Nonlinear Dyn (2024) 112:7093–7111
https://doi.org/10.1007/s11071-024-09377-2

ORIGINAL PAPER

Dynamic output feedback stabilization for a class of nonsmooth stochastic nonlinear systems perturbed by multiple time-varying delays

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Received: 31 August 2023 / Accepted: 1 February 2024 / Published online: 15 March 2024

Abstract In this paper, the output feedback stabilization problem is investigated for a class of low-order stochastic nonlinear time-delay systems with the lower-triangular form, where the powers of chained integrators are arbitrary real numbers between 0 and 1, and the multiple time-vary delays act on each system state. Because of the existence of low-order nonlinear terms, the system is not feedback linearizable and differentiable. Based on an extended adding a power integrator approach and a stability theory of stochastic continuous systems, an output feedback controller is systematically designed to ensure the global strong asymptotic stability of the closed-loop system. In the controller design, the negative effect of the multiple time-varying delays is counteracted by skillfully constructing a novel Lyapunov–Krasovskii functional, and the observer gains are determined by developing a recursive selection procedure. Finally, two numerical examples are provided to verify the effectiveness of the proposed method.

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Keywords Output feedback stabilization · Low-order nonlinearities · Stochastic nonlinear systems · State observer · Multiple time-varying delays

Abbreviations

LKF	Lyapunov–Krasovskii functional
SNTDS	Stochastic nonlinear time-delay system
HOS	High-order system
AAPI	Adding a power integrator
LOS	Low-order systems
GSS	Global strong stability
NGC	Nonlinear growth condition
SWP	Standard Wiener process
GSSP	Globally strongly stable in probability
GSASP	Globally strongly asymptotically stable in probability

1 Introduction

There are many kinds of reasons that can lead to time lag in a practical system, such as delayed measurements, signal transmissions and some intrinsic properties of the system. For the sake of reflecting the real time lag and achieving some special objectives, time delays are often used in actual system modeling [1]. Therefore, the study of time-delay dynamic systems is not just of great significance in engineering applications but also poses a fundamental challenge for mathematical theory. There are many important results in this filed. For example, Lyapunov stability theory was extended to time-delay systems by Krasovskii in 1959, which results in the well-known Lyapunov–Krasovskii functional (LKF)-based method [2]. Later, to avoid the manipulation of functionals, another important stability analysis approach called Razumikhin theorem-based method was proposed by Razumikhin [3]. Noteworthily, the early study on time-delay systems mainly focused on linear systems (e.g., [4,5]). In the last two decades, nonlinear time-delay systems have attracted more attention. Lots of research results have emerged for solving the relevant problems with the help of Krasovskii's or Razumikhin's approach. Examples include [6–8] and reference therein. Furthermore, because stochastic phenomena exist widely in the real world, stochastic nonlinear time-delay systems (SNTDSs) have gradually become a research hotspot in nonlinear control field in recent years; see, for instance [9–14].

As we known, the systems with lower-triangular structure are very important research objectives in nonlinear systems domain due to the fact that not only they are able to model many practical systems but also lots of general nonlinear systems can be transformed into a system with lower-triangular form by a differential homeomorphism transformation under some conditions [15]. The backstepping method proposed by Kokotovic et al. (see [16,17]), is one of the most useful techniques for solving control problems of the lower-triangular nonlinear systems with the strict-feedback structure in various scenarios including stochastic and time-delay cases [18–23]. However, the systems processed by using this technique must meet the requirement of fully or partially feedback linearization and have linear virtual control inputs. The high-order systems (HOS) [24,25] that have been focused on for many years are a typical class of lower-triangular nonlinear systems not meeting the above requirements. Fortunately, by using the homogeneous domination approach [26] and the adding a power integrator (AAPI) technique [27], many interesting results on the feedback control problem of the HOS with stochastic disturbance have been obtained. Particularly, combining the LKF-based method, stabilization of high-order SNTDSs by state feedback control was considered by authors in [28–31]. Meanwhile, by introducing state-observer approaches the stabilizing problem of high-order SNTDSs via output feedback control was investigated in [32–35]. In the present paper, stabilization of low-order SNTDSs will be further studied, which is generally considered to be the nonsmooth counterpart of high-order SNTDSs.

It should be noted that several practical systems can be modeled by using the low-order system (LOS) in engineering practice, such as the liquid-level system with interaction [36], the regenerative chatter system [37] and the cascade chemical system [38]. Because the LOS cannot be feedback linearized, stabilization of such system cannot be handled by using the standard backstepping technique. In addition, due to the powers of chained integrator being less than one, the LOS is only continuous but completely nondifferentiable even not satisfying the Lipschitz condition. From a technique point of view, it is more difficult to stabilize a LOS than stabilize a HOS. Nevertheless, there have been lots of important results focusing on this problem. For example, in [39], Qian et al. described the conditions of achieving global strong stability (GSS) of continuous systems having multiple solutions, and a planar low-order system was used as an illustrative example. Later, for a class of low-order nonlinear systems, Ref. [40] investigated the finite-time stabilization problem by developing an efficient recursive design procedure, Ref. [41] addressed the GSS of a class of low-order nonlinear systems by proposed multi-rate sampling controller, and Ref. [42] dealt with the output feedback control problem of the LOS described with the p-normal form. In the stochastic setting, Ref. [43] discussed the finite-time stabilization for the LOS with stochastic disturbances by making use of the AAPI technique and stochastic stability criterion. The same problem was investigated in Ref. [44] where stochastic inverse dynamics are considered in the system. However, there are no any results on low-order SNTDSs up to now.

The literature review on the study of SNTDSs mentioned above is summarized in Table 1. Based on this observation, we focus on an interesting problem in the present work. That is, how to construct an output feedback control law to guarantee stability of low-order SNTDSs where each system state is affected by multiple time-varying delays? Clearly, because of the intrinsic characteristics of low-order nonlinear systems, it is a

Table 1 A brief summarization of the literature review

Objects	Refs. [9–14] SNTDSs	Refs. [18–23] Strict-feedback SNTDSs	Refs. [28–35] High-order SNTDSs	Refs. [40–44] Low-order systems
Methods	Qualitative theory of stochastic functional differential equation; Krasovskii's or Razumikhin's approach.	Lyapunov–Krasovskii functional approach; backstepping recursive design method; stochastic stability theory.	Stochastic Lyapunov–Krasovskii stability theory; AAPI technique; homogeneous domination approach.	Global strong stability theory in the sense of Kurzweil; nonsmooth feedback control approach;
Results	Some conditions on the existence of solutions and some criteria on the stability of systems are proposed.	Some sufficient conditions guaranteeing the stability in probability of the resulting systems are provided.	State-feedback or output-feedback control schemes are designed to ensure the stability of the closed-loop systems.	Nonsmooth feedback stabilizers are constructed; there are very few results on lower-order SNTDSs.

difficult task to address the problem. To be specific, the following three difficulties will be encountered when designing an output feedback controller. First, the low-order nonlinear terms lead to that the considered system in our paper is merely continuous but nonsmooth. Thus, the existing methods developed in Refs. [32–35] cannot be directly used to address the problem because some smoothness of systems is required in these methods. Second, compared with delay-free nonlinear systems, there are more difficulties to stabilize a nonlinear system with time delays. Although the some stabilization results with regard to the LOS have been obtained in Refs. [40–44], it is clearly that they cannot be suitable for the LOS with time delays. Third, there is no doubt that the simultaneous appearance of multiple time-varying delays, unmeasured state variables, stochastic disturbances and low-order nonlinear terms will greatly increase the difficulties of solving the problem. In this technique note, we prepare for developing a new design technique to achieve the stabilization of low-order SNTDSs by overcoming the difficulties mentioned above.

The main contributions of our paper include:

- The output feedback stabilization of low-order SNTDSs is first considered in the present paper. Based on the AAPI technique of lower-triangular systems and the stability theory of stochastic continuous systems, an output feedback stabilizer is constructed via developing an efficient recursive design method for low-order SNTDSs.
- For the purpose of counteracting the negative effect of multiple time-varying delays, we skillfully construct an appropriate LKF in the design process which plays a crucial role in system analysis. Note that it is different from the results developed for high-order SNTDSs in [28–35].
- The powers of chained integrators for the low-order systems studied in the published papers, such as [40,41,43,44], are restricted to the positive odd rational numbers. The present paper removes this restriction. That is, the powers of integrators can be any real numbers between 0 and 1.
- Motivated by the work of [40,42], a new reduced-order observer is proposed to produce the estimated values of unmeasured system states. In addition, a recursive selection procedure is developed, and the desired observer gains can be determined by following the procedure.

The present work is organized as follows. The low-order SNTDS considered in our paper, the concepts related to GSASP and some useful lemmas are introduced in Sect. 2. The detailed design process of our output feedback stabilizer and the stability analysis of the closed-loop system are provided in Sect. 3. An example of numerical simulation and an example of application to a practical system are given for the proposed methods in Sect. 4. Some conclusions and future works are stated in Sect. 5.

Throughout the paper, the symbols $\mathbb{R}$, $\mathbb{R}^m$ and $\mathbb{R}^+$ are used to represent the real number set, the m-dimensional Euclidian space and the positive real number set, respectively. For a matrix A, the symbol A^T represents the transpose of A. Furthermore, $Tr(A)$ is used for denoting the trance of A if A is square. The symbol $||\cdot||$ denotes the Frobenius norm of matrixes or vectors, and especially we use the symbol $|\cdot|$ to denote the absolute value. $[\cdot]^r$ is defined as $\mathrm{sgn}(\cdot)|\cdot|^r$ for $r\in\mathbb{R}$. $\mathcal{C}^b_{\mathcal{F}_0}([-h,0];\mathbb{R}^n)$ denotes the family of all $\mathcal{F}_0$-

measurable and continuous $\mathbb{R}^n$-value random variables $X = \{X(\omega) : -h \le \omega \le 0\}$. The symbols $\mathcal{I}_k$ and $\mathcal{N}_k$ denote the index sets $\{2, 3, \ldots, k\}$ and $\{1, 2, \ldots, k\}$, respectively. To be simple, the argument of a function is sometime omitted. For example, we sometimes use $d(\cdot)$ or d to denote $d(s)$.

2 Problem statement and preliminaries

The SNTDS with the following Itô formalism is considered in this paper

$$\begin{cases} dx_1 = \big([x_2]^r + f_1(\bar{x}_1, \bar{x}_{1d})\big)dt + g_1^T(\bar{x}_1, \bar{x}_{1d})dw, \\ dx_2 = \big([x_3]^r + f_2(\bar{x}_2, \bar{x}_{2d})\big)dt + g_2^T(\bar{x}_2, \bar{x}_{2d})dw, \\ \quad \vdots \\ dx_n = \big([u]^r + f_n(\bar{x}_n, \bar{x}_{nd})\big)dt + g_n^T(\bar{x}_n, \bar{x}_{nd})dw, \\ y = x_1, \end{cases} \tag{1}$$

where the power of the chained integrators r $(0 < r < 1)$ is an arbitrary real number, $w \in \mathbb{R}^m$ represents a standard Wiener process (SWP), and u is control input. $x = (x_1, x_2, \ldots, x_n)^T$, which is unmeasurable except for x_1, represents the system state. For every $i \in \mathcal{N}_n$, $\bar{x}_i = (x_1, x_2, \ldots, x_i)^T$, $\bar{x}_{id} = (x_{1d}, x_{2d}, \ldots, x_{id})^T$, where the symbol x_{id} represents the delayed state $x_i\big(t - d_i(t)\big)$. The functions $d_i(t) : \mathbb{R}^+ \to [0, h_i]$, satisfying the conditions $\dot{d}_i(t) \le v_i < 1$ with the positive constant numbers v_i, $i \in \mathcal{N}_n$, denote the time-varying state delays. The initial data of the system are taken as $x(\theta) = \{\phi(\theta) : -h \le \theta \le 0\} \in \mathcal{C}^b_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$, where $h = \max\{h_1, h_2, \ldots, h_n\}$. At the end, for every $i \in \mathcal{N}_n$, $f_i : \mathbb{R}^i \times \mathbb{R}^i \to \mathbb{R}$ satisfying $f_i(0, 0) = 0$ is continuous and called the drift term, and $g_i : \mathbb{R}^i \times \mathbb{R}^i \to \mathbb{R}^m$ satisfying $g_i(0, 0) = 0$ is continuous and called the diffusion term.

