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ORIGINAL PAPER

Dynamics of Rossby wave packets with topographic features via derivative expansion approach

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Abstract The derivative expansion method was successfully proposed by scholars to characterize the propagation and to discuss the dynamics of nonlinear water waves in the last century. The results manifest the great superiority of this method. The present paper aims mainly at the uses of this technical method to describe the evolutionary processes and to explain the dynamical mechanisms of nonlinear Rossby waves for large-scale geophysical fluid motions. We derive a nonlinear Schrödinger equation from describing the evolution of Rossby wave amplitude. Furthermore, the boundary value problem is handled by using the perturbation expansion method. The effects of initial amplitude, frequency and zonal wave number on the amplitude of Rossby solitary waves are analyzed in both the presence and the absence of topography cases. The effects of weak shear current on dipole blocking are discussed in the presence of bottom topographic structures. The results indicate that topography has a significant impact on the size of amplitude and propagation speed of Rossby waves, and sheared background current will benefit the generation of blockings.

Keywords Quasi-geostrophic theory · Derivative expansion method · Schrödinger equation · Rossby solitary waves · Blocking

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1 Introduction

Rossby wave characterizes a kind of slow and large-scale motions in the geophysical fluid mechanics. The existence of Rossby waves can not only well explain the movement of ridge and trough on the high altitude weather map, but also explain the weather change near the ground. In earlier articles, many scholars have explained the relevant theories of Rossby solitary wave in detail. Redekopp [1] proved that the finite amplitude evolution of Rossby solitary wave can be described using the Korteweg–de Vries (KdV) equation or the modified Korteweg–de Vries (mKdV) equation. Furthermore, it is pointed out that the change in horizontal shear can significantly affect the generation of Rossby wave. Maslowe [2] extended the theory. Not limited to single-layer and two-layer models, more complex continuous three-layer models were studied in detail. Yeh [3] first discussed the maintenance and disappearance of atmospheric blocking in the westerlies with the energy dispersion theory of Rossby wave. A few years ago, Malguzzi and Rizzoli [4,5] proposed to use Rossby solitary wave to explain the generation, enhancement, and decay process of dipole in the atmosphere during blocking. Liu and Tan [6] showed the evolution of Rossby wave during changes in β. In addition, the changes of β during latitude changes were discussed, and the plane approximation of β was extended. What is more, the phase velocity formula of Rossby wave including β variation is given under general conditions. Zhao [7] incorporated the influence of topography into the continuity equation. The results indicate that topography effects have a significant impact on the stability and characteristics of Rossby wave. Huang [8] indicated that the only effect of the time-dependent background wind on the Rossby wave is the accumulated motion in the zonal direction. Campbell [9] studied the nonlinear evolution of a barotropic forced Rossby wave in a meridional shear flow. Song [10] derived the mKdV equation to describe the evolution of the amplitude of the Rossby solitary wave and specifically explained that the coefficients of the mKdV equation are affected by β effect, buoyancy frequency, and the basic shear flow. Yang [11] explained that the dissipative effect and time-dependent topography effect cause changes in the mass and energy of solitary waves. He [12] mainly discussed the impact of plateau topography gradients on atmospheric Rossby waves. The results indicate that the topography of the Tibetan Plateau is very favorable for the generation of Rossby wave. The dissipative Petviashvili equation was derived by Chen [13] to describe the two-dimensional Rossby waves. Zhang [14] considered the effects of generalized β effect, dissipation, and topography factors on $(2+1)$-dimensional Rossby wave. The conclusion drawn is that the generalized β effect has a significant impact on the generation of nonlinear Rossby solitary waves, while dissipation and topography factors have an important impact on the evolution of Rossby wave amplitude. Zhao [15] also analyzed the impact of topography on Rossby waves. But the topography under consideration is unstable. Through theoretical analysis, the physical characteristics of solitary waves in terms of mass and energy were discussed. Shi [16] made it clear that background flow shear is required for the existence of Rossby solitary wave. Zhang [17,18] proposed a variable coefficient equation to describe the amplitude evolution of Rossby wave. Qualitative analysis was conducted on the impact of various physical factors on the evolution process of nonlinear Rossby wave. Similarly, Zhang [19] introduced the variable coefficient KdV equation to describe the evolution of the amplitude of Rossby solitary wave. Yang [20] suggested that three-dimensional Rossby solitary wave can better reflect real ocean and atmospheric conditions. On this basis, the dissipation effect of three-dimensional Rossby wave was discussed.

Scholars also have great interest in the instability of Rossby solitary wave [21]. Chen [22] stated that slowly varying topography is an external impact factor for Rossby wave. Wang [23] derived the Gardner equation to simulate the propagation of nonlinear Rossby wave amplitude. Moreover, the numerical results indicate that the size of meridional topography and frequency have a certain impact on the evolution of Rossby wave, but the influence of topography will be greater. What is more, Zhao [24] investigated the instability of two-layer quasi-geostrophic model and gave a nonlinear stability criterion. Yang [25] derived the $(2+1)$-dimensional coupled nonlinear Schrödinger equations. Drawing the conclusion that the range of modulational instable regions is not only affected by amplitude, but also greatly related to the number of waves in the y-direction. This provides strong theoretical support for our further research on stability. In addition, some scholars are not limited to considering the influence of a single physical factor on Rossby solitary wave, but rather analyze the influence of combined physical factors on waves. Gnevyshev [26] explained that the combination of the earth's rotation and topography enhances the β effect, while the combination of the shear flow and topography is the opposite. Yin [27] presented the spatial transmission characteristics of Rossby wave. The above results indicate that, on the one hand, scholars are constantly proposing new model equations to describe the evolution of Rossby wave. On the other hand, the influence of various physical factors on the evolution of Rossby wave has been deeply explored. This has profound implications for the study of Rossby solitary wave models in atmospheric and oceanic dynamics.

Another important issue to consider is finding solutions to various nonlinear equations that satisfy the evolution of the amplitude of Rossby solitary waves. Zhang [28] provided some exact explicit parametric representations of traveling wave solutions of the Klein–Gordon–Zakharov equations. Meng [29] provided another calculation method. By using symbolic computation, the nonlinear equation is transformed into bilinear form. Then, multi-soliton solutions are derived from this. Shi [30] derived the analytic solutions of nonlinear Schrödinger equation with dissipation and external forces, respectively. Based on the obtained solutions, the effects of dissipation and external forces on the evolution of Rossby waves were discussed in detail. Karjanto and van Groesen [31] discussed three different breathing solutions of the Schrödinger equation. The relationship between the breathing solutions is analyzed. Additionally, via Darboux transformation algorithm, rogue wave and semi-rational solutions of the mixed nonlinear Schrödinger equation will be derived [32]. Liu [33] extended the real equation to the complex equation and provided the explicit formulae of the smooth position solutions of the complex modified KdV equation. Some scholars have also made contributions to the existence of solutions for higher-order equations. Ma [34] proved the existence of N-soliton solutions of the $(2+1)$-dimensional KdV equation. Liu [35] proved that the global solution of the sixth-order nonlinear Schrödinger equation not only exists, but also is unique. Bai [36] used a bilinear neural network method to solve the $(2 + 1)$-dimensional Kadomtsev–Petviashvili equation in order to obtain the exact solutions. The method of solving nonlinear equations is expanded. The improved $\tan(\varphi/2)$ expansion method is also a good method for finding analytical solutions to nonlinear equations. Narenmandula [37] used it to obtain the analytical solution of the forced $(2+1)$-dimensional Zakharov–Kuznetsov equation. It is also indicated that the improved $\tan(\varphi/2)$ expansion method can be applied to other nonlinear equations and obtain more accurate solutions.

