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Global robust stabilization for cascaded systems with dynamic uncertainties and asymmetric time-varying constraints

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Abstract This article investigates an adaptive output feedback control problem for a non-holonomic system with integral input-to-state stable inverse dynamics and output constraints. A tan-type barrier Lyapunov function is utilized to handle asymmetric time-varying output constraints, and the full-order observer is constructed to estimate the unmeasurable state. The dynamic uncertainty is eliminated by changing the supply rate of the integral input-to-state stability. It is demonstrated that the closed-loop system is asymptotically stable, and the output does not violate the asymmetric time-varying constraints under this control scheme. A simulation example validates the effectiveness of the proposed controller.

Keywords Non-holonomic systems · Output constraints · Integral input-to-state stable · Stabilization control

1 Introduction

Cascaded systems are a class of nonlinear systems composed of two or more subsystems. Many mechanical systems can be described as cascaded systems, such as robot systems, multi-agent systems, and trailer systems [1–3]. The stability analysis of cascaded systems has attracted extensive attention in recent decades [4–6]. Generally, the stability characteristics of subsystems cannot determine the stability of cascaded systems [7]. To alleviate this difficulty, various control methods for nonlinear cascaded systems have been developed. An intuitive way to address this problem is to design a full-state feedback controller that fully uses the state information of the whole cascaded system [8–10]. Another essential idea is to utilize partial-state feedback control to stabilize the cascaded system [11–13]. In [14], a stepwise constructive partial-state feedback control strategy based on input-to-state stability was proposed. In addition, a smooth controller for a class of nonlinear cascaded systems was designed in [15] by combining the feedback control scheme and variable separation technology. Unlike previous research, we propose a control approach based on integral input-to-state stability, which ensures the stability of the nonlinear cascaded system.

The input-to-state stable (ISS) concept introduced by [16] has been proven to be a valid instrument for studying the robust stability of nonlinear cascaded systems. Its various properties have been thoroughly investigated [17–19]. Compared with the ISS concept, the integral input-to-state stable (iISS) concept [20] is strictly weaker and can contain a broader class of nonlinear systems with practical significance. Jiang et

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al. [21] solved the problem of output feedback robust regulation and proposed a unified framework for a type of nonlinear cascaded systems with inverse dynamics. In [22], a small gain theorem was obtained for interconnected nonlinear systems using state-dependent characterization. The results in [23] removed the technical conditions and covered more common situations than earlier studies. Yu et al. [24] further considered the technology for changing the supply rate and discussed the necessity of the given conditions for the case in which the attenuation rate involves an oscillation function. In this paper, we analyze the iISS subsystem and address a decay rate that includes an oscillation function.

Owing to physical factors, safety requirements, and performance indicators, almost all mechanical equipment operates under output or state constraints. Consequently, it is of great value to investigate the output and state constraints of a system in control theory and practical applications. Various approaches have been adopted to handle state and output constraints [25,26]. In the past few years, the utilization of the barrier Lyapunov function (BLF) has become increasingly common to prevent constraint violations in nonlinear systems [27–29]. In addition, many types of BLF are investigated in [30–32], such as the log-type BLF, the integral-type BLF, and the tan-type BLF. The tan-type BLF has the benefit of integrating constraint analysis into a general method, and it can utilize the system's structural characteristics. However, few results have been obtained regarding the controller design and stability analysis of nonlinear cascaded systems with output constraints and iISS inverse dynamics.

This paper considers the stability of nonlinear cascaded systems with iISS inverse dynamics and asymmetric time-varying output constraints. The proposed control strategy ensures the stability of nonlinear cascaded systems. Moreover, the output satisfies asymmetric time-varying constraints at all times. A BLF-based controller for a nonlinear cascaded system is designed based on backstepping. Dynamic uncertainty is eliminated by changing the iISS supply rate. A tan-type BLF handles asymmetric time-varying output constraints, and a full-order observer estimates the unmeasurable state. Finally, we can guarantee that the system output does not violate the constraints and that all signals in the closed-loop system are bounded. Simulation results show that the method is effective.

Notations The set of natural numbers is denoted by N, and i-dimensional Euclidean space by R^i. The set of all positive real numbers is R_+. A continuous function $\zeta : R_+ \to R_+$ is said to be of class $\mathcal{P}$ and written as $\zeta \in \mathcal{P}$ if $\zeta(s) > 0$ for all $s \in R_+\backslash\{0\}$ and $\zeta(0) = 0$. A class $\mathcal{P}$ function is said to be of class $\mathcal{K}$ if it is strictly increasing. It is of class $\mathcal{K}_\infty$ if, in addition, $\lim_{s\to\infty} \zeta(s) = \infty$ is satisfied. $x(t)$ represents an appropriate time-varying vector, and $||x(t)||$ is defined as the Euclidean norm of x at time t. $||x||_\infty = sup_{t\geq 0}|x(t)|$, and if $||x||_\infty$ is real, state $x \in L_\infty$. For an n-dimensional vector $x = (x_1, \ldots, x_n)^T \in R^n$, we write $x_{[i]} = (x_1, \ldots, x_i)^T$ when $i = 2, \ldots, n-1$.

2 Problem description

Consider the following class of uncertain chained nonholonomic cascaded systems with output constraints:

$$\begin{aligned}
\dot{\eta} &= q(\eta, x)\\
\dot{x}_0 &= u_0 + x_0\varphi_0(x_0)\\
\dot{x}_i &= x_{i+1}u_0 + \phi_i^d(u_0, x_0, x, \eta), 1 \leq i \leq n-1\\
\dot{x}_n &= u + \phi_n^d(u_0, x_0, x, \eta)\\
y &= (x_0, x_1)^T,
\end{aligned} \tag{1}$$

where $(x_0, x^T)^T = (x_0, x_1, x_2, \ldots, x_n)^T$ and (u_0, u_1) stand for system states and control inputs, respectively; $y \in R^2$ is the measurable output of the system; $\eta \in R^r$ represents unmeasured dynamic uncertainty; $\varphi_0(x_0)$ denotes a well-known smooth, nonnegative function related to x_0; and $\phi_i^d(\cdot) \in R(i = 1, \ldots, n)$ is a kind of unknown nonlinear function.

The following are asymmetric time-varying constraints on the output y:

$$\Omega_{x_i} = \{-k_{i1}(t) < x_i(t) < k_{i2}(t)\}, i = 0, 1, \tag{2}$$

where $k_{i1}(t) > 0$ and $k_{i2}(t) > 0$ are preassigned functions.

Assumption 1 [33] There is a positive definite smooth iISS-Lyapunov function $U_0(\eta)$ for the η-subsystem, such that

$$\underline{\alpha}_\eta(||\eta||) \leq U_0(\eta) \leq \overline{\alpha}_\eta(||\eta||) \tag{3}$$

$$\frac{\partial U_0}{\partial \eta} q(\eta, x) \leq -\alpha_0(||\eta||) + \gamma_0(|x_1|), \tag{4}$$

where $\underline{\alpha}_\eta(\cdot)$, $\overline{\alpha}_\eta(\cdot)$, $\gamma_0(\cdot) \in \mathcal{K}_\infty$, $\alpha_0 \in \mathcal{P}$, and $\gamma_0(\cdot)$ satisfies $\gamma_0(s) = \mathcal{O}(s^2)$ as $s \to 0_+$.

Assumption 2 [34] There exist known nonnegative smooth functions $\varphi_i(u_0, x_0, x_1)$ and unknown nonnegative smooth functions $\psi_i(\eta)$ for every $1 \le i \le n$, such that

$$|\phi_i^d(u_0, x_0, x, \eta)| \le |x_1|(\varphi_i(u_0, x_0, x_1) + \psi_i(\eta)), \tag{5}$$

where $\psi_i^2(s) = \mathcal{O}(\alpha_0(s)), s \to 0_+$ holds when $\liminf_{s\to\infty} \alpha_0(s) = \infty$, and $\psi_i^2(s) = \mathcal{O}(\alpha_0(s)), s \to 0_+, s \to \infty$ holds when $\liminf_{s\to\infty} \alpha_0(s) < \infty$.

Remark 1 Assumption 1 is typical in papers related to input-to-state stability. The concept of the iISS-Lyapunov function in Assumption 1 is mainly used to describe the dynamic uncertainty of the system under study [34] and has become a core tool for the analysis of nonlinear cascaded systems. Assumption 2 states that the unmeasurable dynamic uncertainty can be separated from the nonlinear drift term using a separation Lemma [35] and treated separately, which is a more general assumption than [36]. In addition, to overcome the influence of the nonlinear drift term $\psi_i(\eta)$, it is necessary to impose a local small gain condition.

Lemma 1 [37] *For any* $\tau \in [0, 1)$, *the following inequality is established*

$$\tan\left(\frac{\pi\tau}{2}\right) \le \left(\frac{\pi\tau}{2}\right)\sec\left(\frac{\pi\tau}{2}\right) \le \left(\frac{\pi\tau}{2}\right)\sec^2\left(\frac{\pi\tau}{2}\right). \tag{6}$$

3 Design of output feedback controller

We construct output feedback controllers to asymptotically stabilize a cascaded system with output constraints.

3.1 Discontinuous input state scaling transformation

We first consider the case that $x_0 \ne 0$ and choose the following form of control rate:

$$u_0 = -\lambda_0 x_0 - x_0\varphi_0(x_0), \tag{7}$$

where λ_0 is a positive design parameter. Using the Gronwall–Bellman inequality, we can obtain that for any initial instant $t_0 \ge 0$ and any initial condition $x_0(t_0) \in R$, the corresponding solution $x_0(t) \ne 0$ for each $t \ge t_0$ [6]. Therefore, $u_0 \ne 0$ at any time $t \ge t_0$.