If $g_i(\cdot) = 0$ and $d_i(t) = 0$, system (1) will be a deterministic LOS, whose chained integrators have powers greater than 0 but less than 1. Notably, it is a nonsmooth counterpart of a deterministic HOS. Since the powers of chained integrator are between 0 and 1, the system is not feedback linearizable and differentiable. This characteristic coupling with the effects of stochastic disturbances, unmeasured states and time delays will make the controller design more difficult. In this paper, a new design technique is developed to overcome the difficulties and achieve the output feedback control of system (1). For achieving this aim, we have Assumption 1 for system (1).

Assumption 1 There are constants $a_i > 0$ and $b_i > 0$, $i \in \mathcal{N}_n$, such that

$$|f_i(\bar{x}_i, \bar{x}_{id})| \le a_i \sum_{j=1}^{i} (|x_j|^r + |x_{jd}|^r),$$

$$||g_i(\bar{x}_i, \bar{x}_{id})|| \le b_i \sum_{j=1}^{i} (|x_j|^{\frac{r+1}{2}} + |x_{jd}|^{\frac{r+1}{2}}).$$

Remark 1 In Liu [43] and Huang [45], the drift terms $f_i(\bar{x}_i, \bar{x}_{id}) = f_i(\bar{x}_i)$ and the diffusion terms $g_i(\bar{x}_i, \bar{x}_{id}) = g_i(\bar{x}_i)$, which are independent of time delays. That is, Assumption 3.1 in [43] and Assumption 2 in [45], which is called the nonlinear growth condition (NGC), are special cases of Assumption 1 of the present paper when $x_{id} = 0$, $i \in \mathcal{N}_n$. Clearly, our assumption is capable of being used for more general stochastic systems and it is a more general NGC. It should be noted that NGC are commonly used in the study of nonlinear systems. Many papers have discussed the significance and rationality of the NGC, to name just a few, see [43,45,46] etc.

Consider the general SNTDS

$$dx(t) = \mu\big(x(t), x(t - \tau(t)), t\big)dt + \sigma^T\big(x(t), x(t - \tau(t)), t\big)dw(t), \quad t \ge 0, \tag{2}$$

with $\tau(t)$ satisfying $0 \le \tau(t) \le \tau_M$. $w(t)$ is an m-dimensional SWP. $\mu : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{m \times n}$ are continuous functions and satisfy $\mu(0, 0, t) = 0$, $\sigma(0, 0, t) = 0$. The initial condition is taken as $\{x(\theta) : -\tau_M \le \theta \le 0\} = \varphi \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau_M, 0]; \mathbb{R}^n)$. System (2) always has a solution in the weak sense, but it may not be unique [47] because the (local) Lipschitz condition cannot be satisfied by the coefficients. This implies that the stability problem of system (2) cannot be solved by using the classical stochastic system theory (see [48,49]), since the uniqueness of solution is always needed in the application of this theory. Inspired by the deterministic analogs in [50,51] and the delay-free analog in [52,53], the classical stochastic stability concept has been slightly extended in [54] so that it can be applied to more general SNTDSs. In what follows, GSSP and GSASP represent the abbreviation of "globally strongly stable in probability" and "globally strongly asymptotically stable in probability," respectively.

Definition 1 ([54]). The trivial solution $x = 0$ of system (2) is GSSP, if there is a class-$\mathcal{K}$ function $\gamma_x(\cdot)$ such that $P_x\{|x(t)| \leq \gamma_x(||\varphi||)\} \geq 1 - \epsilon$ holds for any $\epsilon > 0$ and weak solution $x(t)$, where $t \geq 0$, $\varphi \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau_M, 0]; \mathbb{R}^n)\backslash\{0\}$ with $||\varphi|| = \sup\{|x(\theta)| : -\tau_M \leq \theta \leq 0\}$. Furthermore, if $P_x\{\lim_{t\to+\infty} ||x(t)|| = 0\} = 1$ holds for any weak solution $x(t)$, it is said to be GSASP.

Lemma 1 ([54]). *Suppose that there exist a positive definite function $V(x) \in \mathcal{C}^2$ satisfying $V(x) \to \infty$ as $||x|| \to \infty$, and a nonnegative function $W(x) \in \mathcal{C}^0$, such that*

$$\mathscr{L}V(x) \triangleq \frac{\partial V(x)}{\partial x}\mu\big(x(t), x(t-\tau(t))\big) + \frac{1}{2}Tr\left\{\sigma\big(x(t), x(t-\tau(t))\big)\frac{\partial^2 V(x)}{\partial x^2} \sigma^T\big(x(t), x(t-\tau(t))\big)\right\} \leq -W\big(x(t)\big), \quad (3)$$

then system (2) is GSSP at $x = 0$. Furthermore, it is GSASP at $x = 0$ in the case of $W(x)$ being positive definite.

Remark 2 In classical stochastic system theory, there are two types of available approaches to deal with the problem of stability and stabilization for SNTDSs, namely the LKF approach [48] and the Lyapunov–Razumikhin function approach [49]. When one uses these two methods, the existence and uniqueness of strong solutions for SNTDSs are always required, that is, the considered systems must satisfy the (locally) Lipschitz condition. Clearly, they cannot be applied to system (2), because the drift and diffusion terms of system (2) are only continuous but non-Lipschitz. Definition 1 and Lemma 1 can be viewed as an extension of stability theory for SNTDSs developed in [48], which are applicable to more general SNTDSs.

Remark 3 Definition 1 provides two stability notions for nonsmooth stochastic system (2), namely the global strong stability in probability and the global strong asymptotic stability in probability. Both types of stability describe the asymptotic behavior of the trajectories of the system or the stability behavior of the trivial solution $x = 0$ of the system as time goes to infinity. The difference between them is that the latter not only requires each weak solution to satisfy the former, but also requires them to eventually converge to zero.

The following lemmas provide several useful inequalities which are necessary in what follows.

Lemma 2 ([55]). *Let $a, b > 1$ with $\frac{1}{a} + \frac{1}{b} = 1$. For any $v, w \in \mathbb{R}$, we have*

$$|vw| \leq \frac{1}{a}|v|^a + \frac{1}{b}|w|^b. \quad (4)$$

Lemma 3 ([55]). *Let $0 < \alpha < 1$. For any $v \geq -1$, we have*

$$(1 + v)^\alpha \leq 1 + \alpha v. \quad (5)$$

Lemma 4 ([51]). *For any $v, w \in \mathbb{R}$ and $a, b, c \in \mathbb{R}^+$, we have*

$$|v|^a|w|^b \leq \frac{a}{a+b}c|v|^{a+b} + \frac{b}{a+b}c^{-\frac{a}{b}}|w|^{a+b}. \quad (6)$$

Lemma 5 ([56]). *For any $v, w \in \mathbb{R}$ and $\gamma \geq 1$, the following inequalities hold*

$$v + w|^\gamma \leq 2^{\gamma-1}|v^\gamma + w^\gamma|, \quad (7)$$

$$|[v]^{\frac{1}{\gamma}} - [w]^{\frac{1}{\gamma}}| \leq 2^{1-\frac{1}{\gamma}}|v - w|^{\frac{1}{\gamma}}. \quad (8)$$

Lemma 6 ([57]). *For any $v_1, v_2, \ldots, v_n, c \in \mathbb{R}^+$, it is true that*

$$(v_1 + v_2 + \cdots + v_n)^c \leq \max\{n^{c-1}, 1\}(v_1^c + v_2^c + \cdots + v_n^c). \quad (9)$$

In particular, when $c = 2$, we have $(v_1 + v_2 + \cdots + v_n)^2 \leq n(v_1^2 + v_2^2 + \cdots + v_n^2)$.

Lemma 7 *Let $\varepsilon \in (0, 1)$ and $v \in \mathbb{R}$. For any real number $s \in (0, 1)$, we have*

$$[v]^s + [1 - v]^s > -\varepsilon^2|v|^{1+s} + (2^s - 1)\varepsilon^{1-s}. \quad (10)$$

Proof First, consider the case when $|v| \leq 1$. Letting $F(v) = [v]^s + [1 - v]^s$, we have $F'(v) = s|v|^{s-1} - s(1 - v)^{s-1}$. Clearly, when $-1 \leq v < \frac{1}{2}$, $F'(v) > 0$ and when $\frac{1}{2} \leq v < 1$, $F'(v) < 0$, that is to say, $F(v)$ is strictly monotonically increasing on $[-1, \frac{1}{2})$ and strictly monotonically decreasing on $[\frac{1}{2}, 1)$. Thus, it follows from $F(0) = F(1)$ that $F(v)$ have a minimum at $v = -1$ on the set $\{v : |v| \leq 1\}$. Since $0 < \varepsilon^{1-s} < 1$, a direct calculation yields

$$F(v)+\varepsilon^2|v|^{1+s} \geq F(v) \geq F(-1) = 2^s-1 > \varepsilon^{1-s}(2^s-1).$$

Namely, inequality (10) holds when $|v| \leq 1$.

Next, we consider the case when $|v| > 1$. According to Lemma 2, we have

$$\begin{aligned}&[v]^s+[1-v]^s+\varepsilon^2|v|^{s+1}\\&=\left(([v]^s+[1-v]^s)^{\frac{1+s}{2}}\right)^{\frac{2}{1+s}}+\left((\varepsilon^2|v|^{1+s})^{\frac{1-s}{2}}\right)^{\frac{2}{1-s}}\\&\geq \frac{1+s}{2}\left(([v]^s+[1-v]^s)^{\frac{1+s}{2}}\right)^{\frac{2}{1+s}}+\frac{1-s}{2}\left((\varepsilon^2|v|^{1+s})^{\frac{1-s}{2}}\right)^{\frac{2}{1-s}}\\&\geq ([v]^s+[1-v]^s)^{\frac{1+s}{2}}(\varepsilon^2|v|^{1+s})^{\frac{1-s}{2}}\\&=\left(([v]^s+[1-v]^s)|v|^{1-s}\right)^{\frac{1+s}{2}}\varepsilon^{1-s}.\end{aligned} \tag{11}$$

Let $G(v)=([v]^s+[1-v]^s)|v|^{1-s}$. Then, the derivative of $G(v)$ is capable of being calculated as

$$\begin{aligned}G'(v)&=\left(\mathrm{sgn}(v)|v|+\mathrm{sgn}(1-v)|1-v|^s|v|^{1-s}\right)'\\&=1-s\left|\frac{1}{v}-1\right|^{s-1}-(1-s)\left|\frac{1}{v}-1\right|^s.\end{aligned} \tag{12}$$

Since $\frac{1}{v}-1<0$, Eq. (12) can be further calculated as $G'(v)=1+|\frac{1}{v}-1|^{s-1}(\frac{1}{v}-\frac{s}{v}-1)$. From Lemma 3, it is easy to see $(1-\frac{1}{v})^{1-s} \leq 1-\frac{1}{v}+\frac{s}{v}$. Thus, we can get that $G'(v)<0$ on the set $\{v: |v|>1\}$. This implies that $G(v) \geq \min\{G(-1), G(+\infty)\} = \min\{2^s-1, s\}$. Clearly, Lemma 3 yields $2^s-1<s$. So, we have $G(v) \geq 2^s-1$. Applying this result to inequality (11), we can get that inequality (10) holds when $|v|>1$. □

Remark 4 It should be pointed that an inequality similar to inequality (10) has been introduced in Ref. [58] where the power s must be expressed as the fraction form with positive odd numerators and denominators. In Lemma 7, we provide a new version without this requirement on the power s and give the corresponding proof. Inequality (10) is crucial in the observer design for system (1).