These studies have laid the foundation for us to better understand the spread and evolution of Rossby wave and for further research. However, most of the nonlinear equations describing the amplitude of Rossby solitary wave in these studies are obtained by reductive perturbation method or multiple-scale method. It is worth mentioning that Karjanto [38] derived the Schrödinger equation from the KdV equation using the multiple time scale method. The intertransformation between different nonlinear equations is realized. This paper uses the derivative expansion method to derive the nonlinear Schrödinger equation to describe the evolution of solitary wave amplitude. Besides, based on the presence and absence of topography, the effects of various physical factors on the propagation of Rossby solitary waves and the effects of shear on atmospheric blocking are analyzed. The paper is organized in the following. In Sect. 2, the derivative expansion method is briefly introduced and we derive nonlinear Schrödinger equation dissipative effect to describe the evolution of Rossby wave amplitude. In Sect. 3, the analytical solution of the nonlinear Schrödinger equation is given. As well, the influence of some physical factors on solitary waves is discussed through numerical analysis. Finally, the research results of this paper are summarized in Sect. 4.

2 Method and derivation

2.1 Derivative expansion method and perturbation expansion method

Many methods have been proposed to study nonlinear equations, and the derivative expansion method is a relatively effective method. Its core idea is to expand the derivative in the equation. Next, we will use the derivative expansion method to derive the $(1+1)$-dimensional Schrödinger equation. The amplitude A satisfying the Schrödinger equation is a function of time t and space x. Then, we can choose the following different time and space scales

$$\begin{cases} x_0, x_1, x_2, \ldots, x_n, \\ t_0, t_1, t_2, \ldots, t_n, \end{cases} \tag{1}$$

where $x_m = \epsilon^m x$, $t_m = \epsilon^m t$. Using the chain rule, the derivatives become

$$\begin{cases} \dfrac{\partial}{\partial x} = \displaystyle\sum_{m=0}^{n} \epsilon^m \dfrac{\partial}{\partial x_m}, \\ \dfrac{\partial}{\partial t} = \displaystyle\sum_{m=0}^{n} \epsilon^m \dfrac{\partial}{\partial t_m}, \end{cases} \tag{2}$$

and this increases the number of independent variables in the original equation. Substitute the expanded derivative into the original equation for solution.

And perturbation expansions

$$\Psi' = \delta\Psi^{(1)} + \delta^2\Psi^{(2)} + \cdots, \tag{3}$$

where ϵ and δ are both small parameters in the present problem, specific relations are obtained according to specific situations in the following derivations.

2.2 Nonlinear Schrödinger equation for Rossby wave packet

Begin with the following simple dimensionless barotropic potential vorticity equation as [39]

$$\left(\frac{\partial}{\partial t} + \frac{\partial\Psi}{\partial x}\frac{\partial}{\partial y} - \frac{\partial\Psi}{\partial y}\frac{\partial}{\partial x}\right)\left(\nabla^2\Psi + f + h\right) = 0, \tag{4}$$

where Ψ is the total stream function, $f = f_0 + \beta y$ is the vertical component of Coriolis parameter with $f_0 = 2\Omega \sin\varphi_0$, $h = h(y)$ is the topographic function,

Ω is the angular velocity of the earth's rotation, φ_0 is the local latitude, β is the Rossby parameter, ∇^2 is the Laplace operator, and here it is two dimensional.

Because the flow is impermeable, our boundary conditions can be obtained as

$$\left.\frac{\partial \Psi}{\partial x}\right|_{y=0} = \left.\frac{\partial \Psi}{\partial x}\right|_{y=1} = 0. \tag{5}$$

The total stream function is presented to be basic part and perturbed part as follows:

$$\Psi = -\int_0^y (\bar{u}(s) - c_0)\mathrm{d}s + \Psi'(x, y, t). \tag{6}$$

Substituting Eq. (6) into Eqs. (4) and (5) yields

$$\left[\frac{\partial}{\partial t} + (\bar{u} - c_0)\frac{\partial}{\partial x}\right]\nabla^2\Psi' + p(y)\frac{\partial \Psi'}{\partial x} + J\left[\Psi', \nabla^2\Psi'\right] = 0, \tag{7}$$

$$\left.\frac{\partial \Psi'}{\partial x}\right|_{y=0} = \left.\frac{\partial \Psi'}{\partial x}\right|_{y=1} = 0. \tag{8}$$

where $p(y) = \beta - \bar{u}'' + h'$. It includes the β effect, the shear effect and the topographic effect. $J[a, b] = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ is the Jacobian operator.

The derivation-expansion method suggests us to extend the independent variables x and t as

$$\begin{cases} x_0, x_1, x_2, \\ t_0, t_1, t_2. \end{cases} \tag{9}$$

The derivatives become

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial x_0} + \epsilon\dfrac{\partial}{\partial x_1} + \epsilon^2\dfrac{\partial}{\partial x_2}, \\ \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial t_0} + \epsilon\dfrac{\partial}{\partial t_1} + \epsilon^2\dfrac{\partial}{\partial t_2}; \end{cases} \tag{10}$$

$$\delta = \epsilon \tag{11}$$

is required, which means the balance between dispersion and nonlinearity.

Substituting Eqs. (3) and (10) into Eq. (7) yields approximation equations as follows:

$$o(\epsilon^{(1)}) \quad L^{(0)}\left(\Psi^{(1)}\right) = 0, \tag{12}$$

$$o(\epsilon^{(2)}) \quad L^{(0)}\left(\Psi^{(2)}\right) + L^{(1)}\left(\Psi^{(1)}\right) = M^{(0)}\left(\Psi^{(1)}, \Psi^{(1)}\right), \tag{13}$$

$$o\left(\epsilon^{(3)}\right) \quad L^{(0)}\left(\Psi^{(3)}\right) + L^{(1)}\left(\Psi^{(2)}\right) + L^{(2)}\left(\Psi^{(1)}\right) = M^{(0)}\left(\Psi^{(1)}, \Psi^{(2)}\right) + M^{(0)}\left(\Psi^{(2)}, \Psi^{(1)}\right) + M^{(1)}\left(\Psi^{(1)}, \Psi^{(1)}\right). \tag{14}$$

