article is as follows. In Sect. 2, we provided a detailed exposition on the derivation and simplification of the solution to KD equations from a modified KP equation. Subsequently, in Sect. 3, we conducted an analysis of the dynamics of solitons in both constant and periodic backgrounds. The Sect. 4 focused on the investigation of breathing solutions. Moving forward, in Sect. 5, we explored various dynamic behaviors, including algebraic solitons, multiple rational solutions. Lastly, in Sect. 6, we summarized the research findings and engaged in a discussion.

2 The determinant structure

Through the dependent variable transformations

$$U = \frac{2}{\alpha}\left(\ln\frac{G}{F}\right)_x, \qquad V = \frac{2}{\alpha}\left(\ln\frac{G}{F}\right)_y, \tag{2.1}$$

then the Eq. (1.1) can be transformed into the following bilinear forms

$$\begin{aligned}&(D_t - D_x^3 + 3D_xD_y + 6\mathbb{A}D_y)G\cdot F = 0,\\&(D_y + D_x^2 + 2\mathbb{A}D_x)G\cdot F = 0,\end{aligned} \tag{2.2}$$

where $\mathbb{A} = -\frac{\beta}{\alpha}$, G and F are real functions of variables x, y and t, D is the Hirota derivative [16] defined by

$$\begin{aligned}&D_x^{m_1}D_y^{m_2}D_t^{m_3}g(x,y,t)\cdot f(x,y,t) \equiv\\&\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^{m_1}\left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^{m_2}\\&\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^{m_3} g(x,y,t)f(x',y',t')\,|_{x=x',y=y',t=t'},\end{aligned}$$

with f being a differentiable function of x, y, and t; g being a differentiable function of the formal variables x', y', and t', while m_1, m_2, and m_3 being the non-negative integers.

We start from the bilinear equations of the KP hierarchy [30]

$$\begin{aligned}&(D_x^3 + 3D_xD_y - 4D_t + 3a(D_x^2 + D_y) + 6a^2D_x)\\&\quad\tau_{n+1}\cdot\tau_n = 0,\\&(D_x^2 - D_y + 2aD_x)\tau_{n+1}\cdot\tau_n = 0.\end{aligned} \tag{2.3}$$

Under the variable transformation $x_1 = x$, $x_2 = -y$, $x_{-1} = 4t$, $G = \tau_{n+1}$ and $F = \tau_n$, the modified KP equation (2.3) turns into bilinear KD equations (2.2)

$$\begin{aligned}&(D_{x_1}^3 + 3D_{x_1}D_{x_2} - 4D_{x_{-1}} + 6\mathbb{A}D_{x_2})\tau_{n+1}\cdot\tau_n = 0,\\&(D_{x_1}^2 - D_{x_2} + 2\mathbb{A}D_{x_1})\tau_{n+1}\cdot\tau_n = 0,\end{aligned} \tag{2.4}$$

and taking $F = \tau_0$ and $G = \tau_1$, the above bilinear equation become

$$\begin{aligned}&(D_{x_1}^3 + 3D_{x_1}D_{x_2} - 4D_{x_{-1}} + 6\mathbb{A}D_{x_2})g\cdot f = 0,\\&(D_{x_1}^2 - D_{x_2} + 2\mathbb{A}D_{x_1})g\cdot f = 0,\end{aligned} \tag{2.5}$$

in the algebraic solutions, the bilinear equations (2.5) are reduced to the bilinear equations (2.2). For simplicity of calculation, let parameter $\alpha = 2$ in the following.

2.1 Gram determinant solution for the bilinear equation

In this subsection, we first derive the Gram solutions for the higher-dimensional bilinear system. According to lemma 3.1 of [28], we get the following lemma 1:

Lemma 1 *The KP hierarchy have been proved to have the Gram-type determinant solution*

$$\tau_n = \det_{1\le i,j\le N}(m_{ij}^{(n)}) \tag{2.6}$$

where N is any positive integer. Which are satisfied the next differential and difference relations,

$$\begin{aligned}
\partial_{x_1} m_{ij}^{(n)} &= \phi_i^{(n)}\varphi_j^{(n)},\\
\partial_{x_2} m_{ij}^{(n)} &= (\partial_{x_1}\phi_i^{(n)})\varphi_j^{(n)} - \phi_i^{(n)})(\partial_{x_1}\varphi_j^{(n)}),\\
\partial_{x_3} m_{ij}^{(n)} &= (\partial_{x_1}^2\phi_i^{(n)})\varphi_j^{(n)} + \phi_i^{(n)}(\partial_{x_1}^2\varphi_j^{(n)})\\
&\quad - (\partial_{x_1}\phi_i^{(n)})(\partial_{x_1}\varphi_i^{(n)}),\\
m_{ij}^{(n+1)} &= m_{ij}^{(n)} + \phi_i^{(n)}\varphi_j^{(n+1)},\\
\phi_i^{(n+1)} &= (\partial_{x_1} - \mathbb{A}))\phi_i^{(n)},\\
\varphi_i^{(n-1)} &= -(\partial_{x_1} + \mathbb{A}))\varphi_j^{(n)},\\
\partial_{x_2}\phi_i^{(n)} &= \partial_{x_1}^2\phi_i^{(n)},\\
\partial_{x_2}\varphi_j^{(n)} &= -\partial_{x_1}^2\varphi_j^{(n)},\\
\partial_{x_3}\phi_i^{(n)} &= \partial_{x_1}^3\phi_i^{(n)},\\
\partial_{x_3}\varphi_j^{(n)} &= \partial_{x_1}^3\varphi_j^{(n)},
\end{aligned}$$

with $m_{ij}^{(n)}$, $\phi_i^{(n)}$ and $\varphi_j^{(n)}$ as the functions of x_1, x_2 and x_3, i and j as the integers, and the superscript "(n)" as a note for distinguishing the n-th power. Substituting elements $m_{ij}^{(n)}$ into Gram determinant, we obtain N-soliton solutions for KD equations.

In order to construct soliton solutions and semi-rational soluions to the bilinear equations defined in (2.5), we choose the special ϕ and φ functions

$$\phi_i^{(n)} = (u_i - \mathbb{A})^n e^{\xi_i}\varphi_j^{(n)} = (-v_j + \mathbb{A})^{-n} e^{\eta_j} \tag{2.7}$$

$$m_{ij}^{(n)} = c_j\delta_{ij} + \int^{x_1}\phi_i^{(n)}\varphi_j^{(n)}dx_1 = c_j\delta_{ij} + \frac{1}{u_i+v_j}\left(-\frac{u_i-\mathbb{A}}{v_j+\mathbb{A}}\right)^n e^{\xi_i+\eta_j} \tag{2.8}$$

where

$$\begin{aligned}
\xi_i &= u_i x + u_i^2 x_2 + u_i^3 x_3 + \xi_{i0},\\
\eta_j &= v_j x - v_j^2 x_2 + v_j^3 x_3 + \eta_{j0},
\end{aligned} \tag{2.9}$$

u_i, v_j as the constants, δ_{ij} as the Kronecker delta notation. By substituting the above elements $m_{ij}^{(n)}$ into the Gram determinant, the n-soliton solution of the KD equation is obtained.

According Ref. [30], the differential operators A_i and B_j are introduced

$$\begin{aligned}
A_i &= \sum_{k=0}^{n_i} a_{ik}\left(u_i\partial_{u_i}\right)^{n_i-k},\\
B_j &= \sum_{h=0}^{n_j} b_{jh}\left(v_j\partial_{v_j}\right)^{n_j-h},
\end{aligned} \tag{2.10}$$

to generate rational solutions. Here a_{ik} and b_{jl} are arbitrary complex constants, and n_i and n_j are arbitrary positive integers. We apply the differential A_i and B_j to $m_{ij}^{(n)}$ and denote

$$\begin{aligned}
\widetilde{m}_{ij}^{(n)} &= A_i B_j m_{ij}^{(n)}\\
&= c_j\delta_{ij} + (-\frac{u_i-\mathbb{A}}{v_j+\mathbb{A}})e^{\xi_i+\eta_j}\sum_{k=0}^{n_i} a_{ik}(u_i\partial_{u_i}\\
&\quad + \frac{nu_i}{u_i-\mathbb{A}} + \widetilde{\xi}_i)^{n_i-k}\\
&\sum_{h=0}^{n_j} b_{jh}(v_j\partial_{v_j} - \frac{nv_j}{v_j+\mathbb{A}} + \widetilde{\eta}_j)^{n_j-h}\frac{1}{u_i+v_j},
\end{aligned} \tag{2.11}$$

where

$$\widetilde{\xi}_i = u_i x_1 + 2u_i^2 x_2 + 3u_i^3 x_3,\ \widetilde{\eta}_j = v_j x_1 - 2v_j^2 x_2 + 3v_j^3 x_3.$$

Since these operators apply only to the parameters u_1 and v_j, the determinant $\tau_n = \det_{1\le i,j\le N}(m_{ij}^{(n)})$ also solves bilinear equations (2.4).

When c_j is zero in equation (2.11), the determinant elements are polynomials in x, y, and t, such that τ_n is a polynomial, resulting in a rational solution for the KD equation. When c_j is not zero, we can obtain semi-rational solutions expressed as a combination of exponential functions and polynomials. According to the above analysis, we have the following theorem.

Theorem 2.1 *KD equations (1.1) adimts solutions transformations*

$$U = \frac{2}{\alpha}\left(\ln\frac{\tau_{n+1}}{\tau_n}\right)_x,\ V = \frac{2}{\alpha}\left(\ln\frac{\tau_{n+1}}{\tau_n}\right)_y, \tag{2.12}$$

where the determinant $\tau_n = \det_{1\le i,j\le N}(\widetilde{m}_{ij}^{(n)})$ has elements

$$\widetilde{m}_{ij}^{(n)} = c_j\delta_{ij} + (-\frac{u_i-\mathbb{A}}{v_j+\mathbb{A}})e^{\xi_i+\eta_j}$$

$$\sum_{k=0}^{n_i} a_{ik}(u_i\partial_{u_i} + \frac{nu_i}{u_i - \mathbb{A}} + \widetilde{\xi}_i)^{n_i-k}$$
$$\sum_{h=0}^{n_j} b_{jh}(v_j\partial_{v_j} - \frac{nv_j}{v_j + \mathbb{A}} + \widetilde{\eta}_j)^{n_j-h} \frac{1}{u_i + v_j}, \quad (2.13)$$

and

$$\widetilde{\xi}_i = u_i x_1 + 2u_i^2 x_2 + 3u_i^3 x_3,$$
$$\widetilde{\eta}_j = v_j x_1 - 2v_j^2 x_2 + 3v_j^3 x_3,$$
$$\xi_i = u_i x + u_i^2 x_2 + u_i^3 x_3 + \xi_{i0},$$
$$\eta_j = v_j x - v_j^2 x_2 + v_j^3 x_3 + \eta_{j0},$$

where u_i, v_j, and c_j are arbitrary complex constants, and n_i and n_j are the arbitrary positive integers. By choosing different types of functions of $m_{ij}^{(n)}$, ϕ_i, φ_j, and $\widetilde{m}_{ij}^{(n)}$, the soliton, breather, and (semi-)rational solutions to KD equations (1.1) can be constructed.

