Nonlinear Dyn (2023) 111:18985–19001
https://doi.org/10.1007/s11071-023-08880-2

ORIGINAL PAPER

Novel methods of finite-time synchronization of fractional-order delayed memristor-based Cohen–Grossberg neural networks

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Received: 27 April 2022 / Accepted: 26 August 2023 / Published online: 20 September 2023

Abstract This paper aims to study the finite-time synchronization (FTS) of fractional-order delayed memristor-based Cohen–Grossberg neural networks (FODMCGNNs). Firstly, on the basis of the inequality on fractional-order derivative of the composite function, a novel fractional-order finite-time inequality is established; it extends the existing one and can be employed to discuss the FTS of fractional-order differential systems. More importantly, it is demonstrated theoretically that the estimated settling time by this inequality is more accurate than that with the existing one. Subsequently, on the basis of this novel inequality, the designed feedback controllers, and the fractional-order power law inequality, two novel criteria are obtained to ensure the FTS of FODMCGNNs. Finally, three examples are given to verify the correctness and advantage of the obtained results.

Keywords Finite-time synchronization · Fractional-order · Memristor-based Cohen–Grossberg neural network · Time delay

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1 Introduction

Over the past decades, neural networks (NNs) have attracted much attention on account of their potential applications in agriculture, medical image-based diagnosis, manipulator motion generation, fault diagnosis of mechanical intelligence, and so on. Cohen–Grossberg neural network (CGNN) was first proposed in 1983 [7], which includes numerous models such as evolutionary, population biology, and neurobiology theory. It can be seen as the generalization of some classical NNs such as cellular NNs, Hopfield NNs, and bidirectional associative memory NNs [3]. Hence, CGNNs have been extensively investigated by many researchers [17].

Fractional-order calculus has attracted more and more interest due to its wide application prospects in uncertain financial market, cancer treatment, blood ethanol concentration systems, and so on. Fractional-order calculus possesses more degrees of freedom and infinite memory in comparison with integer-order calculus. Hence, it can describe many systems better. Consequently, fractional-order calculus has been incorporated into CGNNs to form fractional-order Cohen–Grossberg neural networks (FOCGNNs). Recently, FOCGNNs have been widely explored and many valuable contributions on them have been made [11,26].

The memristor is a two-terminal electrical component relating magnetic flux and electric charge. It was first put forward by Chua [6] and realized by

HP labs [25]. The memristor has been extensively investigated for hardware implementation of NNs on account of the superiorities of functional similarity to the biological synapse, low power consumption, nanometer size, and fast switching speed [34]. Hence, the memristor is introduced to simulate the synapse in the FOCGNNs to form fractional-order memristor-based Cohen–Grossberg neural networks (FOMCGNNs). Recently, FOMCGNNs have been a hot topic and some remarkable works on FOMCGNNs have been reported [1,28].

Synchronization of FOMCGNNs has been paid much attention owing to its wide practical applications in image encryption, cryptography, HIV infection model, secure communications, and so on. Up to now, plenty of contributions have been made on synchronization, such as Mittag–Leffler synchronization, quasi-synchronization, asymptotic synchronization, and exponential synchronization, which are all concerned with synchronization error within the infinite-time interval. As reported in [29], most real systems only operate within a limited time interval. Hence, more and more efforts have been made to investigate the finite-time synchronization (FTS) of fractional-order systems and a great deal of valuable research achievements have been reported until now. In [2], the FTS of fractional-order systems was first investigated by using fractional nonsingular terminal sliding mode technique. In [9], the FTS of fractional-order real valued memristor-based neural networks was studied by the fractional-order Gronwall inequality. In [35], the FTS of complex-valued fractional-order neural networks was investigated by some developed fractional-order inequalities. In [31], the FTS of quaternion-valued fractional-order neural networks was investigated on the basis of some established fractional-order differential inequalities. Overall, the above discussed FTS can be divided into two categories. The first category of FTS means that the synchronization error tends to zero in a limited time. The second category of FTS means that the synchronization error does not exceed the given bounds within a limited time interval. The first category of FTS is investigated in this article.

Due to finite processing speeds and information transmission among the units, time delays are universally involved in neural networks, which may bring about some undesirable phenomenon such as chaos, oscillation, and even instability [8]. Therefore, it is necessary and significant to study the FTS of fractional-order delayed memristor-based Cohen–Grossberg neural networks (FODMCGNNs). Recently, some contributions have been made to the FTS of FODMCGNNs. In [36], on the basis of the fractional-order Gronwall inequality, the first kind of FTS of FODMCGNNs was investigated. In [16], the second kind of FTS of FODMCGNNs was investigated with the aid of inequality skills. However, the results in [16] are connected with a certain imperfection. Specifically, the FTS criteria were established on the basis of unreliable inequality and equality, respectively. The detailed discussion can be found in Remarks 7 and 8.

It is widely known that the integer-order finite-time inequality (IOFTI) is a main tool of FTS theory for ordinary differential systems [19], a very natural idea is to extend the IOFTI to the corresponding FO case to explore the FTS of fractional-order differential systems (FODSs). Nowadays, various kinds of IOFTIs have been generalized to the fractional-order finite-time inequalities (FOFTIs) to study the FTS of FODSs [32]. For example, the FOFTI ${}_{t_0}^{c}D_t^{\nu}V(t) \leq -b$ has been established to investigate the FTS of various FO complex networks [23]. The FOFTI ${}_{t_0}^{c}D_t^{\nu}V(t) \leq -\delta V(t) - \epsilon$ has been developed to study the FTS of various FO neural networks [21]. Recently, an attempt [15, Property 3] has been made to generalize the classical first-order inequality $V'(t) \leq -\delta V(t) - \epsilon V^{\vartheta}(t)$ with $0 < \vartheta < 1$ in [24] to the fractional-order case ${}_{t_0}^{c}D_t^{\nu}V(t) \leq -\delta V(t) - \epsilon V^{\vartheta}(t)$ with $0 < \vartheta < 1$. However, the FOFTI has additional features over the IOFTI [33]. The commonly used theoretical basis, which was used in the proof of [15, Property 3], ${}_{t_0}^{c}D_t^{\nu}h^{\alpha}(t) = \frac{\Gamma(2-\nu)\Gamma(1+\alpha)}{\Gamma(1+\alpha-\nu)} h^{\alpha-1}(t){}_{t_0}^{c}D_t^{\nu}h(t)$ may be not applicable. As a result, the FOFTI, which is based on this particular equality, appears to be uncertain and raises questions about its validity. In summary, the theory of FOFTI is still in its infancy and remains to be developed further. The main difficulty is that the existing widely used theoretical basis is difficult to establish the novel FOFTI. It is still a challenging task to develop the novel FOFTI by studying the FTS of FODMCGNNs and reducing the conservativeness of the results.

Motivated by the aforementioned discussions, the FTS of FODMCGNNs will be investigated by the novel FOFTI, the designed feedback controllers, and the fractional-order power law inequality. The key points for our contributions include:

(i) A novel fractional-order finite-time inequality is established; it extends the existing one and can be employed to investigate the FTS of FODSs.
(ii) With the help of the novel fractional-order finite-time inequality, the designed feedback controllers, and the fractional-order power law inequality, two novel criteria are obtained to ensure the FTS of FODMCGNNs.
(iii) It is demonstrated theoretically that the estimated settling time obtained by our developed novel inequality is more accurate than the existing one (see Remark 5).

The rest of this article is arranged as follows. Preliminaries on fractional-order calculus are provided in Sect. 2. Besides, the considered FODMCGNNs are also presented. In Sect. 3, a novel fractional-order finite-time inequality is established. Based on this developed inequality, the designed feedback controllers, and the fractional-order power law inequality, two novel FTS criteria of FODMCGNNs are obtained. In Sect. 4, three examples are provided to support the theoretical results. Conclusions are given in Sect. 5.

Notations: $\mathbb{N}_\imath = \{\imath, \imath+1, \imath+2, \ldots\}$, $\mathbb{N}_\imath^\jmath = \{\imath, \imath+1, \imath+2, \ldots, \jmath\}$, where $\jmath, \imath \in \mathbb{R}$ and $\jmath - \imath \in \mathbb{N}_1$. For $w = (w_1, w_2, \ldots, w_m)^{\mathrm{T}} \in \mathbb{R}^m$, $\|w\| = \sum_{i=1}^m |w_i|$. $\mathrm{sign}(w)$ is the sign function of $w \in \mathbb{R}$. $\mathbb{R}$, $\mathbb{R}^+$ are the set of real numbers, the set of nonnegative numbers, respectively. $\mathbb{R}^m$ is the m-dimensional Euclidean space. $\mathscr{C}^m([0, +\infty), \mathbb{R})$ represents a set of continuous m-order differentiable functions from $[0, +\infty)$ into $\mathbb{R}$.

2 Preliminaries and problem formulation

2.1 Preliminaries

Definition 1 [22] The Caputo derivative of the function $w \in \mathscr{C}^1([t_0, t], \mathbb{R})$ with κ-order is given by

$$ {}_{t_0}^{c}D_t^{\kappa} w(t) = \int_{t_0}^{t} \frac{(t-\theta)^{-\nu}}{\Gamma(1-\nu)} w'(\theta)\mathrm{d}\theta, $$

where $\kappa \in (0, 1)$, $\Gamma(\cdot)$ is the gamma function.

Lemma 1 [27] *Assume $w \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ and $\upsilon \in \mathscr{C}^1([t_0, +\infty), \mathbb{R})$. If w is convex on $\mathbb{R}$ and $\kappa \in (0, 1)$, then*

$$ {}_{t_0}^{c}D_t^{\kappa} w(\upsilon(t)) \le \frac{\mathrm{d}w}{\mathrm{d}\upsilon} {}_{t_0}^{c}D_t^{\kappa} \upsilon(t), \quad t \ge t_0. $$

Lemma 2 [21] *Suppose that $V \in \mathscr{C}^1([t_0, +\infty), \mathbb{R})$,*

$$ {}_{t_0}^{c}D_t^{\kappa} V(t) \le \lambda V(t) + \mu, \quad t \ge t_0, $$

where $0 < \kappa < 1$, $\lambda \ne 0$ and μ are constants. Then

$$ V(t) \le \Big(V(t_0) + \frac{\mu}{\lambda}\Big) E_\kappa(\lambda(t-t_0)^\kappa) - \frac{\mu}{\lambda}, \quad t \ge t_0, $$

where $E_\kappa(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i\kappa+1)}$ is the Mittag–Leffler function.

