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ORIGINAL PAPER

On a generalized Broer-Kaup-Kupershmidt system for the long waves in shallow water

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Abstract Describing the long waves in shallow water, a generalized Broer-Kaup-Kupershmidt system is investigated in this paper. With respect to the horizontal velocity of the water wave and the height of the water surface, we use symbolic computation to build up (A) a scaling transformation, (B) a set of the hetero-Bäcklund transformations, from that generalized system to a known linear partial differential equation, as well as (C) two sets of the similarity reductions, each of which from that generalized system to a known ordinary differential equation. Our results depend on all the shallow-water coefficients for that generalized system.

Keywords Shallow water · Long waves · Generalized Broer-Kaup-Kupershmidt system · Scaling transformation · Hetero-Bäcklund transformations · Similarity reductions · Symbolic computation

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1 Introduction

Being attractive, many researchers are now paying attention to the shallow water waves [1–15]. For example, Ref. [1], a recent paper in Nonlinear Dyn., has presented some multi-soliton solutions of a generalized Broer-Kaup system for the shallow water waves.

In this paper, we plan to investigate a generalized Broer-Kaup-Kupershmidt system, which describes, e.g., the long waves in shallow water, i.e.,

$$u_t = \left(-\alpha u_x + \frac{\beta}{2} u^2 + \beta v\right)_x + \gamma u_x \quad , \tag{1a}$$

$$v_t = (\alpha v_x + \beta u v)_x + \gamma v_x \quad , \tag{1b}$$

with the real differentiable functions $v(x,t)$ and $u(x,t)$ denoting, e.g., the horizontal velocity of the water wave and the height of the water surface, respectively, x and t implying, e.g., the scaled space and time variables, α, $\beta \neq 0$ and γ as the real constants, while the subscripts being the partial derivatives. Shallow-water special cases of System (1) have been seen:

- for the long waves in the shallow water, a Broer-Kaup-Kupershmidt system,

$$u_t - u_{xx} + 2v_x + 2uu_x = 0 \quad , \tag{2a}$$

$$v_t + v_{xx} + 2\,(uv)_x = 0 \quad , \tag{2b}$$

when $\gamma = 0$, $\beta = -2$ and $\alpha = -1$, with t and x being the scaled time and space variables, $v(x,t)$

representing the horizontal velocity of the water wave while $u(x,t)$ meaning the height of the water surface [15];

- for the diffusion-involved shallow water waves, a generalized Broer-Kaup system,

$$u_t = \left(-\alpha u_x + \frac{1}{2}u^2 + v\right)_x \quad , \tag{3a}$$

$$v_t = (\alpha v_x + uv)_x \quad , \tag{3b}$$

when $\gamma = 0$ and $\beta = 1$, with t meaning the time variable, x indicating the propagation direction, $v(x,t)$ relevant to both the wave profile and the tangential fluid velocity at the surface while $u(x,t)$ representing the tangential fluid velocity at the surface [16–18];

- for the long waves in the shallow water, a classical dispersiveless long-wave system, i.e.,

$$u_t + uu_x + v_x = 0 \quad , \tag{4a}$$

$$v_t + (uv)_x = 0 \quad , \tag{4b}$$

when $\alpha = \gamma = 0$ and $\beta = -1$, with $u(x,t)$ meaning the tangential fluid velocity at the surface, $v(x,t)$ representing the wave profile, x denoting the propagation direction, while t being the time variable [17,18] (and references therein);

- for the long waves in the shallow water, a Broer-Kaup system, i.e.,

$$u_t + \frac{1}{2}u_{xx} - v_x - uu_x = 0 \quad , \tag{5a}$$

$$v_t - \frac{1}{2}v_{xx} - (uv)_x = 0 \quad , \tag{5b}$$

when $\alpha = \frac{1}{2}$, $\beta = 1$ and $\gamma = 0$, with $u(x,t)$ standing for the scaled wave horizontal velocity, while $v(x,t)$ related to the wave height and wave horizontal velocity [19,20] (and references therein);

- for the dispersion water waves in the shallow water, a generalized Broer-Kaup system [21] (and references therein), i.e.,

$$u_t = -\alpha\left(u_x - u^2 - 2v\right)_x + \gamma u_x \quad , \tag{6a}$$

$$v_t = \alpha\,(v_x + 2uv)_x + \gamma v_x \quad , \tag{6b}$$

when $\beta = 2\alpha$;

- for the dispersion water waves in the shallow water, a Broer-Kaup system [21] (and references therein), i.e.,

$$u_t = \left(-u_x + u^2 + 2v\right)_x \quad , \tag{7a}$$

$$v_t = (v_x + 2uv)_x \quad , \tag{7b}$$

when $\alpha = 1$, $\beta = 2$ and $\gamma = 0$.