3 Dynamics of the soliton solutions

In this section, we construct and analyze the dynamics of soliton solutions for the KD equation (1.1) separately with constant and periodic background.

3.1 One-soliton solutions

Setting $N = 1$ in equation (2.7) and get the first order determinant

$$m_{ij}^{(n)} = c_j\delta_{ij} + \frac{1}{u_i + v_j}\left(-\frac{u_i - \mathbb{A}}{v_j + \mathbb{A}}\right)^n e^{\xi_i + \eta_j},$$
$$\xi_1 + \eta_1 = (u_1 + v_1)x + (v_1^2 - u_1^2)y + 4(u_1^3 + v_1^3)t + \xi_{10} + \eta_{10}, \quad (3.1)$$

here u_1, v_1, c_1, ξ_{10}, η_{10} are all parameters.

Case 1 On the zero background
If we take $c_1 = 1$, u_1, v_1, ξ_{10}, and η_{10} are arbitrary constants, functions F and G have the form

$$F = 1 + \frac{1}{v_1 + u_1} e^{\xi_1 + \eta_1},$$
$$G = 1 + \left(-\frac{u_1 - \mathbb{A}}{v_1 + \mathbb{A}}\right)\frac{1}{v_1 + u_1} e^{\xi_1 + \eta_1}, \quad (3.2)$$

and the potential u and v are the form

$$U = \frac{2}{\alpha}\left(\ln\frac{g}{f}\right)_x = -\frac{(u_1 + v_1)^3 e^{\xi_1 + \eta_1}}{(v_1 + \mathbb{A})(u_1 + v_1 + e^{\xi_1 + \eta_1})^2},$$
$$V = \frac{2}{\alpha}\left(\ln\frac{g}{f}\right)_y = \frac{(u_1 + v_1)^3 e^{\xi_1 + \eta_1}(u_1 - v_1)}{(v_1 + \mathbb{A})(u_1 + v_1 + e^{\xi_1 + \eta_1})^2}. \quad (3.3)$$

Plots of one-bright-soliton solution and one-dark-soliton solution are shown in Fig. 1.

Case 2 On the periodic background
If we take $c_1 = \mathrm{i}c$, $u_1 = \vartheta + \mathrm{i}\theta$ and $v_1 = -u_1^*$, where ϑ, θ, c are real numbers and $c\theta \neq 0$, then functions F and G have the form

$$F = c_1 + \frac{1}{v_1 + u_1} e^{\xi_1 + \eta_1} = \mathrm{i}c - \frac{i}{2\theta} e^{2\mathrm{i}\theta x - 4\mathrm{i}\vartheta\theta y + 8(\vartheta + \mathrm{i}\theta)^3 t + \xi_{10} + \eta_{10}}, \quad (3.4)$$
$$G = c_1 + \left(-\frac{u_1 - \mathbb{A}}{v_1 + \mathbb{A}}\right)\frac{1}{v_1 + u_1} e^{\xi_1 + \eta_1} = \mathrm{i}c + \frac{\mathrm{i}(\vartheta + \mathrm{i}\theta - \mathbb{A})}{2(-\vartheta + \mathrm{i}\theta + \mathbb{A})\theta} e^{2\mathrm{i}\theta x - 4\mathrm{i}\vartheta\theta y + 8(\vartheta + \mathrm{i}\theta)^3 t + \xi_{10} + \eta_{10}},$$

substituting (3.4) into (2.1), we obtain the one-soliton solution

$$U = \frac{2}{\alpha}\left(\ln\frac{G}{F}\right)_x = \frac{8\theta^3 c e^{2\mathrm{i}\theta x - 4\mathrm{i}\vartheta\theta y + 8(\vartheta + \mathrm{i}\theta)^3 t + \xi_{10} + \eta_{10}}}{(-\vartheta + \mathrm{i}\theta + \mathbb{A})(-2c\theta + e^{2\mathrm{i}\theta x - 4\mathrm{i}\vartheta\theta y + 8(\vartheta + \mathrm{i}\theta)^3 t + \xi_{10} + \eta_{10}})^2} \quad (3.5)$$

when u_1 is a pure imaginary number, the solutions (3.5) are periodic in both x and t with periods $-\frac{\mathrm{i}}{\theta}$ and $\frac{\mathrm{i}\pi}{4\theta^3}$, respectively. It will be seen later that it plays an important role when constructing soliton and (semi-) rational solutions of KD equation (1.1) in the periodic context. Plot of this periodic one-soliton solutions is depicted in Fig. 2. In this paper, regular solutions (3.5) provide the periodic background for higher-order soliton solution (see Fig. 2).

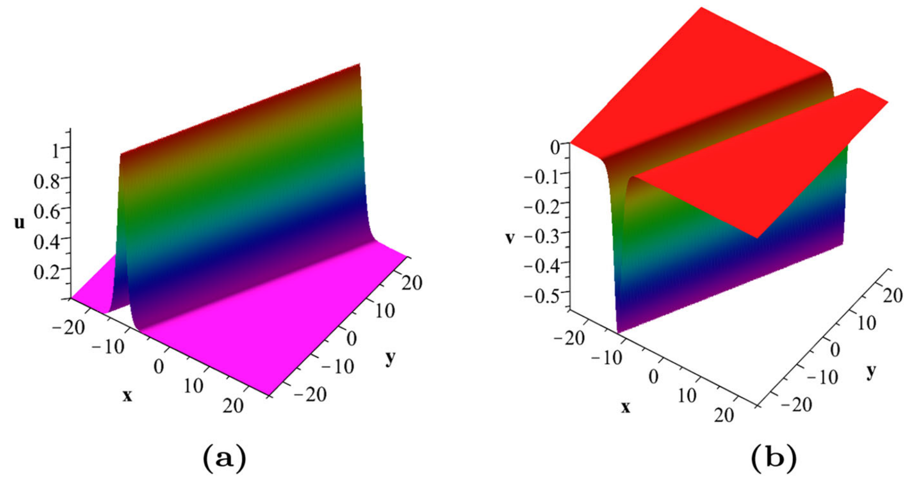


Fig. 1 One-soliton solution u and v (3.3) with parameters $c=1$, $\mathbb{A}=-1$, $u_1=1$, $v_1=\frac{1}{2}$, $\xi_{10}=\eta_{10}=0$, $t=0$

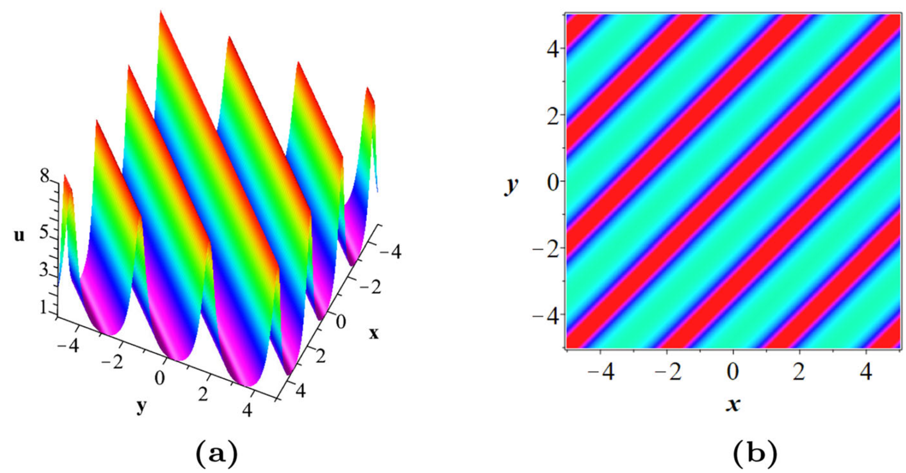


Fig. 2 One periodic solution u (3.5) with suitable parameters $c=1$, $\vartheta=\frac{1}{2}$, $\theta=1$, $\mathbb{A}=\frac{1}{2}$, $\xi_{10}=\eta_{10}=0$, $t=0$

3.2 Two-soliton solutions

Case 1 On the constant background

In order to derive two-soliton solution, by taking $N=2$. For the convenience of calculation, we choose $c_1=c_2=1$. The F and G in 2×2 determinant

$$
\begin{aligned}
F &= \begin{vmatrix} 1+\frac{1}{u_1+v_1}e^{\xi_1+\eta_1} & \frac{1}{u_1+v_2}e^{\xi_1+\eta_2} \\ \frac{1}{u_2+v_1}e^{\xi_2+\eta_1} & 1+\frac{1}{u_2+v_2}e^{\xi_2+\eta_2} \end{vmatrix} \\
&= 1+\frac{1}{v_1+u_1}e^{\varsigma_1}+\frac{1}{v_2+u_2}e^{\varsigma_2} \\
&\quad +\frac{(u_1-u_2)(v_1-v_2)}{(v_1+u_1)(v_2+u_1)(v_1+u_2)(v_2+u_2)}e^{\varsigma_1+\varsigma_2}, \qquad (3.6)\\
G &= \begin{vmatrix} 1+\frac{1}{u_1+v_1}\left(-\frac{u_1-\mathbb{A}}{v_1+\mathbb{A}}\right)e^{\xi_1+\eta_1} & \frac{1}{u_1+v_2}\left(-\frac{u_1-\mathbb{A}}{v_2+\mathbb{A}}\right)e^{\xi_1+\eta_2} \\ \frac{1}{u_2+v_1}\left(-\frac{u_2-\mathbb{A}}{v_1+\mathbb{A}}\right)e^{\xi_2+\eta_1} & 1+\frac{1}{u_2+v_2}\left(-\frac{u_2-\mathbb{A}}{v_2+\mathbb{A}}\right)e^{\xi_2+\eta_2} \end{vmatrix} \\
&= 1-\frac{u_1-\mathbb{A}}{(v_1+\mathbb{A})(v_1+u_1)}e^{\varsigma_1}-\frac{u_2-\mathbb{A}}{(v_2+\mathbb{A})(v_2+u_2)}e^{\varsigma_2} \\
&\quad +\frac{(u_1-\mathbb{A})(u_2-\mathbb{A})(u_1-u_2)(v_1-v_2)}{(v_1+\mathbb{A})(v_2+\mathbb{A})(u_1+v_1)(u_1+v_2)(u_2+v_1)(u_2+v_2)}e^{\varsigma_1+\varsigma_2}, \qquad (3.7)
\end{aligned}
$$

in which

$$
\begin{aligned}
\varsigma_1 &= \xi_1+\eta_1=(u_1+v_1)x+(v_1^2-u_1^2)y \\
&\quad +4(u_1^3+v_1^3)t+\xi_{10}+\eta_{10}, \\
\varsigma_2 &= \xi_2+\eta_2=(u_2+v_2)x+(v_2^2-u_2^2)y \\
&\quad +4(u_2^3+v_2^3)t+\xi_{20}+\eta_{20}.
\end{aligned}
$$