Lemma 3 [14] *For $a, b > 0$, $\xi, \eta > 1$, and $1/\xi + 1/\eta = 1$, one has*

$$ ab \le \frac{a^\xi}{\xi} + \frac{b^\eta}{\eta}. $$

Lemma 4 [32] (Fractional-order power law inequality) *Suppose that $\kappa \in (0, 1)$, $\widetilde{p} \ge 1$, and the function $f \in \mathscr{C}^1(\mathbb{R}, \mathbb{R})$. Then one has*

$$ {}_{t_0}^{c}D_t^{\kappa} f^{\widetilde{p}}(t) \le \widetilde{p}\,\mathrm{sign}(f(t))|f(t)|^{\widetilde{p}-1} {}_{t_0}^{c}D_t^{\kappa} f(t). $$

Lemma 5 [20] *If there exists a positive definite function $V \in \mathscr{C}^1([t_0, +\infty), \mathbb{R}^+)$ such that*

$$ {}_{t_0}^{c}D_t^{\kappa} V(t) \le -\mu, \ \ V(t) \in \mathbb{R}^+\backslash\{0\}, \tag{1} $$

where $0 < \kappa < 1$, $\mu > 0$, then one has $\lim_{t\to t^\star} V(t) = 0$ and $V(t) = 0$ for all $t \ge t^\star$, where

$$ t^\star \le \left[\frac{\Gamma(1+\kappa)V(t_0)}{\mu}\right]^{\frac{1}{\kappa}} + t_0. \tag{2} $$

Lemma 6 [12, Lemma 1] *Assume that $\delta > 0$, $\epsilon > 0$, $0 < \mu < 1$, and $\widetilde{U} \in \mathbb{R}^n$ is a neighborhood of the origin. If there exists a positive definite function $V \in \mathscr{C}^1(\widetilde{U}, \mathbb{R}^+)$ such that*

$$ V'(t) \le -\delta V(t) - \epsilon V^\mu(t), \ \ V(t) \in \mathbb{R}^+\backslash\{0\}, $$

then one has $\lim_{t\to t^*} V(t) = 0$ and $V(t) = 0$, $t \ge t^*$, where

$$ t^* \le t_0 + \frac{\ln\left(1 + \frac{\delta}{\epsilon} V^{1-\mu}(t_0)\right)}{\delta(1-\mu)}. $$

2.2 Problem formulation

Consider the following FODMCGNNs

$$\begin{cases} {}_{t_0}^{c}D_t^{\kappa} x_k(t) = -d_k(x_k(t))\Big\{c_k(x_k(t)) - \mathcal{J}_k \\ \qquad - \sum_{i=1}^{n} a_{ki}(x_i(t))h_i(x_i(t)) \\ \qquad - \sum_{i=1}^{n} b_{ki}(x_i(t-\rho))g_i(x_i(t-\rho))\Big\}, \\ x_k(t) = \psi_k(t), \ t \in [-\rho, 0], \end{cases} \tag{3}$$

and

$$\begin{cases} {}_{t_0}^{c}D_t^{\kappa} y_k(t) = -d_k(y_k(t))\Big\{c_k(y_k(t)) - \mathcal{J}_k \\ \qquad - \sum_{i=1}^{n} a_{ki}(y_i(t))h_i(y_i(t)) \\ \qquad - \sum_{i=1}^{n} b_{ki}(y_i(t-\rho))g_i(y_i(t-\rho))\Big\} \\ \qquad + u_k(t), \\ y_k(t) = \phi_k(t), \ t \in [-\rho, 0], \end{cases} \tag{4}$$

where $k \in \mathbb{N}_1^n$, the fractional-order $\kappa \in (0, 1)$; $d_k(\cdot)$ denotes the amplification function, $c_k(\cdot)$ is the well-behaved function, $x_k(\cdot)$, $y_k(\cdot)$ represent the states of the k-th neuron, $h_i(\cdot)$ and $g_i(\cdot)$ stand for the activation functions, $a_{ki}(\cdot)$ and $b_{ki}(\cdot)$ are the memristive synaptic connection weights, the constant $\rho > 0$ is the transmission delay, J_k represents constant external inputs.

On the basis of the feature of memristor [36], let

$$a_{ki}(x_i(t)) = \begin{cases} a_{ki}^{\diamond\diamond}, & |x_i(t)| > \mathcal{T}_k, \\ a_{ki}^{\diamond}, & |x_i(t)| \le \mathcal{T}_k, \end{cases}$$
$$b_{ki}(x_i(t-\rho)) = \begin{cases} b_{ki}^{\diamond\diamond}, & |x_i(t-\rho)| > \mathcal{T}_k, \\ b_{ki}^{\diamond}, & |x_i(t-\rho)| \le \mathcal{T}_k, \end{cases}$$

$$a_{ki}(y_i(t)) = \begin{cases} a_{ki}^{\diamond\diamond}, & |y_i(t)| > \mathcal{T}_k, \\ a_{ki}^{\diamond}, & |y_i(t)| \le \mathcal{T}_k, \end{cases}$$
$$b_{ki}(y_i(t-\rho)) = \begin{cases} b_{ki}^{\diamond\diamond}, & |y_i(t-\rho)| > \mathcal{T}_k, \\ b_{ki}^{\diamond}, & |y_i(t-\rho)| \le \mathcal{T}_k, \end{cases}$$

where $a_{ki}^{\diamond}$, $a_{ki}^{\diamond\diamond}$, $b_{ki}^{\diamond}$, $b_{ki}^{\diamond\diamond}$, $k, i \in \mathbb{N}_1^n$ are constants, the switching jumps $\mathcal{T}_k > 0$.

Since the FODMCGNNs (3) and (4) are discontinuous, the solution of the FODMCGNNs (3) and (4) cannot be defined via the classical solutions. To investigate the solutions of the FODMCGNNs (3) and (4), the solutions of the FODMCGNNs (3) and (4) have to be considered in Filippov's sense [13]. Hence, the FODMCGNNs (3) and (4) can be transformed into the their differential inclusion form via differential inclusions and the set-valued maps. Based on this, one defines the set-valued maps as follows:

$$K[a_{ki}(x_i(t))] = \begin{cases} a_{ki}^{\diamond\diamond}, & |x_i(t)| > \mathcal{T}_i, \\ \mathrm{co}\left\{a_{ki}^{\diamond}, a_{ki}^{\diamond\diamond}\right\}, & |x_i(t)| = \mathcal{T}_i, \\ a_{ki}^{\diamond}, & |x_i(t)| < \mathcal{T}_i, \end{cases}$$
$$K[b_{ki}(x_i(t-\rho))] = \begin{cases} b_{ki}^{\diamond\diamond}, & |x_i(t-\rho)| > \mathcal{T}_i, \\ \mathrm{co}\left\{b_{ki}^{\diamond}, b_{ki}^{\diamond\diamond}\right\}, & |x_i(t-\rho)| = \mathcal{T}_i, \\ b_{ki}^{\diamond}, & |x_i(t-\rho)| < \mathcal{T}_i, \end{cases}$$

$$K[a_{ki}(y_i(t))] = \begin{cases} a_{ki}^{\diamond\diamond}, & |y_i(t)| > \mathcal{T}_i, \\ \mathrm{co}\left\{a_{ki}^{\diamond}, a_{ki}^{\diamond\diamond}\right\}, & |y_i(t)| = \mathcal{T}_i, \\ a_{ki}^{\diamond}, & |y_i(t)| < \mathcal{T}_i, \end{cases}$$
$$K[b_{ki}(y_i(t-\rho))] = \begin{cases} b_{ki}^{\diamond\diamond}, & |y_i(t-\rho)| > \mathcal{T}_i, \\ \mathrm{co}\left\{b_{ki}^{\diamond}, b_{ki}^{\diamond\diamond}\right\}, & |y_i(t-\rho)| = \mathcal{T}_i, \\ b_{ki}^{\diamond}, & |y_i(t-\rho)| < \mathcal{T}_i, \end{cases}$$

where the convex closure of a set is denoted as co; $K[a_{lk}(x_k(t))]$, $K[b_{lk}(x_k(t-\hbar))]$, $K[a_{lk}(y_k(t))]$, and $K[b_{lk}(y_k(t-\hbar))]$ are all closed, convex and compact about $x_k(t)$, $x_k(t-\hbar)$, $y_k(t)$, and $y_k(t-\hbar)$.

On the basis of the theory of differential inclusions and the set-valued maps, the FODMCGNNs (3) and (4) can be rewritten as

$$\begin{cases} {}_{t_0}^{c}D_t^{\kappa} x_k(t) \in -d_k(x_k(t))\Big\{c_k(x_k(t)) - \mathcal{J}_k \\ \qquad - \sum_{i=1}^{n} K[a_{ki}(x_i(t))]h_i(x_i(t)) \\ \qquad - \sum_{i=1}^{n} K[b_{ki}(x_i(t-\rho))]g_i(x_i(t-\rho))\Big\}, \\ x_k(t) = \psi_k(t), \ t \in [-\rho, 0] \end{cases} \tag{5}$$

and

$$\begin{cases} {}_{t_0}^{c}D_t^{\kappa} y_k(t) \in -d_k(y_k(t))\Big\{c_k(y_k(t)) - \mathcal{J}_k \\ \qquad - \sum_{i=1}^{n} K[a_{ki}(y_i(t))]h_i(y_i(t)) \\ \qquad - \sum_{i=1}^{n} K[b_{ki}(y_i(t-\rho))]g_i(y_i(t-\rho))\Big\} \\ \qquad + u_k(t), \\ y_k(t) = \phi_k(t), \quad t \in [-\rho, 0], \end{cases} \tag{6}$$

respectively.

According to differential inclusions theory and the set-valued map, there exist $\widetilde{a}_{ki}(x_i(t)) \in K[a_{ki}(x_i(t))]$, $\widetilde{b}_{ki}(x_i(t-\rho)) \in K[b_{ki}(x_i(t-\rho))]$, $\widehat{a}_{ki}(y_i(t)) \in K[a_{ki}(y_i(t))]$ and $\widehat{b}_{ki}(y_i(t-\rho)) \in K[b_{ki}(y_i(t-\rho))]$ such that the FODMCGNNs (3) and (4) can be rewritten as

$$\begin{cases} {}_{t_0}^{c}D_t^{\kappa} x_k(t) = -d_k(x_k(t))\Big\{c_k(x_k(t)) - \mathcal{J}_k \\ \qquad - \sum_{i=1}^{n} \widetilde{a}_{ki}(x_i(t))h_i(x_i(t)) \\ \qquad - \sum_{i=1}^{n} \widetilde{b}_{ki}(x_i(t-\rho))g_i(x_i(t-\rho))\Big\}, \\ x_k(t) = \psi_k(t), \ t \in [-\rho, 0] \end{cases} \tag{7}$$

and

$$\begin{cases} {}_{t_0}^{c}D_t^{\kappa} y_k(t) = -d_k(y_k(t))\Big\{c_k(y_k(t)) - \mathcal{J}_k \\ \qquad - \sum_{i=1}^{n} \widehat{a}_{ki}(y_i(t))h_i(y_i(t)) \\ \qquad - \sum_{i=1}^{n} \widehat{b}_{ki}(y_i(t-\rho))g_i(y_i(t-\rho))\Big\} \\ \qquad + u_k(t), \\ y_k(t) = \phi_k(t), \ t \in [-\rho, 0]. \end{cases} \tag{8}$$

Let $z_k(t) = y_k(t) - x_k(t)$. Then the synchronization error system is

$$\begin{cases} {}_{t_0}^{c}D_t^{\kappa} z_k(t) = -d_k(y_k(t))\Big\{c_k(y_k(t)) - \mathcal{J}_k \\ \qquad - \sum_{i=1}^{n} \widehat{a}_{ki}(y_i(t))h_i(y_i(t)) \\ \qquad - \sum_{i=1}^{n} \widehat{b}_{ki}(y_i(t-\rho))g_i(y_i(t-\rho))\Big\} \\ \qquad + u_k(t) \\ \qquad + d_k(x_k(t))\Big\{c_k(x_k(t)) - \mathcal{J}_k \\ \qquad - \sum_{i=1}^{n} \widetilde{a}_{ki}(x_i(t))h_i(x_i(t)) \\ \qquad - \sum_{i=1}^{n} \widetilde{b}_{ki}(x_i(t-\rho))g_i(x_i(t-\rho))\Big\}, \\ z_k(t) = \varphi_k(t), \ t \in [-\rho, 0]. \end{cases} \tag{9}$$

Definition 2 FODMCGNNs (3) and (4) are finite-time synchronized if there exists a constant $\tilde{t} > 0$ satisfying

$$\lim_{t \to \tilde{t}} z_k(t) = 0 \text{ and } z_k(t) = 0, \quad t > \tilde{t}, \ k \in \mathbb{N}_1^n.$$

In addition, $t^* = \inf\{t | z_k(t) = 0, \ t \geq t_0\}$ is the settling time.