3 Dynamic output feedback stabilizer design

Because only output signal $y=x_1$ is measured, we will construct an observer-based stabilizer to deal with the stabilization of low-order SNTDS (1) in this section. Under Assumption 1, the construction process of the controller is divided into three parts.

3.1 State feedback design

In the first part, a state feedback control law is designed by extending the AAPI technique. Notably, to solve the obstacles arisen from the multiple delays, a novel LKF will be introduced in the controller design.

For the x_1-subsystem of (1), choose a Lyapunov–Krasovskii functional $V_1(x_1)=\frac{k_1}{2}x_1^2+\frac{a_1}{1-v_1}\int_{t-d_1}^t |x_1(s)|^{r+1}ds$, $k_1>0$, and two coordinate transformations $\xi_1=x_1$, $\xi_{1d}=x_{1d}$. According to Lemmas 2 and 6, we have

$$\begin{aligned}&\mathscr{L}V_1(x_1)\\&=k_1\xi_1([x_2]^r+f_1)+\frac{1}{2}k_1||g_1||^2\\&\quad+\frac{a_1}{1-v_1}(|\xi_1|^{r+1}-|\xi_{1d}|^{r+1}(1-\dot{d}_1))\\&\leq k_1\xi_1([x_2]^r-[x_2^*]^r)+k_1\xi_1[x_2^*]^r\\&\quad+k_1a_1|\xi_1|(|\xi_1|^r+|\xi_{1d}|^r)\\&\quad+\frac{1}{2}k_1b_1^2(|\xi_1|^{\frac{r+1}{2}}+|\xi_{1d}|^{\frac{r+1}{2}})^2+\frac{a_1}{1-v_1}|\xi_1|^{r+1}-a_1|\xi_{1d}|^{r+1}\\&\leq k_1\xi_1([x_2]^r-[x_2^*]^r)+k_1\xi_1[x_2^*]^r+(k_1a_1+k_1b_1^2+\frac{a_1}{1-v_1})|\xi_1|^{r+1}\\&\quad+k_1a_1|\xi_1||\xi_{1d}|^r-P_{11}|\xi_{1d}|^{r+1}\\&\leq k_1\xi_1([x_2]^r-[x_2^*]^r)+k_1\xi_1[x_2^*]^r+l_{111}|\xi_1|^{r+1},\end{aligned} \tag{13}$$

where $P_{11}=k_1b_1^2-a_1$ and $l_{111}=k_1a_1+k_1b_1^2+\frac{a_1}{1-v_1}+\frac{k_1a_1}{1+r}(\frac{(a_1-k_1b_1^2)(1+r)}{rk_1a_1})^{-r}$. Design $x_2^*=-\beta_1\xi_1$ with $\beta_1=\left(\frac{c_{11}+l_{111}}{k_1}\right)^{\frac{1}{r}}>0$ and the arbitrary constant $c_{11}>0$. Since $k_1\xi_1[x_2^*]^r = k_1\mathrm{sgn}(\xi_1)|\xi_1|\mathrm{sgn}(x_2^*)|x_2^*|^r = -(c_{11}+l_{111})|x_1|^{r+1}$, it follows from (13) that

$$\mathscr{L}V_1(x_1) \leq -c_{11}|\xi_1|^{r+1}+k_1\xi_1([x_2]^r-[x_2^*]^r). \tag{14}$$

Generally, for the $(x_1, x_2, \ldots, x_{i-1})$-subsystem of system (1), assume that we can find $V_{i-1}(x_1, x_2, \ldots, x_i)$ and two sets of coordinate transformations

$$\xi_j=x_j-x_j^*,\quad \xi_{jd}=x_{jd}-x_{jd}^*,\quad j\in\mathcal{N}_i, \tag{15}$$

where $x_j^*=-\beta_{j-1}\xi_{j-1}$, $x_{jd}^*=-\beta_{j-1}\xi_{j-1,d}$ with $\beta_0=0$ and $\beta_j, c_{i-1,j}>0$, $j\in\mathcal{N}_{i-1}$, such that

$$\begin{aligned}&\mathscr{L}V_{i-1}(x_1, x_2, \ldots, x_{i-1})\\&\leq -\sum_{j=1}^{i-1}c_{i-1,j}|\xi_j|^{r+1}+k_{i-1}\xi_{i-1}([x_i]^r-[x_i^*]^r).\end{aligned} \tag{16}$$

Next, we will prove that formula (16) still holds for the $(x_1, x_2, \ldots, x_i)$-subsystem. In fact, choose a Lyapunov–Krasovskii functional for the subsystem

$$\begin{aligned}&V_i(x_1, x_2, \ldots, x_i)\\&=V_{i-1}+\frac{k_i}{2}\xi_i^2+K\sum_{j=1}^{i-1}\int_{t-d_j}^{t}|\xi_j|^{r+1}(s)ds\\&+\frac{a_i}{1-v_i}\int_{t-d_i}^{t}|\xi_i|^{r+1}(s)ds\end{aligned}$$

with arbitrary positive constants k_i and K. Based on coordinate transformation (15), a simple calculation obtains that

$$\begin{aligned}&\mathscr{L}V_i(x_1, x_2, \ldots, x_i)\\&=\mathscr{L}V_{i-1}+k_i\xi_i\Big([x_{i+1}]^r+f_i+\sum_{j=1}^{i-1}B_{ij}([x_{j+1}]^r+f_j)\Big)\\&\quad+\frac{k_i}{2}||g_i+\sum_{j=1}^{i-1}B_{ij}g_j||^2+K\sum_{j=1}^{i-1}|\xi_j|^{r+1}-K\sum_{j+1}^{i-1}|\xi_{jd}|^{r+1}(1-\dot{d}_j)\\&\quad+\frac{a_i}{1-v_i}|\xi_i|^{r+1}-\frac{a_i}{1-v_i}|\xi_{id}|^{r+1}(1-\dot{d}_i)\\&\le-\sum_{j=1}^{i-1}c_{i-1,j}|\xi_j|^{r+1}+k_{i-1}\xi_{i-1}\big([x_i]^r-[x_i^*]^r\big)+k_i\xi_i\big([x_{i+1}]^r-[x_{i+1}^*]^r\big)\\&\quad+k_i\xi_i[x_{i+1}^*]^r+k_i\xi_i\Big(f_i+\sum_{j=1}^{i}B_{ij}([x_{j+1}]^r+f_j)\Big)\\&\quad+\frac{k_i}{2}||g_i+\sum_{j=1}^{i-1}B_{ij}g_j||^2+K\sum_{j=1}^{i-1}|\xi_j|^{r+1}\\&\quad-K\sum_{j=1}^{i-1}(1-v_j)|\xi_{jd}|^{r+1}+\frac{a_i}{1-v_i}|\xi_i|^{r+1}-a_i|\xi_{id}|^{r+1},\end{aligned}\tag{17}$$

where $B_{ij}=\beta_{i-1}\beta_{i-2}\cdots\beta_j,\ j\in\mathcal{N}_{i-1}$.

We will investigate the terms on the right of inequality (17). According to coordinate transformation (15), Assumption 1, Lemmas 5 and 6, it follows that

$$\begin{aligned}&\frac{k_i}{2}||g_i+\sum_{j=1}^{i-1}B_{ij}g_j||^2\\&\le k_i||g_i||^2+k_i(i-1)\sum_{j=1}^{i-1}B_{ij}||g_j||^2\\&\le 2ik_ib_i^2\sum_{j=1}^{i}(|x_j|^{r+1}+|x_{jd}|^{r+1})\\&\quad+2(i-1)k_i\sum_{j=1}^{i-1}jb_j^2B_{ij}^2\sum_{m=1}^{j}(|x_m|^{r+1}+|x_{md}|^{r+1})\\&\le 2^rk_i\sum_{j=1}^{i-1}(\bar{b}_{ij}+\beta_j^{r+1}\bar{b}_{i,j+1})(|\xi_j|^{r+1}+|\xi_{jd}|^{r+1})\\&\quad+2^rk_i\bar{b}_{ii}(|\xi_i|^{r+1}+|\xi_{id}|^{r+1}).\end{aligned}\tag{18}$$

where $\bar{b}_{ij}=2ik_ia_i^2+2(i-1)\sum_{m=j}^{i-1}mB_{im}^2b_m^2,\ j\in\mathcal{N}_i$. Substituting (18) into (17) yields

$$\begin{aligned}&\mathscr{L}V_i(x_1, x_2, \ldots, x_i)\\&\le-\sum_{j=1}^{i-1}c_{i-1,j}|\xi_j|^{r+1}+k_{i-1}\xi_{i-1}([x_i]^r-[x_i^*]^r)\\&\quad+k_i\xi_i([x_{i+1}]^r-[x_{i+1}^*]^r)+k_i\xi_i[x_{i+1}^*]^r\\&\quad+k_i\xi_i\Big(f_i+\sum_{j=1}^{i}B_{ij}([x_{j+1}]^r+f_j)\Big)\\&\quad+\sum_{j=1}^{i-1}l_{ij1}\xi_j^{r+1}+h_{i1}\xi_i^{r+1}-\sum_{j=1}^{i-1}P_{ij}\xi_{jd}^{r+1}-P_{ii}\xi_{id}^{r+1},\end{aligned}\tag{19}$$

where $l_{ij1}=K+2^rk_i\sum(\bar{b}_{ij}+\beta_j^{r+1}\bar{b}_{i,j+1}),\ h_{i1}=\frac{a_i}{1-v_i}+2^rk_i\bar{b}_{ii},\ P_{ij}=K(1-v_i)-2^rk_i(\bar{b}_{ij}+\beta_j^{r+1}\bar{b}_{i,j+1}),\ j\in\mathcal{N}_{i-1}$, and $P_{ii}=a_i-2^rk_i\bar{b}_{ii}$. For the second term, by using (6) and (8), we can get

$$\begin{aligned}&k_{i-1}\xi_{i-1}([x_i]^r-[x_i^*]^r)\\&\le 2^{1-r}k_{i-1}|\xi_{i-1}||\xi_i|^r\le l_{i,i-1,2}|\xi_{i-1}|^{r+1}+h_{i2}|\xi_i|^{r+1},\end{aligned}\tag{20}$$

where $l_{i,i-1,2}, h_{i2}>0$ are two constants. Similarly, it follows from Assumption 1 and Lemma 4 that

$$\begin{aligned}&k_i\xi_i\Big(f_i+\sum_{j=1}^{i}B_{ij}([x_{j+1}]^r+f_j)\Big)\\&\le k_i|\xi_i|\Big(a_i\sum_{j=1}^{i}|x_j|^r+|x_{jd}|^r)+\sum_{j=1}^{i-1}B_{ij}\big([x_{j+1}]^r\\&\quad+a_j\sum_{m=1}^{j}(|x_m|^r+|x_{md}|^r)\big)\Big)\end{aligned}$$