However, to our knowledge, for System (1), there have been no scaling-transformation work, Bäcklund-transformation work and similarity-reduction work published as yet. Hereby, for System (1), we employ symbolic computation [22–28], to construct a scaling transformation, a set of the hetero-Bäcklund transformations and two sets of the similarity reductions.

2 Scaling and hetero-Bäcklund transformations for System (1)

Similar to those in Refs. [29,30], we work out a scaling transformation:

$$\begin{aligned}&\alpha \to \rho^0\alpha\,, \quad \beta \to \rho^0\beta\,, \quad \gamma \to \rho^{-1}\gamma\,, \quad x \to \rho^1 x\,,\\ &t \to \rho^2 t\,, \quad u \to \rho^{-1}u\,, \quad v \to \rho^{-2}v\,,\end{aligned} \tag{8}$$

in which ρ stands for a positive constant.

Next, on the score of Scaling Transformation (8), assuming that[1]

$$\begin{aligned}u(x,t) &= \zeta_1 w_x(x,t) + \zeta_2,\\ v(x,t) &= \zeta_3 w_{xx}(x,t),\end{aligned} \tag{9}$$

making use of symbolic computation, integrating Eq. (1a) once with respect to x with the integration function equal to zero and making a choice of

$$\beta\zeta_3 - \alpha\zeta_1 = \frac{\beta}{2}\zeta_1^2 \quad , \tag{10}$$

we obtain

$$Y_t(w) - \frac{\beta}{2}\zeta_1 Y_{2x}(w) - (\beta\zeta_2 + \gamma)\,Y_x(w) = 0\ , \tag{11}$$

with $\zeta_1 \neq 0$, ζ_2 and $\zeta_3 \neq 0$ as three real constants, the Bell polynomials reported by Refs. [31,32] as

[1] similar to those in Refs. [29,30]

$$
\begin{aligned}
&Y_{mx,nt}(w)\\
&\quad\equiv Y_{m,n}(w_{0,0},\cdots,w_{0,n},\cdots,w_{m,0},\cdots,w_{m,n})\\
&\quad= e^{-w}\partial_x^m\partial_t^n e^w,
\end{aligned}
$$

$w(x,t)$ meaning a C^∞ function of x and t, $w_{k,l} = \partial_x^k\partial_t^l w$, $k=0,\cdots,m$, $l=0,\cdots,n$, while m and n representing two non-negative integers.

Similarly, we use symbolic computation and Assumptions (9), integrate Eq. (1b) twice in relation to x with the integration functions equal to zero and choose

$$\alpha = \frac{\beta}{2}\zeta_1\ , \tag{12}$$

so as to find

$$Y_t(w) - \alpha Y_{2x}(w) - (\beta\zeta_2+\gamma)\,Y_x(w) = 0\ , \tag{13}$$

which is the same as Expression (11).

Then, the assumption,

$$w(x,t) = \ln\left[h(x,t)\right]\ , \tag{14}$$

helps us simplify System (1) into a linear partial differential equation, i.e.,

$$h_t(x,t) - \alpha h_{xx}(x,t) - (\beta\zeta_2+\gamma)\,h_x(x,t) = 0, \tag{15}$$

with $h(x,t)$ representing a positive differentiable function.

Thinking about all the above together, under the constraint

$$\alpha \neq 0\ , \tag{16}$$

for System (1), we construct one set of the hetero-Bäcklund transformations, i.e.,

$$u(x,t) = \frac{2\alpha}{\beta}\frac{h_x(x,t)}{h(x,t)} + \zeta_2\ , \tag{17a}$$

$$v(x,t) = \frac{4\alpha^2}{\beta^2}\left[\frac{h_{xx}(x,t)}{h(x,t)} - \frac{h_x(x,t)^2}{h(x,t)^2}\right]\ , \tag{17b}$$

$$h_t(x,t) - \alpha h_{xx}(x,t) - (\beta\zeta_2+\gamma)\,h_x(x,t) = 0\ . \tag{17c}$$

We note that

- Eqs. (17) are one set of the hetero-Bäcklund transformations, which could couple the solutions $h(x,t)$ of Eq. (17c) and the solutions $u(x,t)$ and $v(x,t)$ of System (1);
- Eq. (17c) is an already-investigated linear partial differential equation, as seen in Refs. [33,34];
- Hetero-Bäcklund Transformations (17) are related to γ, β and α, the shallow-water coefficients for System (1), under Constraint (16);
- the above work is on the long waves in the shallow water, concerning the wave profile and tangential fluid velocity at the surface.