In order to realize the FTS between the FODMCGNNs (3) and (4), the following assumptions are given.

Assumption 1 [20] There exist positive constants $\underline{d}_k$, $\bar{d}_k$, $\widetilde{d}_k$ such that

$$0 < \underline{d}_k < d_k(x(t)) \leq \bar{d}_k < \infty$$

and

$$|d_k(y(t)) - d_k(x(t))| \leq \widetilde{d}_k |y(t) - x(t)|$$

for $x(t), y(t) \in \mathbb{R}$, $k \in \mathbb{N}_1^n$.

Assumption 2 [20] There exists a constant $\zeta_k > 0$ such that

$$|d_k(y(t))c_k(y(t)) - d_k(x(t))c_k(x(t))| \leq \zeta_k |y(t) - x(t)|$$

for $x(t), y(t) \in \mathbb{R}$, $k \in \mathbb{N}_1^n$.

Assumption 3 [20] There exist positive constants $\bar{h}_k$, $\bar{g}_k$, H_k, G_k satisfying

$$|h_k(x(t))| \le \bar{h}_k,$$
$$|g_k(x(t))| \le \bar{g}_k,$$
$$|h_k(y(t)) - h_k(x(t))| \le H_k|y(t) - x(t)|,$$
$$|g_k(y(t)) - g_k(x(t))| \le G_k|y(t) - x(t)|$$

for $x(t), y(t) \in \mathbb{R}$, $k \in \mathbb{N}_1^n$.

Lemma 7 [10] *Let* $\breve{a}_{ki} = \max\{|a_{ki}^{\diamond}|, |a_{ki}^{\diamond\diamond}|\}$. $\breve{b}_{ki} = \max\{|b_{ki}^{\diamond}|, |b_{ki}^{\diamond\diamond}|\}$. *Then one has*

$$\begin{aligned}&|d_k(y_k(t))\widehat{a}_{ki}(y_i(t)) - d_k(x_k(t))\widetilde{a}_{ki}(x_i(t))| \\ &\quad \le \breve{a}_{ki}\widetilde{d}_k|x_k(t) - y_k(t)| + 2\breve{a}_{ki}\overline{d}_k, \qquad (10)\\ &|d_k(y_k(t))\widehat{b}_{ki}(y_i(t-\rho)) - d_k(x_k(t))\widetilde{b}_{ki}(x_i(t-\rho))| \\ &\quad \le \breve{b}_{ki}\widetilde{d}_k|x_k(t) - y_k(t)| + 2\breve{b}_{ki}\overline{d}_k. \qquad (11)\end{aligned}$$

for $x_k(t), x_i(t), x_i(t-\rho), y_k(t), y_i(t), y_i(t-\rho) \in \mathbb{R}$, $k, i \in \mathbb{N}_1^n$.

The objective of this paper is to establish two novel FTS criteria between FODMCGNN (3) and FODMCGNN (4) under the designed feedback controllers.

3 Main results

3.1 Novel fractional-order finite-time inequality

Remark 1 In [15, Property 3], to discuss the FTS of memristive neural networks with the fractional-order $0 < \nu < 1$, the inequality

$$ {}_{t_0}^{c}D_t^{\nu}V(t) \le -\delta V(t) - \epsilon V^{\vartheta}(t), 0 < \vartheta < 1, \delta, \epsilon > 0 \qquad (12)$$

was established on the basis of the commonly used equality

$$ {}_{t_0}^{c}D_t^{\nu}h^{\alpha}(t) = \frac{\Gamma(2-\nu)\Gamma(1+\alpha)}{\Gamma(1+\alpha-\nu)}h^{\alpha-1}(t){}_{t_0}^{c}D_t^{\nu}h(t), \qquad (13)$$

where $0 < \nu < 1$, $\alpha \ge 1$. In fact, the equality (13) may be not applicable. For example, let $h(t) = t$, $\alpha = 2$, $t = 1$, $\nu = 0.6$, $t_0 = 0$. Then one has

$$\begin{aligned}{}_{t_0}^{c}D_t^{\nu}h^{\alpha}(t) &= {}_{0}^{c}D_t^{0.6}t^2|_{t=1}\\ &= \int_0^1 \frac{(1-s)^{-0.6}}{\Gamma(0.4)}2s\mathrm{d}s\\ &= 1.6100 \qquad (14)\end{aligned}$$

and

$$\begin{aligned}&\frac{\Gamma(2-\nu)\Gamma(1+\alpha)}{\Gamma(1+\alpha-\nu)}h^{\alpha-1}(t){}_{t_0}^{c}D_t^{\nu}h(t)\\ &= \frac{\Gamma(1.4)\Gamma(3)}{\Gamma(2.4)}{}_{0}^{c}D_t^{0.6}t|_{t=1}\\ &= \frac{\Gamma(1.4)\Gamma(3)}{\Gamma(2.4)}\int_0^1 \frac{(1-s)^{-0.6}}{\Gamma(0.4)}\mathrm{d}s\\ &= 0.8050. \qquad (15)\end{aligned}$$

It can be seen from (14) and (15) that the equality (13) may be not feasible. Consequently, the inequality (12) seems questionable to be used to investigate the FTS of FODMCGNNs.

Now that the equality (12) seems questionable, it is interesting and important to establish a novel one. With the help of the inequality on fractional-order derivative of composite function (Lemma 1), a novel fractional-order finite-time inequality can be derived as follows.

Lemma 8 *If there exists a positive definite function* $V \in \mathscr{C}^1([t_0, +\infty), \mathbb{R}^+)$ *such that*

$$ {}_{t_0}^{c}D_t^{\nu}V(t) \le -\delta V(t) - \epsilon V^{-\vartheta}(t), \ \ V(t) \in \mathbb{R}^+\backslash\{0\}, \qquad (16)$$

where $0 < \nu < 1$, $\delta \neq 0$, $\epsilon > 0$, $\epsilon + \delta V^{1+\vartheta}(t_0) > 0$, *and* $\vartheta \ge 0$, *then one has* $\lim_{t\to t^*} V(t) = 0$ *and* $V(t) = 0$, $t \ge t^*$, *where*

$$t^* \le T_1(\vartheta) = t_0 + \left(\frac{-T_1^*}{\delta(1+\vartheta)}\right)^{\frac{1}{\nu}}, \qquad (17)$$

T_1^* *is the unique root of the equation*

$$E_{\nu}(z) = \frac{\epsilon}{\epsilon + \delta V^{1+\vartheta}(t_0)}. \qquad (18)$$

Proof Let $\hbar(\upsilon) = \upsilon^{1+\vartheta}$ and $\upsilon(t) = V(t)$. It follows from $\vartheta \ge 0$ that $\hbar(\upsilon)$ is convex on $\mathbb{R}$. Therefore, from Lemma 1 one has

$$ {}_{t_0}^{c}D_t^{\nu}V^{1+\vartheta}(t) \le (1+\vartheta)V^{\vartheta}(t){}_{t_0}^{c}D_t^{\nu}V(t)$$

$$\leq (1+\vartheta)V^{\vartheta}(t)[-\delta V(t)-\epsilon V^{-\vartheta}(t)]$$
$$=-\delta(1+\vartheta)V^{1+\vartheta}(t)-\epsilon(1+\vartheta).$$

With the help of Lemma 2, one has for $t \geq t_0$

$$V^{1+\vartheta}(t) \leq \left(\frac{\epsilon}{\delta}+V^{1+\vartheta}(t_0)\right)E_\nu\big(-\delta(1+\vartheta)(t-t_0)^\nu\big)-\frac{\epsilon}{\delta},$$

i.e., for $t \geq t_0$

$$V(t) \leq \left[\left(\frac{\epsilon}{\delta}+V^{1+\vartheta}(t_0)\right)E_\nu\big(-\delta(1+\vartheta)(t-t_0)^\nu\big)-\frac{\epsilon}{\delta}\right]^{\frac{1}{1+\vartheta}}. \tag{19}$$

Let

$$\Psi(t) = \left[\left(\frac{\epsilon}{\delta}+V^{1+\vartheta}(t_0)\right)E_\nu\left(-\delta(1+\vartheta)(t-t_0)^\nu\right)-\frac{\epsilon}{\delta}\right]^{\frac{1}{1+\vartheta}}.$$

For $\delta>0$, one has that $E_\nu\big(-\delta(1+\vartheta)(t-t_0)^\nu\big)$ is a monotonically decreasing function. In addition, it follows from $\epsilon+\delta V^{1+\vartheta}(t_0)>0$ that $\frac{\epsilon}{\delta}+V^{1+\vartheta}(t_0)>0$. Then $\Psi(t)$ is a monotonically decreasing function.

For $\delta<0$, one has that $E_\nu\big(-\delta(1+\vartheta)(t-t_0)^\nu\big)$ is a monotonically increasing function. In addition, it follows from $\epsilon+\delta V^{1+\vartheta}(t_0)>0$ that $\frac{\epsilon}{\delta}+V^{1+\vartheta}(t_0)<0$. Then $\Psi(t)$ is a monotonically decreasing function.

Therefore, $\Psi(t)$ is monotonically decreasing function for any $\delta \neq 0$. Through verification, one has $\Psi(T_1)=0$. It follows from the inequality (19) that

$$0 \leq \lim_{t\to T_1} V(t)=V(T_1)\leq \Psi(T_1)=0.$$

Furthermore, from squeeze theorem in [4, Theorem 4.2.7], one has

$$\lim_{t\to T_1} V(t)=0,$$

where

$$T_1=t_0+\left(\frac{-T_1^*}{\delta(1+\vartheta)}\right)^{\frac{1}{\nu}},$$

T_1^* is the unique root of the equation

$$E_\nu(z)=\frac{\epsilon}{\epsilon+\delta V^{1+\vartheta}(t_0)}. \tag{20}$$

If there exists $\tilde{t} \geq T_1$ such that $V(\tilde{t})>0$, then from the inequality (19) and the fact that $\Psi(t)$ is monotonically decreasing function, one has

$$0<V(\tilde{t})\leq \Psi(\tilde{t})\leq \Psi(T_1)=0,$$

which is a contradiction.

Thus, one has

$$V(t)=0,\ \text{for } t\geq T_1.$$

The proof of Lemma 8 is completed. □

Remark 2 If $\delta>0$, then from (20) one has $0<E_\nu(z)<1$. Furthermore, from [30, Lemma 3] one has $T_1^*<0$, which leads to $\frac{-T_1^*}{\delta(1+\vartheta)}>0$. If $\delta<0$, then from (20) one has $E_\nu(z)>1$. Furthermore, from [30, Lemma 3] one has $T_1^*>0$, which also leads to $\frac{-T_1^*}{\delta(1+\vartheta)}>0$. All in all, for these two cases one both has $T_1>t_0$.