$$\leq k_i|\xi_i|\sum_{j=1}^{i}\Big(\tilde{b}_{ij}(|\xi_j|^r+\beta_{j-1}^r|\xi_{j-1}|^r)$$
$$+\hat{b}_{ij}(|\xi_{jd}|^r+\beta_{j-1}^r|\xi_{j-1,d}|^r)\Big)$$
$$\leq\sum_{j=1}^{i-1}l_{ij3}|\xi_j|^{r+1}+h_{i3}|\xi_i|^{r+1}+\sum_{j=1}^{i-1}P_{ij}|\xi_{jd}|^{r+1}+P_{ii}|\xi_{id}|^{r+1}, \tag{21}$$

where for any $j \in \mathcal{N}_{i-1}$, $\hat{b}_{ij} = a_i + \sum_{m=j}^{i-1} a_m B_{im}$, $\hat{b}_{ii} = a_i$, $\tilde{b}_{ij} = a_i + B_{i,j-1} + \sum_{m=j}^{i-1} a_m B_{im}$, $\tilde{b}_{ii} = a_i + B_{i,i-1}$, and l_{ij3}, $h_{i3} > 0$ are constants. Then, substituting (20) and (21) into (19) results in

$$\mathscr{L}V_i(x_1,x_2,\ldots,x_i)$$
$$\leq -\sum_{j=1}^{i-1}c_{ij}|\xi_j|^{r+1}+k_i\xi_i([x_{i+1}]^r-[x_{i+1}^*]^r)$$
$$+(h_{i1}+h_{i2}+h_{i3})|\xi_i|^{r+1}+k_i\xi_i[x_{i+1}^*]^r, \tag{22}$$

where $c_{ij} = c_{i-1,j} - l_{ij1} - l_{ij3} > 0$, $j \in \mathcal{N}_{i-2}$, and $c_{i,i-1} = c_{i-1,i-1} - l_{i,i-1,1} - l_{i,i-1,2} - l_{i,i-1,3} > 0$. By designing $x_{i+1}^* = -\beta_i\xi_i$ with $\beta_i = (\frac{c_{ii}+h_{i1}+h_{i2}+h_{i3}}{k_i})^{\frac{1}{r}} > 0$ and the arbitrary positive constant c_{ii}, it follows that

$$\mathscr{L}V_i(x_1,x_2,\ldots,x_i)$$
$$\leq -\sum_{j=1}^{i}c_{ij}|\xi_j|^{r+1}+k_i\xi_i\big([x_{i+1}]^r-[x_{i+1}^*]^r\big). \tag{23}$$

The induction proof above shows that (16) holds for every $i \in \mathcal{N}_n$. Thus, for system (1), we can choose

$$V_n(x_1,x_2,\ldots,x_n)$$
$$=V_{n-1}+\frac{k_n}{2}\xi_n^2+K\sum_{j=1}^{n-1}\int_{t-d_j}^{t}|\xi_j(s)|^{r+1}ds$$
$$+\frac{a_n}{1-v_n}\int_{t-d_n}^{t}|\xi_n(s)|^{r+1}ds$$

with $k_n > 0$, and design $x_{n+1}^* = -\beta_n\xi_n$ with $\beta_n = (\frac{c_{nn}+h_{n1}+h_{n2}+h_{n3}}{k_n})^{\frac{1}{r}} > 0$ and the arbitrary positive constant c_{nn}, such that

$$\mathscr{L}V_n(x_1,x_2,\ldots,x_n)$$
$$\leq -\sum_{j=1}^{n}c_{nj}|\xi_j|^{r+1}+k_n\xi_n([u]^r-[x_{n+1}^*]^r). \tag{24}$$

Remark 5 It is noted that the multiple time-varying delays $d_i(t)$ need to satisfy $\dot{d}(t) \leq v_i < 1$ with the positive constant numbers v_i, $i \in \mathcal{N}_n$. This constraint condition on time delays is necessary in our controller design, which has wide applications in the study of time-delay systems, such as [21,30]. It means that the change of the time delays is relatively slow. From the design process above, we can see that v_i, $i \in \mathcal{N}_n$ greatly effect the performance of the controller and cannot be too close to one in practice.

3.2 Reduced-order observer design

In this part, a reduced-order state observer will be introduced to estimate the unmeasurable system states x_i, $i \in \mathcal{I}_n$. Let $\hat{z}_i = \hat{x}_i - s_{i-1}\hat{x}_{i-1}$, $i \in \mathcal{I}_n$, where $\hat{x}_1 = x_1$ and $s_j > 1$, $j \in \mathcal{N}_{n-1}$ are dynamic gains to be determined later. Construct the following dynamic observer equation to generate $\hat{z}_i$, $i \in \mathcal{I}_n$

$$\begin{cases}\dot{\hat{z}}_i=[\hat{x}_{i+1}]^r-s_{i-1}[\hat{x}_i]^r, \quad i\in\mathcal{I}_{n-1},\\ \dot{\hat{z}}_n=[u]^r-s_{n-1}[\hat{x}_n]^r.\end{cases} \tag{25}$$

For any $i \in \mathcal{I}_n$, the estimator $\hat{x}_i$ of x_i can be obtained by $\hat{x}_i = \hat{z}_i + s_{i-1}\hat{x}_{i-1}$. Meanwhile, let $z_i = x_i - s_{i-1}x_{i-1}$, and define an error variable $e_i = z_i - \hat{z}_i$. According to dynamic systems (1) and (25), we can get that

$$\begin{cases}de_i=\big(([x_{i+1}]^r-[\hat{x}_{i+1}]^r)+(f_i-s_{i-1}f_{i-1})-s_{i-1}([x_i]^r\\ \qquad -[\hat{x}_i]^r)\big)dt+\big(g_i-s_{i-1}g_{i-1}\big)^Tdw,\ i\in\mathcal{I}_{n-1},\\ de_n=\big((f_n-s_{n-1}f_{n-1})-s_{n-1}([x_n]^r-[\hat{x}_n]^r)\big)dt\\ \qquad +\big(g_n-s_{n-1}g_{n-1}\big)^Tdw.\end{cases} \tag{26}$$

For error system (26), construct a Lyapunov–Krasovskii functional

$$U(e_2,e_3,\ldots,e_n)=\sum_{i=2}^{n}\frac{m_i}{2}e_i^2$$
$$+\sum_{i=1}^{n}\frac{w_i}{1-v_i}\int_{t-d_i}^{t}|\xi_i(s)|^{r+1}ds$$

with arbitrary positive constants m_i and w_i. A simple calculation yields

$$\mathscr{L}U(e_2,e_3,\ldots,e_n)$$
$$=\sum_{i=2}^{n-1}m_ie_i\big([(x_{i+1}]^r-[\hat{x}_{i+1}]^r\big)$$

$$-\sum_{i=2}^{n} m_i e_i s_{i-1}\big([x_i]^r-[\hat{x}_i]^r\big)$$
$$+\sum_{i=2}^{n} m_i e_i\big(f_i-s_{i-1}f_{i-1}\big)$$
$$+\frac{1}{2}\sum_{i=2}^{n} m_i||g_i-s_{i-1}g_{i-1}||^2$$
$$+\sum_{i=1}^{n}\frac{w_i}{1-v_i}\big(|\xi_i|^{r+1}-|\xi_{id}|^{r+1}(1-\dot{d}_i)\big). \qquad (27)$$

Consider the first term of (27). According to inequality (8) we obtain that

$$\sum_{i=2}^{n-1} m_i e_i\big([(x_{i+1}]^r-[\hat{x}_{i+1}]^r\big)$$
$$\le 2^{1-r}\sum_{i=2}^{n-1} m_i|e_i|\,|x_{i+1}-\hat{x}_{i+1}|^r. \qquad (28)$$

For the second term of (27), we rewrite it as the following form

$$-\sum_{i=2}^{n} m_i e_i s_{i-1}\big([x_i]^r-[\hat{x}_i]^r\big)$$
$$=-\sum_{i=2}^{n} m_i e_i s_{i-1}\big([x_i]^r-[x_i-e_i]^r)\big)$$
$$-\sum_{i=2}^{n} m_i e_i s_{i-1}\big([x_i-e_i]^r-[\hat{x}_i]^r\big). \qquad (29)$$

For any $i\in\mathcal{I}_n$, assuming that $e_i\neq 0$, let $v=\frac{x_i}{e_i}$, $\varepsilon=s_{i-1}^{-\frac{1}{r+1}}$ and $s=r$. Then, according to Lemma 7, it follows that

$$-m_i e_i s_{i-1}\big([x_i]^r-[x_i-e_i]^r\big)$$
$$\le m_i s_{i-1}^{\frac{r-1}{r+1}}|x_i|^{r+1}-(2^r-1)m_i s_{i-1}^{\frac{2r}{r+1}}|e_i|^{r+1}. \qquad (30)$$

Clearly, inequality (30) holds when $e_i=0$. In addition, with the help of $\hat{x}_i=x_i-e_i-s_{i-1}(x_{i-1}-\hat{x}_{i-1})$ and Lemma 5, we can get that

$$-m_i e_i s_{i-1}\big([x_i-e_i]^r-[\hat{x}_i]^r\big)$$
$$\le 2^{1-r}m_i s_{i-1}^{r+1}|e_i|\,|x_{i-1}-\hat{x}_{i-1}|^r. \qquad (31)$$

Substituting (30) and (31) into (29) gives rise to

$$-\sum_{i=2}^{n} m_i e_i s_{i-1}\big([x_i]^r-[\hat{x}_i]^r\big)$$
$$\le\sum_{i=2}^{n} m_i s_{i-1}^{\frac{r-1}{r+1}}|x_i|^{r+1}-\sum_{i=2}^{n} m_i H(s_{i-1})|e_i|^{r+1}$$
$$+2^{1-r}\sum_{i=2}^{n} m_i s_{i-1}^{r+1}|e_i|\,|x_{i-1}-\hat{x}_{i-1}|^r, \qquad (32)$$

where $H(s_{i-1})=(2^r-1)s_{i-1}^{\frac{2r}{r+1}}$. For the forth term of (27), by using Assumption 1, Lemma 6 and coordinate transformation (15), we have the following estimation

$$\frac{1}{2}\sum_{i=2}^{n} m_i||g_i-s_{i-1}g_{i-1}||^2$$
$$\le\sum_{i=2}^{n} m_i\big(||g_i||^2+s_{i-1}^2||g_{i-1}||^2\big)$$
$$\le\sum_{i=2}^{n} m_i\Big(2ib_i^2\sum_{j=1}^{i}(|x_j|^{\frac{r+1}{2}}+|x_{jd}|^{\frac{r+1}{2}})^2$$
$$+2(i-1)b_{i-1}^2 s_{i-1}^2\sum_{j=1}^{i-1}(|x_j|^{\frac{r+1}{2}}+|x_{jd}|^{\frac{r+1}{2}})^2\Big)$$
$$\le\sum_{i=2}^{n}\Big(\sum_{j=1}^{i-1}R_{ij}(s_{i-1})|\xi_j|^{r+1}+R_{ii}|\xi_i|^{r+1}$$
$$+\sum_{j=1}^{i-1}R_{ij}(s_{i-1})|\xi_{jd}|^{r+1}+R_{ii}|\xi_{id}|^{r+1}\Big), \qquad (33)$$