When $u_0 \ne 0$, the state scaling is defined as

$$z_i = \frac{x_i}{u_0^{n-i}}, i = 1, \ldots, n. \tag{8}$$

Accordingly, the following systems are obtained:

$$\begin{aligned} \dot z_i &= z_{i+1} - (n-i)\frac{\dot u_0}{u_0} z_i + \bar\phi_i^d(u_0, x_0, x, \eta), \\ &\quad 1 \le i \le n-1 \\ \dot z_n &= u + \bar\phi_n^d(u_0, x_0, x, \eta), \end{aligned} \tag{9}$$

where $\bar\phi_i^d(u_0, x_0, x, \eta) = \frac{\phi_i^d(u_0,x_0,x,\eta)}{u_0^{n-i}}$, $i = 1, \ldots, n$, and $z_1(0) \in \Omega_{z_1}$, $\Omega_{z_1} = \{z_1 \in R : -b_{11}(t) < z_1(t) < b_{12}(t)\}$, with pre-allocated functions $b_{11}(t) > 0, b_{12}(t) > 0$.

Assumption 3 The time-varying constraints $k_{ij}(t)(i = 0, 1, j = 1, 2)$ on the output y and the time-varying constraints $b_{1j}(t)(j = 1, 2)$ on z_1 are continuous and bounded, and there are positive constants $\underline{k}_{i1}, \underline{k}_{i2}, \underline{b}_{11}, \underline{b}_{12}, \overline{k}_{i1}, \overline{k}_{i2}, \overline{b}_{11}$, and $\overline{b}_{12}$ such that $\underline{k}_{i1} \le k_{i1}(t)$, $|\dot k_{i1}(t)| \le \overline{k}_{i1}$, $\underline{k}_{i2} \le k_{i2}(t)$, $|\dot k_{i2}(t)| \le \overline{k}_{i2}$, $\underline{b}_{11} \le b_{11}(t)$, $|\dot b_{11}(t)| \le \overline{b}_{11}$, $\underline{b}_{12} \le b_{12}(t)$, $|\dot b_{12}(t)| \le \overline{b}_{12}$.

Remark 2 Assumption 3 slightly relaxes the corresponding assumptions imposed on the constrained nonlinear system in [38] by removing the upper bound of the constraint. These constraints are commonly used in practice to ensure that the conditions are bounded when the output has restrictions [37].

Because $u_0 = -\lambda_0 x_0 - x_0\varphi_0(x_0)$ and $\dot x_0 = -\lambda_0 x_0$, it is easy to see that

$$\begin{aligned} \frac{\dot u_0}{u_0} &= \frac{\lambda_0^2 x_0 + \lambda_0 x_0 \varphi_0(x_0) - x_0\dot\varphi_0(x_0)}{-\lambda_0 x_0 - x_0\varphi_0(x_0)} \\ &= -\lambda_0 + \frac{\dot\varphi_0(x_0)}{\lambda_0 + \varphi_0(x_0)} := \omega + \bar\varphi_0(x_0), \end{aligned} \tag{10}$$

where $\bar\varphi_0(x_0)$ is a known smooth continuous function, and ω is a known constant.

From Assumption 2, for each $\bar\phi_i^d(u_0, x_0, x, \eta)$ $(i = 1, \ldots, n)$,

$$\bar\phi_i^d(u_0, x_0, x, \eta) \le |u_0^{i-1}||z_1|(\varphi_i(u_0, x_0, x_1) + \psi_i(\eta)). \tag{11}$$

3.2 Constructing a full-order observer

For convenience of discussion, we write (9) as

$$\dot z = (E - \frac{\dot u_0}{u_0} F)z + bu + \bar\phi^d(u_0, x_0, x, \eta), \tag{12}$$

where $E=\begin{bmatrix}0 & I_{n-1}\\ 0 & 0\end{bmatrix}$, $\bar{\phi}^d(u_0,x_0,x,\eta)=\begin{bmatrix}\bar{\phi}_1^d(\cdot)\\ \vdots \\ \bar{\phi}_n^d(\cdot)\end{bmatrix}$,

$b=\begin{bmatrix}0\\ \vdots\\ 1\end{bmatrix}$, and $F=diag\{n-1,\ldots,1,0\}$.

The controller is designed by using the full-order observer

$$\dot{\hat{z}}=\left(E-\frac{\dot{u}_0}{u_0}F\right)\hat{z}+bu-GCC^T\hat{z}, \tag{13}$$

where $C^T=[1,0,\ldots,0]$ and the gain matrix $G=(g_{ij})_{n\times n}$, with $g_{ij}=g_{ji}(i,j=1,\ldots,n)$ derived from

$$\begin{cases}\dot{G}=G\left(E-\frac{\dot{u}_0}{u_0}F\right)^T+\left(E-\frac{\dot{u}_0}{u_0}F\right)G-GCC^TG+I_n\\ G(0)=G_0>0\end{cases} \tag{14}$$

The following Lemma guarantees that (13) and (14) make sense.

Lemma 4 [39] *There exist two strictly positive real numbers, g_{min} and g_{max}, so that the solution $G(t)$ of the following equation satisfies $g_{min}I_n\le G(t)\le g_{max}I_n$, $t\ge 0$, for any continuous function $\mu_0(t)$:*

$$\begin{cases}\dot{G}=G(E-\mu_0F)^T+(E-\mu_0F)G-GCC^TG+I_n\\ G(0)=G_0>0\end{cases} \tag{15}$$

Define the error $\varepsilon=z-\hat{z}$. From (12) and (13), it follows that

$$\begin{aligned}\dot{\varepsilon}&=\left(E-\frac{\dot{u}_0}{u_0}F\right)\varepsilon+\bar{\phi}^d(u_0,x_0,x,\eta)+GCC^T\hat{z}\\ &=(E-\frac{\dot{u}_0}{u_0}F-GCC^T)\varepsilon+z_1GC\\ &\quad+\bar{\phi}^d(u_0,x_0,x,\eta).\end{aligned} \tag{16}$$

Lemma 5 *Select $V_\varepsilon(\varepsilon,G)=\varepsilon^TG^{-1}(t)\varepsilon$ to respond to the error system. Then the time derivative of V_ε along (14) is*

$$\begin{aligned}\dot{V}_\varepsilon(\varepsilon,G)\le&-\frac{1}{2}\varepsilon^TG^{-2}(t)\varepsilon+z_1^2\varphi_{z_1}(x_0,x_1)\\ &+2\Sigma_{i=1}^n\psi_i^4(\eta),\end{aligned} \tag{17}$$

where $\varphi_{z_1}(x_0,x_1)$ is given in the following proof.

Proof Based on $\overset{\cdot}{\overbrace{G^{-1}(t)}}=-G^{-1}(t)\dot{G}(t)G^{-1}(t)$, the time derivative of $V_\varepsilon(\varepsilon,G)$ satisfies

$$\begin{aligned}\dot{V}_\varepsilon&=\dot{\varepsilon}^TG^{-1}(t)\varepsilon+\varepsilon^TG^{-1}(t)\dot{\varepsilon}+\varepsilon^T\overset{\cdot}{\overbrace{G^{-1}(t)}}\varepsilon\\ &=\Bigg[\left(E-\frac{\dot{u}_0}{u_0}F-GCC^T\right)\varepsilon+z_1GC\\ &\quad+\bar{\phi}^d(u_0,x_0,x,\eta)\Bigg]^TG^{-1}(t)\varepsilon+\varepsilon^T\overset{\cdot}{\overbrace{G^{-1}(t)}}\varepsilon\\ &\quad+\varepsilon^TG^{-1}(t)\Bigg[\left(E-\frac{\dot{u}_0}{u_0}F-GCC^T\right)\varepsilon+z_1GC\\ &\quad+\bar{\phi}^d(u_0,x_0,x,\eta)\Bigg]\\ &=\varepsilon^T\Bigg[\left(E-\frac{\dot{u}_0}{u_0}F\right)^TG^{-1}(t)\\ &\quad+G^{-1}(t)\left(E-\frac{\dot{u}_0}{u_0}F\right)-2CC^T\Bigg]\varepsilon\\ &\quad+2z_1C^T\varepsilon+2\varepsilon^TG^{-1}(t)\bar{\phi}^d(u_0,x_0,x,\eta)\\ &\quad+\varepsilon^T\overset{\cdot}{\overbrace{G^{-1}(t)}}\varepsilon\\ &=\varepsilon^TG^{-1}(t)[-G(t)CC^TG(t)-I_n]G^{-1}(t)\varepsilon+2z_1\varepsilon_1\\ &\quad+2\varepsilon^TG^{-1}(t)\bar{\phi}^d(u_0,x_0,x,\eta)\\ &=-\varepsilon^TCC^T\varepsilon-\varepsilon^TG^{-2}(t)\varepsilon\\ &\quad+2z_1\varepsilon_1+2\varepsilon^TG^{-1}(t)\bar{\phi}^d(u_0,x_0,x,\eta)\\ &=-\varepsilon^TG^{-2}(t)\varepsilon+2z_1\varepsilon_1\\ &\quad-\varepsilon_1^2+2\varepsilon^TG^{-1}(t)\bar{\phi}^d(u_0,x_0,x,\eta).\end{aligned} \tag{18}$$

Applying Young's inequality,

$$2z_1\varepsilon_1\le\varepsilon_1^2+z_1^2 \tag{19}$$

$$\begin{aligned}&2\varepsilon^TG^{-1}(t)\bar{\phi}^d(u_0,x_0,x,\eta)\\ &\quad\le 2||\bar{\phi}^d(u_0,x_0,x,\eta)||^2+\frac{1}{2}\varepsilon^TG^{-2}(t)\varepsilon.\end{aligned} \tag{20}$$

From (11), it follows that

$$\begin{aligned}||\bar{\phi}^d(\cdot)||^2&=\bar{\phi}_1^d(\cdot)^2+\bar{\phi}_2^d(\cdot)^2+\cdots+\bar{\phi}_n^d(\cdot)^2\\ &\le[z_1(\varphi_1(\cdot)+\psi_1(\cdot))]^2\\ &\quad+[u_0z_1(\varphi_2(\cdot)+\psi_2(\cdot))]^2\\ &\quad+\cdots+[u_0^{n-1}z_1(\varphi_n(\cdot)+\psi_n(\cdot))]^2\\ &\le 2\Sigma_{i=1}^nz_1^2u_0^{2i-2}\varphi_i^2(\cdot)\\ &\quad+\Sigma_{i=1}^nz_1^4u_0^{4i-4}+\Sigma_{i=1}^n\psi_i^4(\cdot).\end{aligned} \tag{21}$$

After analysis, it holds that

$$\begin{aligned}\dot{V}_\varepsilon(\varepsilon,G)&\le-\frac{1}{2}\varepsilon^TG^{-2}(t)\varepsilon+z_1^2(1+4\Sigma_{i=1}^nu_0^{2i-2}\varphi_i^2(\cdot)\\ &\quad+2\Sigma_{i=1}^nz_1^2u_0^{4i-4})+2\Sigma_{i=1}^n\psi_i^4(\cdot)\\ &\le-\frac{1}{2}\varepsilon^TG^{-2}(t)\varepsilon+z_1^2\varphi_{z_1}(x_0,x_1)+2\Sigma_{i=1}^n\psi_i^4(\eta),\end{aligned} \tag{22}$$

where $\varphi_{z_1}(x_0, x_1) = 1 + 4\Sigma_{i=1}^{n} u_0^{2i-2}\varphi_i^2(\cdot) + 2\Sigma_{i=1}^{n} z_1^2 u_0^{4i-4}$. The proof is completed. □

3.3 Stability design of x_0 subsystem

For the control design, we construct a tan-type BLF

$$V_0(x_0) = \frac{k_{b0}^2}{\pi} \tan\left(\frac{\pi x_0^2}{2k_{b0}^2}\right), \tag{23}$$

where $k_{b0} = k_{02}$ if $x_0 > 0$, and otherwise $k_{b0} = k_{01}$.