When $\vartheta=0$ in Lemma 8, one has the following Corollary 1.

Corollary 1 *If there exists a positive definite function* $V\in\mathscr{C}^1([t_0,+\infty),\mathbb{R}^+)$ *such that*

$${}_{t_0}^{c}D_t^\nu V(t)\leq -\delta V(t)-\epsilon,\ \ V(t)\in\mathbb{R}^+\backslash\{0\}, \tag{21}$$

where $0<\nu<1$, $\delta>0$, $\epsilon>0$, *and* $\epsilon+\delta V(t_0)>0$, *then one has* $\lim_{t\to t^*}V(t)=0$ *and* $V(t)=0$, $t\geq t^*$, *where*

$$t^* \leq T_2=t_0+\left(\frac{-T_2^*}{\delta}\right)^{\frac{1}{\nu}}, \tag{22}$$

T_2^* *is the unique root of the equation*

$$E_\nu(z)=\frac{\epsilon}{\epsilon+\delta V(t_0)}. \tag{23}$$

Remark 3 Evidently, the estimation (22) in Corollary 1 is consistent with the result given in [31, Lemma 5]. Therefore, result in Lemma 8 generalizes the one in the existing literature.

Remark 4 When $0<V(t_0)\leq 1$, $\epsilon>0$, and $\delta>0$, one has that $\frac{\epsilon}{\epsilon+\delta V^{1+\vartheta}(t_0)}$ is increasing with respect to ϑ. On the other hand, $E_\nu(\cdot)$ is a monotonically increasing function, so one also has that T_1^* is monotonically increasing with respect to $E_\nu(T_1^*)(=\frac{\epsilon}{\epsilon+\delta V^{1+\vartheta}(t_0)})$. Therefore, when $0<V(t_0)\leq 1$, $\epsilon>0$, and $\delta>0$, T_1^* is monotonically increasing with respect to ϑ.

Remark 5 From Remark 2 one has $T_1^* < 0$ when $\delta > 0$, which combined with the conclusion of Remark 4, one has that for $0 < V(t_0) \leq 1$, $\epsilon > 0$, and $\delta > 0$, the estimated settling time $T_1(\vartheta) = t_0 + \left(\frac{-T_1^*}{\delta(1+\vartheta)}\right)^{\frac{1}{\nu}}$ is monotonically decreasing with respect to ϑ. Furthermore, since $\vartheta \geq 0$, one has $T_1(\vartheta) \leq T_1(0)$. That is to say, when $0 < V(t_0) \leq 1$, $\epsilon > 0$, and $\delta > 0$, the estimated settling time $T_1(\vartheta)$ given by the inequality (16) is more accurate than the one $T_1(0) = T_2$ given by Corollary 1 [31, Lemma 5].

When $\nu = 1$ in Lemma 8, one has the following Corollary 2.

Corollary 2 *If there exists a positive definite function $V \in \mathscr{C}^1([t_0, +\infty), \mathbb{R}^+)$ such that*

$$V'(t) \leq -\delta V(t) - \epsilon V^{-\vartheta}(t), \;\; V(t) \in \mathbb{R}^+ \backslash \{0\},$$

where $\delta \neq 0$, $\epsilon > 0$, $\epsilon + \delta V^{1+\vartheta}(t_0) > 0$, and $\vartheta \geq 0$, then one has $\lim_{t \to t^*} V(t) = 0$ and $V(t) = 0$, $t \geq t^*$, where

$$t^* \leq t_0 + \frac{\ln\left(1 + \frac{\delta}{\epsilon} V^{1+\vartheta}(t_0)\right)}{\delta(1+\vartheta)}.$$

Remark 6 From the proof of Lemma 8 one has that the result in Lemma 8 is also valid for $\delta < 0$ and $\epsilon + \delta V^{1-\mu}(t_0) > 0$. Therefore, combing Lemma 6 and Corollary 2, one has the following Corollary 3.

Corollary 3 *If there exists a positive definite function $V \in \mathscr{C}^1(\widetilde{U}, \mathbb{R}^+)$ such that*

$$V'(t) \leq -\delta V(t) - \epsilon V^{\vartheta}(t), \;\; V(t) \in \mathbb{R}^+ \backslash \{0\}, \tag{24}$$

where $\widetilde{U} \in \mathbb{R}^n$ is a neighborhood of the origin, $\delta \neq 0$, $\epsilon > 0$, $\epsilon + \delta V^{1+\vartheta}(t_0) > 0$, and $\vartheta < 1$, then one has $\lim_{t \to t^*} V(t) = 0$ and $V(t) = 0$, $t \geq t^*$, where

$$t^* \leq t_0 + \frac{\ln\left(1 + \frac{\delta}{\epsilon} V^{1-\vartheta}(t_0)\right)}{\delta(1-\vartheta)}.$$

3.2 FTS of FODMCGNNs

In this subsection, two novel FTS criteria of FODMCGNN are obtained by the novel fractional-order finite-time inequality, the designed feedback controllers, and the fractional-order power law inequality.

In order to achieve the FTS between the FODMCGNN (3) and FODMCGNN (4), the following feedback controller is designed,

$$u_k(t) = \begin{cases} -\gamma \dfrac{z_k(t)}{|z_k(t)|^{\widetilde{p}}} |z_k(t-\rho)|^{\widetilde{p}} - \ell z_k(t) - \eta \dfrac{z_k(t)}{|z_k(t)|} \\ \quad - \chi \dfrac{z_k(t)}{|z_k(t)|^{\varrho}}, \quad |z_k(t)| \neq 0, \\ 0, \;\; |z_k(t)| = 0, \end{cases} \tag{25}$$

where $k \in \mathbb{N}_1^n$, $\varrho \geq \widetilde{p} \geq 1$, γ, ℓ, χ, η are positive constants.

Theorem 1 *Under the feedback controller (25) and Assumptions 1–3, the FODMCGNNs (3) and (4) can be finite-time synchronized if*

$$\widetilde{p} n \chi + \varsigma \widehat{V}_{\widetilde{p}}^{\varrho/\widetilde{p}}(t_0) > 0, \tag{26}$$

where

$$\begin{aligned} \widehat{V}_{\widetilde{p}}(t) &= \sum_{k=1}^{n} |z_k(t)|^{\widetilde{p}}, \;\; \widetilde{p} \geq 1, \\ \widetilde{p}\gamma &= \max_{1 \leq k \leq n} \left\{ \sum_{i=1}^{n} \left(\bar{d}_i G_k \breve{b}_{ik} + \overline{g}_i \breve{b}_{ki} \widetilde{d}_k \right) \right\}, \\ \widetilde{p}\eta &= \max_{1 \leq k \leq n} \left\{ \sum_{i=1}^{n} \left(\widetilde{p} \overline{h}_i 2 \breve{a}_{ki} \bar{d}_k + \widetilde{p} \overline{g}_i 2 \breve{b}_{ki} \bar{d}_k \right) \right\}, \\ \Xi_k &= (\zeta_k + \widetilde{d}_k |\mathcal{J}_k|) \widetilde{p} + \sum_{i=1}^{n} \Big(\bar{d}_k H_i \breve{a}_{ki} (\widetilde{p} - 1) + \bar{d}_i H_k \breve{a}_{ik} \\ &\quad + \widetilde{p} \overline{h}_i \breve{a}_{ki} \widetilde{d}_k + \bar{d}_k G_i \breve{b}_{ki} (\widetilde{p} - 1) + \overline{g}_i \breve{b}_{ki} \widetilde{d}_k (\widetilde{p} - 1) \Big), \end{aligned}$$

and

$$\varsigma = -\max_{1 \leq k \leq n} \{\Xi_k\} + \widetilde{p}\ell.$$

In addition, the settling time t^* is evaluated as

$$t^* \leq t_1^* = t_0 + \left(\frac{-T_1^*}{\varsigma \varrho / \widetilde{p}} \right)^{\frac{1}{\kappa}},$$

where T_1^* is the unique root of the equation

$$E_\kappa(z) = \frac{\widetilde{p} n \chi}{\widetilde{p} n \chi + \varsigma \widehat{V}_{\widetilde{p}}^{\varrho/\widetilde{p}}(t_0)}.$$

Proof Construct the Lyapunov function

$$\widehat{V}_{\widetilde{p}}(t)=\sum_{k=1}^{n}|z_k(t)|^{\widetilde{p}},\ \ \widetilde{p}\geq 1.$$

Then from Lemma 4 one has

$$\begin{aligned}
&{}_{t_0}^{c}D_t^{\kappa}\widehat{V}_{\widetilde{p}}(t)\\
&\quad\leq\sum_{k=1}^{n}\widetilde{p}\operatorname{sign}(z_k(t))|z_k(t)|^{\widetilde{p}-1}{}_{t_0}^{c}D_t^{\kappa}z_k(t)\\
&\quad=\sum_{k=1}^{n}\widetilde{p}\operatorname{sign}(z_k(t))|z_k(t)|^{\widetilde{p}-1}\Bigg\{\\
&\qquad-[d_k(y_k(t))c_k(y_k(t))-d_k(x_k(t))c_k(x_k(t))]\\
&\qquad-\gamma\frac{z_k(t)}{|z_k(t)|}|z_k(t-\rho)|\\
&\qquad-\ell z_k(t)-\eta\frac{z_k(t)}{|z_k(t)|}-\chi\frac{z_k(t)}{|z_k(t)|^{\varrho}}\\
&\qquad+[d_k(y_k(t))-d_k(x_k(t))]J_k\\
&\qquad+\sum_{i=1}^{n}[d_k(y_k(t))\widehat{a}_{ki}(y_i(t))h_i(y_i(t))\\
&\qquad-d_k(y_k(t))\widehat{a}_{ki}(y_i(t))h_i(x_i(t))]\\
&\qquad+\sum_{i=1}^{n}[d_k(y_k(t))\widehat{a}_{ki}(y_i(t))h_i(x_i(t))\\
&\qquad-d_k(x_k(t))\widetilde{a}_{ki}(x_i(t))h_i(x_i(t))]\\
&\qquad+\sum_{i=1}^{n}[d_k(y_k(t))\widehat{b}_{ki}(y_i(t-\rho))g_i(y_i(t-\rho))\\
&\qquad-d_k(y_k(t))\widehat{b}_{ki}(y_i(t-\rho))g_i(x_i(t-\rho))]\\
&\qquad+\sum_{i=1}^{n}[d_k(y_k(t))\widehat{b}_{ki}(y_i(t-\rho))g_i(x_i(t-\rho))\\
&\qquad-d_k(x_k(t))\widetilde{b}_{ki}(x_i(t-\rho))g_i(x_i(t-\rho))]\Bigg\}.
\end{aligned}\tag{27}$$

From Assumptions 1–2 one has

$$\begin{aligned}
&\sum_{k=1}^{n}\widetilde{p}\operatorname{sign}(z_k(t))|z_k(t)|^{\widetilde{p}-1}\Bigg\{-[d_k(y_k(t))c_k(y_k(t))\\
&\qquad-d_k(x_k(t))c_k(x_k(t))]\\
&\qquad+[d_k(y_k(t))-d_k(x_k(t))]\mathcal{J}_k\Bigg\}\\
&\quad\leq\sum_{k=1}^{n}\widetilde{p}|z_k(t)|^{\widetilde{p}-1}(\zeta_k|z_k(t)|+\widetilde{d}_k|\mathcal{J}_k||z_k(t)|)\\
&\quad=\sum_{k=1}^{n}(\zeta_k+\widetilde{d}_k|\mathcal{J}_k|)\widetilde{p}|z_k(t)|^{\widetilde{p}}.
\end{aligned}\tag{28}$$