where $R_{ij}(s_{i-1})=2^{r+1}m_i\big(ib_i^2+(i-1)s_{i-1}^2b_{i-1}^2\big)\big(1+\beta_j\big)^{r+1}$, $j\in\mathcal{N}_{i-2}$, $R_{i,i-1}(s_{i-1})=2^{r+1}m_i\big(ib_i^2+(i-1)s_{i-1}^2b_{i-1}^2+ib_i^2\beta_{i-1}^{r+1}\big)$ and $R_{ii}=2^{r+1}m_i ib_i^2$. Let $\lambda_{i1}(s_i, s_{i+1},\ldots,s_{n-1})=R_{ii}+R_{i+1,i}(s_i)+R_{i+2,i}(s_{i+1})+\cdots+R_{ni}(s_{n-1})$, $i\in\mathcal{N}_{n-1}$ with $R_{11}=0$ and $\lambda_{n1}=R_{nn}$. Then, formula (33) can be further expressed as

$$\frac{1}{2}\sum_{i=2}^{n} m_i||g_i-s_{i-1}g_{i-1}||^2$$
$$\le\sum_{i=1}^{n}\lambda_{i1}(s_i,s_{i+1},\ldots,s_{n-1})\big(|\xi_i|^{r+1}+|\xi_{id}|^{r+1}\big), \qquad (34)$$

where $\lambda_{i1}(s_i, s_{i-1}, \ldots, s_{n-1})$ are nonnegative functions of $s_i, s_{i-1}, \ldots, s_{n-1}$, $i \in \mathcal{N}_{n-1}$ and λ_{n1} is a positive constant. For the last term of (27), it follows from $\dot{d}_i(t) \le v_i < 1$, $i \in \mathcal{N}_n$ that

$$\sum_{i=1}^{n} \frac{w_i}{1-v_i}\left(|\xi_i|^{r+1} - |\xi_{id}|^{r+1}(1-\dot{d}_i)\right) \le \sum_{i=1}^{n} \lambda_{i2}|\xi_i|^{r+1} - \sum_{i=1}^{n} w_i|\xi_{id}|^{r+1}, \tag{35}$$

where $\lambda_{i2} = \frac{w_i}{1-v_i} > 0$, $i \in \mathcal{N}_n$. Substituting (28), (32), (34) and (35) into (27), we have

$$\begin{aligned}
&\mathscr{L}U(e_2, e_3, \ldots, e_n) \\
&\le \sum_{i=2}^{n} m_i s_{i-1}^{\frac{r-1}{r+1}}|x_i|^{r+1} - \sum_{i=2}^{n} m_i H(s_{i-1})|e_i|^{r+1} \\
&\quad + 2^{1-r}\Big(\sum_{i=2}^{n-1} m_i|e_i||x_{i+1}-\hat{x}_{i+1}|^r + \sum_{i=2}^{n} m_i s_{i-1}^{r+1}|e_i||x_{i-1}-\hat{x}_{i-1}|^r\Big) \\
&\quad + \sum_{i=1}^{n}\left(\lambda_{i1}(s_i, s_{i+1}, \ldots, s_{n-1}) + \lambda_{i2}\right)|\xi_i|^{r+1} \\
&\quad - \sum_{i=1}^{n} Q_i|\xi_{id}|^{r+1} + \sum_{i=2}^{n} m_i e_i\left(f_i - s_{i-1}f_{i-1}\right),
\end{aligned} \tag{36}$$

where $Q_i = w_i - \lambda_{i1}(s_i, s_{i+1}, \ldots, s_{n-1})$, $i \in \mathcal{N}_n$.

For the first term of (36), according to (7) and (15) it follows that

$$\begin{aligned}
&\sum_{i=2}^{n} m_i s_{i-1}^{\frac{r-1}{r+1}}|x_i|^{r+1} \\
&\le 2^r \sum_{i=2}^{n} m_i s_{i-1}^{\frac{r-1}{r+1}}\left(|\xi_i|^{r+1} + \beta_{i-1}^{r+1}|\xi_{i-1}|^{r+1}\right) \\
&= \sum_{i=1}^{n} \lambda_{i3}(s_{i-1}, s_i)|\xi_i|^{r+1},
\end{aligned} \tag{37}$$

where $\lambda_{i3}(s_{i-1}, s_i) = 2^r\left(m_i s_{i-1}^{\frac{r-1}{r+1}} + m_{i+1}s_i^{\frac{r-1}{r+1}}\beta_i^{r+1}\right)$, $i \in \mathcal{N}_n$ with $s_0 = s_n = 0$. By using the variable transformations $x_i = z_i + s_{i-1}x_{i-1}$, $\hat{x}_i = \hat{z}_i + \hat{s}_{i-1}\hat{x}_{i-1}$ and $e_i = z_i - \hat{z}_i$, for any $i \in \mathcal{I}_n$, we have

$$x_i - \hat{x}_i = e_i + s_{i-1}e_{i-1} + s_{i-1}s_{i-2}e_{i-2} + s_{i-1}s_{i-2}\cdots s_2e_2. \tag{38}$$

This means that for any $i \in \mathcal{I}_n$, $x_i - \hat{x}_i$ can be expressed as the linear combination of $e_2, e_3, \ldots, e_i$. Based on (6), (9) and (38), for the third term of (36), we have

$$\begin{aligned}
&2^{1-r}\Big(\sum_{i=2}^{n-1} m_i|e_i||x_{i+1}-\hat{x}_{i+1}|^r + \sum_{i=2}^{n} m_i s_{i-1}^{r+1}|e_i||x_{i-1}-\hat{x}_{i-1}|^r\Big) \\
&= 2^{1-r}m_2|e_2||e_3+s_2e_2|^r + 2^{1-r}\sum_{i=3}^{n-1} m_i|e_i|\big(|e_{i+1}+s_ie_i \\
&\quad + s_is_{i-1}e_{i-1} + \cdots + s_is_{i-1}\cdots s_2e_2|^r + s_{i-1}^{r+1}|e_{i-1} \\
&\quad + s_{i-2}e_{i-2} + s_{i-2}s_{i-3}e_{i-3} + \cdots + s_{i-2}s_{i-3}\cdots s_2e_2|^r\big) \\
&\quad + 2^{1-r}m_ns_{n-1}^{r+1}|e_n||e_{n-1}+s_{n-2}e_{n-2}+s_{n-2}s_{n-3}e_{n-3} \\
&\quad + \cdots + s_{n-2}s_{n-3}\cdots s_2e_2|^r \\
&\le \sum_{i=2}^{n-1}\sigma_{i1}(s_i, s_{i+1}, \ldots, s_{n-1})|e_i|^{r+1} + \sigma_{n1}|e_n|^{r+1} \\
&\triangleq \sum_{i=2}^{n}\sigma_{i1}(s_i, s_{i+1}, \ldots, s_{n-1})|e_i|^{r+1},
\end{aligned} \tag{39}$$

where $\sigma_{i1}(s_i, s_{i+1}, \ldots, s_{n-1})$ are nonnegative functions of the gains $s_i, s_{i+1}, \ldots, s_{n-1}$, $i \in \mathcal{N}_{n-1}$ and σ_{n1} is a positive constant. Next, we concentrate on the last term of (36). In fact, by using Assumption 1, coordinate transform (15) and Lemma 5, it follows that

$$\begin{aligned}
&\sum_{i=2}^{n} m_ie_i(f_i - s_{i-1}f_{i-1}) \\
&\le \sum_{i=2}^{n} m_i|e_i|\Big(a_i\sum_{j=1}^{i}\left(|x_j|^r+|x_{jd}|^r\right) + s_{i-1}a_{i-1}\sum_{j=1}^{i-1}\left(|x_j|^r+|x_{jd}|^r\right)\Big) \\
&\le \sum_{i=2}^{n} m_i|e_i|\Big(\sum_{j=1}^{i-1}A_{ij}(s_{i-1})\left(|\xi_j|^r+|\xi_{jd}|^r\right) + A_{ii}\left(|\xi_i|^r+|\xi_{id}|^r\right)\Big),
\end{aligned} \tag{40}$$

where $A_{ij}(s_{i-1}) = (a_i + s_{i-1}a_{i-1})(1+\beta_j^r)$, $j \in \mathcal{N}_{i-2}$, $A_{i,i-1}(s_{i-1}) = s_{i-1}a_i + a_i(1+\beta_{i-1}^r)$ and $A_{ii} = a_i$. According to Lemma 4, inequality (40) can be further calculated as

$$\begin{aligned}
&\sum_{i=2}^{n} m_ie_i(f_i - s_{i-1}f_{i-1}) \\
&\le \sum_{i=2}^{n}\Big(\sum_{j=1}^{i-1}\left(\bar{A}_{ij}(s_{i-1})|\xi_j|^{r+1}\tilde{A}_{ij}(s_{i-1})|\xi_{jd}|^{r+1}\right) \\
&\quad + \bar{A}_{ii}|\xi_i|^{r+1} + \tilde{A}_{ii}|\xi_{id}|^{r+1} + \sum_{j=1}^{i}\hat{A}_{ij}|e_i|^{r+1}\Big),
\end{aligned} \tag{41}$$

where $\bar{A}_{ij}(s_{i-1}), \tilde{A}_{ij}(s_{i-1}), j \in \mathcal{N}_{i-1}$ are appropriate nonnegative functions of s_{i-1}, and $\bar{A}_{ii}, \tilde{A}_{ii}, \hat{A}_{ij}, j \in \mathcal{N}_i$ are appropriate positive constants. For any $i \in \mathcal{N}_n$, let $\lambda_{i4}(s_i, s_{i+1}, \ldots, s_{n-1}) = \sum_{j=i+1}^{n} \bar{A}_{ji}(s_{j-1}) + \bar{A}_{ii}$ with $\bar{A}_{11} = 0$ and $\lambda_{n4} = \bar{A}_{nn}$, and choose appropriate $\tilde{A}_{ij}(s_{i-1})$ and $\tilde{A}_{ii}$ such that $Q_i = \sum_{j=i+1}^{n} \tilde{A}_{ji}(s_{j-1}) + \tilde{A}_{ii}$ with $\tilde{A}_{11} = 0$ and $Q_n = \tilde{A}_{nn}$. Then, it follows that

$$\begin{aligned}&\sum_{i=2}^{n} m_i e_i (f_i - s_{i-1} f_{i-1}) \\ &\le \sum_{i=1}^{n} \lambda_{i4}(s_i, s_{i+1}, \ldots, s_{n-1})|\xi_i|^{r+1} + \sum_{i=1}^{n} Q_i |\xi_{id}|^{r+1} \\ &\quad + \sum_{i=2}^{n} \sigma_{i2} |e_i|^{r+1},\end{aligned} \tag{42}$$

where $\sigma_{i2} = \sum_{j=1}^{i} \hat{A}_{ij}, i \in \mathcal{I}_n$. Substituting (37), (39) and (42) into (36) yields

$$\begin{aligned}&\mathscr{L}U(e_2, e_3, \ldots, e_n) \\ &\le \sum_{i=1}^{n} \big(\lambda_{i1}(s_i, s_{i+1}, \ldots, s_{n-1}) + \lambda_{i2} + \lambda_{i3}(s_{i-1}, s_i) \\ &\quad + \lambda_{i4}(s_i, s_{i+1}, \ldots, s_{n-1})\big)|\xi_i|^{r+1} + \sum_{i=2}^{n} \big(-m_i H(s_{i-1}) \\ &\quad + \sigma_{i1}(s_i, s_{i+1}, \ldots, s_{n-1}) + \sigma_{i2}\big)|e_i|^{r+1}.\end{aligned} \tag{43}$$

3.3 Main results

Based on the design obtained in Sects. 3.1 and 3.2 above, we give the main stabilization results for system (1) in the third part. Output feedback stabilizer, stability analysis and the procedure of selecting dynamic gains $s_i, i \in \mathcal{N}_{n-1}$ will be provided here.