The linear and nonlinear operators of each order are

$$L^{(0)} = \left[\frac{\partial}{\partial t_0} + (\bar{u} - c_0)\frac{\partial}{\partial x_0}\right]\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}\right) + p(y)\frac{\partial}{\partial x_0},$$

$$L^{(1)} = \left[\frac{\partial}{\partial t_1} + (\bar{u} - c_0)\frac{\partial}{\partial x_1}\right]\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}\right) + p(y)\frac{\partial}{\partial x_1} + \left[\frac{\partial}{\partial t_0} + (\bar{u} - c_0)\frac{\partial}{\partial x_0}\right]\left(2\frac{\partial^2}{\partial x_0\partial x_1}\right),$$

$$L^{(2)} = \left[\frac{\partial}{\partial t_2} + (\bar{u} - c_0)\frac{\partial}{\partial x_2}\right]\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}\right) + p(y)\frac{\partial}{\partial x_2} + \left[\frac{\partial}{\partial t_1} + (\bar{u} - c_0)\frac{\partial}{\partial x_1}\right] \times \left(2\frac{\partial^2}{\partial x_0\partial x_1}\right) + \left[\frac{\partial}{\partial t_0} + (\bar{u} - c_0)\frac{\partial}{\partial x_0}\right] \times \left(\frac{\partial^2}{\partial x_1^2} + 2\frac{\partial^2}{\partial x_0\partial x_2}\right),$$

$$M^{(0)}(*, \otimes) = -\frac{\partial(*)}{\partial x_0}\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}\right)(\otimes) + \frac{\partial(*)}{\partial y}\frac{\partial}{\partial x_0}\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}\right)(\otimes),$$

$$M^{(1)}(*, \otimes) = -\frac{\partial(*)}{\partial x_1}\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}\right)(\otimes) + \frac{\partial(*)}{\partial y}\frac{\partial}{\partial x_1}\left(\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2}\right)(\otimes) - \frac{\partial(*)}{\partial x_0}\frac{\partial}{\partial y}\left(2\frac{\partial^2}{\partial x_0\partial x_1}\right)(\otimes) + \frac{\partial(*)}{\partial y}\frac{\partial}{\partial x_0}\left(2\frac{\partial^2}{\partial x_0\partial x_1}\right)(\otimes).$$

Formal solution of first-order equation is assumed to be

$$\Psi^{(1)} = A(x_1, x_2, t_1, t_2)\varphi(y)e^{i(kx_0-wt_0)} + c.c, \quad (15)$$

where c.c represents the conjugation of the former part. The following equation holds after substituting Eq. (15) into Eq. (12), when $\bar{u} - c_0 - c \neq 0$. Add boundary conditions, and it becomes

$$\begin{cases} \varphi''(y) + (\dfrac{p(y)}{\bar{u} - c_0 - c} - k^2)\varphi(y) = 0, \\ \varphi(0) = \varphi(1) = 0. \end{cases} \quad (16)$$

It is the Rayleigh–Kuo equation [40] with varying coefficients. $c = \frac{w}{k}$ is the phase velocity of the wave. Substituting Eq. (16) into Eq. (13) yields the nonsecular condition as

$$\frac{\partial A}{\partial t_1} + c_g \frac{\partial A}{\partial x_1} = 0, \quad (17)$$

where $c_g = c + \frac{2k^2(\bar{u}-c_0-c)^2}{p(y)}$ is the group velocity of wave. Meanwhile, Eq. (13) becomes the following form

$$L^{(0)}\left(\Psi^{(2)}\right) = ikA^2\left(\frac{p(y)}{\bar{u} - c_0 - c}\right)' \varphi^2(y)e^{2i(kx_0-wt_0)} + c.c. \quad (18)$$

Assuming $\Psi^{(2)} = B(x_1, x_2, t_1, t_2)\varphi_1(y)e^{2i(kx_0-wt_0)} + c.c$, it is easily obtained that

$$[\varphi_1'' - 4k^2\varphi_1]B + \frac{p(y)}{\bar{u} - c_0 - c}\varphi_1(y)B = G(y)A^2. \quad (19)$$

where $G(y) = \frac{1}{2(\bar{u}-c_0-c)}\left(\frac{p(y)}{\bar{u}-c_0-c}\right)' \varphi^2(y)$. Without loss of generality, let $B = A^2$, and the modified meridional structure is

$$\begin{cases} \varphi_1'' - 4k^2\varphi_1 + \dfrac{p(y)}{\bar{u} - c_0 - c}\varphi_1(y) \\ = \dfrac{1}{2(\bar{u} - c_0 - c)}\left(\dfrac{p(y)}{\bar{u} - c_0 - c}\right)' \varphi^2(y), \\ \varphi_1(0) = \varphi_1(1) = 0. \end{cases} \quad (20)$$

Higher-order equation is still needed to ensure the evolution of amplitude A. Substituting $\Psi^{(1)}$ and $\Psi^{(2)}$ into Eq. (14), nonsecular condition is satisfied as follows:

$$i\left(\frac{\partial A}{\partial t_2} + c_g\frac{\partial A}{\partial x_2}\right) + \alpha\frac{\partial^2 A}{\partial x_1^2} + \gamma|A|^2A = 0, \quad (21)$$

where $\alpha = \frac{I_1}{I}$, $\gamma = \frac{I_2}{I}$, $I = \int_0^1 \frac{p(y)}{(\bar{u}-c_0-c)^2}\varphi^2(y)\mathrm{d}y$, $I_1 = -\int_0^1 \frac{2kc_g+w-3k(\bar{u}-c_0)}{\bar{u}-c_0-c}\varphi^2(y)\mathrm{d}y$, $I_2 = -\int_0^1 \frac{k\varphi}{\bar{u}-c_0-c}\Big((\frac{p(y)}{\bar{u}-c_0-c})'\varphi(y)\varphi_1(y) + \varphi(y)G' + 2\varphi'(y)G\Big)\mathrm{d}y$.

Equation (21) is the nonlinear Schrödinger equation which describes the evolution of nonlinear Rossby wave packet. Introducing transforms $T = t_2$, $X = x_1 - c_g t_1 = \frac{1}{\epsilon}(x_2 - c_g t_2)$, Eq. (21) becomes the standard form

$$i\frac{\partial A}{\partial T} + \alpha\frac{\partial^2 A}{\partial X^2} + \gamma|A|^2A = 0. \quad (22)$$

3 Obtaining solutions and discussion

First, let

$$\bar{u}(y) = \bar{u}_0 + \nu Q(y), \quad (23)$$

$$h(y) = a_1 y + a_0, \quad (24)$$

where a_1, a_0 are constant and φ in Eq. (16) is expanded as a series

$$\varphi = \varphi_{00} + \nu\varphi_{01} + \cdots. \quad (25)$$

Substituting Eq. (23) and Eq. (25) into Eq. (16) yields the following boundary value problem approximatively

$$\begin{cases} \dfrac{\mathrm{d}^2\varphi_{00}}{\mathrm{d}y^2} - k^2\varphi_{00} = -\dfrac{\beta + a_1}{\bar{u}_0 - c_0 - c}\varphi_{00} \\ \varphi_{00}(0) = \varphi_{00}(1) = 0. \end{cases} \quad (26)$$

Solving Eq. (26), we can obtain $\varphi_{00} = \sqrt{2}\sin my$. What is more, it satisfies that $m^2 + k^2 = \frac{\beta+a_1}{\bar{u}_0-c_0-c}$. So, the approximate solution of φ is

$$\varphi = \sqrt{2}\sin my. \quad (27)$$

Let

$$Q(y) = \cos my, \quad (28)$$

and φ_1 is expanded as a series

$$\varphi_1 = \nu\varphi_{10} + \nu^2\varphi_{11} + \cdots. \quad (29)$$

Substituting Eq. (28) and Eq. (29) into Eq. (20) and boundary conditions yield the following boundary value problem approximatively

$$\begin{cases} \frac{d^2\varphi_{10}}{dy^2} - 4k^2\varphi_{10} + \frac{\beta + a_1}{\bar{u}_0 - c_0 - c}\varphi_{10} \\ = \frac{(m^2+k^2)^2k^2m}{4(\beta+a_1)^2}(3\sin my - \sin 3my), \\ \varphi_{10}(0) = \varphi_{10}(1) = 0. \end{cases} \tag{30}$$