From Lemma 7 and Assumption 3, one has

$$\begin{aligned}
&\sum_{k=1}^{n}\widetilde{p}\operatorname{sign}(z_k(t))|z_k(t)|^{\widetilde{p}-1}\\
&\qquad\times\Bigg\{\sum_{i=1}^{n}[d_k(y_k(t))\widehat{a}_{ki}(y_i(t))h_i(y_i(t))\\
&\qquad-d_k(y_k(t))\widehat{a}_{ki}(y_i(t))h_i(x_i(t))]\\
&\qquad+\sum_{i=1}^{n}[d_k(y_k(t))\widehat{a}_{ki}(y_i(t))h_i(x_i(t))\\
&\qquad-d_k(x_k(t))\widetilde{a}_{ki}(x_i(t))h_i(x_i(t))]\Bigg\}\\
&\quad\leq\sum_{k=1}^{n}\sum_{i=1}^{n}\Bigg(\widetilde{p}\bar{d}_kH_i\breve{a}_{ki}|z_k(t)|^{\widetilde{p}-1}|z_i(t)|\\
&\qquad+\widetilde{p}\overline{h}_i\Big(\breve{a}_{ki}\widetilde{d}_k|z_k(t)|^{\widetilde{p}}+2\breve{a}_{ki}\overline{d}_k|z_k(t)|^{\widetilde{p}-1}\Big)\Bigg).
\end{aligned}\tag{29}$$

Similarly, one has

$$\begin{aligned}
&\sum_{k=1}^{n}\widetilde{p}\operatorname{sign}(z_k(t))|z_k(t)|^{\widetilde{p}-1}\\
&\qquad\times\Bigg\{\sum_{i=1}^{n}[d_k(y_k(t))\widehat{b}_{ki}(y_i(t-\rho))g_i(y_i(t-\rho))\\
&\qquad-d_k(y_k(t))\widehat{b}_{ki}(y_i(t-\rho))g_i(x_i(t-\rho))]\\
&\qquad+\sum_{i=1}^{n}[d_k(y_k(t))\widehat{b}_{ki}(y_i(t-\rho))g_i(x_i(t-\rho))\\
&\qquad-d_k(x_k(t))\widetilde{b}_{ki}(x_i(t-\rho))g_i(x_i(t-\rho))]\Bigg\}\\
&\quad\leq\sum_{k=1}^{n}\sum_{i=1}^{n}\Bigg(\widetilde{p}\bar{d}_kG_i\breve{b}_{ki}|z_k(t)|^{\widetilde{p}-1}|z_i(t-\rho)|\\
&\qquad+\widetilde{p}\overline{g}_i\Big(\breve{b}_{ki}\widetilde{d}_k|z_k(t-\rho)||z_k(t)|^{\widetilde{p}-1}\\
&\qquad+2\breve{b}_{ki}\overline{d}_k|z_k(t)|^{\widetilde{p}-1}\Big)\Bigg).
\end{aligned}\tag{30}$$

It follows from Lemma 3 that

$$\sum_{k=1}^{n}\sum_{i=1}^{n}\widetilde{p}\bar{d}_k H_i\breve{a}_{ki}|z_k(t)|^{\widetilde{p}-1}|z_i(t)|$$
$$\leq \sum_{k=1}^{n}\sum_{i=1}^{n}\widetilde{p}\bar{d}_k H_i\breve{a}_{ki}\Big(\frac{\widetilde{p}-1}{\widetilde{p}}|z_k(t)|^{\widetilde{p}}+\frac{1}{\widetilde{p}}|z_i(t)|^{\widetilde{p}}\Big)$$
$$=\sum_{k=1}^{n}\sum_{i=1}^{n}\Big(\bar{d}_k H_i\breve{a}_{ki}(\widetilde{p}-1)+\bar{d}_i H_k\breve{a}_{ik}\Big)|z_k(t)|^{\widetilde{p}}, \tag{31}$$

$$\sum_{k=1}^{n}\sum_{i=1}^{n}\widetilde{p}\bar{d}_k G_i\breve{b}_{ki}|z_k(t)|^{\widetilde{p}-1}|z_i(t-\rho)|$$
$$\leq \sum_{k=1}^{n}\sum_{i=1}^{n}\widetilde{p}\bar{d}_k G_i\breve{b}_{ki}\Big(\frac{\widetilde{p}-1}{\widetilde{p}}|z_k(t)|^{\widetilde{p}}+\frac{1}{\widetilde{p}}|z_i(t-\rho)|^{\widetilde{p}}\Big)$$
$$=\sum_{k=1}^{n}\sum_{i=1}^{n}\bar{d}_k G_i\breve{b}_{ki}(\widetilde{p}-1)|z_k(t)|^{\widetilde{p}}$$
$$+\sum_{k=1}^{n}\sum_{i=1}^{n}\bar{d}_i G_k\breve{b}_{ik}|z_k(t-\rho)|^{\widetilde{p}}, \tag{32}$$

and

$$\sum_{k=1}^{n}\sum_{i=1}^{n}\widetilde{p}\overline{g}_i\breve{b}_{ki}\widetilde{d}_k|z_k(t-\rho)||z_k(t)|^{\widetilde{p}-1}$$
$$\leq \sum_{k=1}^{n}\sum_{i=1}^{n}\widetilde{p}\overline{g}_i\breve{b}_{ki}\widetilde{d}_k\Big(\frac{\widetilde{p}-1}{\widetilde{p}}|z_k(t)|^{\widetilde{p}}+\frac{1}{\widetilde{p}}|z_k(t-\rho)|^{\widetilde{p}}\Big)$$
$$=\sum_{k=1}^{n}\sum_{i=1}^{n}\overline{g}_i\breve{b}_{ki}\widetilde{d}_k\Big((\widetilde{p}-1)|z_k(t)|^{\widetilde{p}}+|z_k(t-\rho)|^{\widetilde{p}}\Big). \tag{33}$$

In addition,

$$\sum_{k=1}^{n}\widetilde{p}\,\text{sign}(z_k(t))|z_k(t)|^{\widetilde{p}-1}$$
$$\times\Big\{-\gamma\frac{z_k(t)}{|z_k(t)|^{\widetilde{p}}}|z_k(t-\rho)|^{\widetilde{p}}-\ell z_k(t)$$
$$-\eta\frac{z_k(t)}{|z_k(t)|}-\chi\frac{z_k(t)}{|z_k(t)|^{\varrho}}\Big\}$$
$$=-\sum_{k=1}^{n}\Big(\widetilde{p}\gamma|z_k(t-\rho)|^{\widetilde{p}}+\widetilde{p}\ell|z_k(t)|^{\widetilde{p}}$$
$$+\widetilde{p}\eta|z_k(t)|^{\widetilde{p}-1}+\widetilde{p}\chi|z_k(t)|^{\widetilde{p}-\varrho}\Big). \tag{34}$$

From (27)–(34) one has

$${}^{c}_{t_0}D^{\kappa}_{t}\widehat{V}_{\widetilde{p}}(t)$$
$$\leq \sum_{k=1}^{n}(\zeta_k+\widetilde{d}_k|\mathcal{J}_k|)\widetilde{p}|z_k(t)|^{\widetilde{p}}$$
$$-\sum_{k=1}^{n}\Big(\widetilde{p}\gamma|z_k(t-\rho)|^{\widetilde{p}}+\widetilde{p}\ell|z_k(t)|^{\widetilde{p}}$$
$$+\widetilde{p}\eta|z_k(t)|^{\widetilde{p}-1}+\widetilde{p}\chi|z_k(t)|^{\widetilde{p}-\varrho}\Big)$$
$$+\sum_{k=1}^{n}\sum_{i=1}^{n}\Big(\bar{d}_k H_i\breve{a}_{ki}(\widetilde{p}-1)+\bar{d}_i H_k\breve{a}_{ik}\Big)|z_k(t)|^{\widetilde{p}}$$
$$+\sum_{k=1}^{n}\sum_{i=1}^{n}\widetilde{p}\overline{h}_i\Big(\breve{a}_{ki}\widetilde{d}_k|z_k(t)|^{\widetilde{p}}+2\breve{a}_{ki}\overline{d}_k|z_k(t)|^{\widetilde{p}-1}\Big)$$
$$+\sum_{k=1}^{n}\sum_{i=1}^{n}\bar{d}_k G_i\breve{b}_{ki}(\widetilde{p}-1)|z_k(t)|^{\widetilde{p}}$$
$$+\sum_{k=1}^{n}\sum_{i=1}^{n}\bar{d}_i G_k\breve{b}_{ik}|z_k(t-\rho)|^{\widetilde{p}}$$
$$+\sum_{k=1}^{n}\sum_{i=1}^{n}\overline{g}_i\breve{b}_{ki}\widetilde{d}_k\Big((\widetilde{p}-1)|z_k(t)|^{\widetilde{p}}+|z_k(t-\rho)|^{\widetilde{p}}\Big)$$
$$+\sum_{k=1}^{n}\sum_{i=1}^{n}\widetilde{p}\overline{g}_i2\breve{b}_{ki}\overline{d}_k|z_k(t)|^{\widetilde{p}-1}$$
$$=\sum_{k=1}^{n}\Big[(\zeta_k+\widetilde{d}_k|\mathcal{J}_k|)\widetilde{p}$$
$$+\sum_{i=1}^{n}\Big(\bar{d}_k H_i\breve{a}_{ki}(\widetilde{p}-1)+\bar{d}_i H_k\breve{a}_{ik}+\widetilde{p}\overline{h}_i\breve{a}_{ki}\widetilde{d}_k$$
$$+\bar{d}_k G_i\breve{b}_{ki}(\widetilde{p}-1)+\overline{g}_i\breve{b}_{ki}\widetilde{d}_k(\widetilde{p}-1)\Big)-\widetilde{p}\ell\Big]|z_k(t)|^{\widetilde{p}}$$
$$+\sum_{k=1}^{n}\Big[\sum_{i=1}^{n}\Big(\widetilde{p}\overline{h}_i2\breve{a}_{ki}\overline{d}_k+\widetilde{p}\overline{g}_i2\breve{b}_{ki}\overline{d}_k\Big)-\widetilde{p}\eta\Big]|z_k(t)|^{\widetilde{p}-1}$$
$$+\sum_{k=1}^{n}\Big[\sum_{i=1}^{n}\Big(\bar{d}_i G_k\breve{b}_{ik}+\overline{g}_i\breve{b}_{ki}\widetilde{d}_k\Big)-\widetilde{p}\gamma\Big]|z_k(t-\rho)|^{\widetilde{p}}$$
$$-\sum_{k=1}^{n}\widetilde{p}\chi|z_k(t)|^{\widetilde{p}-\varrho}. \tag{35}$$