Theorem 1 *Consider system* (1) *under Assumption* 1. *There exists an output feedback control law* $u = u(y, \hat{x}_2, \ldots, \hat{x}_n)$ *with dynamic observer* (25)*, such that the closed-loop system is GSASP at* $x = 0$.

Proof By using the system output y and the state estimations $\hat{x}_i, i \in \mathcal{I}_n$ generated by (25), we construct the following controller for system (1)

$$\begin{aligned}u &= -\beta_n \xi_n(y, \hat{x}_2, \ldots, \hat{x}_n) \\ &= -(\beta_n \hat{x}_n + \beta_n\beta_{n-1}\hat{x}_{n-1} + \cdots + \beta_n\beta_{n-1}\cdots\beta_2\hat{x}_2 \\ &\quad + \beta_n\beta_{n-1}\cdots\beta_1 y)\end{aligned} \tag{44}$$

To investigate the stochastic stability of (1)–(44), the Lyapunov–Krasovskii functional $V = V_n(x_1, x_2, \cdots, x_n) + U(e_1, e_2, \ldots, e_n)$ is chosen for the system. According to (24) and (43), the differential of V is given as

$$\begin{aligned}\mathscr{L}V &= \mathscr{L}V_n(x_1, x_2, \ldots, x_n) + \mathscr{L}U(e_1, e_2, \ldots, e_n) \\ &\le \sum_{i=1}^{n} \big(-c_{ni} + \lambda_{i1}(s_i, s_{i+1}, \ldots, s_{n-1}) + \lambda_{i2} \\ &\quad + \lambda_{i3}(s_{i-1}, s_i)\lambda_{i4}(s_i, s_{i+1}, \ldots, s_{n-1})\big)|\xi_i|^{r+1} \\ &\quad + \sum_{i=2}^{n} \big(-m_i H(s_{i-1}) + \sigma_{i1}(s_i, s_{i+1}, \ldots, s_{n-1}) \\ &\quad + \sigma_{i2}\big)|e_i|^{r+1} + k_n\xi_n([u]^r - [x_{n+1}^*]^r).\end{aligned} \tag{45}$$

From the state feedback controller design, we can see that $x_{n+1}^* = -\beta_n\xi_n = -(\beta_n x_n + \beta_n\beta_{n-1}x_{n-1} + \cdots + \beta_n\beta_{n-1}\cdots\beta_2 x_2 + \beta_n\beta_{n-1}\cdots\beta_1 y$. Thus, according to Lemma 5, (44) and (38), we have

$$\begin{aligned}&k_n\xi_n([u]^r - [x_{n+1}^*]^r) \\ &\le 2^{1-r}k_n|\xi_n|\,|u - x_{n+1}^*|^r \\ &= 2^{1-r}k_n|\xi_n|\,|\beta_n(x_n - \hat{x}_n) + \beta_n\beta_{n-1}(x_{n-1} - \hat{x}_{n-1}) \\ &\quad + \cdots + \beta_n\beta_{n-1}\cdots\beta_2(x_2 - \hat{x}_2)|^r \\ &= 2^{1-r}k_n|\xi_n|\Big|\sum_{i=2}^{n-1} L_i(s_i, s_{i+1}, \ldots, s_{n-1})e_i + L_n e_n\Big|^r,\end{aligned} \tag{46}$$

where $L_i(s_i, s_{i+1}, \ldots, s_{n-1}) = \beta_n s_{n-1}s_{n-2}\cdots s_i + \beta_n\beta_{n-1}$
$s_{n-2}\cdots s_i + \cdots + \beta_n\beta_{n-1}\beta_{n-2}\cdots\beta_i, i \in \mathcal{I}_{n-1}$ and $L_n = \beta_n$. It follows from Lemmas 4 and 6 that inequality (46) is further rewritten as

$$\begin{aligned}&k_n\xi_n([u]^r - [x_{n+1}^*]^r) \\ &\le \lambda_{n5}|\xi_n|^{r+1} + \sum_{i=2}^{n} \sigma_{i3}(s_i, s_{i+1}, \ldots, s_{n-1})|e_i|^{r+1},\end{aligned} \tag{47}$$

where $\lambda_{n5}, \sigma_{n3}$ are two appropriate positive constants and $\sigma_{i3}(s_i, s_{i+1}, \ldots, s_{n-1}), i \in \mathcal{I}_{n-1}$ are nonnegative functions of $s_i, s_{i+1}, \ldots, s_{n-1}$. Let $\lambda_{i5} = 0, i \in \mathcal{N}_{n-1}$. Then, substituting (47) into (45) yields

$$\begin{aligned}\mathscr{L}V \le \sum_{i=1}^{n}\big(&-c_{ni}+\lambda_{i1}(s_i,s_{i+1},\ldots,s_{n-1})+\lambda_{i2}\\&+\lambda_{i3}(s_{i-1},s_i)+\lambda_{i4}(s_i,s_{i+1},\cdots,s_{n-1})+\lambda_{i5}\big)|\xi_i|^{r+1}\\&+\sum_{i=2}^{n}\big(-m_iH(s_{i-1})+\sigma_{i1}(s_i,s_{i+1},\ldots,s_{n-1})\\&+\sigma_{i2}+\sigma_{i3}(s_i,s_{i+1},\ldots,s_{n-1})\big)|e_i|^{r+1}.\end{aligned} \quad (48)$$

Now, by using inequality (48), we recursively select the dynamic gains $s_i, i \in \mathcal{N}_{n-1}$ such that $\mathscr{L}W < 0$ holds. First, for numbers $\mu_n > 0$ and $\rho_n > 0$ we select appropriate s_{n-1} such that the following condition is to be satisfied

$$\begin{cases}-c_{nn}+\lambda_{n1}+\lambda_{n2}+\lambda_{n3}(s_{n-1})+\lambda_{n4}+\lambda_{n5}\le-\mu_n,\\-m_nH(s_{n-1})+\sigma_{n1}+\sigma_{n2}+\sigma_{n3}\le-\rho_n.\end{cases}$$

Based on s_{n-1} fixed above, taking two real numbers $\mu_{n-1} > 0$ and $\rho_{n-1} > 0$ we select appropriate s_{n-2} such that

$$\begin{cases}-c_{n,n-1}+\lambda_{n-1,1}(s_{n-1})+\lambda_{n-1,2}+\lambda_{n-1,3}(s_{n-2},s_{n-1})\\\qquad+\lambda_{n-1,4}(s_{n-1})\le-\mu_{n-1},\\-m_{n-1}H(s_{n-2})+\sigma_{n-1,1}(s_{n-1})+\sigma_{n2}+\sigma_{n-1,3}(s_{n-1})\le-\rho_{n-1}.\end{cases}$$

Following this process until step $n-2$, we have fixed $n-2$ dynamic gains $s_{n-1}, s_{n-2}, \ldots, s_2$. Now, we take three real numbers $\mu_1 > 0$, $\mu_2 > 0$ and $\sigma_2 > 0$, and select appropriate s_1 such that the following three conditions hold

$$\begin{cases}-c_{n2}+\lambda_{21}(s_2,s_3\ldots,s_{n-1})+\lambda_{22}+\lambda_{23}(s_1,s_2)\\\qquad+\lambda_{24}(s_2,s_3\ldots,s_{n-1})\le-\mu_2,\\-m_2H(s_1)+\sigma_{21}(s_2,s_3,\ldots,s_{n-1})+\sigma_{22}\\\qquad+\sigma_{23}(s_2,s_3,\ldots,s_{n-1})\le-\rho_2,\\-c_{n1}+\lambda_{11}(s_1,s_2\ldots,s_{n-1})+\lambda_{12}+\lambda_{13}(s_1)\\\qquad+\lambda_{14}(s_1,s_2\ldots,s_{n-1})\le-\mu_1.\end{cases}$$

Based on the selection of the gains above, (48) becomes

$$\mathscr{L}V \le -\sum_{i=1}^{n}\mu_i|\xi_i|^{r+1}-\sum_{i=2}^{n}\rho_j|e_i|^{r+1}<0. \quad (49)$$

Let $W = \sum_{i=1}^{n}\mu_i|\xi_i|^{r+1}+\sum_{i=2}^{n}\rho_j|e_i|^{r+1}$. Clearly, W is nonnegative, continuous and positive definite. So, from formula (49) and Lemma 1 we can see that closed-loop system (1)–(44) is GSASP by selecting the dynamic gains appropriately.

Remark 6 For system (1), Sect. 3 provides a systemical design method of the output feedback controller and a recursive selection procedure of the dynamic observer gains. According to the proof of Theorem 1, the $n-1$ pending gains $s_i, i \in \mathcal{N}_{n-1}$ can be effectively selected, under which the stability of the resulting system will be achieved. It should be noted that the $4n-2$ adjustable parameters m_i, w_i, μ_i and σ_i can make the selection of the gains more flexible. Furthermore, for the sake of increasing the flexibility and convenience of the controller, the $2n+1$ adjustable parameters K, k_i and c_{ii} are introduced in the construction of LKF candidates $V_i, i \in \mathcal{N}_n$. From the construct process, we can see that a better performance of the proposed control strategy may be obtained by properly selecting them. The system variables and these parameters are described in Table 2. In addition, it follows from Sect. 3 that the information of the bounds a_i and b_i in Assumption 1 is only required rather than the functions $f_i(\cdot)$ and $g_i(\cdot), i \in \mathcal{N}_n$ in the design of our controller (44). This indicates that even if the functions $f_i(\cdot)$ and $g_i(\cdot)$ have some uncertainty, it will not affect the design process of the controller. That is to say, the proposed control scheme in our paper has a certain degree of robustness to uncertainties of the system.

Remark 7 When the system parameters are determined, the proposed control algorithm can be divided into three main steps. First, the estimators $\hat{x}_i(k)$ of unmeasurable system states are computed by the equation $\hat{x}_i(k) = \hat{z}_i(k)+s_{i-1}\hat{x}_{i-1}(k), i \in \mathcal{I}_n$. Second, by using the measured values $y(k)$ of the system output and the estimators $\hat{x}_i(k), i \in \mathcal{I}_n$, the control input $u(k)$ can be computed from (44). Third, according to Eq. (25), the values of the state observer $\hat{z}_i(k+1), i \in \mathcal{I}_n$, at time $k+1$ are calculated in regard to $u(k)$, $y(k)$ and $\hat{x}_i(k), i \in \mathcal{I}_n$. Meanwhile, the states of the original system at time $k+1$ are updated by using the control input $u(k)$. From the calculation process above, it can be seen that the computational cost of the proposed method in our paper is low. To illustrate this point, the function tic-toc in MATLAB is used to measure the computation time in simulation examples of Sect. 4.