Then $\varphi_{10} = -\frac{(m^2+k^2)^2mk^2}{4(\beta+a_1)^2}(\frac{1}{k^2}\sin my - \frac{1}{3k^2+8m^2}\sin 3my)$. So, the approximate solution of φ_1 is

$$\varphi_1 = -\frac{(m^2+k^2)^2mk^2}{4(\beta+a_1)^2} \left(\frac{1}{k^2}\sin my - \frac{1}{3k^2+8m^2}\sin 3my\right). \tag{31}$$

As the solution of the standard Schrödinger equation obtained by Yin [41] is

$$A(X,T) = \sqrt{\frac{2\alpha}{\gamma}}M \operatorname{sech}M\,(X - 2\alpha k_0 T)e^{i\left[k_0X - \alpha(k_0^2 - M^2)T\right]} + c.c, \tag{32}$$

where c.c represents the conjugate of the previous term and M and k_0 are the amplitude and moving speed of the envelope Rossby solitary waves determined by the initial state at $A(X,T) = A(0,0)$, the stream function is

$$\Psi = -\int_0^y (\bar{u}_0 + \nu\cos(my) - c_0)ds + \epsilon\sqrt{\frac{2\alpha}{\gamma}}M\operatorname{sech}[\epsilon M(x - (c_g + 2\epsilon\alpha k_0)t)] \times \sqrt{2}\sin(my)e^{i[(k+\epsilon k_0)x - (\omega + \epsilon k_0 c_g + \epsilon^2\alpha(k_0^2 - M^2))t]} + c.c. \tag{33}$$

We assume that the topography function is $h = a_1 + a_0$, and only a_1 appears in our model equation without a_0. Therefore, when $a_1 = 0$, it is equivalent to ignoring the influence of topography. The following analysis will be divided into two situations: one is that the terrain does not exist and the other is that it includes the influence of topographic factors.

According to Eq. (32), the amplitude $A(x,t)$ is not only related to time and space, but also affected by the initial state A(0,0). Therefore, take different values for M and k_0 to represent the effect of different initial states. Let us take a look at the effect of different values of M on amplitude. The effect of different values of M on amplitude is shown in Fig. 1 when the parameters in the amplitude are $\bar{u}_0 = 0.7$, $\nu = 0.1$, $m = \pi$, $\epsilon = 0.2$, $\omega = 1$, $k = 4$ and $k_0 = 4$, respectively. And subgraph (a) represents the impact of different M on amplitude when there is no topography ($a_1 = 0$), and subgraph (b) represents the situation when topography exists ($a_1 = 20$). It is worth noting that we have taken three different values for M, namely 1, 3 and 5.

Subgraph (a) in Fig. 1 shows the change of amplitude A with space x and time t when M is taken as 1, 3 and 5. It shows that the initial state has a great influence on the amplitude when there is no topography present. When M is taken as 1, the amplitude changes slowly with time and space. However, compared with M which is taken as 3 or 5, it is easy to find that the amplitude changes more and more significantly with the increase of M. What is more, it has a certain periodicity. From subgraph (b), it can be seen that the conclusion is the same when topography exists. But it should be noted that the range of our coordinate axes is not exactly the same. From Fig. 1, it can be seen that the amplitude is greater when the terrain exists than when it does not.

As shown in Figs. 2 and 3, let us fix M to 3 and observe how different values of k_0 affect the amplitude. Similarly, in Fig. 2, we consider the three-dimensional images of amplitude changes when the topography does not exist (subgraph (a), $a_1 = 0$) and when it exists (subgraph (b), $a_1 = 20$). As well, in Fig. 3, we consider the two-dimensional images of amplitude changes when the topography does not exist (subgraph (a), $a_1 = 0$) and when it exists (subgraph (b), $a_1 = 20$).

Here, we take the values of k_0 as 2, 4 and 6, respectively, in subgraphs (a) and (b) of Fig. 2. It is obvious that the amplitude changes more significantly when k_0 increases, which is the same as the conclusion in Fig. 1. Furthermore, the amplitude is greater when topography exists. This is also consistent with the conclusion obtained in Fig. 1. I will not say more than is needed. In addition, we take k_0 as 0, 2, 4, and 6, respectively, in subgraphs (a) and (b) of Fig. 3. It can be seen more clearly from the figure that when the time t is 0, 3, 6 and 9, the amplitude of solitary wave decreases as k_0 increases. As time goes by, the propagation speed of amplitude also increases with the increase of k_0. In addition, we have given different ranges of the coordinate axis for subgraphs (a) and (b) to better reflect the changes in amplitude. Therefore, by comparing subgraph (a) with subgraph (b), it can be found that the propagation speed of the amplitude of Rossby solitary