Using the inequality skill in [18, Theorem 1], one has

$$\begin{aligned}
&\sum_{k=1}^{n}|z_k(t)|^{\widetilde{p}-\varrho}\\
&=\sum_{k=1}^{n}\frac{1}{(|z_k(t)|^{\widetilde{p}})^{\frac{\varrho-\widetilde{p}}{\widetilde{p}}}}\\
&\geq\sum_{k=1}^{n}\frac{1}{\left(\sum_{k=1}^{n}|z_k(t)|^{\widetilde{p}}\right)^{\frac{\varrho-\widetilde{p}}{\widetilde{p}}}}\\
&=n\left(\sum_{k=1}^{n}|z_k(t)|^{\widetilde{p}}\right)^{\frac{\widetilde{p}-\varrho}{\widetilde{p}}}. \qquad (36)
\end{aligned}$$

Let

$$\begin{aligned}
\Xi_k&=(\zeta_k+\widetilde{d}_k|\mathcal{J}_k|)\widetilde{p}+\sum_{i=1}^{n}\Big(\bar{d}_kH_i\breve{a}_{ki}(\widetilde{p}-1)+\bar{d}_iH_k\breve{a}_{ik}\\
&\quad+\widetilde{p}\overline{h}_i\breve{a}_{ki}\widetilde{d}_k+\bar{d}_kG_i\breve{b}_{ki}(\widetilde{p}-1)+\overline{g}_i\breve{b}_{ki}\widetilde{d}_k(\widetilde{p}-1)\Big),\\
\widetilde{p}\eta&=\max_{1\leq k\leq n}\left\{\sum_{i=1}^{n}\Big(\widetilde{p}\overline{h}_i2\breve{a}_{ki}\overline{d}_k+\widetilde{p}\overline{g}_i2\breve{b}_{ki}\overline{d}_k\Big)\right\},\\
\widetilde{p}\gamma&=\max_{1\leq k\leq n}\left\{\sum_{i=1}^{n}\Big(\bar{d}_iG_k\breve{b}_{ik}+\overline{g}_i\breve{b}_{ki}\widetilde{d}_k\Big)\right\},
\end{aligned}$$

and

$$\varsigma=-\max_{1\leq k\leq n}\{\Xi_k\}+\widetilde{p}\ell.$$

From (35) and (36) one has

$$\begin{aligned}
&{}_{t_0}^{c}D_t^{\kappa}\widehat{V}_{\widetilde{p}}(t)\\
&\leq(\Xi_k-\widetilde{p}\ell)\widehat{V}_{\widetilde{p}}(t)-\widetilde{p}\chi n\left(\sum_{k=1}^{n}|z_k(t)|^{\widetilde{p}}\right)^{\frac{\widetilde{p}-\varrho}{\widetilde{p}}}\\
&\leq\left(\max_{1\leq k\leq n}\{\Xi_k\}-\widetilde{p}\ell\right)\widehat{V}_{\widetilde{p}}(t)-\widetilde{p}\chi n\widehat{V}_{\widetilde{p}}^{\frac{\widetilde{p}-\varrho}{\widetilde{p}}}(t)\\
&=-\varsigma\widehat{V}_{\widetilde{p}}(t)-\widetilde{p}\chi n\widehat{V}_{\widetilde{p}}^{-\frac{\varrho-\widetilde{p}}{\widetilde{p}}}(t). \qquad (37)
\end{aligned}$$

It follows from Lemma 8 that the FODMCGNNs (3) and (4) can be finite-time synchronized under the feedback controller (25) if the condition (26) is satisfied. The settling time t^* is evaluated as

$$t^*\leq t_1^*=t_0+\left(\frac{-T_1^*}{\varsigma\varrho/\widetilde{p}}\right)^{\frac{1}{\kappa}},$$

where T_1^* is the unique root of the equation

$$E_\kappa(z)=\frac{\widetilde{p}n\chi}{\widetilde{p}n\chi+\varsigma\widehat{V}_{\widetilde{p}}^{\varrho/\widetilde{p}}(t_0)}.$$

The proof of Theorem 1 is completed. □

Especially, for $\varrho=\widetilde{p}$, the inequality (42) is reduced to the inequality ${}_{t_0}^{c}D_t^{\kappa}\widehat{V}_{\widetilde{p}}(t)\leq-\varsigma\widehat{V}_{\widetilde{p}}(t)-\widetilde{p}n\chi$, which has the same form with the inequality (21) in [31, Lemma 5]. Using the similar skills in the proof of Theorem 1, one has the following corollary.

Corollary 4 *Under Assumptions 1–3 and the feedback controller (25), the FODMCGNNs (3) and (4) can be finite-time synchronized if*

$$\widetilde{p}n\chi+\varsigma\widehat{V}_{\widetilde{p}}(t_0)>0 \text{ and } \widetilde{p}=\varrho. \qquad (38)$$

The settling time t^* is evaluated by

$$t^*\leq t_2^*=t_0+\left(\frac{-T_2^*}{\varsigma}\right)^{\frac{1}{\kappa}},$$

where T_2^* is the unique root of the equation

$$E_\kappa(z)=\frac{\widetilde{p}n\chi}{\widetilde{p}n\chi+\varsigma\widehat{V}_{\widetilde{p}}(t_0)}.$$

Remark 7 In [16, Theorem 2], the fractional-order derivative of Lyapunov function $V(t)=\sum_{k=1}^{n}|z_k(t)|^{\widetilde{p}}$ was estimated by the equality (13). However, the equality (13) may be not applicable and the corresponding counterexample can be found in Remark 1. Therefore, the FTS criterion in [16, Theorem 2] may also not be feasible.

Remark 8 In [16, Theorem 1], the FTS of FODMCGNNs was discussed by the following inequality $E_\kappa(t)\leq\frac{1}{\kappa}e^{t^{\frac{1}{\kappa}}}$. However, this inequality may be not applicable. For example, letting $\kappa=0.322$, $t=3$, one has $E_\kappa(t)-\frac{1}{\kappa}e^{t^{\frac{1}{\kappa}}}=4.9297>0$. Therefore, the FTS criterion in [16, Theorem 1] may not be feasible.

Especially, for $\varrho=\widetilde{p}=1$, motivated by [20], the following modified feedback controller with the sign function is designed to realize the FTS between the FODMCGNNs (3) and the FODMCGNNs (4):

$$\begin{aligned}
\widetilde{u}_k(t)=&-\gamma\,\mathrm{sign}(z_k(t))|z_k(t-\rho)|-\ell z_k(t)\\
&-\eta\,\mathrm{sign}(z_k(t)), \qquad (39)
\end{aligned}$$

where $k\in\mathbb{N}_1^n$, γ,ℓ,η are positive constants. Similar to the proof of Theorem 1, one has the following result.

Theorem 2 *Under Assumptions 1–3 and the feedback controller (39), the FODMCGNNs (3) and (4) can be finite-time synchronized if*

$$\varsigma \widehat{V}_1(t_0) + \omega > 0, \tag{40}$$

where

$$\widehat{V}_1(t) = \sum_{k=1}^{n} |z_k(t)|,$$

$$\varsigma = -\max_{1\le k\le n}\{\Xi_k\} + \ell,$$

$$\omega = \sum_{k=1}^{n}(\eta - \Theta_k) > 0,$$

$$\Xi_k = (\zeta_k + \widetilde{d}_k|\mathcal{J}_k|) + \sum_{i=1}^{n}\left(\bar{d}_i H_k \breve{a}_{ik} + \overline{h}_i \breve{a}_{ki}\widetilde{d}_k\right),$$

$$\Theta_k = \sum_{i=1}^{n}\left(\overline{h}_i 2\breve{a}_{ki}\overline{d}_k + \overline{g}_i 2\breve{b}_{ki}\overline{d}_k\right),$$

$$\gamma = \max_{1\le k\le n}\left\{\sum_{i=1}^{n}\left(\bar{d}_i G_k \breve{b}_{ik} + \overline{g}_i \breve{b}_{ki}\widetilde{d}_k\right)\right\}.$$

The settling time t^* is evaluated by

$$t^* \le t_3^* = t_0 + \left(\frac{-T_3^*}{\varsigma}\right)^{\frac{1}{\kappa}},$$

where T_3^* is the unique root of the equation

$$E_\kappa(z) = \frac{\omega}{\varsigma \widehat{V}_1(t_0) + \omega}. \tag{41}$$

Proof Let

$$\widehat{V}_1(t) = \sum_{k=1}^{n} |z_k(t)|.$$

Similar to the proof of Theorem 1, one has

$$\begin{aligned}
{}_{t_0}^{c}D_t^{\kappa} \widehat{V}_1(t)
&\le (\Xi_k - \ell)\widehat{V}_1(t) + \sum_{k=1}^{n}(\Theta_k - \eta) \\
&\quad + \sum_{k=1}^{n}\left[\sum_{i=1}^{n}\left(\bar{d}_i G_k \breve{b}_{ik} + \overline{g}_i \breve{b}_{ki}\widetilde{d}_k\right) - \gamma\right]|z_k(t-\rho)| \\
&\le \left(\max_{1\le k\le n}\{\Xi_k\} - \ell\right)\widehat{V}_1(t) + \sum_{k=1}^{n}(\Theta_k - \eta) \\
&= -\varsigma \widehat{V}_1(t) - \omega.
\end{aligned} \tag{42}$$

It follows from Corollary 1 that the FODMCGNNs (3) and (4) can be finite-time synchronized under the feedback controller (39) if the condition (40) is satisfied. The settling time t^* is evaluated as

$$t^* \le t_3^* = t_0 + \left(\frac{-T_3^*}{\varsigma}\right)^{\frac{1}{\kappa}},$$

where T_3^* is the unique root of the equation

$$E_\kappa(z) = \frac{\omega}{\varsigma \widehat{V}_1(t_0) + \omega}.$$

□

From the proof of Theorem 2, one has the following corollary in terms of Lemma 5.

Corollary 5 *Under Assumptions 1–3 and the feedback controller (39), the FODMCGNNs (3) and (4) can be finite-time synchronized if*

$$\ell > \max_{1\le k\le n}\{\Xi_k\} \tag{43}$$

and

$$\eta > \max_{1\le k\le n}\left\{\sum_{i=1}^{n}\left(\bar{d}_i G_k \breve{b}_{ik} + \overline{g}_i \breve{b}_{ki}\widetilde{d}_k\right)\right\}. \tag{44}$$

The settling time t^* is evaluated by

$$t^* \le t_4^* = t_0 + \left(\frac{\Gamma(1+\kappa)V(t_0)}{\omega}\right)^{\frac{1}{\kappa}}.$$

4 Numerical examples

In this section, on the basis of predictor-corrector algorithm [5], three examples are presented to demonstrate the correctness and advantage of the obtained results.