Remark 3 The analysis form of traditional BLFs [38] has the shape $V_0(x_0) = \frac{1}{2}\log\frac{k_{b0}^2}{k_{b0}^2 - x_0^2}$, which is not suitable for integrating the constraint analysis into a general approach. Compared with the traditional log-type BLF, the tan-type BLF can make full use of the structural characteristics of system (1) and has more attractive characteristics. When there are no constraints on x_0 (i.e., constraint k_{b0} is infinite), we can obtain $\lim_{k_{b0}\to\infty} \frac{k_{b0}^2}{\pi}\tan(\frac{\pi x_0^2}{2k_{b0}^2}) = \frac{1}{2}x_0^2$. Therefore, the proposed tan-type BLF can handle the stability problem with or without constraints.

Taking the derivation of $V_0(x_0)$ yields

$$\dot V_0(x_0) = \sec^2\left(\frac{\pi x_0^2}{2k_{b0}^2}\right) x_0\dot x_0 + \frac{2k_{b0}}{\pi}\tan\left(\frac{\pi x_0^2}{2k_{b0}^2}\right)\dot k_{b0} - \frac{x_0^2}{k_{b0}}\sec^2\left(\frac{\pi x_0^2}{2k_{b0}^2}\right)\dot k_{b0}. \tag{24}$$

For convenience of calculation, we define

$$\phi_{b0}(x_0) = \begin{cases} \phi_{k_{02}}(x_0) = \sec^2\left(\frac{\pi x_0^2}{2k_{02}^2}\right), x_0 > 0 \\ \phi_{k_{01}}(x_0) = \sec^2\left(\frac{\pi x_0^2}{2k_{01}^2}\right), x_0 \le 0. \end{cases} \tag{25}$$

By the definition of $\phi_{b0}(x_0)$ and (23),

$$\begin{aligned}\dot V_0(x_0) &\le \phi_{b0}(x_0)x_0(u_0 + x_0\varphi_0(x_0)) - \frac{x_0^2}{k_{b0}}\phi_{b0}(x_0)\dot k_{b0} + \frac{x_0^2}{k_{b0}}\phi_{b0}(x_0)|\dot k_{b0}| \\ &\le \phi_{b0}(x_0)x_0(u_0 + x_0\varphi_0(x_0)) + \frac{2}{k_{b0}}\phi_{b0}(x_0)x_0^2|\dot k_{b0}|.\end{aligned} \tag{26}$$

Substituting (7) into (26) yields

$$\begin{aligned}\dot V_0(x_0) &\le -\phi_{b0}(x_0)\lambda_0 x_0^2 + \frac{2}{k_{b0}}\phi_{b0}(x_0)x_0^2|\dot k_{b0}| \\ &\le -(\lambda_0 - \nu_1)x_0^2\phi_{b0}(x_0) \\ &\le -(\lambda_0 - \nu_1)x_0^2 V_0 \\ &\le 0,\end{aligned} \tag{27}$$

where $\lambda_0 > \nu_1 \ge \frac{2\bar k_{b0}}{\underline{k}_{b0}}$, $\bar k_{b0} = \max\{\bar k_{01}, \bar k_{02}\}$, $\underline{k}_{b0} = \min\{\underline{k}_{01}, \underline{k}_{02}\}$. Because $\dot V_0(x_0) \le -(\lambda_0 - \nu_1)x_0^2 V_0 \le 0$, $x_0(t)$ exponentially converges to zero.

Next, we prove that x_0 satisfies time-varying constraints.

$$V_0(x_0) = \frac{k_{b0}^2}{\pi}\tan\left(\frac{\pi x_0^2}{2k_{b0}^2}\right) \le V_0(x_0(0)) \tag{28}$$

$$\frac{\pi x_0^2}{2k_{b0}^2} \le \arctan\left(\frac{V_0(x_0(0))\pi}{k_{b0}^2}\right) < \frac{\pi}{2}, \tag{29}$$

and correspondingly,

$$x_0^2 < k_{b0}, -k_{01}(t) < x_0(t) < k_{02}(t). \tag{30}$$

3.4 Backstepping design

The controller is devised by applying the backstepping method. Consider the system

$$\begin{aligned}\dot z_1 &= \hat z_2 + \varepsilon_2 - (n-1)\frac{\dot u_0}{u_0}z_1 + \bar\phi_1^d(u_0, x_0, x, \eta) \\ \dot{\hat z}_i &= \hat z_{i+1} - (n-i)\frac{\dot u_0}{u_0}\hat z_i - C_i^T GCC^T\hat z, 2 \le i \le n-1 \\ \dot{\hat z}_n &= u - C_n^T GCC^T\hat z,\end{aligned} \tag{31}$$

where $C_2 = [0, 1, \ldots, 0]^T, \ldots, C_n = [0, 0, \ldots, 1]^T$.

Suppose $z(0) \in \{z(t) \in R^n | -b_{11}(t) < z_1(t) < b_{12}(t)\}$ and define the Lyapunov function

$$V_{k_{b1}}(z_1) = \frac{k_{b1}^2}{\pi}\tan\left(\frac{\pi z_1^2}{2k_{b1}^2}\right), \tag{32}$$

where $k_{b1} = b_{12}$ if $x_0 > 0$, and otherwise $k_{b1} = b_{11}$.

Alternatively, as shown previously, we may calculate the derivative of $V_{k_{b1}}$ to obtain

$$\dot V_{k_{b1}}(z_1) = \sec^2\left(\frac{\pi z_1^2}{2k_{b1}^2}\right) z_1\dot z_1 + \frac{2k_{b1}}{\pi}\tan\left(\frac{\pi z_1^2}{2k_{b1}^2}\right)\dot k_{b1} - \frac{z_1^2}{k_{b1}}\sec^2\left(\frac{\pi z_1^2}{2k_{b1}^2}\right)\dot k_{b1}, \tag{33}$$

and

$$\phi_{b1}(z_1) = \begin{cases} \phi_{b_{12}}(z_1) = \sec^2\left(\frac{\pi z_1^2}{2b_{12}^2}\right), z_1 > 0 \\ \phi_{b_{11}}(z_1) = \sec^2\left(\frac{\pi z_1^2}{2b_{11}^2}\right), z_1 \le 0. \end{cases} \quad (34)$$

Applying the inequality of Lemma 1, it follows that

$$\dot{V}_{k_{b1}}(z_1) \le \phi_{b1}(z_1)z_1\dot{z}_1 + \nu_2 z_1^2\phi_{b1}(z_1), \quad (35)$$

where $\nu_2 \ge \frac{2\bar{k}_{b1}}{\underline{k}_{b1}}$, $\bar{k}_{b1} = \max\{\bar{b}_{11}, \bar{b}_{12}\}$, and $\underline{k}_{b1} = \min\{\underline{b}_{11}, \underline{b}_{12}\}$. We next explore the design of the back-stepping method.

Step 1 Let $\xi_1 = z_1$ and $\xi_2 = \hat{z}_2 - \alpha_1$, where α_1 is the virtual control law, and ξ_2 is the error variable. We choose the candidate BLF as

$$V_1 = V_\varepsilon + V_{k_{b1}}(\xi_1). \quad (36)$$

Then, from Lemma 5 and (35),

$$\begin{aligned} \dot{V}_1 &= \dot{V}_\varepsilon + \dot{V}_{k_{b1}}(\xi) \\ &\le -\frac{1}{2}\varepsilon^T G^{-2}(t)\varepsilon + \xi_1^2\varphi_{z_1}(x_0, x_1) \\ &\quad + 2\Sigma_{i=1}^n \psi_i^4(\eta) + \phi_{b1}(\xi_1)\xi_1[\hat{z}_2 + \varepsilon_2 \\ &\quad - (n-1)\frac{\dot{u}_0}{u_0}\xi_1 + \bar{\phi}_1^d(u_0, x_0, x, \eta)] \\ &\quad + \nu_2\xi_1^2\phi_{b1}(\xi_1), \end{aligned} \quad (37)$$

and from (11)

$$\phi_{b1}(\xi_1)\xi_1\varepsilon_2 \le \delta\varepsilon^T G^{-2}\varepsilon + \frac{1}{4\delta}g_{max}^2\xi_1^2\phi_{b1}^2(\xi_1) \quad (38)$$

$$\begin{aligned} \phi_{b1}(\xi_1)\xi_1\bar{\phi}_1^d(u_0, x_0, x, \eta) &\le \phi_{b1}(\xi_1)\xi_1^2\varphi_1(u_0, x_0, x_1) \\ &\quad + \frac{1}{4}\xi_1^4\phi_{b1}^2(\xi_1) + \psi_1^2(\eta). \end{aligned} \quad (39)$$