3 Similarity reductions for System (1)

Making use of symbolic computation and substituting the assumptions,[2]

$$u(x,t) = \theta(x,t) + \omega(x,t)p[z(x,t)]\ , \tag{18a}$$

$$v(x,t) = \delta(x,t) + \kappa(x,t)q[z(x,t)]\ , \tag{18b}$$

into System (1) result in

$$
\begin{aligned}
&p''\alpha\omega z_x^2 - pp'\beta\omega^2 z_x - p^2\beta\omega\omega_x - q'\beta\kappa z_x - q\beta\kappa_x\\
&+ p'\left[\omega z_t + \alpha\left(2\omega_x z_x + \omega z_{xx}\right) - \gamma\omega z_x - \beta\omega\theta z_x\right]\\
&+ p\left[\omega_t + \alpha\omega_{xx} - \beta\left(\omega\theta_x + \omega_x\theta\right) - \gamma\omega_x\right]\\
&+ \theta_t + \alpha\theta_{xx} - \beta\delta_x - \beta\theta\theta_x - \gamma\theta_x = 0\ ,
\end{aligned} \tag{19a}
$$

$$
\begin{aligned}
&- q''\alpha\kappa z_x^2 - \left(p'q + pq'\right)\beta\omega\kappa z_x - pq\beta\left(\omega\kappa_x + \omega_x\kappa\right)\\
&+ q'\left[\kappa z_t - \alpha\left(2\kappa_x z_x + \kappa z_{xx}\right) - \gamma\kappa z_x - \beta\theta\kappa z_x\right]\\
&+ q\left[\kappa_t - \alpha\kappa_{xx} - \beta\left(\theta\kappa_x + \theta_x\kappa\right) - \gamma\kappa_x\right]\\
&- p'\beta\delta\omega z_x - p\beta\left(\delta_x\omega + \delta\omega_x\right) + \delta_t - \alpha\delta_{xx}\\
&- \beta\left(\theta\delta_x + \theta_x\delta\right) - \gamma\delta_x = 0\ ,
\end{aligned} \tag{19b}
$$

with $z(x,t) \neq 0$, $\kappa(x,t) \neq 0$, $\delta(x,t)$, $\omega(x,t) \neq 0$ and $\theta(x,t)$ meaning the to-be-determined differentiable real functions with $z_x z_t \neq 0$, the prime sign standing the differentiation on z, while $q(z)$ and $p(z)$ denoting for two differentiable non-zero real functions of z.

Considering Eqs. (19) as a couple of the ordinary differential equations (ODEs) concerning $q(z)$ and $p(z)$, one might require the ratios of the coefficients of different derivatives/powers of $q(z)$ and $p(z)$ to merely represent certain functions of z.

The coefficients of p'' in Eq. (19a) and q'' in Eq. (19b), as the normalizing coefficients in Eqs. (19), respectively, come to

$$\Omega_1(z)\alpha\omega z_x^2 = -\beta\omega^2 z_x\ , \tag{20a}$$

$$\Omega_2(z)\alpha\omega z_x^2 = -\beta\omega\omega_x\ , \tag{20b}$$

$$\Omega_3(z)\alpha\omega z_x^2 = -\beta\kappa z_x\ , \tag{20c}$$

$$\Omega_4(z)\alpha\omega z_x^2 = -\beta\kappa_x\ , \tag{20d}$$

[2] similar to those in Refs. [35–41]

$$\Omega_5(z)\alpha\omega z_x^2 = \omega z_t + \alpha\left(2\omega_x z_x + \omega z_{xx}\right) - \gamma\omega z_x - \beta\omega\theta z_x \ , \tag{20e}$$

$$\Omega_6(z)\alpha\omega z_x^2 = \omega_t + \alpha\omega_{xx} - \beta\left(\omega\theta_x + \omega_x\theta\right) - \gamma\omega_x \ , \tag{20f}$$

$$\Omega_7(z)\alpha\omega z_x^2 = \theta_t + \alpha\theta_{xx} - \beta\delta_x - \beta\theta\theta_x - \gamma\theta_x \ , \tag{20g}$$

and

$$-\Gamma_1(z)\alpha\kappa z_x^2 = -\beta\omega\kappa z_x \ , \tag{21a}$$

$$-\Gamma_2(z)\alpha\kappa z_x^2 = -\beta\left(\omega\kappa_x + \omega_x\kappa\right) \ , \tag{21b}$$

$$-\Gamma_3(z)\alpha\kappa z_x^2 = -\beta\delta\omega z_x \ , \tag{21c}$$

$$-\Gamma_4(z)\alpha\kappa z_x^2 = \kappa z_t - \alpha\left(2\kappa_x z_x + \kappa z_{xx}\right) - \gamma\kappa z_x - \beta\theta\kappa z_x \ , \tag{21d}$$

$$-\Gamma_5(z)\alpha\kappa z_x^2 = -\beta\left(\delta_x\omega + \delta\omega_x\right) \ , \tag{21e}$$

$$-\Gamma_6(z)\alpha\kappa z_x^2 = \kappa_t - \alpha\kappa_{xx} - \beta\left(\theta\kappa_x + \theta_x\kappa\right) - \gamma\kappa_x \ , \tag{21f}$$

$$-\Gamma_7(z)\alpha\kappa z_x^2 = \delta_t - \alpha\delta_{xx} - \beta\left(\theta\delta_x + \theta_x\delta\right) - \gamma\delta_x \ , \tag{21g}$$

with $\Omega_i(z)$'s (i=1, $\cdots$, 7) and $\Gamma_j(z)$'s (j=1, $\cdots$, 7) meaning certain real to-be-determined functions of z only.