Example 1 Consider the FODMCGNNs (3) and (4) with the following parameters: $\kappa = 0.95$, $n = 2$, $t_0 = 0$, $\rho = 0.2$, $\mathcal{J}_k = 0.01$, $d_k(x_k(t)) = 0.1\sin(x_k(t)) + 0.2$, $c_k(x_k(t)) = 0.2\cos(x_k(t))$, $h_i(x_i(t)) = 0.1\tanh(x_i(t))$, $g_i(x_i(t-\rho)) = 0.1\tanh(x_i(t-\rho))$,

$\psi_1(t) = 0.1, \psi_2(t) = -0.2, \phi_1(t) = -0.1, \phi_2(t) = 0.2, t \in [-0.2, 0], k, i \in \mathbb{N}_1^2$,

$$a_{11}(x_1(t)) = \begin{cases} 1.0, & |x_1(t)| > 1, \\ 1.3, & |x_1(t)| \le 1, \end{cases}$$

$$a_{12}(x_2(t)) = \begin{cases} -0.9, & |x_2(t)| > 1, \\ -1.2, & |x_2(t)| \le 1, \end{cases}$$

$$a_{21}(x_1(t)) = \begin{cases} -1.1, & |x_1(t)| > 1, \\ -0.3, & |x_1(t)| \le 1, \end{cases}$$

$$a_{22}(x_2(t)) = \begin{cases} 1.7, & |x_2(t)| > 1, \\ 1.5, & |x_2(t)| \le 1, \end{cases}$$

$$b_{11}(x_1(t-\rho)) = \begin{cases} 3.1, & |x_1(t-\rho)| > 1, \\ 2.2, & |x_1(t-\rho)| \le 1, \end{cases}$$

$$b_{12}(x_2(t-\rho)) = \begin{cases} -1.2, & |x_2(t-\rho)| > 1, \\ -1.4, & |x_2(t-\rho)| \le 1, \end{cases}$$

$$b_{21}(x_1(t-\rho)) = \begin{cases} -1.0, & |x_1(t-\rho)| > 1, \\ -1.1, & |x_1(t-\rho)| \le 1, \end{cases}$$

$$b_{22}(x_2(t-\rho)) = \begin{cases} 2.3, & |x_2(t-\rho)| > 1, \\ 2.1, & |x_2(t-\rho)| \le 1. \end{cases}$$

By a simple calculation, one has $\breve{a}_{11} = 1.3, \breve{a}_{12} = 1.2, \breve{a}_{21} = 1.1, \breve{a}_{22} = 1.7, \breve{b}_{11} = 3.1, \breve{b}_{12} = 1.4, \breve{b}_{21} = 1.1, \breve{b}_{22} = 2.3$. It is easy to verify that Assumptions 1–3 are satisfied. In addition, one can gain $H_k = G_k = 0.1$, $\bar{d}_k = 0.3, \widetilde{d}_k = 0.1, \zeta_k = 0.06, \bar{h}_k = \bar{g}_k = 0.1$.

In the following, we show the advantage of Theorem 1 in comparison with Corollary 4 when $\widetilde{p} = 1$. For the controller parameters γ, ℓ, η, χ given in Table 1, by a simple calculation one has $\widehat{V}_{\widetilde{p}}(0) = 0.6, \varsigma = 0.8240$. If one chooses different parameters $\varrho = 1.1$ and $\varrho = 1$, then one has $\widetilde{p}n\chi + \varsigma\widehat{V}_{\widetilde{p}}^{\varrho/\widetilde{p}}(0) = 2.4698 > 0$ and $\widetilde{p}n\chi + \varsigma\widehat{V}_{\widetilde{p}}(0) = 2.4944 > 0$, respectively. Hence, the conditions (26) and (38) in Theorem 1 and Corollary 4 are satisfied, respectively. According to Theorem 1 and Corollary 4, one both has the FODMCGNNs (3) and (4) are finite-time synchronized, and the estimated settling times are $t_1^* = 0.2123$ and $t_2^* = 0.2463$, respectively. It is obvious that t_1^* in Theorem 1 is less than t_2^* in Corollary 4. Therefore, for $\widetilde{p} = 1$ Theorem 1 obtained by Lemma 8 is less conservative than Corollary 4 obtained by [31, Lemma 5].

Similarly, we can illustrate the advantage of Theorem 1 in comparison with Corollary 4 when $\widetilde{p} > 1$. For the controller parameters $\widetilde{p}, \gamma, \ell, \eta, \chi$ given in Table 2, by a simple calculation one has $\widehat{V}_{\widetilde{p}}(0) = 0.5352$, $\varsigma = 0.8931$. If one chooses different parameters $\varrho = 1.2$ and $\varrho = 1.1$, then one has $\widetilde{p}n\chi + \varsigma\widehat{V}_{\widetilde{p}}^{\varrho/\widetilde{p}}(0) = 2.6516 > 0$ and $\widetilde{p}n\chi + \varsigma\widehat{V}_{\widetilde{p}}(0) = 2.6780 > 0$, respectively. Hence, the conditions (26) and (38) in Theorem 1 and Corollary 4 are satisfied, respectively. According to Theorem 1 and Corollary 4, one both has the FODMCGNNs (3) and (4) are finite-time synchronized, and the estimated settling times are $\widetilde{t}_1^* = 0.1728$ and $\widetilde{t}_2^* = 0.2001$, respectively. It is obvious that $\widetilde{t}_1^*$ in Theorem 1 is less than $\widetilde{t}_2^*$ in Corollary 4. Therefore, for $\widetilde{p} > 1$ Theorem 1 obtained by Lemma 8 is less conservative than Corollary 4 obtained by [31, Lemma 5].

Table 1 The comparisons among t_1^* and t_2^* in Example 1

Results	$\widetilde{p}$	ϱ	γ	ℓ	η	χ	$t_i^*(i = 1, 2)$
Theorem 1	1	1.1	0.171	1	0.42	1	$t_1^* = 0.2123$
Corollary 4	1	1	0.171	1	0.42	1	$t_2^* = 0.2463$

Table 2 The comparisons among $\widetilde{t}_1^*$ and $\widetilde{t}_2^*$ in Example 1

Results	$\widetilde{p}$	ϱ	γ	ℓ	η	χ	$\widetilde{t}_i^*(i = 1, 2)$
Theorem 1	1.1	1.2	0.1555	1	0.42	1	$\widetilde{t}_1^* = 0.1728$
Corollary 4	1.1	1.1	0.1555	1	0.42	1	$\widetilde{t}_2^* = 0.2001$

To show the correctness of Theorem 1 for $\widetilde{p} = 1$ and $\widetilde{p} > 1$, synchronization errors of FODMCGNNs (3) and (4) with different parameters p in Example 1 are plotted in Figs. 1 and 2, respectively. One can observe that synchronization is achieved at 0.1914 and 0.1664, which are less than the estimated settling time $t_1^* = 0.2123$ and $\widetilde{t}_1^* = 0.1728$, respectively. This validates the correctness of Theorem 1 for $\widetilde{p} = 1$ and $\widetilde{p} > 1$.

Example 2 Consider the FODMCGNNs (3) and (4) with the following parameters: $\kappa = 0.98, n = 2, t_0 = 0, \rho = 0.4, \mathcal{J}_k = 0.02, d_k(x_k(t)) = 0.2\cos(x_k(t)) + 0.1, c_k(x_k(t)) = 0.1\sin(x_k(t)), h_i(x_i(t)) = 0.2\tanh(x_i(t)), g_i(x_i(t-\rho)) = 0.2\tanh(x_i(t-\rho))$, $\psi_1(t) = -0.2, \psi_2(t) = 0.1, \phi_1(t) = 0.2, \phi_2(t) = -0.1, t \in [-0.4, 0], k, i \in \mathbb{N}_1^2$,

$$a_{11}(x_1(t)) = \begin{cases} 1.2, & |x_1(t)| > 1, \\ 1.1, & |x_1(t)| \le 1, \end{cases}$$

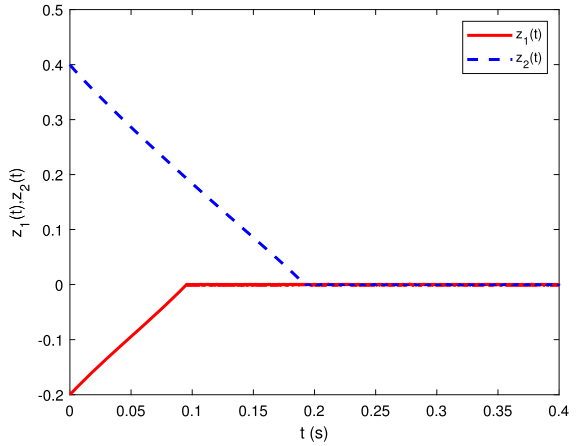

Fig. 1 Synchronization errors $z_1(t)$, $z_2(t)$ with $\widetilde{p} = 1$ in Example 1

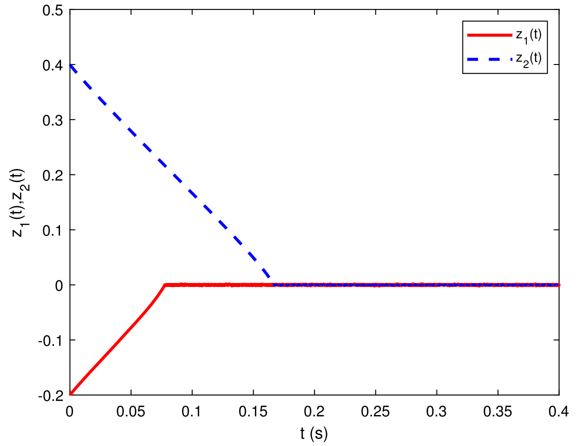

Fig. 2 Synchronization errors $z_1(t)$, $z_2(t)$ with $\widetilde{p} = 1.1$ in Example 1

$$a_{12}(x_2(t)) = \begin{cases} -1.0, |x_2(t)| > 1, \\ -1.1, |x_2(t)| \leq 1, \end{cases}$$

$$a_{21}(x_1(t)) = \begin{cases} -1.2, & |x_1(t)| > 1, \\ -0.5, & |x_1(t)| \leq 1, \end{cases}$$

$$a_{22}(x_2(t)) = \begin{cases} 1.5, & |x_2(t)| > 1, \\ 1.7, & |x_2(t)| \leq 1, \end{cases}$$

$$b_{11}(x_1(t-\rho)) = \begin{cases} 2.1, & |x_1(t-\rho)| > 1, \\ 1.2, & |x_1(t-\rho)| \leq 1, \end{cases}$$

$$b_{12}(x_2(t-\rho)) = \begin{cases} -1.5, & |x_2(t-\rho)| > 1, \\ -1.2, & |x_2(t-\rho)| \leq 1, \end{cases}$$

Table 3 The comparisons among t_3^* and t_4^* in Example 2

Results	γ	ℓ	η	$t_i^*(i=3,4)$
Theorem 2	1.0950	1	1.2740	$t_3^* = 0.5787$
Corollary 5	1.0950	1	1.2740	$t_4^* = 0.5888$

$$b_{21}(x_1(t-\rho)) = \begin{cases} -1.6, & |x_1(t-\rho)| > 1, \\ -1.2, & |x_1(t-\rho)| \leq 1, \end{cases}$$

$$b_{22}(x_2(t-\rho)) = \begin{cases} 2.5, & |x_2(t-\rho)| > 1, \\ 2.2, & |x_2(t-\rho)| \leq 1. \end{cases}$$

By a simple calculation, one has $\breve{a}_{11} = 1.2$, $\breve{a}_{12} = 1.1$, $\breve{a}_{21} = 1.2$, $\breve{a}_{22} = 1.7$, $\breve{b}_{11} = 2.1$, $\breve{b}_{12} = 1.5$, $\breve{b}_{21} = 1.6$, $\breve{b}_{22} = 2.5$. It is easy to verify that Assumptions 1–3 are satisfied. In addition, one can gain $H_k = G_k = 0.2$, $\bar{d}_k = 0.3$, $\widetilde{d}_k = 0.2$, $\zeta_k = 0.03$, $\bar{h}_k = \bar{g}_k = 0.2$.