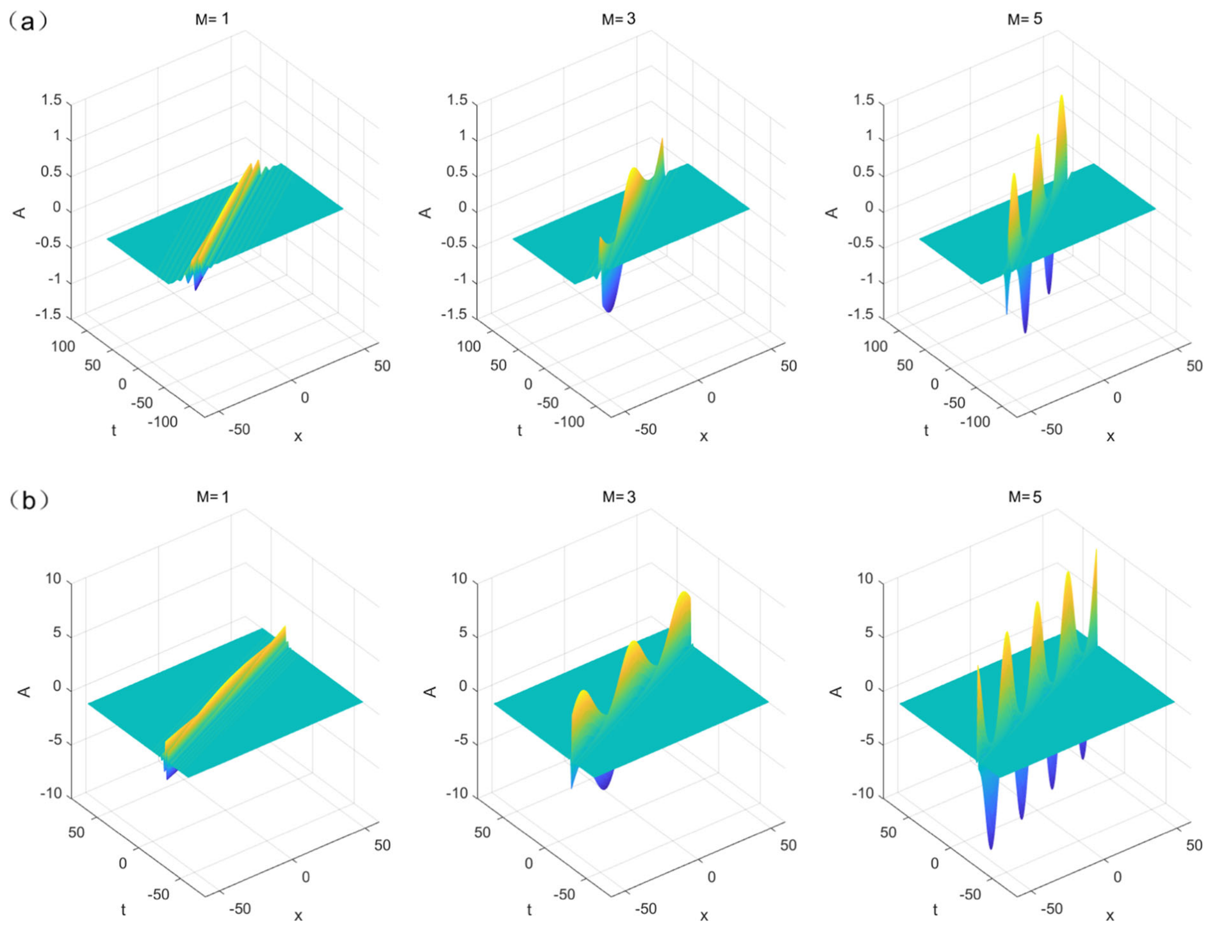


Fig. 1 The effect of M on amplitude: **a** flat bottom topography; **b** sloped topography

wave is significantly accelerated and the magnitude is significantly increased when topography exists.

To sum up, the given initial state of amplitude A will greatly affect magnitude of amplitude and propagation of solitary waves with time and space. Therefore, selecting an appropriate initial amplitude is more conducive to the formation of large amplitude and fast propagating Rossby solitary waves.

Figure 4 shows the effect of frequency on the amplitude of solitary waves. Subgraph (a) represents the situation when the topography does not exist ($a_1 = 0$). Subgraph (b) represents the situation when the topography exists ($a_1 = 20$). The frequency ω is taken as 1, 1.5, 2 and 2.5, respectively. The values of other parameters are the same as those in Fig. 1. By observing the image, it is not difficult to find that the frequency does not affect the magnitude of the amplitude of the Rossby solitary waves, but rather has a significant impact on the propagation speed. As the frequency increases, amplitude will propagate at a faster speed. Similarly, the range of the coordinate axis of subgraphs (a) and (b) in Fig. 4 is also different. By comparing subgraphs (a) and (b) at a fixed time t, it can be observed that the amplitude propagates faster in the presence of topography. Besides, the amplitude obtained is much larger. This is the same as in Fig. 3.

Similarly, based on the values of the parameters in Fig. 1, only the magnitude of the zonal wave number is changed. Figure 5 analyzes the impact of the zonal wave number on the amplitude of the Rossby wave. The situation when the topography does not exist is shown in subgraph (a). Subgraph (b) represents the situation when the topography exists. The range of the coordinate axes in subgraph (a) and subgraph (b) is different, which still needs to be noted. We can draw the following conclusions: Increasing the wave num-

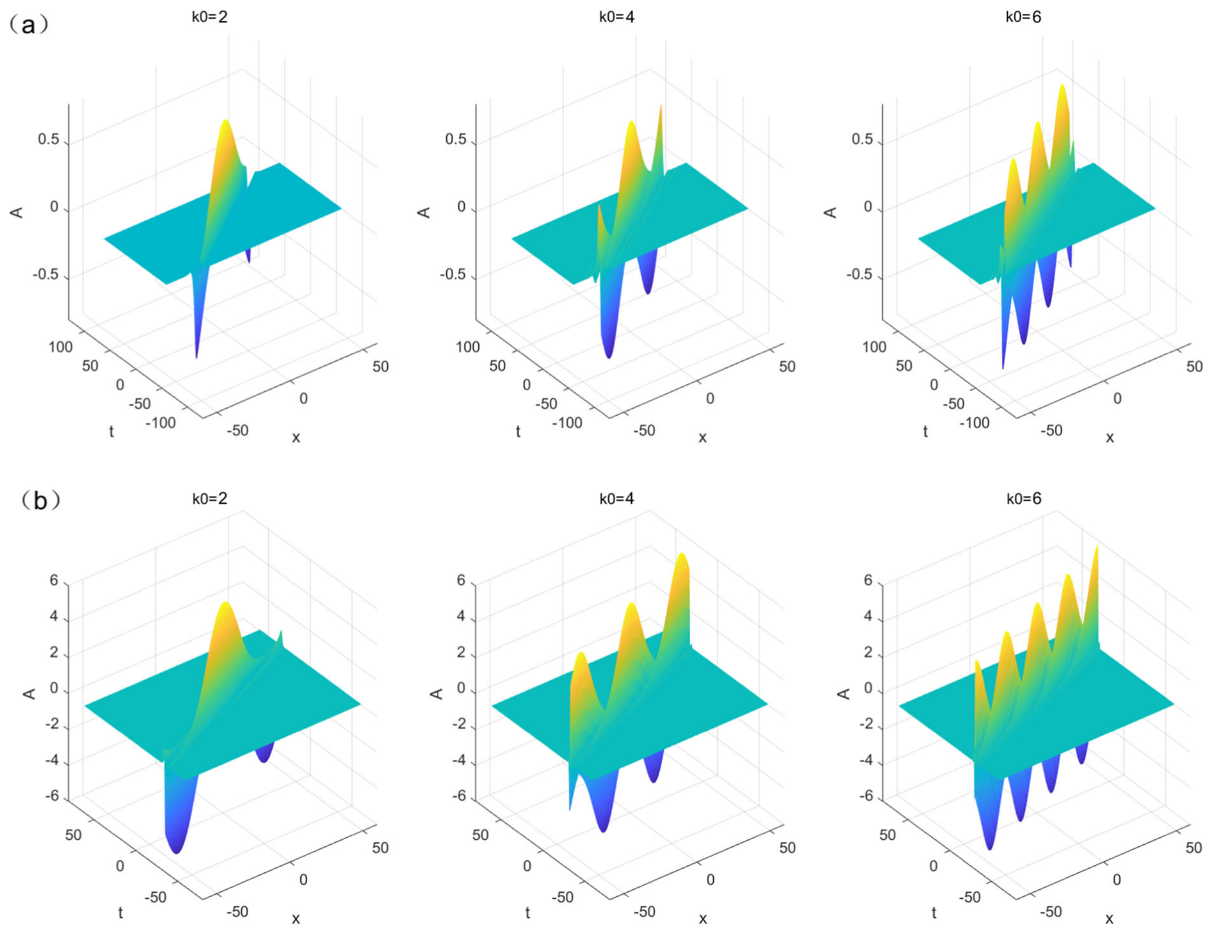


Fig. 2 The effect of k_0 on amplitude: **a** flat bottom topography; **b** sloped topography

ber will reduce the amplitude of the Rossby solitary wave and slow down the propagation speed of the solitary wave. This phenomenon is more pronounced in the presence of topography. Additionally, by comparing subgraphs (a) and (b), it can be found that at the same time, the amplitude in the presence of topography is larger and propagates further. As a consequence, the wave number is also an important factor affecting the evolution of Rossby waves. Observing graphics is more conducive to our analysis and understanding of actual physical phenomena. Therefore, as shown in Fig. 6, we analyze the propagation of nonlinear Rossby solitary waves through the evolution of the total flow field. The absence of topography and the presence of topography are shown in subgraphs (a) and (b), respectively.