After simple algebra calculation, we obtain two solitons. We analyzed the collision of two solitons and provided the asymptotic behavior of the collision when $t=0$. The two-soliton solution of U is expressed as

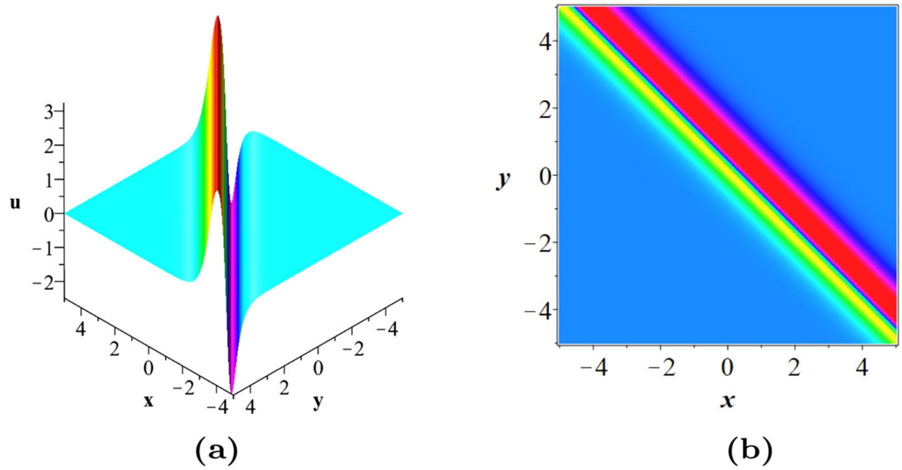


Fig. 3 Two soliton solution of U with suitable parameters $\mathbb{A}=-1$, $u_1=1, u_2=3, v_1=2$, $v_2=4, t=0$

$$U=\frac{-N_{112}e^{2\zeta_1+\zeta_2}-N_{122}e^{\zeta_1+2\zeta_2}-N_{12}e^{\zeta_1+\zeta_2}-N_1e^{\zeta_1}-N_2e^{\zeta_2}}{D_1e^{\zeta_1}+D_2e^{\zeta_2}+D_{22}e^{2\zeta_2}+D_{12}e^{\zeta_1+\zeta_2}+D_{122}e^{\zeta_1+2\zeta_2}+D_{112}e^{2\zeta_1+\zeta_2}+D_{1122}e^{2\zeta_1+2\zeta_2}+D_{11}e^{2\zeta_1}+1}, \tag{3.8}$$

where

$$N_1=\frac{-u_1-v_1}{v_1+\mathbb{A}}, N_2=\frac{u_2+v_2}{v_2+\mathbb{A}},$$
$$N_{112}=-\frac{(v_1-v_2)(u_1-u_2)(-u_1+\mathbb{A})(u_2+v_2)}{(u_2+v_1)(u_1+v_2)(v_2+\mathbb{A})(v_1+\mathbb{A})(u_1+v_1)^2},$$
$$N_{122}=-\frac{(v_1-v_2)(u_1-u_2)(-u_2+\mathbb{A})(u_1+v_1)}{(u_2+v_1)(u_1+v_2)(v_2+\mathbb{A})(v_1+\mathbb{A})(u_2+v_2)^2},$$
$$N_{12}=\frac{(-u_1+\mathbb{A})(u_1-u_2+v_1-v_2)}{(u_1+v_1)(v_1+\mathbb{A})(u_2+v_2)}+\frac{(u_1+u_2+v_1+v_2)(u_1-u_2)(-u_2+\mathbb{A})(-u_1+\mathbb{A})(v_1-v_2)}{(u_1+v_1)(v_1+\mathbb{A})(u_2+v_2)(v_2+\mathbb{A})(u_1+v_2)(u_2+v_1)}-\frac{(-u_2+\mathbb{A})(u_1-u_2+v_1-v_2)}{(u_2+v_2)(v_2+\mathbb{A})(u_1+v_1)}-\frac{(u_1+u_2+v_1+v_2)(v_1-v_2)(u_1-u_2)}{(u_1+v_1)(u_2+v_2)(u_1+v_2)(u_2+v_1)},$$
$$D_1=\frac{2\mathbb{A}-u_1+v_1}{(u_1+v_1)(v_1+\mathbb{A})},\quad D_2=\frac{2\mathbb{A}-u_2+v_2}{(u_2+v_2)(v_2+\mathbb{A})},$$
$$D_{22}=\frac{-u_2+\mathbb{A}}{(u_2+v_2)^2(v_2+\mathbb{A})},\quad D_{11}=\frac{-u_1+\mathbb{A}}{(u_1+v_1)^2(v_1+\mathbb{A})},$$
$$D_{122}=\frac{2\mathbb{A}-u_2+v_2}{(u_2+v_2)(v_2+\mathbb{A})},\quad D_{112}=\frac{-u_2+\mathbb{A}}{(u_2+v_2)^2(v_2+\mathbb{A})},$$
$$D_{12}=\frac{(u_1-u_2)(-u_2+\mathbb{A})(-u_1+\mathbb{A})(v_1-v_2)}{(u_1+v_1)(v_1+\mathbb{A})(u_2+v_2)(v_2+\mathbb{A})(u_1+v_2)(u_2+v_1)}+\frac{(-u_2+\mathbb{A})}{(u_2+v_2)(v_2+\mathbb{A})(u_1+v_1)}+\frac{(-u_1+\mathbb{A})}{(u_1+v_1)(v_1+\mathbb{A})(u_2+v_2)}+\frac{(v_1-v_2)(u_1-u_2)}{(u_1+v_1)(u_2+v_2)(u_1+v_2)(u_2+v_1)},$$
$$D_{1122}=\frac{(u_1-u_2)^2(-u_1+\mathbb{A})(-u_2+\mathbb{A})(v_1-v_2)^2}{(u_1+v_1)^2(v_1+\mathbb{A})(u_2+v_2)^2(v_2+\mathbb{A})(u_1+v_2)^2(u_2+v_1)^2}.$$

Figures 3, 4, and 5 respectively depict the interaction and corresponding density of two solitons under different parameters in a zero background. Figures 3 and 4 illustrate the interaction between a bright soliton and a dark soliton for different values of parameter u_2. Figure 5 shows the interaction between two bright solitons. We define the left-moving soliton as soliton 1, located on the plane $\zeta_1=0$, and the right-moving soliton as soliton 2, located on the plane $\zeta_2=0$. Subsequently, we present the asymptotic forms of the solutions for these two solitons.

We have the following asymptotic forms for the two-soliton solutions.

Before collision ($y\to-\infty$)
Soliton 1 ($\zeta_1\approx 0, \zeta_2\to+\infty$)

$$U\to 2\frac{-N_{122}e^{\zeta_1}}{D_{22}+D_{122}e^{\zeta_1}+D_{1122}e^{2\zeta_1}}$$
$$=-2\frac{(u_1+v_1)^2}{(2\mathbb{A}-u_1+v_1)+2\sqrt{(-u_1+\mathbb{A})(v_1+\mathbb{A})}cosh(\varsigma_1-\varrho_1)},$$
$$e^{\varrho_1}=\frac{(u_2+v_1)(u_1+v_2)(u_1+v_1)}{(u_1-u_2)(v_1-v_2)}\sqrt{\frac{v_1+\mathbb{A}}{-u_1+\mathbb{A}}}.$$

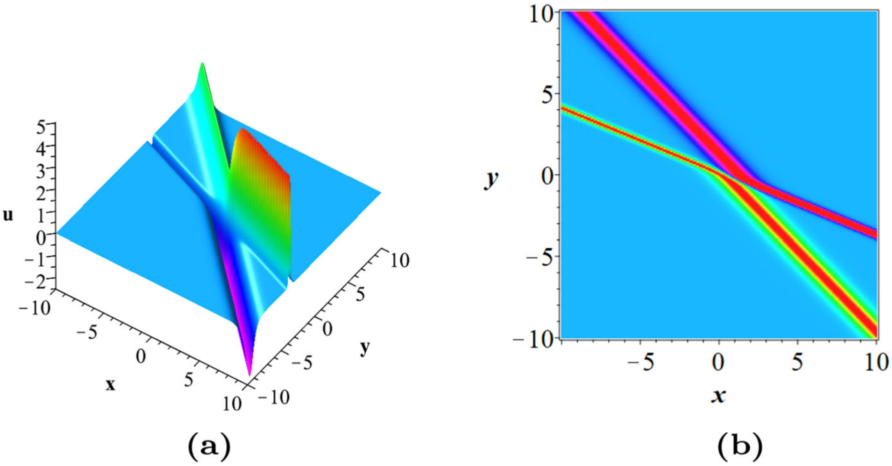


Fig. 4 Two soliton solution of U with suitable parameters $\mathbb{A} = -1, u_1 = 1, u_2 = \frac{3}{2}, v_1 = 2, v_2 = 4, t = 0$