Substituting (38) and (39) into (37), $\dot{V}_1$ satisfies

$$\begin{aligned} \dot{V}_1 &\le -\left(\frac{1}{2} - \delta\right)\varepsilon^T G^{-2}(t)\varepsilon + \xi_1^2\varphi_{z_1}(x_0, x_1) \\ &\quad + 2\Sigma_{i=1}^n \psi_i^4(\eta) + \psi_1^2(\eta) + \phi_{b1}(\xi_1)\xi_1\xi_2 \\ &\quad + \phi_{b1}(\xi_1)\xi_1[\alpha_1 + \frac{1}{4\delta}g_{max}^2\xi_1\phi_{b1}(\xi_1) \\ &\quad + \xi_1\varphi_1(u_0, x_0, x_1) \\ &\quad - (n-1)\frac{\dot{u}_0}{u_0}\xi_1 + \frac{1}{4}\xi_1^3\phi_{b1}(\xi_1) + \nu_2\xi_1]. \end{aligned} \quad (40)$$

Because $\xi_1 = z_1$, α_1 is selected as

$$\begin{aligned} \alpha_1(x_0, z_1, k_{b1}) &= -\iota_1(x_0, \xi_1)\xi_1 - \frac{1}{4\delta}g_{max}^2\xi_1\phi_{b1}(\xi_1) \\ &\quad - \xi_1\varphi_1(u_0, x_0, x_1) + (n-1)\frac{\dot{u}_0}{u_0}\xi_1 \\ &\quad - \frac{1}{4}\xi_1^3\phi_{b1}(\xi_1) - \nu_2\xi_1, \end{aligned} \quad (41)$$

where $\iota_1(x_0, \xi_1)$ is a smooth, positive function dependent on (x_0, ξ_1), which we give below. Taking (41) into (40), we get

$$\begin{aligned} \dot{V}_1 &\le -\left(\frac{1}{2} - \delta\right)\varepsilon^T G^{-2}(t)\varepsilon + \xi_1^2\varphi_{z_1}(x_0, x_1) \\ &\quad + 2\Sigma_{i=1}^n \psi_i^4(\eta) + \psi_1^2(\eta) + \phi_{b1}(\xi_1)\xi_1\xi_2 \\ &\quad - \iota_1(x_0, \xi_1)\xi_1^2\phi_{b1}(\xi_1). \end{aligned} \quad (42)$$

Step i $(2 \le i \le n)$: Suppose a few virtual control rates $\alpha_j(x_0, \xi_1, \hat{z}_{[j]}, G, k_{b1})$ with $\xi_j = \hat{z}_j - \alpha_{j-1}(2 \le j \le i)$, and some Lyapunov functions V_{i-1} have been structured in step $i-1$,

$$V_{i-1} = V_\varepsilon + V_{k_{b1}}(\xi) + \frac{1}{2}\Sigma_{j=2}^{i-1}\xi_j^2 \quad (43)$$

The derivative of V_{i-1} satisfies

$$\begin{aligned} \dot{V}_{i-1} &\le -\left(\frac{1}{2} - (i-1)\delta\right)\varepsilon^T G^{-2}(t)\varepsilon + \xi_1^2\varphi_{z_1}(x_0, x_1) \\ &\quad + 2\Sigma_{i=1}^n \psi_i^4(\eta) + (i-1)\psi_1^2(\eta) + (i-2)\xi_1^2 \\ &\quad + \xi_{i-1}\xi_i - \iota_1(x_0, \xi_1)\xi_1^2\phi_{b1}(\xi_1) - \Sigma_{j=2}^{i-1}\iota_j\xi_j^2, \end{aligned} \quad (44)$$

where the design parameter satisfy $\iota_j > 0$.

We plan to establish a similar property at step i. Suppose $\xi_{i+1} = \hat{z}_{i+1} - \alpha_i$, and we select V_i as

$$V_i = V_{i-1} + \frac{1}{2}\xi_i^2. \quad (45)$$

We first take the derivative of ξ_i to facilitate the calculation and obtain that

$$\begin{aligned} \dot{\xi}_i &= \dot{\hat{z}}_i - \dot{\alpha}_{i-1} \\ &= \xi_{i+1} + \alpha_i - (n-i)\frac{\dot{u}_0}{u_0}\hat{z}_i - C_i^T GC\hat{z}_1 - \frac{\partial\alpha_{i-1}}{\partial x_0}\dot{x}_0 \\ &\quad - \frac{\partial\alpha_{i-1}}{\partial k_{b1}}\dot{k}_{b1} - \frac{\partial\alpha_{i-1}}{\partial z_1}\hat{z}_2 + (n-1)\frac{\partial\alpha_{i-1}}{\partial z_1}\frac{\dot{u}_0}{u_0}z_1 \\ &\quad - \frac{\partial\alpha_{i-1}}{\partial z_1}\bar{\phi}_1^d(u_0, x_0, x, \eta) - \frac{\partial\alpha_{i-1}}{\partial z_1}\varepsilon_2 \\ &\quad - \Sigma_{k,l=1}^n\frac{\partial\alpha_{i-1}}{\partial g_{kl}}\dot{g}_{kl} - \Sigma_{j=2}^{i-1}\frac{\partial\alpha_{i-1}}{\partial\hat{z}_j}\dot{\hat{z}}_j. \end{aligned} \quad (46)$$

From (11), one obtains

$$-\xi_i\frac{\partial\alpha_{i-1}}{\partial z_1}\varepsilon_2 \le \delta\varepsilon^T G^{-2}\varepsilon + \frac{1}{4\delta}g_{max}^2\xi_i^2\left(\frac{\partial\alpha_{i-1}}{\partial z_1}\right)^2 \quad (47)$$

$$-\xi_i \frac{\partial \alpha_{i-1}}{\partial z_1} \bar{\phi}_1^d(u_0, x_0, x, \eta) \le \frac{1}{4}\xi_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial z_1}\right)^2 \varphi_1^2(u_0, x_0, x_1) + \xi_1^2 + \frac{1}{4}\xi_i^2 \left(\frac{\partial \alpha_{i-1}}{\partial z_1}\right)^2 \xi_1^2 + \psi_1^2(\eta). \quad (48)$$

Relations (46)–(48) yield

$$\begin{aligned}\dot{V}_i &= \dot{V}_{i-1} + \xi_i \dot{\xi}_i \\ &\le -\left(\frac{1}{2} - i\delta\right) \varepsilon^T G^{-2}(t)\varepsilon + \xi_1^2 \varphi_{z_1}(x_0, x_1) \\ &\quad + 2\Sigma_{i=1}^n \psi_i^4(\eta) + i\psi_1^2(\eta) - \iota_1(x_0, \xi_1)\xi_1^2 \phi_{b1}(\xi_1) \\ &\quad + (i-1)\xi_1^2 + \xi_i \xi_{i+1} - \Sigma_{j=2}^{i-1} \iota_j \xi_j^2 + \xi_i [\alpha_i \\ &\quad + \xi_{i-1} - (n-i)\frac{\dot{u}_0}{u_0}\hat{z}_i - \frac{\partial \alpha_{i-1}}{\partial z_1}\hat{z}_2 - C_i^T G C \hat{z}_1 \\ &\quad - \Sigma_{k,l=1}^n \frac{\partial \alpha_{i-1}}{\partial g_{kl}} \dot{g}_{kl} - \Sigma_{j=2}^{i-1} \frac{\partial \alpha_{i-1}}{\partial \hat{z}_j} \dot{\hat{z}}_j - \frac{\partial \alpha_{i-1}}{\partial x_0}\dot{x}_0 \\ &\quad - \frac{\partial \alpha_{i-1}}{\partial k_{b1}} \dot{k}_{b1} + (n-1)\frac{\partial \alpha_{i-1}}{\partial z_1}\frac{\dot{u}_0}{u_0} z_1 \\ &\quad + \frac{1}{4}\xi_i \left(\frac{\partial \alpha_{i-1}}{\partial z_1}\right)^2 \xi_1^2 + \frac{1}{4\delta} g_{max}^2 \xi_i \left(\frac{\partial \alpha_{i-1}}{\partial z_1}\right)^2 \\ &\quad + \frac{1}{4}\xi_i \left(\frac{\partial \alpha_{i-1}}{\partial z_1}\right)^2 \varphi_1^2(u_0, x_0, x_1)], \end{aligned} \quad (49)$$

form which a virtual control rate is designed as

$$\begin{aligned}\alpha_i(x_0, \xi_1, \hat{z}_{[i]}, G, k_{b1}) &= -\iota_i \xi_i - \xi_{i-1} \\ &\quad + (n-i)\frac{\dot{u}_0}{u_0}\hat{z}_i + C_i^T G C \hat{z}_1 + \frac{\partial \alpha_{i-1}}{\partial x_0}\dot{x}_0 + \frac{\partial \alpha_{i-1}}{\partial k_{b1}}\dot{k}_{b1} \\ &\quad + \Sigma_{k,l=1}^n \frac{\partial \alpha_{i-1}}{\partial g_{kl}}\dot{g}_{kl} + \Sigma_{j=2}^{i-1}\frac{\partial \alpha_{i-1}}{\partial \hat{z}_j}\dot{\hat{z}}_j + \frac{\partial \alpha_{i-1}}{\partial z_1}\hat{z}_2 \\ &\quad - (n-1)\frac{\partial \alpha_{i-1}}{\partial z_1}\frac{\dot{u}_0}{u_0} z_1 - \frac{1}{4\delta} g_{max}^2 \xi_i \left(\frac{\partial \alpha_{i-1}}{\partial z_1}\right)^2 \\ &\quad - \frac{1}{4}\xi_i \left(\frac{\partial \alpha_{i-1}}{\partial z_1}\right)^2 \varphi_1^2(u_0, x_0, x_1) - \frac{1}{4}\xi_i \left(\frac{\partial \alpha_{i-1}}{\partial z_1}\right)^2 \xi_1^2. \end{aligned} \quad (50)$$

Then (49) can be rewritten as

$$\begin{aligned}\dot{V}_i &\le -\left(\frac{1}{2} - i\delta\right)\varepsilon^T G^{-2}(t)\varepsilon + \xi_1^2 \varphi_{z_1}(x_0, x_1) \\ &\quad + 2\Sigma_{i=1}^n \psi_i^4(\eta) + i\psi_1^2(\eta) + (i-1)\xi_1^2 \\ &\quad + \xi_i \xi_{i+1} - \iota_1(x_0, \xi_1)\xi_1^2 \phi_{b1}(\xi_1) \\ &\quad - \Sigma_{j=2}^i \iota_j \xi_j^2. \end{aligned} \quad (51)$$