Remark 8 In Sect. 3, we developed a recursive design method that yields an output feedback stabilizer for low-order SNTDSs. This stabilizer is comprised of a state feedback controller and a reduced-order observer. It should be noted that the present paper is first to provide a solution to the stabilization problem of low-order SNTDSs. Compared with the recent works in [43,44], where low-order stochastic nonlinear systems are con-

Table 2 List of system variables and adjustable parameters

System variables		Adjustable parameters	
x_i	System states	K	Affect the value of β_j
x_{id}	Delayed states	k_i	Affect the value of β_j
$\hat{x}_i$	Estimated states	c_{ii}	Affect the value of β_j
x_i^*	Virtual controllers	s_i	Affect the value of u
$\hat{z}_i$	Observer states	m_i	Affect selection of s_j
e_i	Error variables	w_i	Affect selection of s_j
y	System output	μ_i	Affect selection of s_j
u	Control input	σ_i	Affect selection of s_j

sidered and the corresponding stabilization problems in different scenarios have been solved, time-varying delays of the system states are taken into account in our work. This is one of main reasons for the difficulty in solving the problem. Although there are lots of research results on some special stochastic time-delay systems, such as [21–23] for strict-feedback SNTDSs, and [32–35] for high-order SNTDSs, it is still a difficult task to address the problem for low-order SNTDSs due to the intrinsic characteristics of such systems. For example, the backstepping-based design method commonly used in Refs. [21,22] cannot be applied to our work because low-order SNTDSs are not feedback linearizable. Meanwhile, the AAPI-based design method used in Refs. [33,35] also cannot be directly applied to our work because low-order SNTDS (1) is continuous but nondifferentiable. An appropriate LKF and a new reduced-order observer are developed to successfully overcome the obstacles in the present paper. In a sense, compared with the existing achievements mentioned above, the advantage of our work is to be able to solve the output feedback stabilization problem of low-order SNTDSs under some conditions.

4 Simulation example

In order to verify the effectiveness of the feedback control method developed in Sect. 3, we provide an illustrative example in this section.

Example 1 Consider a planar low-order SNTDS

$$\begin{cases} dx_1 = [x_2]^{\frac{3}{4}}dt + \frac{1}{16}\cos(x_1x_{1d})dt + \frac{1}{8}[x_1]^{\frac{7}{8}}dw, \\ dx_2 = [u]^{\frac{3}{4}}dt + \frac{1}{9}\sin(x_2)\cos(x_{2d})dt \\ \qquad +\frac{1}{10}\sin(x_1 + x_{2d})dw, \\ y = x_1, \end{cases} \tag{50}$$

where $x_i = x_i(t)$ are system states, $x_{id} = x_i\big(t - d_i\big)$ are delayed states, and $d_i = d_i(t)$ are time-varying delays of the system, $i = 1, 2$. Take $d_1(t) = 0.1(1 + \sin(t))$ and $d_2(t) = 0.2(1 - \cos(t))$. Clearly, system (50) satisfies Assumption 1 with $r = \frac{3}{4}$, $a_1 = \frac{1}{16}$, $b_1 = \frac{1}{8}$, $a_2 = \frac{1}{9}$, $b_2 = \frac{1}{10}$, and the conditions $\dot{d}_i(t) \leq v_i < 1$, $i = 1, 2$ with $v_1 = \frac{1}{10}$, $v_2 = \frac{1}{5}$. Assuming that only the output signal y can be measurable, we now apply the proposed control method to construct a stabilizer for system (50).

For the x_1-subsystem of (50), choose $V_1(x_1) = \frac{k_1}{2}x_1^2 + \frac{5}{72}\int_t^{t-d_1}|x_1(s)|^{\frac{7}{4}}ds$ with $k_1 > 0$. It follows that $x_2^* = -\beta_1 x_1$ with $\beta_1 = (\frac{c_{11}+l_{111}}{k_1})^{\frac{4}{3}}$ where $c_{11} > 0$ and $l_{111} = \frac{5}{75} + \frac{5k_1}{64} + \frac{k_1}{28}(\frac{7}{3k_1} - \frac{7}{12})^{-\frac{3}{4}}$. For whole system (50), let $\xi_2 = x_2 - x_2^*$, $\xi_{2d} = x_{2d} - x_{2d}^*$, and choose $V_2(x_1, x_2) = V_1(x_1) + \frac{k_2}{2}\xi_2^2 + K\int_t^{t-d_1}|\xi_1(s)|^{\frac{7}{4}}ds + \frac{5}{36}\int_t^{t-d_2}|\xi_2(s)|^{\frac{7}{4}}ds$ with $k_2 > 0$, $K > 0$. We construct a state feedback controller $u = u(x_1, x_2) = -\beta_2 x_2 - \beta_2\beta_1 x_1$ with $\beta_2 = (\frac{c_{22}+h_{21}+h_{22}+h_{23}}{k_2})^{\frac{4}{3}}$ where c_{22} is an arbitrary positive constant, $h_{21} = \frac{5}{36} + \frac{1}{25}2^{\frac{3}{4}}k_2$, $h_{22} = \frac{3}{7}2^{\frac{9}{12}}k_1$ and $h_{23} = \left(\frac{2}{63}(1+\beta_1^{\frac{3}{4}}) + \frac{1}{9} + \beta_1 + \frac{4}{7}\beta_1(1+\beta_1^{\frac{3}{4}})\right)k_2 + \frac{4}{63}(1+\beta_1^{\frac{3}{4}})$ $k_2\Big(\frac{21}{2k_2(1+\beta_1^{\frac{3}{4}})}\big(\frac{9}{10}K - (\frac{1}{25} + \frac{1}{5}\beta_1^2 + \frac{1}{25}2^{\frac{3}{4}}\beta_1^{\frac{7}{4}})k_2\big)\Big)^{-\frac{3}{4}} + \frac{4}{63}k_2\big(\frac{7}{3k_2} - \frac{21}{25}2^{\frac{3}{4}}\big)^{-\frac{3}{4}} + \frac{1}{28}k_1\beta_1\Big(\frac{56}{3k_2\beta_1}\big(\frac{9}{10}K - (\frac{1}{25} + \frac{1}{5}\beta_1^2 + \frac{1}{25}2^{\frac{3}{4}}\beta_1^{\frac{7}{4}})k_2\big)\Big)^{-\frac{3}{4}}$.

Next, we construct an observer for unmeasurable state x_2. The symbol $\hat{x}_2$ denotes the estimated value of x_2. Let $\hat{z}_2 = \hat{x}_2 - s_1 y$, where $s_1 > 1$ is the dynamic gain needing to be selected. Consider the following observer system

$$\dot{\hat{z}}_2 = [u]^{\frac{3}{4}} + s_1[\hat{z}_2 + s_1 x_1]^{\frac{3}{4}}. \tag{51}$$

By using the variable substitution $\hat{x}_2 = \hat{z}_2 + s_1 x_1$, dynamic equation (51) generates the estimated state $\hat{x}_2$. Furthermore, substituted $\hat{x}_2$ for x_2 in $u = u(x_1, x_2)$, we get the desired controller

$$u = u(y, \hat{x}_2) = -\beta_2\hat{x}_2 - \beta_2\beta_1 y. \tag{52}$$

The numerical simulation results were obtained according to the Euler-Maruyama technique [59] in the MATLAB environment (MATLAB R2020b). The initial values of the original system and the observer system are set to be $(x_1(0), x_2(0)) = (-0.6, 0.7)$ and $\hat{z}_2(0) = 0.8$, respectively. In order to illustrate the influence of the adjustable parameters K, k_1, k_2, c_{11} and c_{22} on controller (52), the detailed experimental data with the different parameter values are described in Table 3. It can be seen from the table that the values of β_1, β_2 and s_1 can be changed by adjusting these parameters, which means that better performance of the system may be achieved by selecting proper parameter values.

We choose the first set of parameter values for simulation display. Namely, take $K = 0.05$, $k_1 = 0.1$, $k_2 = 0.01$, $c_{11} = 0.2$ and $c_{22} = 0.1$. Based on the selection procedure provided in Sect. 3.3, the dynamic gain is selected as $s_1 = 20$. β_1 and β_2 can be calculated as 3.9 and 182.01, respectively. Figures 1, 2 and 3 demonstrate the simulation results. From the numerical results, it follows that system (50) is GSASP at $(x_1, x_2) = (0, 0)$ under proposed control scheme (51)–(52). By using the function tic-tok in MATLAB, the computation time is measured as 0.024 s.

Example 2 Consider the liquid-level system with interaction (see [36]) shown in Fig. 4. Suppose that tank 1 and tank 2 have the same capacitances of c. The liquid levels of two tanks are H_1 and H_2, respectively, and their steady-state liquid levels are all $\bar{H}$. Note that the difference of H_1 and H_2 at a certain moment will lead to the change of the flow rates Q_1 and Q_2, which can be described as

$$Q_1 = \begin{cases} p_1|H_2 - H_1|^{\frac{1}{2}}, & H_2 \le H_1 \\ -p_1|H_2 - H_1|^{\frac{1}{2}}, & H_2 > H_1 \end{cases} \tag{53}$$

and

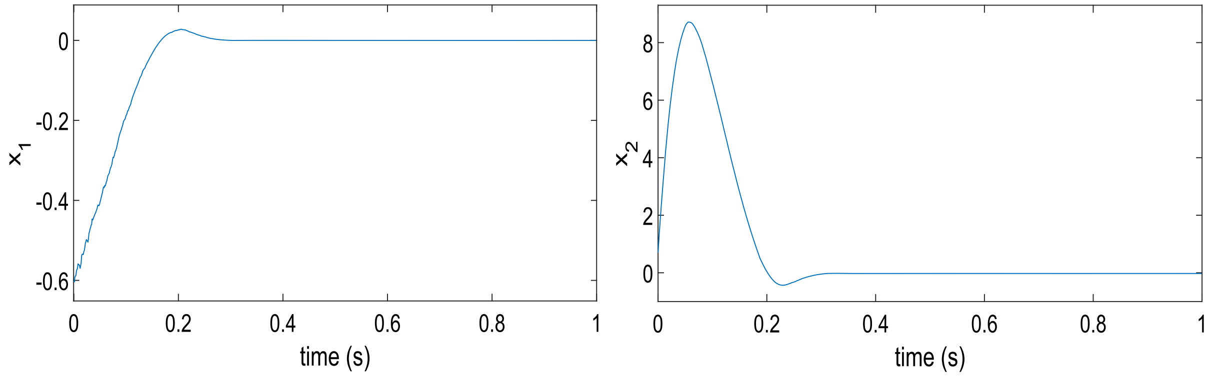
$$Q_2 = p_2 H_2^{\frac{1}{2}}, \tag{54}$$

where p_1 and p_2 are resistance coefficients of the corresponding valves. Assume that Q is inflow rate of the system. The goal is to adjust Q such that both H_1 and H_2 asymptotically converge to the steady state $\bar{H}$. Let $x_1 = H_1 - \bar{H}$, $x_2 = H_2 - H_1$ and $[u]^{\frac{1}{2}} = \frac{1}{c}Q - \frac{1}{c}p_2\bar{H}^{\frac{1}{2}}$. When considering random disturbances and multiple time-varying delays, the dynamic of the liquid levels of the tanks 1 and 2 can be described as

$$\begin{cases} dx_1 = \frac{1}{c}p_1[x_2]^{\frac{1}{2}}dt + \frac{1}{2}|x_{1d}|^{\frac{3}{4}}dw, \\ dx_2 = [u]^{\frac{1}{2}}dt + \varphi(\bar{x}_2, \bar{x}_{2d})dt + \frac{1}{2}|x_{2d}|^{\frac{3}{4}}dw, \\ y = x_1, \end{cases} \tag{55}$$