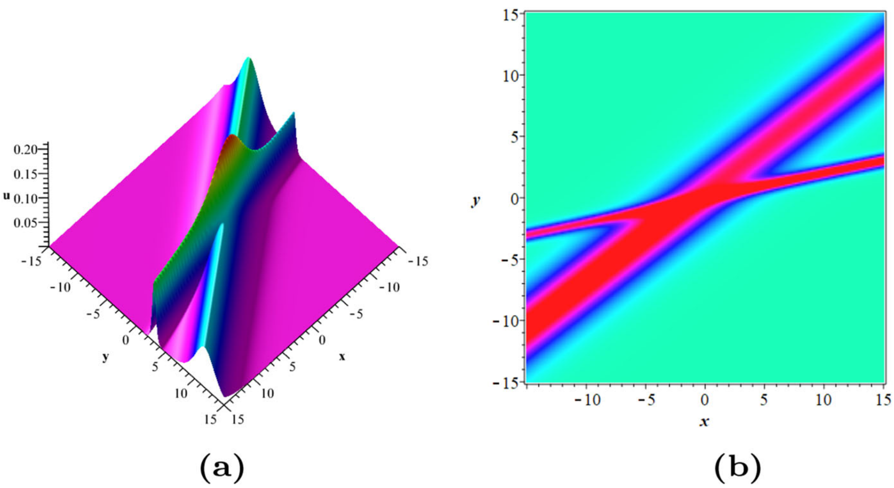


Fig. 5 Two soliton solution of U with suitable parameters $\mathbb{A} = -1, u_1 = 1, u_2 = 3, v_1 = -\frac{1}{3}, v_2 = -\frac{9}{4}, t = 0$

Soliton 2 ($\zeta_2 \approx 0, \zeta_1 \to -\infty$)

$$U \to 2\frac{-N_2 e^{\zeta_2}}{D_2 e^{\zeta_2} + D_{22} e^{2\zeta_2} + 1}$$
$$= -2\frac{(u_2+v_2)^2}{(2\mathbb{A} - u_2 + v_2) + 2\sqrt{(-u_2+\mathbb{A})(v_2+\mathbb{A})}cosh(\varsigma_2 - \varrho_2)},$$
$$e^{\varrho_2} = (u_2+v_2)\sqrt{\frac{v_2+\mathbb{A}}{-u_2+\mathbb{A}}}.$$

After collision ($y \to +\infty$)
Soliton 1 ($\zeta_1 \approx 0, \zeta_2 \to -\infty$)

$$U \to 2\frac{-N_1 e^{\zeta_1}}{D_1 e^{\zeta_1} + +D_{11} e^{2\zeta_1} + 1}$$
$$= -2\frac{(u_1+v_1)^2}{(2\mathbb{A} - u_1 + v_1) + 2\sqrt{(-u_1+\mathbb{A})(v_1+\mathbb{A})}cosh(\varsigma_1 - \varrho_3)},$$
$$e^{\varrho_3} = (u_1+v_1)\sqrt{\frac{v_1+\mathbb{A}}{-u_1+\mathbb{A}}}.$$

Soliton 2 ($\zeta_2 \approx 0, \zeta_1 \to +\infty$)

$$U \to 2\frac{-N_{112} e^{\zeta_2}}{D_{112} e^{\zeta_2} + D_{1122} e^{2\zeta_2} + D_{11}}$$
$$= -2\frac{(u_2+v_2)^2}{(2\mathbb{A} - u_2 + v_2) + 2\sqrt{(-u_2+\mathbb{A})(v_2+\mathbb{A})}cosh(\varsigma_2 - \varrho_4)},$$
$$e^{\varrho_4} = \frac{(u_2+v_1)(u_1+v_2)(u_2+v_2)}{(u_1-u_2)(v_1-v_2)}\sqrt{\frac{v_2+\mathbb{A}}{-u_2+\mathbb{A}}}.$$

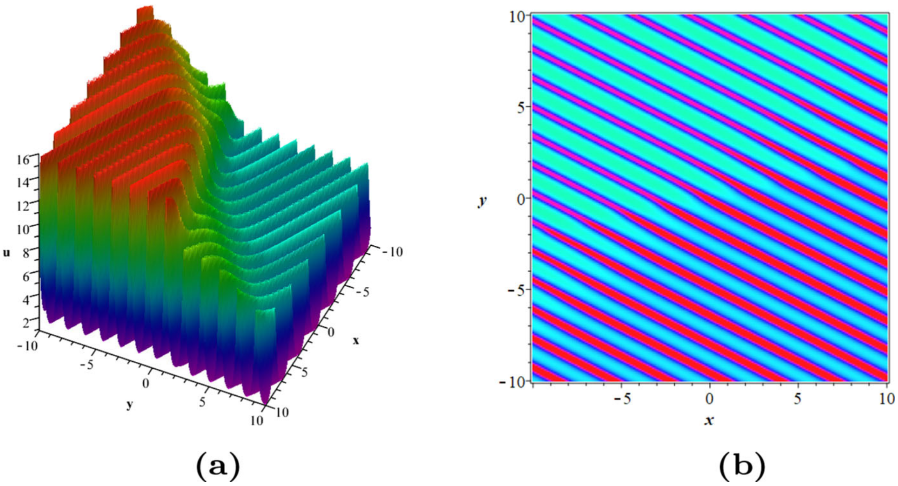


Fig. 6 Two soliton solution of U on the periodic background with suitable parameters $\mathbb{A} = -1$, $u_1 = 1, u_2 = 3, v_1 = -\frac{1}{3}$, $v_2 = -2, t = 0, \vartheta = -1$, $\theta = 1, c_1 = c_2 = 1, c_3 = \mathrm{i}$, $\xi_i = \eta_j = 0\ (i = 1, 2, 3)$

According to the analysis of asymptotic behavior, when two solitons collide, their interaction is completely elastic. This means that after the collision, the shape and velocity of the two solitary waves remain unchanged. In other words, they retain their original waveform and motion state without any deformation or energy loss.

Case 2 On the periodic background
Taking $N = 3$ and $v_3 = -u_3^*$ in Eq. (2.9) to construct two-soiton solutions on the periodic background. The tau functions F and G have the next determinant expressions

$$F = \det_{1\le i,j\le 3}(m_{ij}^{(0)}),$$
$$G = \det_{1\le i,j\le 3}(m_{ij}^{(1)}) \tag{3.9}$$

where the entries are defined by

$$m_{ij}^{(n)} = c_j\delta_{ij} + \frac{1}{u_i + v_j}\left(-\frac{u_i - \mathbb{A}}{v_j + \mathbb{A}}\right)^n e^{\xi_i+\eta_j}, \quad n = 0, 1, \tag{3.10}$$

in which $u_3 = \vartheta + \mathrm{i}\theta$, $v_3 = \mathrm{i}p_3^*$, $c_1 = c_2 = 1$, $\vartheta = a$, $\xi_i + \eta_i = \xi_i + \eta_i = (u_i + v_i)x + (q_i^2 - u_i^2)y + 4(v_i^3 + v_i^3)t + \xi_{i0} + \eta_{i0}$. $i = 1, 2$, $\xi_3 + \eta_3 = -8\mathrm{i}\theta^3 t + (24\mathrm{i}t\vartheta^2 - 4\mathrm{i}y\vartheta + 2\mathrm{i}x)\theta + \xi_{30} + \eta_{30}$.

Parameters $u_1, u_2, v_1, v_2, \vartheta, \theta$, and c are all real numbers. Substituting the functions F and G into Eq. (2.1), we obtain the two soliton solution in a periodic background. Figure 6 depicts the interaction of two solitons against the periodic background and their corresponding densities.

4 Line breather solutions

In this section, we provide linear breather solutions and their propagation evolution diagrams by selecting different values of N and parameters.

Case 1 On the constant background
To construct the traveling breather solutions of the KD equation under a constant background, we first set the size of the determinant to be even in Theorem 2.1. Assuming $N = 2$, $\tilde{u}_1 = \tilde{v}_2 = \mathrm{i}u$, $\tilde{v}_1 = \tilde{u}_2 = \mathrm{i}v$, $c_1 = \vartheta + \mathrm{i}\theta$, $c_2 = -c_1^*$, and choosing $\tilde{u}_i = \tilde{v}_i$ are pure imaginary, u, v, ϑ, and θ are real. In particular, ϑ and θ are not equal to zero. Then, U and V are rewritten as follows

$$U = \frac{2}{\alpha}\left(\ln\frac{G}{F}\right)_x,$$
$$V = \frac{2}{\alpha}\left(\ln\frac{G}{F}\right)_y, \tag{4.1}$$

in which

$$\begin{aligned}F &= \begin{vmatrix} (\vartheta + \mathrm{i}\theta)e^{-\xi_1-\eta_1} + \frac{1}{\mathrm{i}u+\mathrm{i}v} & -\frac{\mathrm{i}}{2u} \\ -\frac{\mathrm{i}}{2v} & (-\vartheta + \mathrm{i}\theta)e^{-\xi_2-\eta_2} + \frac{1}{\mathrm{i}u+\mathrm{i}v} \end{vmatrix} \\ &= \frac{(u-v)^2}{4(u+v)^2uv}e^{\xi_1+\eta_1+\xi_2+\eta_2} + \frac{\mathrm{i}\vartheta + \theta}{p+q}e^{\xi_1+\eta_1} \\ &\quad + \frac{-\mathrm{i}\vartheta + \theta}{u+v}e^{\xi_2+\eta_2} - \vartheta^2 - \theta^2, \end{aligned} \tag{4.2}$$

$$\begin{aligned}G &= \begin{vmatrix} (\vartheta + \mathrm{i}\theta)e^{-\xi_1-\eta_1} - \frac{u}{(\mathrm{i}u+\mathrm{i}v)v} & \frac{\mathrm{i}}{2u} \\ \frac{\mathrm{i}}{2v} & (-\vartheta + \mathrm{i}\theta)e^{-\xi_2-\eta_2} - \frac{v}{(\mathrm{i}u+\mathrm{i}v)u} \end{vmatrix} \\ &= \frac{(\mathrm{i}au + \mathrm{i}av - \mathbb{A}^2 + uv)(u-v)^2}{4(u+v)^2(\mathrm{i}v + \mathbb{A})(\mathrm{i}u + \mathbb{A})uv}e^{\xi_1+\eta_1+\xi_2+\eta_2},\end{aligned}$$

$$+\frac{(\mathrm{i}\vartheta+\theta)(\mathbb{A}^2+u^2)}{(u+v)(\mathrm{i}v+\mathbb{A})(\mathrm{i}u+\mathbb{A})}e^{\xi_1+\eta_1} - \frac{(\mathrm{i}\vartheta-\theta)(\mathbb{A}^2+v^2)}{(u+v)(\mathrm{i}v+\mathbb{A})(\mathrm{i}u+\mathbb{A})}e^{\xi_2+\eta_2} - \frac{(\mathrm{i}\mathbb{A}u+\mathrm{i}\mathbb{A}v+\mathbb{A}^2-uv)(\vartheta^2+\theta^2)}{(\mathrm{i}v+\mathbb{A})(\mathrm{i}u+\mathbb{A})}, \quad (4.3)$$

and

$$\xi_1+\eta_1 = -4\mathrm{i}(u^3+v^3)t+\mathrm{i}(u+v)x + (u^2-v^2)y+\xi_{10}+\eta_{10},$$
$$\xi_2+\eta_2 = -4\mathrm{i}(u^3+v^3)t+\mathrm{i}(u+v)x + (-u^2+v^2)y+\xi_{20}+\eta_{20}.$$

Line breather solution of U at $t=0$ on the constant background is depicted in Fig. 7.