In particular, when $i = n$, the control law $u(t)$ is chosen as

$$\begin{aligned}u &= \alpha_n(x_0, \xi_1, \hat{z}_{[i]}, G) = -\iota_n \xi_n - \xi_{n-1} \\ &\quad + C_n^T G C \hat{z}_1 + \frac{\partial \alpha_{n-1}}{\partial x_0}\dot{x}_0 + \frac{\partial \alpha_{n-1}}{\partial k_{b1}}\dot{k}_{b1} \\ &\quad + \Sigma_{k,l=1}^n \frac{\partial \alpha_{n-1}}{\partial g_{kl}}\dot{g}_{kl} + \Sigma_{j=2}^{n-1}\frac{\partial \alpha_{n-1}}{\partial \hat{z}_j}\dot{\hat{z}}_j + \frac{\partial \alpha_{n-1}}{\partial z_1}\hat{z}_2 \\ &\quad - (n-1)\frac{\partial \alpha_{n-1}}{\partial z_1}\frac{\dot{u}_0}{u_0} z_1 - \frac{1}{4\delta} g_{max}^2 \xi_n \left(\frac{\partial \alpha_{n-1}}{\partial z_1}\right)^2 \\ &\quad - \frac{1}{4}\xi_n \left(\frac{\partial \alpha_{n-1}}{\partial z_1}\right)^2 \varphi_1^2(u_0, x_0, x_1) \\ &\quad - \frac{1}{4}\xi_n \left(\frac{\partial \alpha_{n-1}}{\partial z_1}\right)^2 \xi_1^2, \end{aligned} \quad (52)$$

which guarantees the Lyapunov function

$$V_n = V_\varepsilon + V_{k_{b1}} + \frac{1}{2}\Sigma_{i=2}^n \xi_i^2, \quad (53)$$

satisfying

$$\begin{aligned}\dot{V}_n &\le -\left(\frac{1}{2} - n\delta\right)\varepsilon^T G^{-2}(t)\varepsilon + \xi_1^2 \varphi_{z_1}(x_0, x_1) + n\psi_1^2(\eta) \\ &\quad + 2\Sigma_{i=1}^n \psi_i^4(\eta) - \iota_1(x_0, \xi_1)\xi_1^2 \phi_{b1}(\xi_1) \\ &\quad + (n-1)\xi_1^2 - \Sigma_{j=2}^n \iota_j \xi_j^2. \end{aligned} \quad (54)$$

Next, we will use the technology of changing the supply rate to eliminate the dynamic uncertainty. Therefore, we construct another form iISS-Lyapunov function,

$$\overline{U}_0 = \int_0^{U_0(\eta)} \rho(s)ds, \quad (55)$$

where $\rho(s)$ is a continuously nondecreasing positive function to be selected.

Since η satisfies the iISS condition in Assumption 1, it holds that

$$\begin{aligned}\dot{\overline{U}}_0(\eta) &\le -\frac{1}{2}\alpha_0(||\eta||)\rho \circ \underline{\alpha}_\eta(||\eta||) \\ &\quad + \rho \circ \overline{\alpha}_\eta \circ \alpha_0^{-1} \circ 2\gamma_0(|x_1|)\gamma_0(|x_1|). \end{aligned} \quad (56)$$

(i) In the case that $\liminf_{s\to\infty} \alpha_0(s) = \infty$, it can be known that there exists a $\mathcal{K}_\infty$ function $\hat{\alpha}$ satisfying $\hat{\alpha}(s) \le \alpha_0(s)$ for any $s \ge 0$. Because $\hat{\alpha}(s) \le \alpha_0(s)$ and $\psi_i^2(s) = \mathcal{O}(\alpha_0(s))$, $s \to 0_+$, $i = 1, \ldots, n$, it can be obtained that there exists a continuous nondecreasing positive function $q(s)$ satisfying $\psi_i^2(s) \le q(s)\alpha_0(s)$, $s \ge 0$. Considering the condition of Assumption 2 and continuity, α_0 has a maximum at $[0, s_1]$, which is $\tilde{\alpha}(\tilde{s})(0 \le \tilde{s} \le s_1)$. Consequently, the following inequality holds:

$$\psi_i^4(s) = \psi_i^2(s) \cdot \psi_i^2(s) \le q(s)^2 \tilde{\alpha}(\tilde{s})\alpha_0(s) = q_1(s)\alpha_0(s). \quad (57)$$

We can similarly obtain that $\psi_i^4(s) = \mathcal{O}(\alpha_0(s)), s \to 0_+$, and $\psi_i^4(s) \leq q_1(s)\alpha_0(s), s \geq 0, i = 1, \ldots, n$. Therefore, there exists an appropriate function $\rho(\cdot)$ that satisfies

$$\frac{1}{8}\alpha_0(||\eta||)\rho \circ \underline{\alpha}_\eta(||\eta||) \geq \max\{2\Sigma_{i=1}^n \psi_i^4(\eta), n\psi_1^2(\eta)\}; \tag{58}$$

(ii) If $\liminf_{s\to\infty} \alpha_0(s) < \infty$, given that $\psi_i^2(s) = \mathcal{O}(\alpha_0(s)), s \to 0_+, s \to \infty, i = 1, \ldots, n$, there exists a constant c_1 such that, for any $s > 0, \psi_i^2(s) < c_1\alpha_0(s)$. Since $\alpha_0 \in \mathcal{P}$ and $\liminf_{s\to\infty} \alpha_0(s) < \infty$, α_0 has a maximum value of $\tilde{\alpha}(\tilde{s})$ on $[0, \infty)$. Thus, we can obtain that

$$\psi_i^4(s) = \psi_i^2(s) \cdot \psi_i^2(s) \leq c_1^2\tilde{\alpha}(\tilde{s})\alpha_0(s) = c\alpha_0. \tag{59}$$

Accordingly, (58) still holds.

Remark 4 In previous studies of nonlinear control systems with iISS inverse dynamics, the decay rate $\alpha_0(s)$ is a $\mathcal{K}$ function or a non-oscillatory function [32]. We consider the case of a general oscillation function and deal with unpredictable dynamic uncertainty in two cases. From the above analysis, we can see that, for a general oscillation function, the condition $\psi_i^2(s) = \mathcal{O}(\alpha_0(s)), s \to \infty, i = 1, \ldots, n$ is necessary, i.e., if $\liminf_{s\to\infty} \alpha_0(s) < \infty$, then (i) is not valid for some iISS systems.

We take into account candidate V for the total Lyapunov function in the stability study,

$$V = V_\varepsilon + V_{k_{b1}} + \frac{1}{2}\Sigma_{i=2}^n \xi_i^2 + \overline{U}_0(\eta). \tag{60}$$

Then

$$\begin{aligned}\dot{V} \leq &-\left(\frac{1}{2} - n\delta\right)\varepsilon^T G^{-2}(t)\varepsilon + \xi_1^2\varphi_{z_1}(x_0, x_1)\\ &-\frac{1}{4}\alpha_0(||\eta||)\rho \circ \underline{\alpha}_\eta(||\eta||) - \iota_1(x_0, \xi_1)\xi_1^2\phi_{b1}(\xi_1)\\ &+(n-1)\xi_1^2 - \Sigma_{j=2}^n \iota_j\xi_j^2\\ &+\rho \circ \overline{\alpha}_\eta \circ \alpha_0^{-1} \circ 2\gamma_0(|x_1|)\gamma_0(|x_1|).\end{aligned} \tag{61}$$

For the small gain condition $\gamma_0(s) = \mathcal{O}(s^2), s \to 0_+$ and state scaling (8), there exists a smooth positive and nondecreasing function $\hat{\gamma}_0(s)$ satisfying

$$\gamma_0(|x_1|) = \gamma_0(|u_0^{n-1}\xi_1|) \leq \xi_1^2\hat{\gamma}_0(x_0, \xi_1). \tag{62}$$

Therefore,

$$\begin{aligned}&\rho \circ \overline{\alpha}_\eta \circ \alpha_0^{-1} \circ 2\gamma_0(|x_1|)\gamma_0(|x_1|)\\ &\quad\leq \rho \circ \overline{\alpha}_\eta \circ \alpha_0^{-1} \circ 2\gamma_0(|x_1|)\hat{\gamma}_0(x_0, \xi_1)\xi_1^2.\end{aligned} \tag{63}$$

Furthermore, it holds that

$$\begin{aligned}\dot{V} \leq &-\left(\frac{1}{2} - n\delta\right)\varepsilon^T G^{-2}(t)\varepsilon\\ &-\frac{1}{4}\alpha_0(||\eta||)\rho \circ \underline{\alpha}_\eta(||\eta||)\\ &-\Sigma_{j=2}^n \iota_j\xi_j^2 - [\iota_1(x_0, \xi_1)\phi_{b1}(\xi_1) - (n-1)\\ &-\rho \circ \overline{\alpha}_\eta \circ \alpha_0^{-1} \circ 2\gamma_0(|x_1|)\hat{\gamma}_0(x_0, \xi_1)\\ &-\varphi_{z_1}(x_0, x_1)]\xi_1^2.\end{aligned} \tag{64}$$

Taking the constants $0 < \delta \leq \frac{1}{4n}$, $\iota_j \geq 1 (j = 2, \ldots, n)$, and the smooth function

$$\begin{aligned}\iota_1(x_0, \xi_1)\phi_{b1}(\xi_1) \geq\ &\rho \circ \overline{\alpha}_\eta \circ \alpha_0^{-1} \circ 2\gamma_0(|x_1|)\hat{\gamma}_0(x_0, \xi_1)\\ &+ n + \varphi_{z_1}(x_0, x_1),\end{aligned} \tag{65}$$

we examine the term

$$\begin{aligned}\dot{V} \leq &-\frac{1}{4}\varepsilon^T G^{-2}(t)\varepsilon - \Sigma_{i=1}^n \xi_i^2\\ &-\frac{1}{4}\alpha_0(||\eta||)\rho \circ \underline{\alpha}_\eta(||\eta||).\end{aligned} \tag{66}$$

The above results can be summarized as the following theorem.