In the following, we show the advantage of Theorem 2 in comparison with Corollary 5. For the controller parameters γ, ℓ, η given in Table 3, by a simple calculation one has $\widehat{V}_1(0) = 0.6$, $\varsigma = 0.0535$, and $\omega = 1$, then one has $\varsigma \widehat{V}_1(t_0) + \omega = 1.0321 > 0$, $1.2740 = \eta > \max_{1\leq k\leq n}\left\{\sum_{i=1}^{n}\left(\bar{d}_i G_k \breve{b}_{ik} + \bar{g}_i \breve{b}_{ki} \widetilde{d}_k\right)\right\} = 1.0950$, and $1 = \ell > \max_{1\leq k\leq n}\{\Xi_k\} = 0.9465$. Hence, the conditions (40) and (43)–(44) in Theorem 2 and Corollary 5 are satisfied, respectively. According to Theorem 2 and Corollary 5, one both has the FODMCGNNs (3) and (4) are finite-time synchronized, and the estimated settling times are $t_3^* = 0.5787$ and $t_4^* = 0.5888$, respectively. It is obvious that t_3^* in Theorem 2 is less than t_4^* in Corollary 5. Therefore, Theorem 2 obtained by Corollary 1 is less conservative than Corollary 5 obtained by Lemma 5.

To show the correctness of Theorem 2, synchronization error of FODMCGNNs (3) and (4) in Example 2 are plotted in Fig. 3. One can observe that synchronization is achieved at 0.2074, which is less than the estimated settling time $t_3^* = 0.5787$. This validates the correctness of Theorem 2.

Example 3 Consider the FODMCGNNs (3) and (4) with the following parameters: $\kappa = 0.97$, $n = 3$, $t_0 = 0$, $\rho = 0.2$, $\mathcal{J}_k = 0.015$, $d_k(x_k(t)) = 0.3\sin(x_k(t)) + 0.2$, $c_k(x_k(t)) = 0.2\cos(x_k(t))$, $h_i(x_i(t)) = 0.1\tanh(x_i(t))$, $g_i(x_i(t-\rho)) = 0.1\tanh(x_i(t-\rho))$, $\psi_1(t) = -0.1$, $\psi_2(t) = 0.2$, $\psi_3(t) = 0.15$, $\phi_1(t) = 0.1$, $\phi_2(t) = -0.2$, $\phi_3(t) = 0.25$, $t \in [-0.2, 0]$,

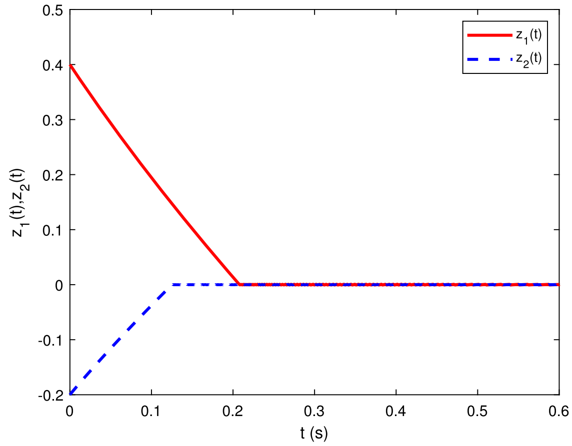

Fig. 3 Synchronization errors $z_1(t)$, $z_2(t)$ in Example 2

$k, i \in \mathbb{N}_1^3$,

$$a_{11}(x_1(t)) = \begin{cases} 1.1, & |x_1(t)| > 1, \\ 1.0, & |x_1(t)| \le 1, \end{cases}$$

$$a_{12}(x_2(t)) = \begin{cases} -1.1, & |x_2(t)| > 1, \\ -1.2, & |x_2(t)| \le 1, \end{cases}$$

$$a_{13}(x_3(t)) = \begin{cases} -5.0, & |x_3(t)| > 1, \\ -4.4, & |x_3(t)| \le 1, \end{cases}$$

$$a_{21}(x_1(t)) = \begin{cases} -1.3, & |x_1(t)| > 1, \\ -0.6, & |x_1(t)| \le 1, \end{cases}$$

$$a_{22}(x_2(t)) = \begin{cases} 1.6, & |x_3(t)| > 1, \\ 1.8, & |x_3(t)| \le 1, \end{cases}$$

$$a_{23}(x_3(t)) = \begin{cases} 4.7, & |x_3(t)| > 1, \\ -6.1, & |x_3(t)| \le 1, \end{cases}$$

$$a_{31}(x_1(t)) = \begin{cases} -1.2, & |x_1(t)| > 1, \\ -0.5, & |x_1(t)| \le 1, \end{cases}$$

$$a_{32}(x_2(t)) = \begin{cases} 1.5, & |x_2(t)| > 1, \\ 1.7, & |x_2(t)| \le 1, \end{cases}$$

$$a_{33}(x_3(t)) = \begin{cases} 2.3, & |x_3(t)| > 1, \\ 2.5, & |x_3(t)| \le 1, \end{cases}$$

$$b_{11}(x_1(t-\rho)) = \begin{cases} 2.2, & |x_1(t-\rho)| > 1, \\ 1.3, & |x_1(t-\rho)| \le 1, \end{cases}$$

$$b_{12}(x_2(t-\rho)) = \begin{cases} -1.6, & |x_2(t-\rho)| > 1, \\ -1.3, & |x_2(t-\rho)| \le 1, \end{cases}$$

$$b_{13}(x_3(t-\rho)) = \begin{cases} -2.7, & |x_3(t-\rho)| > 1, \\ -4.1, & |x_3(t-\rho)| \le 1, \end{cases}$$

$$b_{21}(x_1(t-\rho)) = \begin{cases} -1.7, & |x_1(t-\rho)| > 1, \\ -1.3, & |x_1(t-\rho)| \le 1, \end{cases}$$

$$b_{22}(x_2(t-\rho)) = \begin{cases} 2.7, & |x_2(t-\rho)| > 1, \\ 2.4, & |x_2(t-\rho)| \le 1, \end{cases}$$

$$b_{23}(x_3(t-\rho)) = \begin{cases} -8.3, & |x_3(t-\rho)| > 1, \\ -7.6, & |x_3(t-\rho)| \le 1, \end{cases}$$

$$b_{31}(x_1(t-\rho)) = \begin{cases} -1.5, & |x_1(t-\rho)| > 1, \\ -1.1, & |x_1(t-\rho)| \le 1, \end{cases}$$

$$b_{32}(x_2(t-\rho)) = \begin{cases} 2.6, & |x_2(t-\rho)| > 1, \\ 2.2, & |x_2(t-\rho)| \le 1, \end{cases}$$

$$b_{33}(x_3(t-\rho)) = \begin{cases} -3.8, & |x_3(t-\rho)| > 1, \\ -4.4, & |x_3(t-\rho)| \le 1. \end{cases}$$

By a simple calculation, one has $\breve{a}_{11} = 1.1$, $\breve{a}_{12} = 1.2$, $\breve{a}_{13} = 5$, $\breve{a}_{21} = 1.3$, $\breve{a}_{22} = 1.8$, $\breve{a}_{23} = 6.1$, $\breve{a}_{31} = 1.2$, $\breve{a}_{32} = 1.7$, $\breve{a}_{33} = 2.5$, $\breve{b}_{11} = 2.2$, $\breve{b}_{12} = 1.6$, $\breve{b}_{13} = 4.1$, $\breve{b}_{21} = 1.7$, $\breve{b}_{22} = 2.7$, $\breve{b}_{23} = 8.3$, $\breve{b}_{31} = 1.5$, $\breve{b}_{32} = 2.6$, $\breve{b}_{33} = 4.4$. It is easy to verify that Assumptions 1–3 are satisfied. In addition, one can gain $H_k = G_k = 0.1$, $\bar{d}_k = 0.5$, $\widetilde{d}_k = 0.3$, $\zeta_k = 0.1$, $\bar{h}_k = \bar{g}_k = 0.1$.

In the following, we show the advantage of Theorem 2 in comparison with Corollary 5. For the controller parameters γ, ℓ, η given in Table 4, by a simple calculation one has $\widehat{V}_1(0) = 0.7$, $\varsigma = 0.0535$, and $\omega = 0.3$, then one has $\varsigma \widehat{V}_1(t_0) + \omega = 0.3374 > 0$, $1 = \ell > \max_{1 \le k \le n}\{\Xi_k\} = 0.9465$, and $1.8 = \eta > \max_{1 \le k \le n}\left\{\sum_{i=1}^{n}\left(\bar{d}_i G_k \breve{b}_{ik} + \bar{g}_i \breve{b}_{ki} \widetilde{d}_k\right)\right\} = 1.0950$. Hence, the conditions (40) and (43)–(44) in Theorem 2 and Corollary 5 are satisfied, respectively. According to Theorem 2 and Corollary 5, one both has the FODMCGNNs (3) and (4) are finite-time synchronized, and the estimated settling times are $t_3^* = 2.2286$ and $t_4^* = 2.3649$, respectively. It is obvious that t_3^* in Theorem 2 is less than t_4^* in Corollary 5. Therefore, Theorem 2 obtained by Corollary 1 is less conservative than Corollary 5 obtained by Lemma 5.

To show the correctness of Theorem 2, synchronization error of FODMCGNNs (3) and (4) in Example 3 are plotted in Fig. 4. One can observe that synchronization is achieved at 0.1557, which is less than the estimated settling time $t_3^* = 2.8884$. This validates the correctness of Theorem 2.

Table 4 The comparisons among t_3^* and t_4^* in Example 3

Results	γ	ℓ	η	$t_i^*(i=3,4)$
Theorem 2	1.0950	1	1.8000	$t_3^*=2.2286$
Corollary 5	1.0950	1	1.8000	$t_4^*=2.3649$

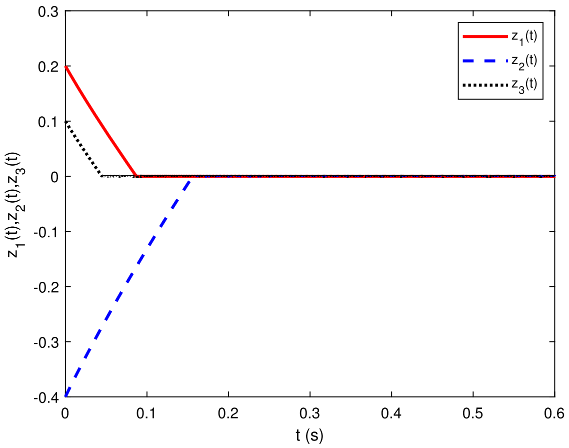

Fig. 4 Synchronization errors $z_1(t)$, $z_2(t)$, $z_3(t)$ in Example 3

5 Conclusions

The FTS has been investigated for a class of FODM-CGNNs. Firstly, a novel fractional-order finite-time inequality (see Lemma 8) has been developed; it generalizes the existing one and can be employed to investigate the FTS of FODSs. More importantly, it has been demonstrated theoretically that the estimated settling time is more accurate than the existing one (see Remark 5). Subsequently, based on this novel inequality, the designed feedback controllers, and the fractional-order power law inequality, two novel criteria have been obtained to ensure the FTS of the FODM-CGNNs. Finally, three examples have been presented to illustrate the correctness and advantage of the derived results.

Acknowledgements This work is supported by the National Natural Science Foundation of China (No. 62073217, No. 61374030, and No. 61533012) and the China Postdoctoral Science Foundation (No. 2018M641991).

Data availability All data generated or analyzed during this study are included in this article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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