Case 2 On the periodic background

In order to derive line breathing solutions on a periodic background, we impose that the determinant in Theorem 2.1 should be an odd number. For simplicity, let's assume the determinant size is $N=3$. Assuming $N=3$, $\tilde{u}_1=\tilde{v}_2=\mathrm{i}u$, $\tilde{v}_1=\tilde{u}_2=\mathrm{i}v$, $\tilde{u}_3=\tilde{v}_3=\mathrm{i}m$, $c_1=\vartheta+\mathrm{i}\theta$, $c_2=-c_1^*$, we can obtain line breathing solutions on the periodic background with the tau function

$$F = \begin{vmatrix} (\vartheta+\mathrm{i}\theta)e^{-\xi_1-\eta_1}+\frac{1}{\mathrm{i}p+\mathrm{i}q} & -\frac{\mathrm{i}}{2p} & \frac{1}{\mathrm{i}p+\mathrm{i}m} \\ -\frac{\mathrm{i}}{2q} & (-\vartheta+i\theta)e^{-\xi_2-\eta_2}+\frac{1}{\mathrm{i}p+\mathrm{i}q} & \frac{1}{\mathrm{i}q+\mathrm{i}m} \\ \frac{1}{\mathrm{i}q+\mathrm{i}m} & \frac{1}{\mathrm{i}p+\mathrm{i}m} & (-\vartheta+\mathrm{i}\theta)e^{-\xi_3-\eta_3}+\frac{\mathrm{i}}{2m} \end{vmatrix}, \quad (4.4)$$

$$G = \begin{vmatrix} (\vartheta+\mathrm{i}\theta)e^{-\xi_1-\eta_1}-\frac{p}{(\mathrm{i}p+\mathrm{i}q)q} & \frac{\mathrm{i}}{2p} & -\frac{p}{(\mathrm{i}p+\mathrm{i}m)m} \\ \frac{\mathrm{i}}{2q} & (-\vartheta+i\theta)e^{-\xi_2-\eta_2}-\frac{q}{(\mathrm{i}p+\mathrm{i}q)p} & -\frac{q}{(\mathrm{i}q+\mathrm{i}m)m} \\ -\frac{m}{(\mathrm{i}q+\mathrm{i}m)q} & -\frac{m}{(\mathrm{i}p+\mathrm{i}m)p} & (-\vartheta+\mathrm{i}\theta)e^{-\xi_3-\eta_3}+\frac{\mathrm{i}}{2m} \end{vmatrix}, \quad (4.5)$$

then we obtain the line breather solution on the periodic background and through the graphic display their transmission dynamics behavior.

Line breather solution of U on the constant background is depicted in Fig. 8 with suitable parameters. Through Figs. 7 and 8, we find that the amplitude of the line respirator may be higher than that on a constant background due to the appearance of the periodic background.

5 Dynamics of the rational solutions

5.1 Fundamental algebraic solitons

Taking $N=1$, $n=0$, $c_1=0$ in Theorem 2.1. Then, the KD equation (1.1) has the solution (2.1) with

$$F=\widetilde{m}_{11}^{(0)}=\frac{1}{u_1+v_1}\left[(\widetilde{\xi}_1+c_{11}-\frac{u_1}{u_1+v_1})(\widetilde{\eta}_1+d_{11} -\frac{v_1}{u_1+v_1})+\frac{u_1v_1}{(u_1+v_1)^2})\right]e^{\xi_1+\eta_1}, \quad (5.1)$$

$$G=\widetilde{m}_{11}^{(1)}=-\frac{u_1-\mathbb{A}}{v_1+\mathbb{A}}\frac{1}{u_1+v_1} \times\left[\left(\frac{u_1}{u_1-\mathbb{A}}+\widetilde{\xi}_1+c_{11}-\frac{u_1}{u_1+v_1}\right) \times\left(-\frac{v_1}{v_1+\mathbb{A}}+\widetilde{\eta}_1+d_{11}-\frac{v_1}{u_1+v_1}\right) +\frac{u_1v_1}{(u_1+v_1)^2}\right]e^{\xi_1+\eta_1}. \quad (5.2)$$

where

$$\widetilde{\xi}_1=u_1x-2u_1^2y+12u_1^3t,$$
$$\widetilde{\eta}_1=v_1x+2v_1^2y+12v_1^3t,$$

Here, choosing parameters $c_{11}=\mathrm{i}c$, $d_{11}=\mathrm{i}d$ and $\widetilde{u}_1$, $\widetilde{v}_1$, $\mathbb{A}$ as real. After a simple calculation, we obtain

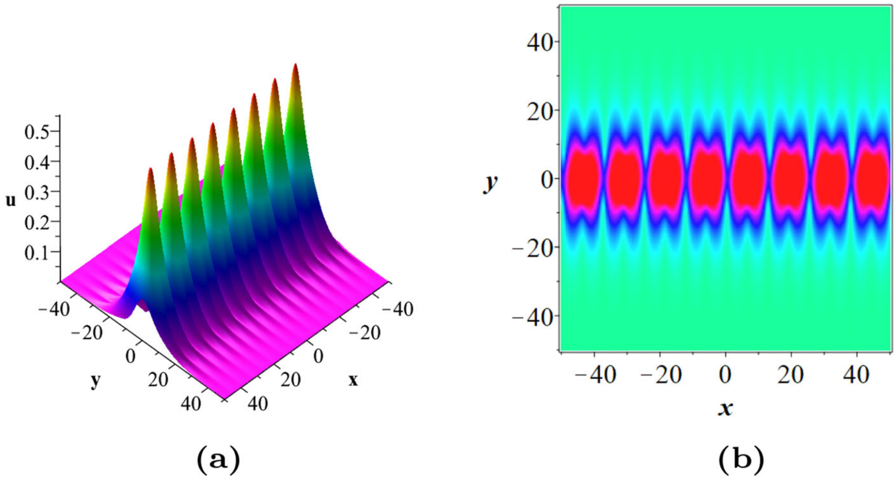


Fig. 7 Line breather solution u on the constant background with suitable parameters $\vartheta = 0, \theta = -2, u = \frac{1}{10}, v = \frac{2}{5}, t = 0, \mathbb{A} = 1, \xi_{i0} = \eta_{j0} = 0\ (i, j = 1, 2)$

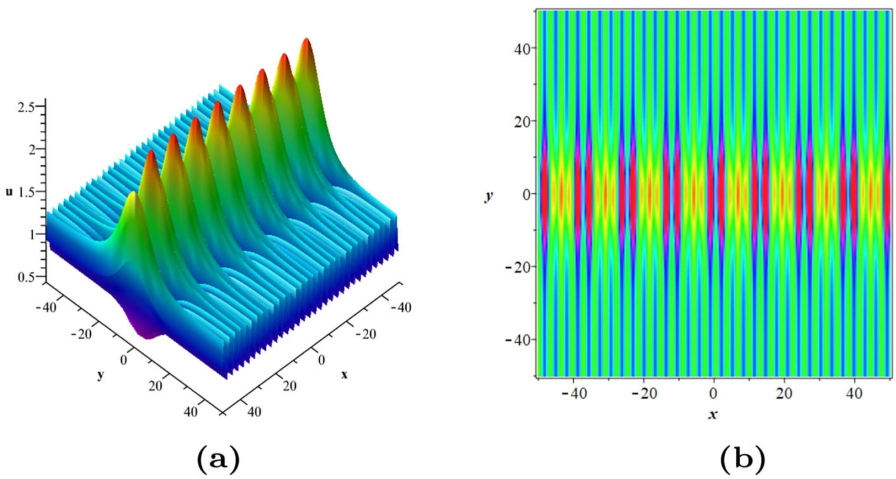


Fig. 8 Line breather solution u on the periodic background with suitable parameters $\vartheta = 0, \theta = -1, p = \frac{1}{10}, q = \frac{2}{5}, t = 0, \mathbb{A} = 1, m = 1, c_3 = -\mathrm{i}, \xi_{i0} = \eta_{j0} = 0\ (i, j = 1, 2, 3)$

$$|F| = \frac{1}{u_1+v_1}\left|\left(\widetilde{\xi}_1 + \mathrm{i}c - \frac{u_1}{u_1+v_1}\right)\left(\widetilde{\eta}_1 + di - \frac{v_1}{u_1+v_1}\right) + \frac{u_1 v_1}{(u_1+v_1)^2}\right| e^{\xi_1+\eta_1},$$

$$= \frac{1}{u_1+v_1}\sqrt{F_1^2+F_2^2}e^{\xi_1+\eta_1},$$

$$F_1 = \left(\widetilde{\xi}_1 - \frac{u_1}{u_1+v_1}\right)\left(\widetilde{\eta}_1 - \frac{v_1}{u_1+v_1}\right) - cd + \frac{u_1 v_1}{(u_1+v_1)^2},$$

$$F_2 = \left(\widetilde{\xi}_1 - \frac{u_1}{u_1+v_1}\right)d + \left(\widetilde{\eta}_1 - \frac{v_1}{u_1+v_1}\right)c.$$

$$|G| = \frac{u_1-\mathbb{A}}{v_1+\mathbb{A}}\frac{1}{u_1+v_1}\left|\left(\frac{u_1}{u_1-\mathbb{A}} + \widetilde{\xi}_1 + \mathrm{i}c - \frac{u_1}{u_1+v_1}\right) \times \left(-\frac{v_1}{v_1+\mathbb{A}} + \widetilde{\eta}_1 + \mathrm{i}d - \frac{v_1}{u_1+v_1}\right) + \frac{u_1 v_1}{(u_1+v_1)^2}\right| e^{\xi_1+\eta_1},$$

$$= \left|\frac{u_1-\mathbb{A}}{v_1+\mathbb{A}}\right| \frac{1}{u_1+v_1}\sqrt{G_1^2+G_2^2}e^{\xi_1+\eta_1},$$

$$G_1 = \left(\frac{u_1}{u_1 - \mathbb{A}} + \widetilde{\xi}_1 - \frac{u_1}{u_1 + v_1}\right) \times \left(-\frac{v_1}{v_1 + \mathbb{A}} + \widetilde{\eta}_1 - \frac{v_1}{u_1 + v_1}\right),$$
$$G_2 = \left(\frac{u_1}{u_1 - \mathbb{A}} + \widetilde{\xi}_1 - \frac{u_1}{u_1 + v_1}\right) d + \left(-\frac{v_1}{v_1 + \mathbb{A}} + \widetilde{\eta}_1 - \frac{v_1}{u_1 + v_1}\right) c.$$

To ensure the regularity of $|U|$ and $|V|$, we need to take parameters so that $F_1^2 + F_2^2 \neq 0$ and $G_1^2 + G_2^2 \neq 0$. The elastic collision of the two-soliton is demonstrated in Fig. 9.