Theorem 1 *Assume that the nonlinear cascaded system meets Assumptions 1–4, and the control rates are given by (7) and (52). If the output meets constraints (2), then the subsequent properties are established:*

(i) *The signals $(x(t), \varepsilon(t), \eta(t), \xi(t))$ of the nonlinear cascaded system are bounded;*

(ii) *The system states and control inputs asymptotically converge to 0, i.e.,*

$$\lim_{t\to\infty} (||\eta(t)|| + ||x(t)|| + ||x_0(t)||) = 0; \tag{67}$$

(iii) *The symmetric time-varying output constraints are not violated, i.e.,*

$$-k_{i1}(t) < x_i(t) < k_{i2}(t), i = 0, 1. \tag{68}$$

Proof (i) According to the definition of V, and because $\dot{V} \leq 0$, the signals $(x_0(t), \varepsilon(t), \eta(t), \xi(t))$ in nonlinear cascaded systems are bounded in the whole control process. Then, because $\varepsilon_1 \in L_\infty$, $z_1 = \xi_1 \in L_\infty$, and $\varepsilon_1 = z_1 - \hat{z}_1$, we can obtain that $\hat{z}_1 \in L_\infty$. Furthermore, we conclude that α_1 is limited due to (41). Because $\xi_2 = \hat{z}_2 - \alpha_1$ and $\xi_2 \in L_\infty$, we can say that $\hat{z}_2 \in L_\infty$. Hence, it is concluded that $\hat{z}_i (i = 1, \ldots, n)$ are bounded. Because $\varepsilon = z - \hat{z}$ and $\hat{z} \in L_\infty$, we can obtain that $z \in L_\infty$. Furthermore, x_i are bounded by $x_i = z_i u_0^{n-i}$. Therefore, the solution exists and is unique on $[0, \infty)$.

(ii) LaSalle's Invariant Theorem states that when t approaches infinity, $(\varepsilon(t), \eta(t), \xi(t))$ converge to 0. Therefore, because $\xi_1 = z_1$, $\lim_{t\to\infty} \xi_1(t) = 0$ then $\lim_{t\to\infty} z_1(t) = 0$, and $\lim_{t\to\infty} \hat{z}_1(t) = 0$. When t goes to infinity, $\alpha_1 = 0$ according to its definition, meaning $\lim_{t\to\infty} \hat{z}_2(t) = 0$, and we obtain similar results for $\lim_{t\to\infty} \hat{z}_i(t) = 0$ $(i = 3, \ldots, n)$. We can determine $\lim_{t\to\infty} z_i(t) = 0$ $(i = 1, \ldots, n)$ from the definition of $\varepsilon = z - \hat{z}$. We can use the definition of x_i to obtain that

$$\lim_{t\to\infty} x_i(t) = 0, i = 1, \ldots, n; \tag{69}$$

(iii) Similar to x_0, we can proved that z_1 also satisfies time-varying constraints

$$V_{k_{b1}}(z_1) = \frac{k_{b1}^2}{\pi} \tan\left(\frac{\pi z_1^2}{2k_{b1}^2}\right) \leq V(t) \leq V(0) \tag{70}$$

$$\frac{\pi z_1^2}{2k_{b1}^2} \leq \arctan\left(\frac{\pi V(0)}{k_{b1}^2}\right) < \frac{\pi}{2}, \tag{71}$$

and

$$z_1^2 < k_{b1}, -b_{11}(t) < z_1(t) < b_{12}(t). \tag{72}$$

Neither x_0 nor z_1 violates the time-varying requirements. From (7), we know that u_0 is a function related to x_0. Therefore, we can obtain from (30) that $-k_{01}(t)\varphi_0(x_0) < \varphi_0(x_0)x_0(t) < k_{02}(t)\varphi_0(x_0)$ and $-\lambda_0 k_{02}(t) < -\lambda_0 x_0(t) < \lambda_0 k_{01}(t)$ can be obtained. Thus, the following inequality holds:

$$-(k_{01}(t)\varphi_0(x_0) + \lambda_0 k_{02}(t)) < u_0(t) < k_{02}(t)\varphi_0(x_0) + \lambda_0 k_{01}(t). \tag{73}$$

Let $\overline{u}_0 = k_{02}(t)\varphi_0(x_0) + \lambda_0 k_{01}(t)$ and $\underline{u}_0 = k_{01}(t)\varphi_0(x_0) + \lambda_0 k_{02}(t)$. Then $-\underline{u}_0 < u_0(t) < \overline{u}_0$, and we can obtain that $-\underline{u}_0^{n-1} < u_0^{n-1}(t) < \max\{\underline{u}_0^{n-1}, \overline{u}_0^{n-1}\}$. Let $\tilde{u}_0 = \max\{\underline{u}_0^{n-1}, \overline{u}_0^{n-1}\}$. Then

$$-\max\{b_{11}(t)\tilde{u}_0, b_{12}(t)\underline{u}_0^{n-1}\} < z_1 u_0^{n-1} < \max\{b_{12}(t)\tilde{u}_0, b_{11}(t)\underline{u}_0^{n-1}\}. \tag{74}$$

After the above calculation, we choose $k_{11}(t) = \max\{b_{11}(t)\tilde{u}_0, b_{12}(t)\underline{u}_0^{n-1}\}$ and $k_{12}(t) = \max\{b_{12}(t)\tilde{u}_0, b_{11}(t)\underline{u}_0^{n-1}\}$. By nonlinear scaling, the constraint range of x_1 is $-k_{11}(t) < x_1(t) < k_{12}(t)$. According to (30), the output y satisfies the asymmetric time-varying constraints. This completes the proof. □

3.5 Stabilization of x-subsystem for $x_0(t_0) = 0$

If $x_0(t_0) = 0$, u_0 is designed as

$$u_0 = c, c > 0. \tag{75}$$

This controlled rate keeps x_0 away from 0. Then

$$\dot{x}_0(t_0) = c + x_0\varphi_0(x_0(t_0)) = c. \tag{76}$$

The initial time t_0 satisfies $x_0\varphi_0(x_0(t_0)) = 0 < c$. Due to the nonnegativity and smoothness of $\varphi_0(x_0)$, there is a small neighborhood of $x_0(t_0) = 0$, so that

$$x_0\varphi_0(x_0) < c. \tag{77}$$

In the period satisfying $x_0\varphi_0(x_0) < c$, we designed an output feedback control u_0 form (75) and u form (52). Since $u_0 = c > 0$ and $\varphi_0(x_0)$ is a smooth nonnegative function, x_0 starts at 0 and grows until $x_0\varphi_0(x_0) = c$. At this time, we let $x_0 = x_0^*$, and when $x_0\varphi_0(x_0) = c$, the control input u_0 is switched to (7).

The following theorem discusses the case where the initial state is zero.

Theorem 2 *According to Assumptions 1–4, the system will reach a steady state and satisfy the constraint criteria in a finite time if the following switch-based output feedback control scheme applies the appropriate design parameters to system (1) with time-varying constraints (2):*

$$u_0 = \begin{cases} c, & x_0(t) < x_0^* \\ -\lambda_0 x_0 - x_0\varphi_0(x_0), & x_0(t) \geq x_0^* \end{cases} \tag{78}$$

$$\begin{aligned} u = & -\iota_n\xi_n - \xi_{n-1} + C_n^T G C \hat{z}_1 + \frac{\partial\alpha_{n-1}}{\partial x_0}\dot{x}_0 + \frac{\partial\alpha_{n-1}}{\partial k_{b1}}\dot{k}_{b1} \\ & + \Sigma_{k,l=1}^{n} \frac{\partial\alpha_{n-1}}{\partial g_{kl}}\dot{g}_{kl} + \Sigma_{j=2}^{n-1}\frac{\partial\alpha_{n-1}}{\partial\hat{z}_j}\dot{\hat{z}}_j + \frac{\partial\alpha_{n-1}}{\partial z_1}\hat{z}_2 \\ & -\frac{1}{4}\xi_n\left(\frac{\partial\alpha_{n-1}}{\partial z_1}\right)^2\varphi_1^2(u_0, x_0, x_1) - \frac{1}{4}\xi_n\left(\frac{\partial\alpha_{n-1}}{\partial z_1}\right)^2\xi_1^2 \\ & -(n-1)\frac{\partial\alpha_{n-1}}{\partial z_1}\frac{\dot{u}_0}{u_0}z_1 - \frac{1}{4\delta}g_{max}^2\xi_n\left(\frac{\partial\alpha_{n-1}}{\partial z_1}\right)^2. \end{aligned} \tag{79}$$

Proof The proof process is given in the above analysis. □

Remark 5 The main work of this article is to add asymmetric time-varying output constraints based on [34] and relax ISS dynamic uncertainty to iISS dynamic uncertainty. The proposed control scheme ensures that the cascaded system is stable and that the asymmetric time-varying output constraints are always satisfied.

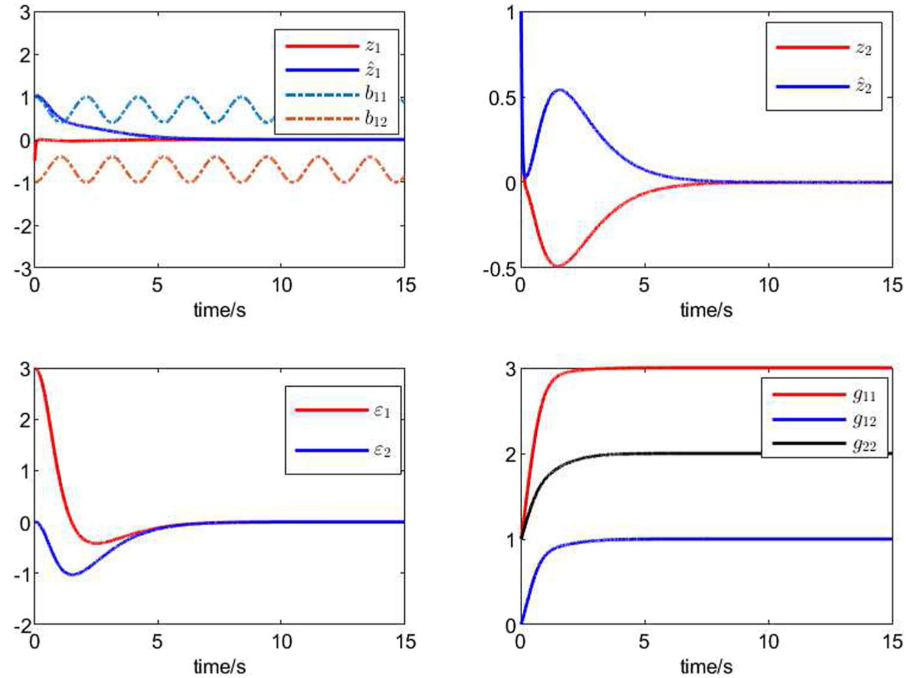


Fig. 1 Value of parameter estimation, error and gain matrix

We expand the scope of the constraints from previous work [37]. Since many practical systems are not always ISS, the cascaded systems studied in this paper with iISS dynamic uncertainty can include more general systems.