M is 3, k_0 is 4, and the other parameters are the same as the previous. The evolution of the total flow function is shown in Fig. 6. It is obvious that according to the current selection of parameter values, there is no dipole generation in the absence of topography. After adding topography factors, multiple pairs of dipoles were generated. This indicates that topography may be an important factor in the generation of dipole blocking. Subgraph (b) of Fig. 6 clearly shows the evolution of dipole blocking. Therefore, in the presence of topography, the impact of basic flow shear ν on dipole blocking is analyzed.

In the previous calculation process, we have assumed that there is weak shear in the background westerly flow. That is, $\bar{u} = \bar{u}_0 + \nu \cos(my)$. So, we can change the magnitude of weak shear by changing the magnitude of ν. The evolution of dipole blocking under weak shear is shown in Fig. 7 by taking different values of ν. Subgraphs (a–d) in Fig. 7 correspond to the evolution of dipole blocking when ν is 0.4, 0.3, 0.2 and 0.1, respectively. Figure 7 shows that shear only affects the

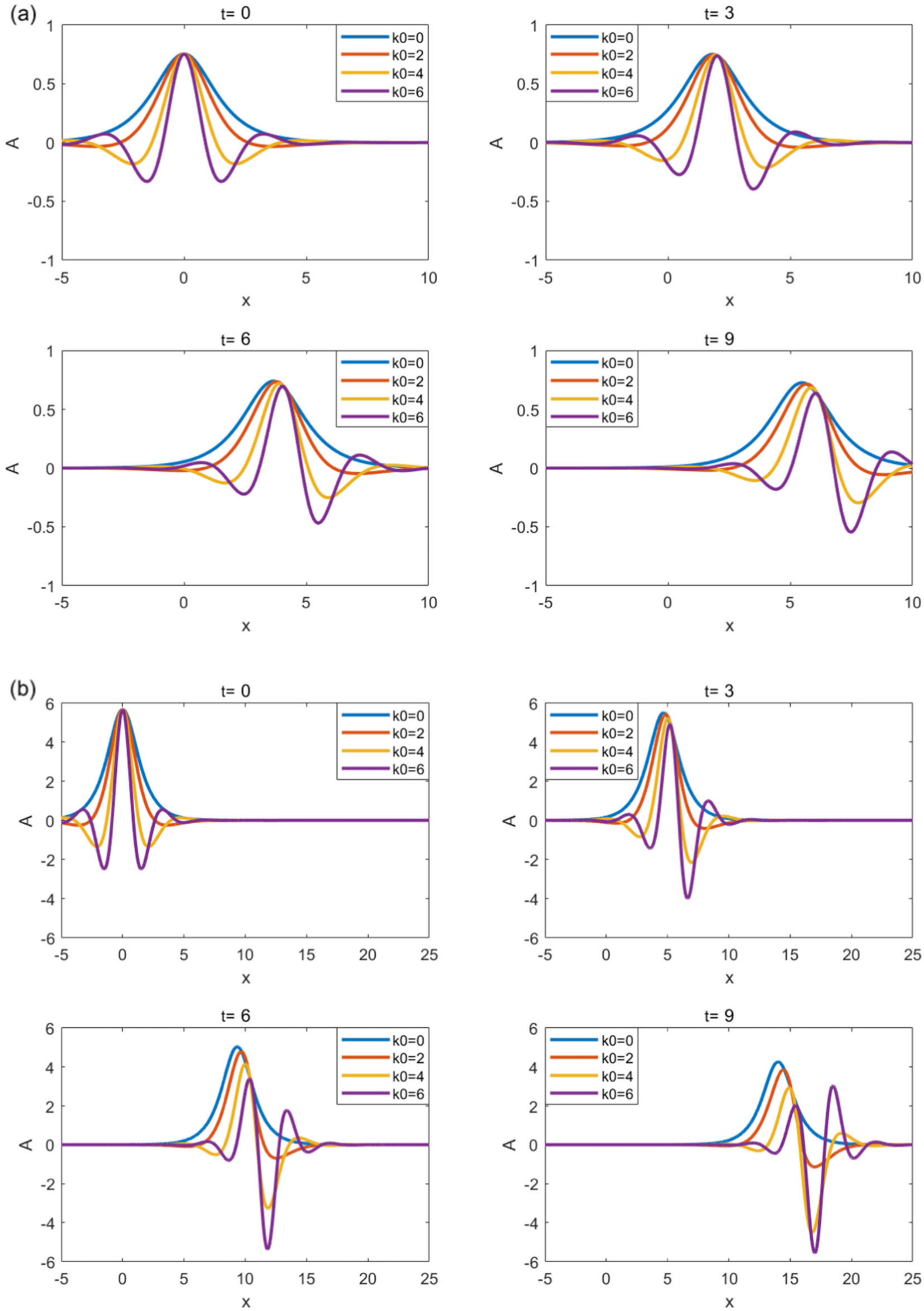


Fig. 3 The effect of different values of k_0 on amplitude (two-dimensional diagram): **a** flat bottom topography; **b** sloped topography