5.2 Algebraic multi-soliton solutions

Taking $N = 2, n_1 = 1, n = 0, c_1 = 0$ in Theorem 2.1. Thus we get

$$F = \begin{vmatrix} \widetilde{m}_{11}^{(0)} & \widetilde{m}_{12}^{(0)} \\ \widetilde{m}_{21}^{(0)} & \widetilde{m}_{22}^{(0)} \end{vmatrix}$$
$$G = \begin{vmatrix} \widetilde{m}_{11}^{(1)} & \widetilde{m}_{12}^{(1)} \\ \widetilde{m}_{21}^{(1)} & \widetilde{m}_{22}^{(1)} \end{vmatrix}, \tag{5.3}$$

where

$$\widetilde{m}_{ij}^{(0)} = \frac{1}{u_i + v_j} \times \left[(\widetilde{\xi}_i + c_{i1} - \frac{u_i}{u_i + v_j})(\widetilde{\eta}_j + d_{j1} - \frac{v_j}{u_i + v_j}) + \frac{u_i v_j}{(u_i + v_j)^2}\right] e^{\xi_i + \eta_j}, \tag{5.4}$$
$$\widetilde{m}_{ij}^{(1)} = -\frac{u_i - \mathbb{A}}{v_j + \mathbb{A}} \frac{1}{u_i + v_j} \times \left[(\frac{u_i}{u_i - \mathbb{A}} + \widetilde{\xi}_i + c_{i1} - \frac{u_i}{u_i + v_j}) \times (-\frac{v_j}{v_j + \mathbb{A}} + \widetilde{\eta}_j + d_{j1} - \frac{v_j}{u_i + v_j}) + \frac{u_i v_j}{(u_i + v_j)^2}\right] e^{\xi_i + \eta_j}, \tag{5.5}$$

On the wave plane, there are three types of waves with different propagation directions and speeds. As time progresses (refer to Fig. 10, the position of the collision center changes. This can be visualized in a two-dimensional image. Figure 10 illustrates that waves with larger amplitudes will catch up to waves with smaller amplitudes. After the collision, the waves will separate from each other due to variations in their heights. If we choose parameters as $u_1 = v_2 = -\frac{1}{4}$, $v_1 = u_2 = \frac{5}{4}$, $c_{11} = d_{11} = 2\mathrm{i}$, $c_{21} = 4\mathrm{i}$, $d_{21} = 10\mathrm{i}$, $\xi_{10} = \eta_{10} = 0$, $\xi_{20} = \eta_{20} = 0$, the image of the potential U degenerates into that of algebraic two-soliton solution in Section 3.2 (see Fig. 11). Plot of the degeneration algebraic two-soliton U is depicted in Fig. 5.

5.3 Multiple rational solutions on periodic background

Taking $N = 3, n_1 = 1, n = 0, c_1 = 0$ in Theorem 2.1.

$$f = \begin{vmatrix} \widetilde{m}_{11}^{(0)} & \widetilde{m}_{12}^{(0)} & \widetilde{m}_{13}^{(0)} \\ \widetilde{m}_{21}^{(0)} & \widetilde{m}_{22}^{(0)} & \widetilde{m}_{23}^{(0)} \\ \widetilde{m}_{31}^{(0)} & \widetilde{m}_{32}^{(0)} & \widetilde{m}_{33}^{(0)} \end{vmatrix}$$
$$g = \begin{vmatrix} \widetilde{m}_{11}^{(1)} & \widetilde{m}_{12}^{(1)} & \widetilde{m}_{13}^{(1)} \\ \widetilde{m}_{21}^{(1)} & \widetilde{m}_{22}^{(1)} & \widetilde{m}_{23}^{(1)} \\ \widetilde{m}_{31}^{(1)} & \widetilde{m}_{32}^{(1)} & \widetilde{m}_{33}^{(1)} \end{vmatrix}, \tag{5.6}$$

where

$$\widetilde{m}_{ij}^{(n)} = \frac{1}{u_i + v_j}\left(-\frac{u_i - \mathbb{A}}{v_j + \mathbb{A}}\right)^n \times \left(\widetilde{\xi}_i + c_{i1} + \frac{n u_i}{u_i - \mathbb{A}} - \frac{u_i}{u_i + v_j}\right) e^{\xi_i + \eta_j},$$
$$\widetilde{m}_{ij}^{(n)} = \frac{1}{u_i + v_j}\left(-\frac{u_i - \mathbb{A}}{v_j + \mathbb{A}}\right)^n \times \left(\widetilde{\eta}_i + c_{i1} - \frac{n v_j}{v_j + \mathbb{A}} - \frac{u_i}{u_i + v_j}\right) e^{\xi_i + \eta_j},$$
$$\widetilde{m}_{ij}^{(n)} = c_j + \frac{1}{u_3 + v_3}\left(-\frac{u_3 - \mathbb{A}}{v_3 + \mathbb{A}}\right) e^{\xi_i + \eta_j},$$

Substituting F and G into Eq. (2.1) yields semi-rational solutions U and V. Since the explicit expressions of U and V are very complicated, we omit the specific forms of U and V here. To better investigate the dynamics of this class of semi-intelligible, we set the parameter to $u_1 = v_2 = -\frac{1}{2}$, $v_1 = u_2 = \frac{5}{2}$, $v_3 = u_3 = -\mathrm{i}$, $c_{11} = 2\mathrm{i}$, $d_{11} = 6\mathrm{i}$, $c_{21} = 4\mathrm{i}$, $d_{21} = 8\mathrm{i}$, $\xi_{i0} = \eta_{j0} = 0$, $(i, j = 1, 2, 3)$, $c_1 = c_2 = 1$, $c_3 = -5\mathrm{i}$, $t = 0$

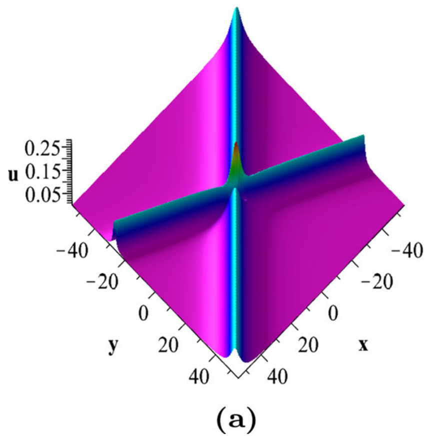


Fig. 9 Algebraic two-soliton solutions of U and V with suitable parameters $u_1 = \frac{1}{2}$, $v_1 = 1$, $c = d = 2$, $\mathbb{A} = 1$, $\xi_{10} = \eta_{10} = 0$, $t = 0$

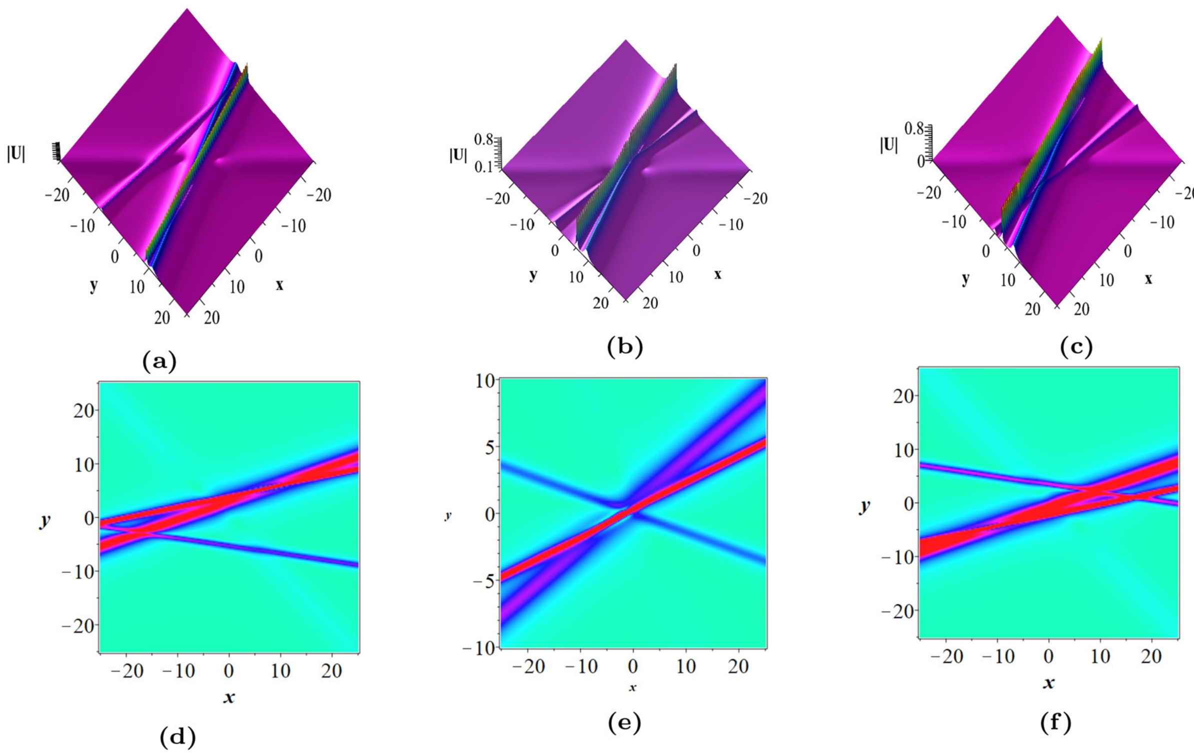
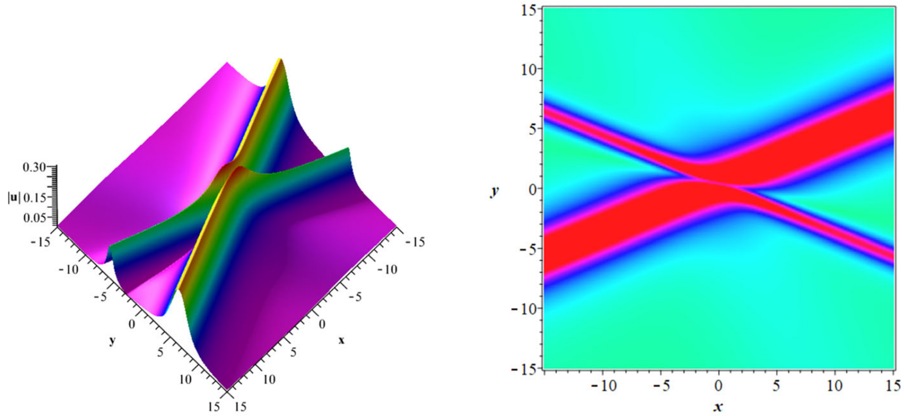