4 Simulation examples

Consider an uncertain nonlinear cascaded system,

$$\begin{cases} \dot{\eta} = -\frac{\eta}{1+\eta^2} + \eta x_1^2 \\ \dot{x}_0 = u_0 + x_0 \\ \dot{x}_1 = u_0 x_2 - x_1 + \frac{\eta x_1}{1+\eta^2} \\ \dot{x}_2 = u \\ y = (x_0, x_1)^T, \end{cases} \tag{80}$$

where the η-subsystem $\dot{\eta} = -\frac{\eta}{1+\eta^2} + \eta x_1^2$ is iISS, with an iISS-Lyapunov function $V(\eta) = \ln(1+\eta^2)$ and $\alpha_0(|\eta|) = \frac{2\eta^2}{(1+\eta^2)^2}$, $\gamma_0(|x_1|) = 2x_1^2$. From (80), we can obtain that $\varphi_0(x_0) = 1$, $\phi_1^d(u_0, x_0, x, \eta) = -x_1 + \frac{\eta x_1}{1+\eta^2}$ and $\phi_2^d(u_0, x_0, x, \eta) = 0$. Under the condition of Assumption 2, $\varphi_1(u_0, x_0, x_1) = 1$, $\psi_1(\eta) = \frac{\eta}{1+\eta^2}$, $\varphi_2(u_0, x_0, x_1) = 0$, and $\psi_2(\eta) = 0$. Therefore, we can concluded that $\frac{\psi_1^2(s)}{\alpha_0(s)} = \frac{1}{2}$ and $\frac{\psi_2^2(s)}{\alpha_0(s)} = 0$. Since $\liminf_{s\to\infty} \alpha_0(s) < \infty$, the condition in Assumption 2 is satisfied.

Assuming that $z_1 = \frac{x_1}{u_0}$, $z_2 = x_2$, then (80) takes the form

$$\begin{cases} \dot{z}_1 = z_2 - \frac{\dot{u}_0}{u_0} z_1 - z_1 + \frac{\eta z_1}{1+\eta^2} \\ \dot{z}_2 = u. \end{cases} \tag{81}$$

According to the above research, the state estimation and full-order observer are written as

$$\begin{cases} \dot{\hat{z}}_1 = \hat{z}_2 - \frac{\dot{u}_0}{u_0}\hat{z}_1 - g_{11}\hat{z}_1 \\ \dot{\hat{z}}_2 = u - g_{12}\hat{z}_1 \\ \dot{g}_{11} = -2\frac{\dot{u}_0}{u_0} g_{11} + 2g_{12} - g_{11}^2 + 1 \\ \dot{g}_{12} = -\frac{\dot{u}_0}{u_0} g_{12} + g_{22} - g_{11}g_{12} \\ \dot{g}_{22} = 1 - g_{22}^2. \end{cases} \tag{82}$$

From the definition of ε, we can obtain that

$$\begin{cases} \dot{\varepsilon}_1 = -\left(\frac{\dot{u}_0}{u_0} + g_{11}\right)\varepsilon_1 + \varepsilon_2 + g_{11}z_1 - z_1 + \frac{\eta z_1}{1+\eta^2} \\ \dot{\varepsilon}_2 = -g_{12}\varepsilon_1 + g_{12}z_1. \end{cases} \tag{83}$$

The control law is based on the backstepping technique. The controlled system can be rewritten as

$$\begin{cases} \dot{z}_1 = \hat{z}_2 + \varepsilon_2 - \frac{\dot{u}_0}{u_0} z_1 - z_1 + \frac{\eta z_1}{1+\eta^2} \\ \dot{\hat{z}}_2 = u - g_{12}\hat{z}_1. \end{cases} \tag{84}$$

The subsequent dynamic output feedback control tactics come from the proposed design scheme:

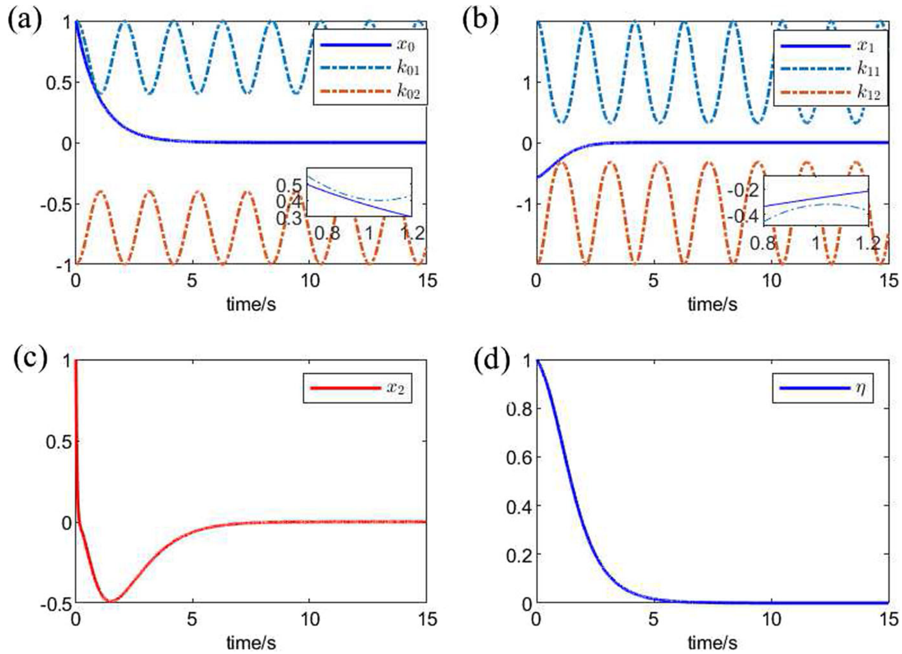


Fig. 2 **a** Value track of x_0 with time-varying constraints; **b** value track of x_1 awith time-varying constraints; **c** value track of x_2; **d** value track of η

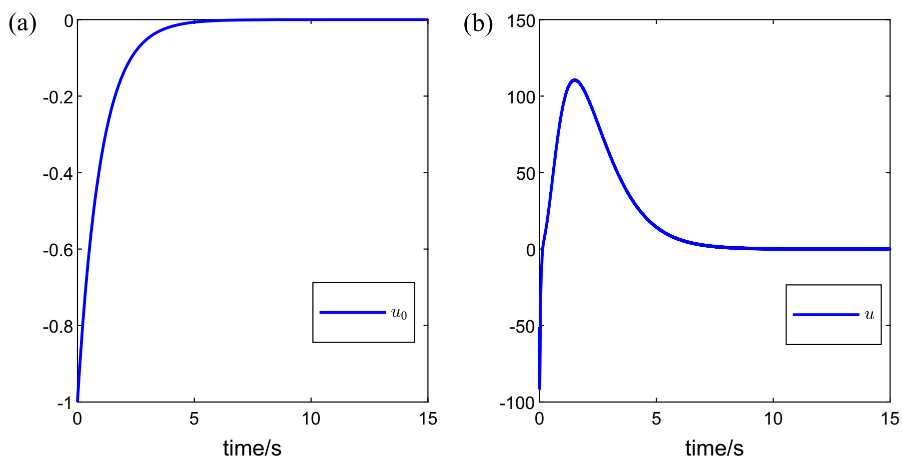


Fig. 3 **a** Value track of input u_0; **b** value track of input u

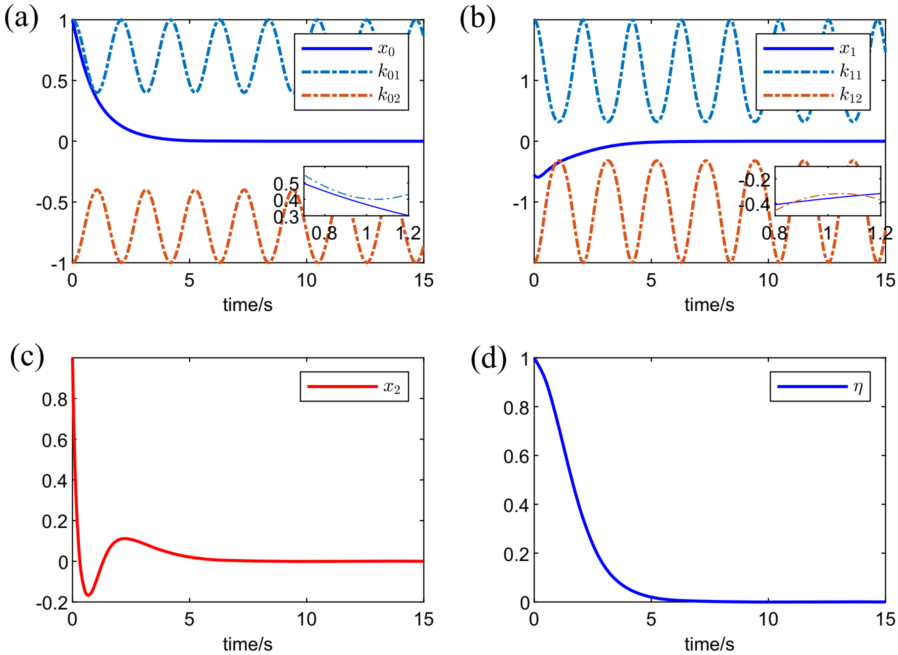
$$u = -\iota_2\xi_2 - \phi_{b1}(\xi_1)\xi_1 + C_2^T GCC^T\hat{z} + \frac{\partial\alpha_1}{\partial u_0}\dot{u}_0 + \frac{\partial\alpha_1}{\partial k_{b1}}\dot{k}_{b1} + \frac{\partial\alpha_1}{\partial\xi_1}\hat{z}_2 - \frac{\partial\alpha_1}{\partial\xi_1}\frac{\dot{u}_0}{u_0}\xi_1 - \frac{1}{4}\xi_2\left(\frac{\partial\alpha_1}{\partial\xi_1}\right)^2 - \frac{1}{4\delta}g_{max}^2\xi_2\left(\frac{\partial\alpha_1}{\partial\xi_1}\right)^2 - \frac{1}{4}\xi_1^2\xi_2\left(\frac{\partial\alpha_1}{\partial\xi_1}\right)^2, \quad (85)$$

where $\alpha_1 = -\iota_1(x_0, \xi_1)\xi_1 - \frac{1}{4\delta}g_{max}^2\xi_1\phi_{b1}(\xi_1) - \xi_1 - \frac{1}{4}\xi^3\phi_{b1}(\xi_1) + \frac{\dot{u}_0}{u_0}\xi_1 - \nu_2\xi_1$, $C^T = [1 \quad 0]$, and $C_2^T = [0 \quad 1]$.