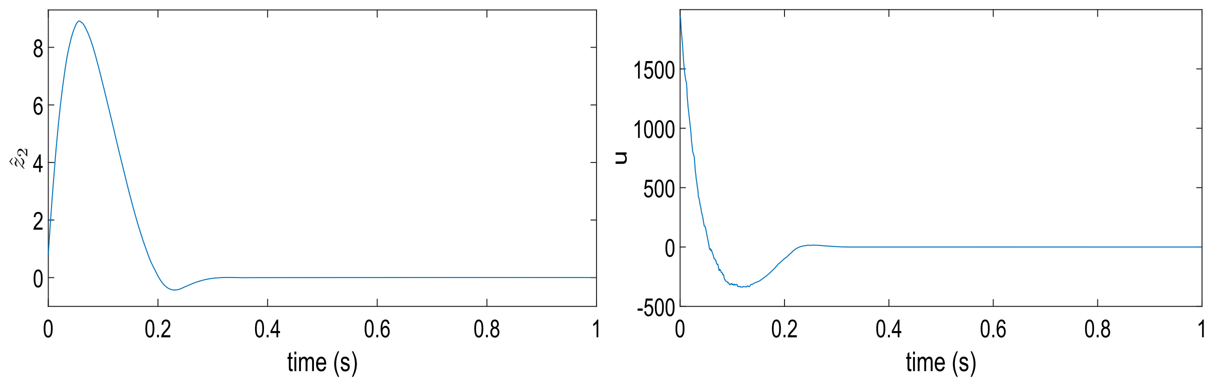
Table 3 Experimental data with the different parameter values

K	k_1	k_2	c_{11}	c_{22}	β_1	β_2	s_1
0.05	0.1	0.01	0.2	0.1	3.9	182.01	20
0.01	0.2	0.05	0.4	0.5	3.2	84.3	55
0.08	0.4	0.08	0.6	0.2	2.13	41.06	30
0.3	0.08	0.03	0.01	1	1.1	155.56	80



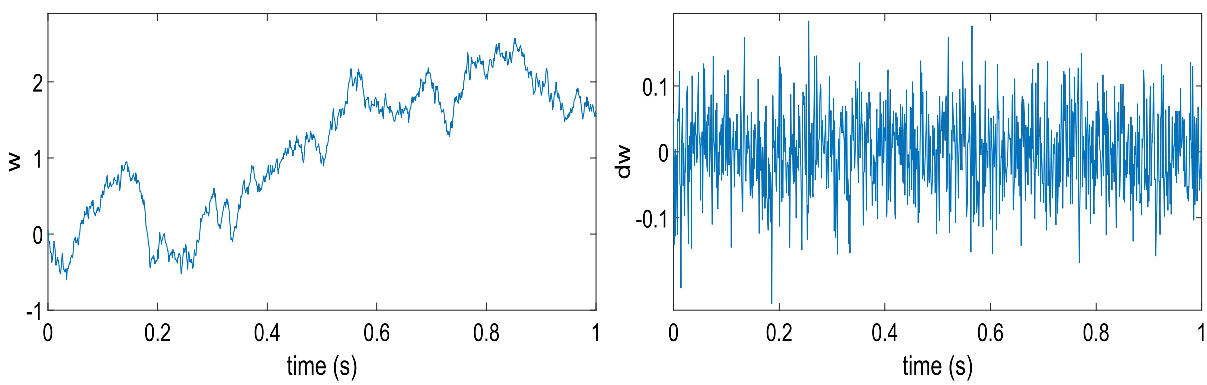
(a) System state x_1 (b) System state x_2

Fig. 1 The state trajectories of the closed-loop system



(a) Observer state $\hat{z}_2$ (b) Control input u

Fig. 2 The state trajectory of the observer system and the controller evolutions



(a) Wiener process (b) White Gaussian noise

Fig. 3 The Wiener process and the associated white Gaussian noise where $\varphi(\bar{x}_2, \bar{x}_{2d}) = -\frac{2}{c}p_1[x_2]^{\frac{1}{2}} - \frac{1}{c}p_2(x_{1d} + x_{2d} + \bar{H})^{\frac{1}{2}} + \frac{1}{c}p_2\bar{H}^{\frac{1}{2}}$ and $x_{1d} = x_1\big(t - d_1(t)\big), x_{2d} = x_2\big(t - d_2(t)\big), d_1(t) = 0.1cos^2(t), d_2(t) = 0.1sin^2(t) + 0.2$. It is clear that system (55) satisfies Assumption 1 with $r = \frac{1}{2}$, and the condition $\dot{d}_i(t) \le v_i < 1, i = 1, 2$ with $v_1 = \frac{1}{5}, v_2 = \frac{1}{5}$. When only the output signal y is measurable, the proposed approach in the previous sections can be used to solve the stabilization problem of the system.

A state feedback controller is designed firstly for system (55) by using the proposed approach, i.e.,

$$u = u(x_1, x_2) = -\beta_2 x_2 - \beta_1\beta_2 x_1, \tag{56}$$

where $\beta_1 = (\frac{c}{k_1 p_1})^{\frac{1}{r}}\big(c_{11} + k_1 b_1^2 + \frac{a_1}{1-v_1} + k_1 a_1 + \frac{k_1 a_1}{1+r}(\frac{(1+r)(a_1-k_1b_1^2)}{k_1 a_1 r})^{-r}\big)^{\frac{1}{r}}$ and $\beta_2 = k_2^{-\frac{1}{r}}(c_{22} + h_{21} + 2^{1-r}\frac{r\bar{k}_1}{1+r} + \frac{a_2}{1-v_2})^{\frac{1}{r}}$ with $h_{21} = \frac{k_2}{r+1}(a_2\beta_1^r + \frac{p_1}{c}\beta_1^{r+1}) + k_2(a_2 + \frac{p_1}{c}\beta_1) + \frac{1}{r+1}k_2 a_2(1 + \beta_1^r)(\frac{r+1}{r})^{-r}(\frac{K(1-v_1)-2^r k_2\beta_1^{r+1}-k_2\beta_1^2}{k_2 a_2(1+\beta_1^r)})^{-r} + \frac{k_2 a_2}{r+1}(\frac{(a_2-2^r k_2)(r+1)}{r k_2 a_2})^{-r}$. For the unmeasurable state x_2, the estimated value can be taken as $\hat{x}_2 = \hat{z}_2 + s_1 x_1$,, where $\hat{z}$ is generated by the dynamic

$$\dot{\hat{z}}_2 = [u]^{\frac{1}{2}} + \frac{1}{c}p_1 k_1 s_1[\hat{z}_2 + s_1 x_1]^{\frac{1}{2}}. \tag{57}$$

Using $\hat{x}_2$ in place of x_2 in (56), we get an output feedback stabilizer $u = u(y, \hat{x}_2) = -\beta_2\hat{x}_2 - \beta_1\beta_2 y$. In the simulation, the initial conditions are given as $x_1(0) = 0.3, x_2(0) - 0.4$ and $\hat{z}_2(0) = 0.6$. The dynamic gain is given as $s_1 = 28$. The other simulation parameters are given as $K = 0.5, k_1 = 0.6, k_2 = 0.08, c_{11} = 1.2, c_{22} = 1.7$. The simulation results are shown in Figs. 5, 6 and 7. It can be seen that system (53) is stabilized by the proposed output feedback control scheme.

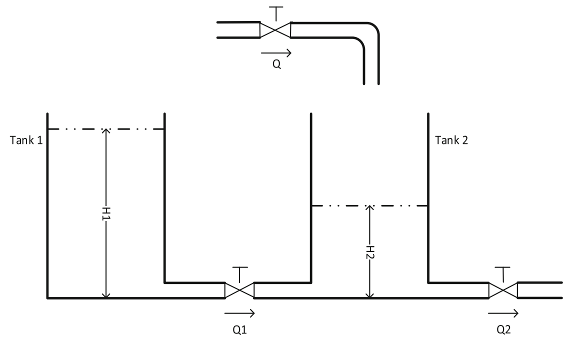


Fig. 4 The liquid-level system with interaction

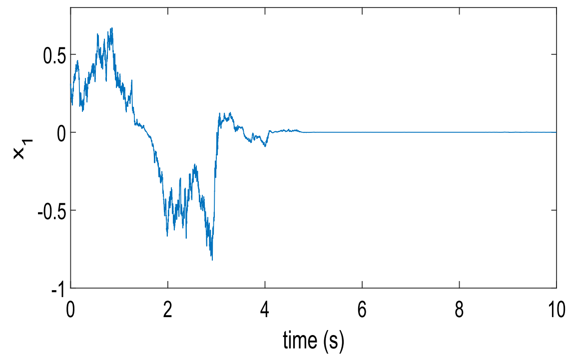


Fig. 5 The trajectory of system output y

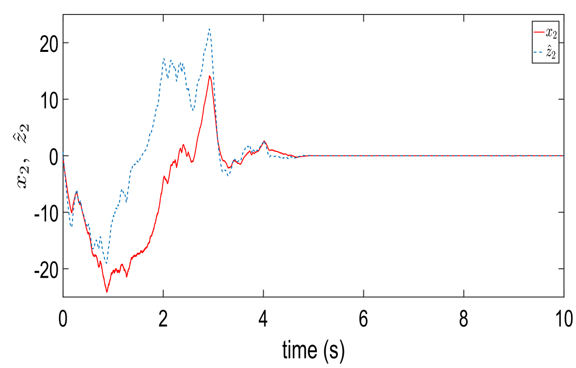


Fig. 6 The trajectories of x_2 and $\hat{z}_2$

5 Conclusions

In this technical paper, we dealt with the stabilization problem by using an output feedback approach for a class of low-order SNTDSs where the powers of integrators can be arbitrarily taken on the interval (0, 1). In order to overcome the difficulties arisen from the low-order nonlinearities, stochastic disturbers and multiple time-varying delays, we generalize the classical AAPI approach [27] and the observer design method introduced in [40] to construct an output feedback stabilizer. A novel LKF was skillfully chosen in the controller design, and the observer gains can be recursively selected to guarantee the GSASP of the resulting system at the trivial solution.

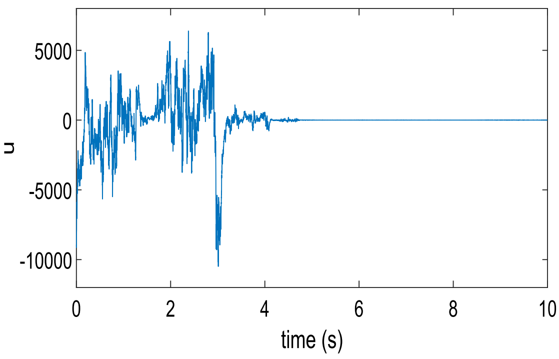


Fig. 7 The trajectory of control input u

A meaningful work for further study is to extend the results obtained in this paper to a more general class of low-order SNTDSs with the different powers of chained integrator. In fact, the authors have attempted to solve the problem. But there are still some technical difficulties that cannot be tackled. Moreover, another work that deserves attention in the future is how to deal with the sampled-data stabilization problem of low-order SNTDSs. Clearly, it is a more difficult problem because some information and properties of the system will be lost due to sampling. A new design scheme based on the method proposed in our paper needs to be developed to solve the problem.

Funding This work is supported by the National Natural Science Foundation of China (Grant Nos. 62063031, 12101454), the College Teachers Innovation Foundation of Gansu Province of China (Grant No. 2023B-132), the Research Project of Tianshui Normal University (Grant No. TDJ2023-02) and the Foundation Project of Chongqing Normal University (Grant No. 23XLB013).

Data availability The data that support the findings of this study are available from the corresponding author, Jinping Jia, upon reasonable request.

Declarations

Conflict of interest We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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