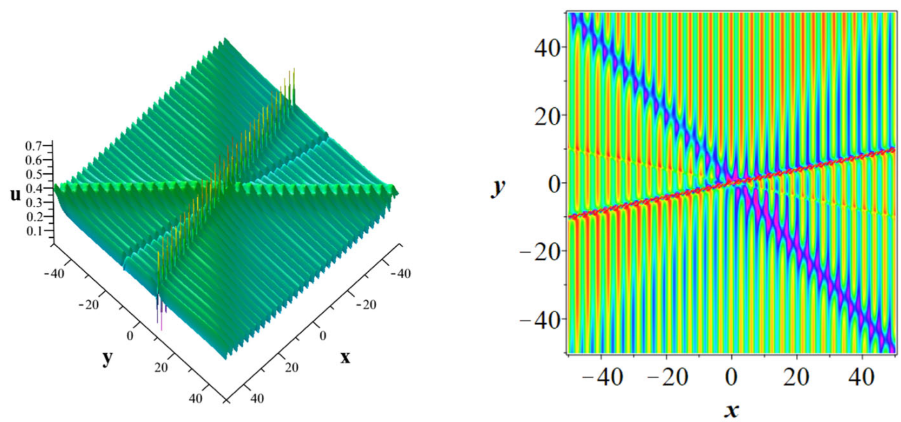
Fig. 10 Algebraic three-soliton solutions U with suitable parameters $u_1 = -\frac{1}{2}$, $v_1 = \frac{3}{2}$, $u_2 = -\frac{5}{2}$, $v_2 = 2$, $c_{11} = \frac{1}{2}\mathrm{i}$, $d_{11} = 2\mathrm{i}$, $c_{21} = \mathrm{i}$, $d_{21} = 40\mathrm{i}$, $\mathbb{A} = 1$, $\xi_{10} = \eta_{10} = 0$, $\xi_{20} = \eta_{20} = 0$, (a, d) $t = \frac{1}{4}$, (b, e), $t = 0$, (c, f) $t = -\frac{1}{6}$

From Fig. 12(a), it can be seen that three periodic waves are displayed on the periodic background surface, of which two waves are anti-dark waves and the third wave is dark waves. When the parameter is taken as $u_1 = u_2 = -\frac{1}{2}$, $v_1 = v_2 = \frac{5}{2}$, $v_3 = u_3 = -\mathrm{i}$, $c_{11} = 2\mathrm{i}$, $d_{11} = 6\mathrm{i}$, $c_{21} = 4\mathrm{i}$, $d_{21} = 8\mathrm{i}$, $\xi_{i0} = \eta_{j0} = 0$, $(i, j = 1, 2, 3)$, $c_1 = c_2 = 1$, $c_3 = -5\mathrm{i}$, $t = 0$, Fig. 13(a) shows two periodic waves on a periodic background, one is anti-dark and the other is dark. It can be seen that the amplitude of the wave increases after the collision, while the propagation direction does not change.



(a) (b)

Fig. 11 Algebraic soliton solution U with suitable parameters $u_1 = v_2 = -\frac{1}{4}$, $v_1 = u_2 = \frac{5}{4}$, $c_{11} = d_{11} = 2\mathrm{i}$, $c_{21} = 4\mathrm{i}$, $d_{21} = 10\mathrm{i}$, $\xi_{10} = \eta_{10} = 0$, $\xi_{20} = \eta_{20} = 0$, $t = 0$



(a) (b)

Fig. 12 Semi-rational soliton solution U with suitable parameters $u_1 = v_2 = -\frac{1}{2}$, $v_1 = u_2 = \frac{5}{2}$, $v_3 = u_3 = -\mathrm{i}$, $c_{11} = 2\mathrm{i}$, $d_{11} = 6\mathrm{i}$, $c_{21} = 4\mathrm{i}$, $d_{21} = 8\mathrm{i}$, $\xi_{i0} = \eta_{j0} = 0, \quad (i, j = 1, 2, 3)$, $c_1 = c_2 = 1$, $c_3 = -5i$, $t = 0$

6 Conclusions and discussions

KP hierarchy reduction technique is an effective method for studying the soliton structure of integrable systems. Based on this method, we provide general soliton solutions and (semi-)rational solutions of the $N \times N$ determinant of the KD equation using bilinear methods. Soliton solutions, line breather solutions, algebraic soliton solutions, and multiple rational solutions are constructed for constant and periodic backgrounds, respectively. The dynamical characteristics of the solutions under different parameters are analyzed. We find that these solutions are located on a constant background for $N = 2n$ and on a periodic background for $N = 2n + 1$. We analyze in detail the local dynamics of rational and semi-rational solutions for $N = 1$, 2, and 3 cases. Using KP reduction to study integrable

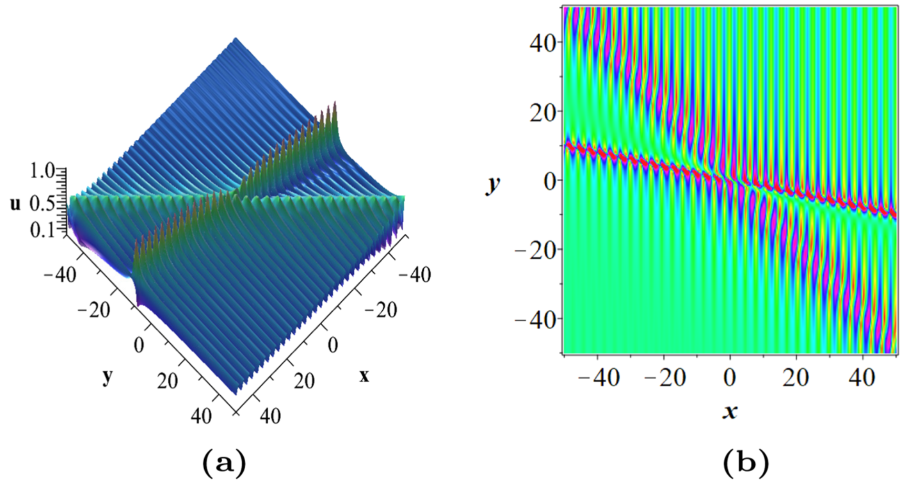


Fig. 13 Semi-rational soliton solution U with suitable parameters $u_1 = u_2 = -\frac{1}{2}$, $v_1 = v_2 = \frac{5}{2}$, $v_3 = u_3 = -\mathrm{i}$, $c_{11} = 2\mathrm{i}$, $d_{11} = 6\mathrm{i}$, $c_{21} = 4\mathrm{i}$, $d_{21} = 8\mathrm{i}$, $\xi_{i0} = \eta_{j0} = 0$, $(i, j = 1, 2, 3)$, $c_1 = c_2 = 1$, $c_3 = -5\mathrm{i}$, $t = 0$

equations is an interesting direction and a topic to be explored in progress.

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Data availability Enquiries about data availability should be directed to the authors.

Declarations

Competing interests The authors have not disclosed any competing interests.

References

1. Zabusky, N.J., Kruskal, M.D.: Interaction of solitons in a collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. **6**, 240–243 (1965)
2. Shen, S., Yang, Z., Li, X., Zhang, S.: Periodic propagation of complex-valued hyperbolic-cosine-Gaussian solitons and breathers with complicated light field structure in strongly nonlocal nonlinear media. Commun. Nonlinear. Sci. **103**, 106005 (2021)
3. Shen, S., Yang, Z.J., Pang, Z.G., Ge, Y.R.: The complex-valued astigmatic cosine-Gaussian soliton solution of the nonlocal nonlinear Schrödinger equation and its transmission characteristics. Appl. Math. Lett. **125**, 107755 (2022)
4. Song, L.M., Yang, Z.J., Li, X.L., Zhang, S.M.: Coherent superposition propagation of Laguerre–Gaussian and Hermite–Gaussian solitons. Appl. Math. Lett. **102**, 106114 (2020)
5. Yang, Z.J., Zhang, S.M., Li, X.L., Pang, Z.G., Bu, H.X.: High-order revivable complex-valued hyperbolic-sine-Gaussian solitons and breathers in nonlinear media with a spatial nonlocality. Nonlinear Dyn. **94**, 2563–2573 (2018)
6. Guo, J.L., Yang, Z.J., Song, L.M., Pang, Z.G.: Propagation dynamics of tripole breathers in nonlocal nonlinear media. Nonlinear Dyn. **101**, 1147–1157 (2020)
7. Sun, B.N., Wazwaz, A.M.: Interaction of lumps and dark solitons in the Melnikov equation. Nonlinear Dyn. **92**, 2049–2059 (2018)
8. Ohta, Y., Wang, D.S., Yang, J.K.: General N dark solitons in the coupled nonlinear Schrödinger equations. Stud. Appl. Math. **127**, 345–371 (2011)
9. Francesco, C., Grinevich, P.G., Santini, P.M.: The periodic *N* breather anomalous wave solution of the Davey–Stewartson equations; first appearance, recurrence, and blow up properties. J. Phys. A Math. Theor. **57**, 1–32 (2023)
10. Gao, Y.T., Tian, B.: Cylindrical Kadomtsev–Petviashvili model, nebulons and symbolic computation for cosmic dust ion-acoustic waves. Phys. Lett. A **349**, 314–319 (2006)
11. Tian, B., Gao, Y.T.: Symbolic-computation study of the perturbed nonlinear Schrödinger model in inhomogeneous optical fibers. Phys. Lett. A **342**, 228–236 (2005)
12. Ablowitz, M.J., Clarkson, P.A.: Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1991)
13. Matveev, V., Salle, M.: Darboux Transformation and Solitons. Springer-Verlag, Berlin (1991)
14. Shen, Y., Bo, T., Zhou, T.Y., Gao, X.T.: N-fold Darboux transformation and solitonic interactions for the Kraenkel–Manna–Merle system in a saturated ferromagnetic material. Nonlinear Dyn. **111**, 2641–2649 (2022)