Then the simulation consequences are used to prove the availability of the projected controller design. We consider the case that $x_0(t_0) \neq 0$, and choose $u_0 = -\lambda_0 x_0 - x_0$. The system parameters are selected as $\iota_1 = \iota_2 = 1$, $\delta = \frac{1}{6}$, $\lambda_0 = 1$, and $\nu_2 = 30$. We take $k_{01} = b_{11} = 0.7 + 0.3\cos(3t)$, $k_{02} = b_{12} = 0.7 + 0.3\cos(3t)$, and $k_{11} = 0.98 + 0.84\cos(3t) + 0.18\cos^2(3t)$, $k_{12} = 0.98 + 0.84\cos(3t) + 0.18\cos^2(3t)$. The initial values are selected as $\eta(0) = 1$, $x_0(0) = 1$, $x_1(0) = -0.55$, $g_{11}(0) = 1$, $g_{12}(0) = 0$, $g_{22}(0) = 1$, $\xi_1(0) = -0.5$, $\xi_2(0) = 0$, $\hat{z}_1(0) = 1$, $\hat{z}_2(0) = 1$, $\varepsilon_1(0) = 3$, $\varepsilon_2(0) = 0$.

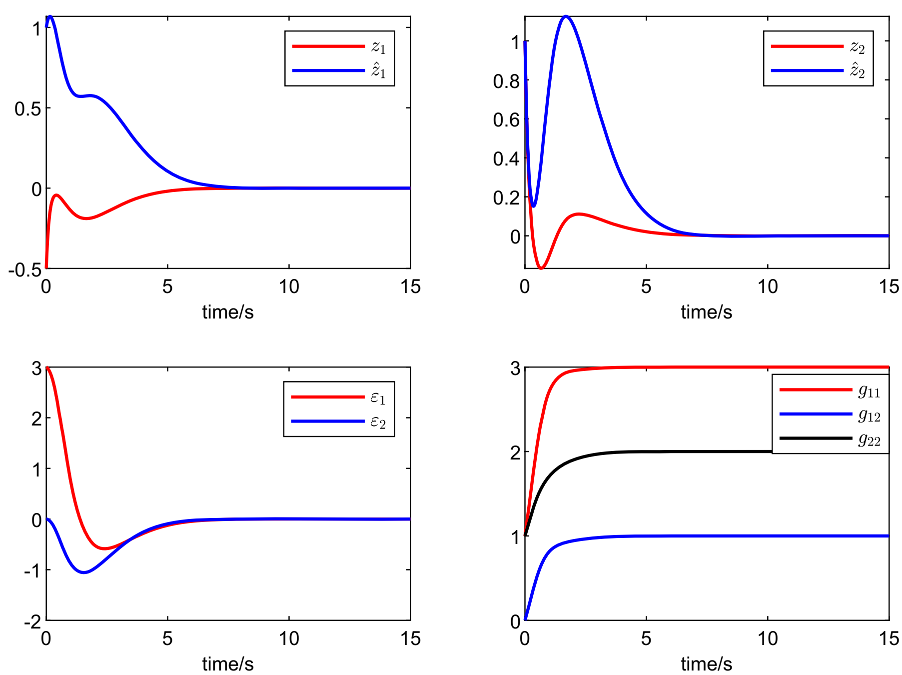
Figures 1, 2 and 3 display the simulation effects, in which the system state and control law reach zero in a limited time. As shown in Fig. 2, the system state and unmeasurable dynamics achieve a steady state in a

Fig. 4 Trajectories of system output: **a** trail of x_0; **b** trail of x_1; **c** trail of x_2; **d** trail of η



limited time. The trajectories of x_0 and x_1 are always within the specified range without violating the output constraints. Figure 1 shows that the system's parameter estimation and error quickly converge to 0. z_1 is always within the specified range. The asymmetric output is not violated. The parameter estimation, error, and gain are limited in the control process and become stable in approximately 8 s. Furthermore, it can be seen that the proposed robust control scheme can effectively handle the output feedback control of non-holonomic systems with output constraints and iISS dynamic uncertainties.

Fig. 5 Parameter estimation, error and gain matrix without constraints



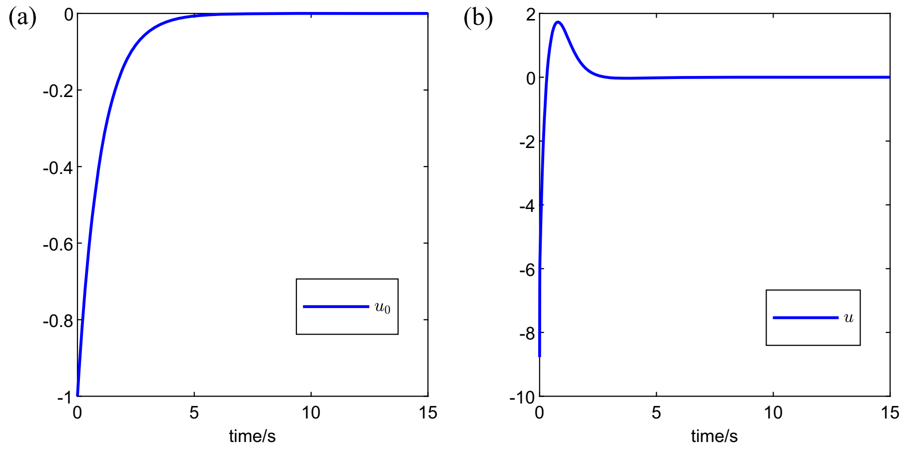


Fig. 6 Control input of the closed loop system: **a** value track of input u_0; **b** value track of input u

We select system (80) for comparison. Choosing identical parameters and initial values can reflect the advantages of the controller. Unlike before, asymmetric output constraints are not considered. Therefore, the control law design [34] is

$$
\begin{aligned}
u &= -\iota_2\xi_2 - \xi_1 + g_{12}\hat{z}_1 + \frac{\partial\alpha_1}{\partial\xi_1}\hat{z}_2 - \frac{\partial\alpha_1}{\partial\xi_1}\frac{\dot{u}_0}{u_0}z_1 + \frac{\partial\alpha_1}{\partial u_0}\dot{u}_0 \\
&\quad - \frac{1}{4u}g_{max}^2\xi_2\left(\frac{\partial\alpha_1}{\partial\xi_1}\right)^2 - \frac{1}{4}\xi_2\left(\frac{\partial\alpha_1}{\partial\xi_1}\right)^2 \\
&\quad - \frac{1}{4}\xi_2\left(\frac{\partial\alpha_1}{\partial\xi_1}\right)^2 z_1^2 \\
\alpha_1 &= -\frac{1}{4u}g_{max}^2\xi_1 + \frac{\dot{u}_0}{u_0}z_1 - \frac{1}{4}\xi_1^3 - 6\xi_1 \\
&\quad - 2\xi_1^3 - 2\xi_1^3u_0^4 - \iota_1\xi_1 \\
\dot{\xi}_1 &= \xi_2 + \alpha_1 + \varepsilon_2 - \frac{\dot{u}_0}{u_0}z_1 - z_1 + z_1\frac{\eta}{1+\eta^2} \\
\dot{\xi}_2 &= -\iota_2\xi_2 - \xi_1 - \frac{\partial\alpha_1}{\partial\xi_1}\varepsilon_2 + \frac{\partial\alpha_1}{\partial\xi_1}z_1 - \frac{\partial\alpha_1}{\partial\xi_1}z_1\frac{\eta}{1+\eta^2} \\
&\quad - \frac{1}{4u}g_{max}^2\xi_2\left(\frac{\partial\alpha_1}{\partial\xi_1}\right)^2 - \frac{1}{4}\xi_2\left(\frac{\partial\alpha_1}{\partial\xi_1}\right)^2 \\
&\quad - \frac{1}{4}\xi_2\left(\frac{\partial\alpha_1}{\partial\xi_1}\right)^2 z_1^2.
\end{aligned}
\tag{86}
$$

The outcomes are shown in Figs. 4, 5 and 6. The system responses to the trajectory in Fig. 4 do not consider asymmetric time-varying output limitations; it is evident that x_1 exceeds the constraint range by about 1 s, reflecting the control rate in this paper. Although Figs. 4 and 5 ensure the stability of the nonlinear cascaded system, it breaks the time-varying output restrictions and exhibits wider response fluctuations. It is clear from comparing the figures mentioned above that the overall control method operates effectively.

5 Conclusion

We explored a group of nonlinear cascaded systems with output constraints and iISS inverse dynamics. To overcome the difficulty that non-holonomic systems are not stabilized by continuous feedback control, we designed a discontinuous time-invariant control law using input-state scaling transformation technology. The unmeasurable state was estimated by constructing a full-order observer with a gain derived from the Riccati matrix differential equation. A tan-type barrier Lyapunov function was introduced to handle asymmetric time-varying output constraints. Future work includes extending our results to cascaded systems with non-vanishing disturbances and output constraints, as well as achieving the stability of the cascaded system within the specified time.

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Data Availability The authors declare that all data analyzed during this paper are included in this article.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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