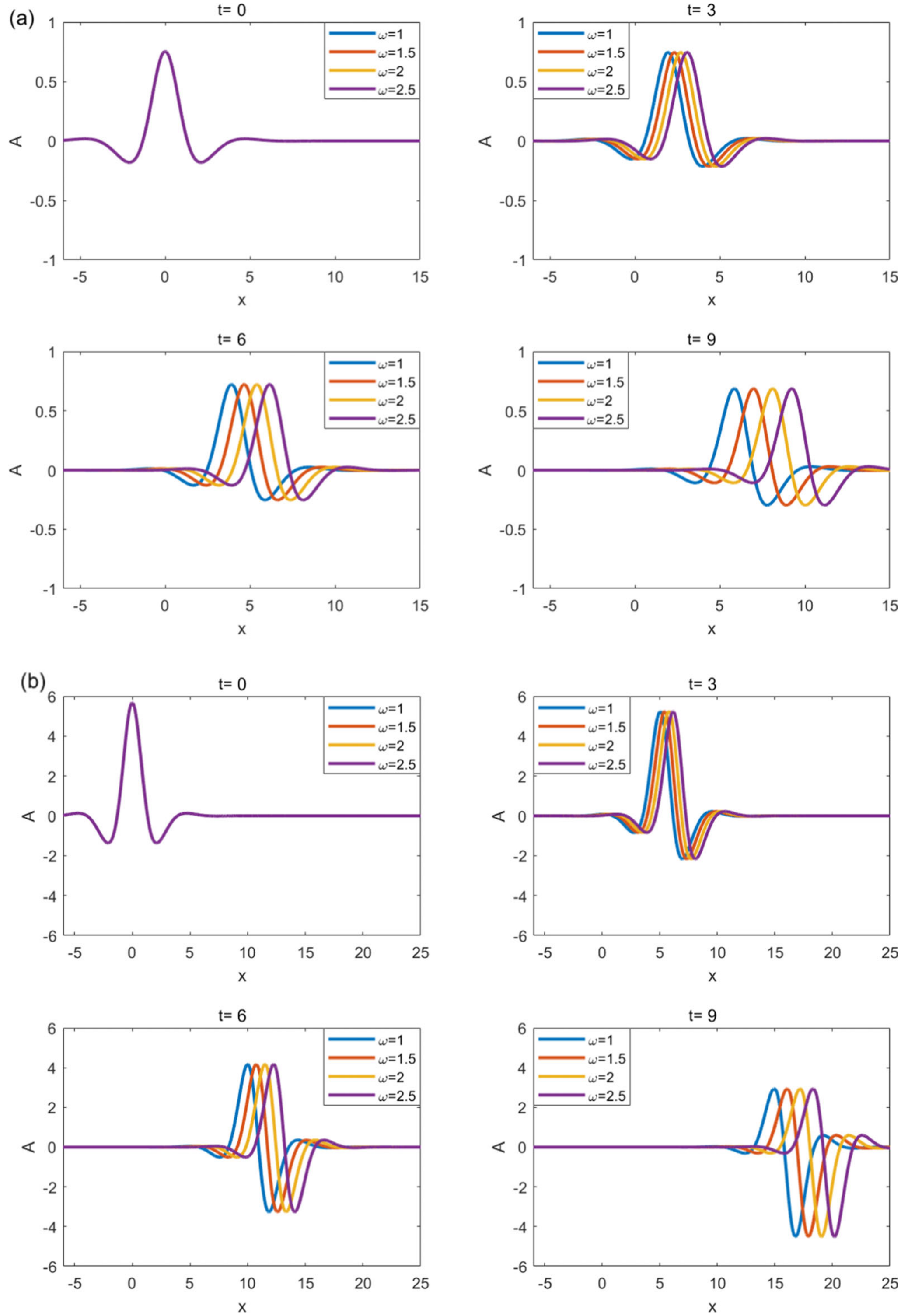


Fig. 4 The effect of frequency on the amplitude of solitary waves: **a** flat bottom topography; **b** sloped topography

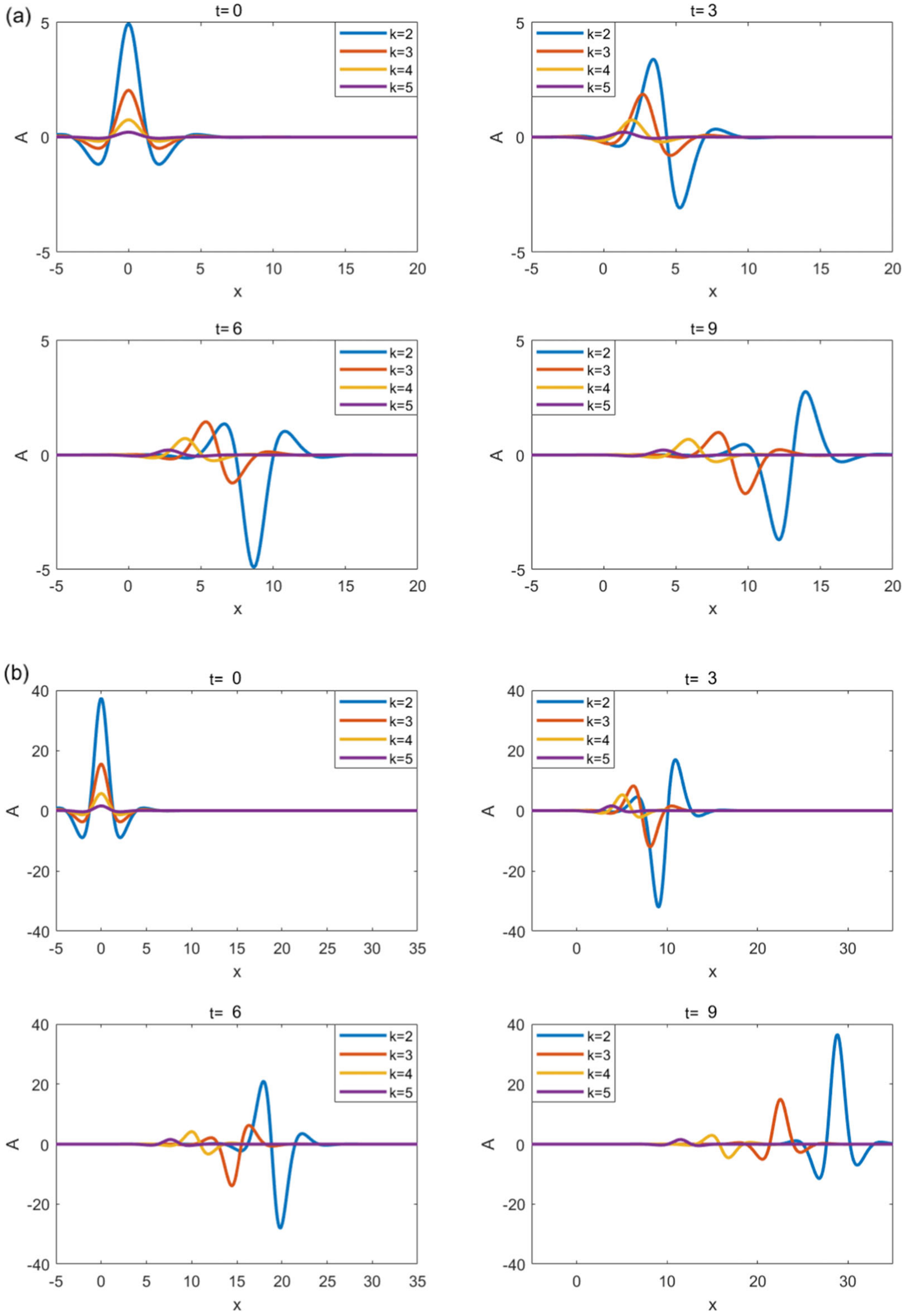


Fig. 5 The effect of zonal wave numbers on the amplitude of solitary waves: **a** flat bottom topography; **b** sloped topography