15. Shen, Y., Bo, T., Zhou, T.Y., Cheng, C.D.: Multi-pole solitons in an inhomogeneous multi-component nonlinear optical medium. Chaos Solitons Fractals **171**, 113497 (2023)
16. Hirota, R.: The Direct Methods in Soliton Theory. Cambridge University Press, Cambridge (2004)
17. Shen, Y., Bo, T., Cheng, C.D., Zhou, T.Y.: N-soliton, Mth-order breather, Hth-order lump, and hybrid solutions of an extended (3+1)-dimensional Kadomtsev–Petviashvili equation. Nonlinear Dyn. **111**, 10407–10424 (2023)
18. Liu, S.H., Bo, T.: Singular soliton, shock-wave, breather-stripe soliton, hybrid solutions and numerical simulations for a (2+1)-dimensional Caudrey–Dodd–Gibbon–Kotera–Sawada system in fluid mechanics. Nonlinear Dyn. **108**, 2471–2482 (2022)
19. An, Y.N., Guo, R.: The mixed solutions of the (2+ 1)-dimensional Hirota–Satsuma–Ito equation and the analysis of nonlinear transformed waves. Nonlinear Dyn. **111**, 18291–18311 (2023)
20. Niu, J.X., Guo, R., Zhang, J.W.: Solutions on the periodic background and transition state mechanisms for the higher-order Chen–Lee–Liu equation. Wave Motion **123**, 103233 (2023)
21. Zhou, Y., Manukure, S., McAnally, M.: Lump and rogue wave solutions to a (2+ 1)-dimensional Boussinesq type equation. J. Geom. Phys. **167**, 104275 (2021)
22. Kumar, S., Malik, S., Rezazadeh, H., Lanre, A.: The integrable Boussinesq equation and its breather, lump and soliton solutions. Nonlinear Dyn. 1–14 (2022)
23. Ntiamoah, D., William, O., Lanre, A.: The higher-order modified Korteweg–de Vries equation: its soliton, breather and approximate solutions. J. Ocean Eng. Sci. (2022)
24. Lanre, A.: Shallow ocean soliton and localized waves in extended (2+1)-dimensional nonlinear evolution equations. Phys. Lett. A **463**, 128668 (2023)
25. Lanre, A., Morazara, E.: Integrability, multi-solitons, breathers, lumps and wave interactions for generalized extended Kadomtsev–Petviashvili equation. Nonlinear Dyn. **111**(5), 4683–4707 (2023)
26. Peng, W.Q., Tian, S.F., Wang, X.B., Zhang, T.T., Fang, Y.: Riemann–Hilbert method and multi-soliton solutions for three-component coupled nonlinear Schrödinger equations. J. Geom. Phys. **146**, 103508 (2019)
27. Ohta, Y.: Dark soliton solution of Sasa–Satsuma equation. AIP Conf. Proc. **1212**(1), 114–121 (2010)
28. Chen, J.C., Yong, C., Feng, B.F., Maruno, K., Ohta, Y.: General high-order rogue waves of the $(1+1)$-dimensional Yajima–Oikawa system. J. Phys. Soc. Jpn. **87**, 094007 (2017)
29. Chen, J.C., Feng, B.F.: General bright and dark soliton solutions to the massive Thirring model via KP hierarchy reductions. Mathematics 1–27 (2021)
30. Ohta, Y., Yang, J.K.: General high-order rogue waves and their dynamics in the nonlinear Schrödinger equation. Proc. R. Soc. A-Math. Phys. **468**, 1716–1740 (2011)
31. Ohta, Y., Yang, J.K.: Dynamics of rogue waves in the Davey–Stewartson II equation. J. Phys. A-Math. Theor. **46**(10), 105202 (2013)
32. Ohta, Y., Yang, J.K.: General rogue waves in the focusing and defocusing Ablowitz–Ladik equations. J. Phys. A-Math. Theor. **47**(25), 255201 (2014)
33. Rao, J., Porsezian, K., He, J.S.: Semi-rational solutions of the third-type Davey–Stewartson equation. Chaos **27**(8), 083115 (2017)
34. Feng, B.F., Luo, X.D., Ablowitz, M.J., Musslimani, Z.H.: General soliton solution to a nonlocal nonlinear Schrödinger equation with zero and nonzero boundary conditions. Nonlinearity **31**, 5385–5409 (2018)
35. Liu, W., Zheng, X.X., Li, X.L.: Bright and dark soliton solutions to the partial reverse space-time nonlocal Melnikov equation. Nonlinear Dyn. **94**, 2177–2189 (2018)
36. Konopelchenko, B.G., Dubrovsky, V.G.: Some new integrable nonlinear evolution equations in $(2+1)$ dimensions. Phys. Lett. A. **102**, 15–17 (1984)
37. Xu, X.G., Meng, X.H., Gao, Y.T.: Analytic N-solitary-wave solution of a variable-coefficient Gardner equation from fluid dynamics and plasma physics. Appl. Math. Comput. **210**(2), 313–320 (2009)
38. Wazwaz, A.M.: New solitons and kink solutions for the Gardner equation. Commun. Nonlinear Sci. Numer. Simul. **12**(8), 1395–1404 (2007)
39. Fu, Z., Liu, S.: New kinds of solutions to Gardner equation. Chaos Solitons Fractals **20**(2), 301–309 (2004)
40. Wang, K.J.: Traveling wave solutions of the Gardner equation in dusty plasmas. Results Phys. **33**, 105207 (2022)
41. Wang, K.L.: Exact travelling wave solution for the local fractional Camassa–Holm–Kadomtsev–Petviashvili equation. Alex. Eng. J. **63**, 371–376 (2023)
42. Soomere, T.: Interaction of Kadomtsev–Petviashvili solitons with unequal amplitudes. Phys. Lett. A. **332**, 74–81 (2004)
43. Yokus, A., Isah, M.A.: Stability analysis and solutions of $(2+1)$-Kadomtsev–Petviashvili equation by homoclinic technique based on Hirota bilinear form. Nonlinear Dyn. **109**, 3029–3040 (2022)
44. Ismael, H.F., Murad, M.A.S., Bulut, H.: M-lump waves and their interaction with multi-soliton solutions for a generalized Kadomtsev–Petviashvili equation in $(3+1)$-dimensions. Chin. J. Phys. **77**, 1357–1364 (2022)
45. Sun, Z.Y., Gao, Y.T., Yu, X.: Inelastic interactions of the multiple-front waves for the modified Kadomtsev–Petviashvili equation in fluid dynamics, plasma physics and electrodynamics. Wave Motion **46**, 511–521 (2009)
46. Xu, T., Zhang, H.Q., Zhang, Y.X., Li, J., Feng, Q., Tian, B.: Two types of generalized integrable decompositions and new solitary-wave solutions for the modified Kadomtsev–Petviashvili equation with symbolic computation. J. Math. Phys. **49**, 013501 (2008)
47. Sheng, Z.: Symbolic computation and new families of exact non-travelling wave solutions of $(2+1)$-dimensional Konopelchenko–Dubrovsky equations. Chaos Solitons Fractals **31**, 951–959 (2007)
48. Wang, D.S., Zhang, H.Q.: Further improved F-expansion method and new exact solutions of Konopelchenko–Dubrovsky equation. Chaos Solitons Fractals **25**(3), 601–610 (2005)
49. Ma, H.C., Bai, Y.X., Deng, A.P.: Multiple lump solutions of the $(2+1)$-dimensional Konopelchenko–Dubrovsky equation. Math. Methods Appl. Sci. **43**(12), 7135–7142 (2020)
50. Yuan, Y.Q., Tian, B., Liu, L., Wu, X.Y., Sun, Y.: Solitons for the $(2+1)$-dimensional Konopelchenko–Dubrovsky equations. J. Math. Anal. Appl. **460**(1), 476–486 (2018)

51. Gu, Y., Manafian, J., Malmir, S.: Lump, lump-trigonometric, breather waves, periodic wave and multi-waves solutions for a Konopelchenko–Dubrovsky equation arising in fluid dynamics. Int. J. Mod. Phys. B **37**, 2350141 (2023)
52. Xu, P.B., Gao, Y.T., Gai, X.L., Meng, D.X., Shen, Y.J., Wang, L.: Soliton solutions, Bäcklund transformation and Wronskian solutions for the extended (2 + 1)-dimensional Konopelchenko–Dubrovsky equations in fluid mechanics. Appl. Math. Comput. **218**, 2489–2496 (2011)
53. Xu, P.B., Gao, Y.T., Yu, X., Wang, L., Lin, G.D.: Painlevé analysis, soliton solutions and Bäcklund transformation for extended (2 + 1)-dimensional Konopelchenko–Dubrovsky equations in fluid mechanics via symbolic computation. Commun. Theor. Phys. **55**, 1017–1023 (2011)
54. Grinevich, P.G., Santini, P.M.: The finite-gap method and the periodic NLS Cauchy problem of anomalous waves for a finite number of unstable modes. Russ. Math. Surv. **74**, 211–263 (2018)
55. Feng, B.F., Ling, L.M., Takahashi, D.A.: Multi-breather and high-order rogue waves for the nonlinear schrödinger equation on the elliptic function background. Stud. Appl. Math. **144**, 46–101 (2020)

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