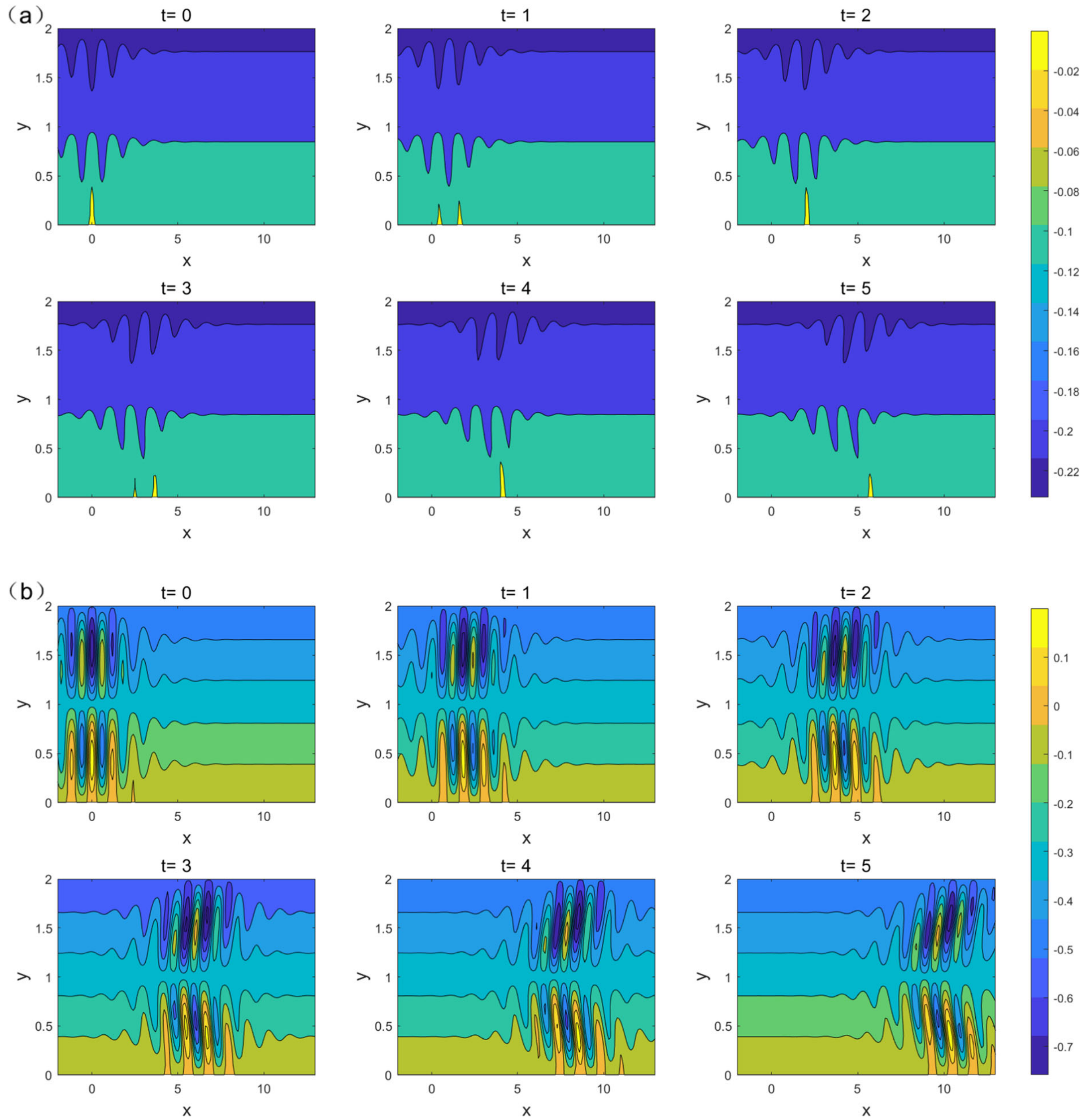


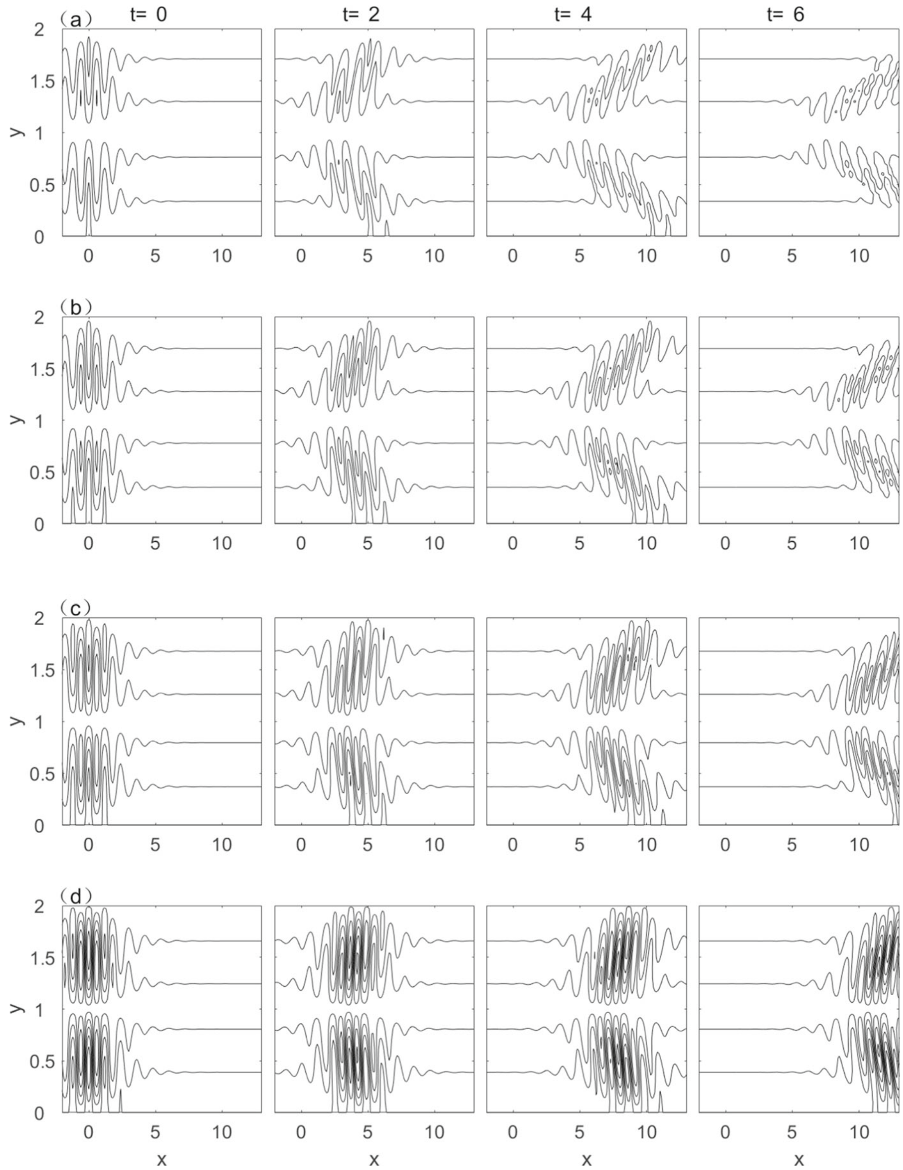
Fig. 6 Evolution of total flow field under the given parameters: **a** flat bottom topography; **b** sloped topography

strength of dipole blocking and does not affect its moving speed. The weaker the shear, the stronger the dipole blocking.

4 Conclusions

The traditional method for deriving the nonlinear Schrödinger equations for the evolution of nonlinear Rossby wave amplitude is the perturbation method or the multiple-scale method. In this paper, we use the derivative expansion method to derive the Schrödinger equation to describe the evolution of Rossby solitary waves. In the bargain, the perturbation method is used to solve the variable coefficient boundary value problem obtained during the derivation process. Based on the solution of the Schrödinger equation, a detailed

Fig. 7 Evolution of dipole blocking under different weak shear. **a** $\nu = 0.4$; **b** $\nu = 0.3$; **c** $\nu = 0.2$; **d** $\nu = 0.1$.



numerical analysis was conducted. What is more, the effects of various physical factors on solitary waves were discussed. The following conclusions are drawn:

Through exploring the amplitude of solitary waves, it is found that Rossby solitary waves are affected by initial amplitude, frequency and zonal wave number in time and space. When the initial amplitude is properly selected, a larger frequency and a smaller wave number are more conducive to the generation of large amplitude Rossby solitary waves. Moreover, compared with the situation without topography, the amplitude of Rossby solitary waves is larger and the propagation speed of solitary wave is faster when topography exists. In the end, the evolution of dipole blocking is analyzed using contour plots of the flow field. It is found that topography also has a certain impact on the generation of blocking. In addition, the impact of shear on blocking was also emphasized. A conclusion is drawn that weak shear is beneficial to the generation of dipole blocking. This is consistent with previous research results by other scholars.

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Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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