On account of the remarks in Ref. [35], each set of $\theta(x,t)$, $\omega(x,t)$, $\delta(x,t)$, $\kappa(x,t)$ and $z(x,t)$ can turn to, at least, a similarity reduction.

Ground on the second freedom in Remark 3 in Ref. [35],[3] Eqs. (20a), (20c) and (21a) give rise to

$$\omega(x,t) = \pm\frac{\alpha}{\beta}z_x, \quad \kappa(x,t) = \mp\frac{\alpha^2}{\beta^2}z_x^2,$$
$$\Omega_1(z) = \mp 1, \quad \Omega_3(z) = 1 \ , \quad \Gamma_1(z) = \pm 1 \ , \tag{22}$$

according to the first freedom in Remark 3 in Ref. [35], Eq. (20b) indicates

$$z(x,t) = \lambda_1 x + \lambda_2 t + \lambda_3 \ , \qquad \Omega_2(z) = 0 \ , \tag{23}$$

and then Eqs. (20d) and (21b) bring about

$$\Omega_4(z) = \Gamma_2(z) = 0 \ , \tag{24}$$

in which $\lambda_1 \neq 0$, $\lambda_2 \neq 0$ and λ_3 mean the real constants.

Because the first freedom in Remark 3 in Ref. [35] makes us transform Eqs. (21c) and (21d) into

$$\delta(x,t) = 0 \ , \quad \theta(x,t) = \frac{\lambda_2 - \gamma\lambda_1}{\beta\lambda_1} \ ,$$
$$\Gamma_3(z) = \Gamma_4(z) = 0 \ , \tag{25}$$

[3] There exist three freedoms in Remark 3, as seen in Ref. [35].

Eqs. (20e),(20f), (20g), (21e), (21f) and (21g) come to

$$\Omega_6(z){=}\Omega_5(z){=}\Omega_7(z){=}\Gamma_5(z){=}\Gamma_6(z){=}\Gamma_7(z){=}0 \ . \tag{26}$$

Since System (1) can be simplified to the following ODEs:

$$p'' \mp pp' + q' = 0 \ , \tag{27a}$$

$$q'' \pm \left(p'q + pq'\right) = 0 \ , \tag{27b}$$

integrating ODEs (27) once with respect to z, separately, we are capable of transforming ODEs (27) into a single ODE, i.e.,

$$p'' - \frac{1}{2}p^3 \mp \phi_1 p + \phi_2 = 0 \ , \tag{28}$$

based on

$$q = -p' \pm \frac{1}{2}p^2 + \phi_1 \ , \tag{29}$$

with ϕ_1 and ϕ_2 representing two real constants of integration.

Taking into consideration all the above in this section, under Constraint (16), we build up the following two sets of the similarity reductions for System (1):

$$u(x,t) = \frac{\lambda_2 - \gamma\lambda_1}{\beta\lambda_1} \pm \frac{\alpha}{\beta}\lambda_1 p[z(x,t)] \ , \tag{30a}$$

$$v(x,t) {=} \pm\frac{\alpha^2}{\beta^2}\lambda_1^2\left\{p'[z(x,t)] \mp \frac{1}{2}p[z(x,t)]^2 - \phi_1\right\} , \tag{30b}$$

$$z(x,t) = \lambda_1 x + \lambda_2 t + \lambda_3 \ , \tag{30c}$$

$$p'' - \frac{1}{2}p^3 \mp \phi_1 p + \phi_2 = 0 \ . \tag{30d}$$

Each of ODEs (30d), as a known ODE, has been presented in Refs. [42,43]. The reason for the appearance of two sets of Similarity Reductions (30) is that the "±" signs present.

As for the long waves in the shallow water, with respect to the wave profile and tangential fluid velocity at the surface, under Constraint (16), Similarity Reductions (30) are dependent on α, β and γ, the shallow-water coefficients for System (1).

4 Conclusions

People are studying hard the shallow water waves. For example, a recent paper in Nonlinear Dyn., Ref. [1], has presented a generalized Broer-Kaup system for the shallow water waves.

In this paper, we have investigated System (1), a generalized Broer-Kaup-Kupershmidt system for the long waves in shallow water. Concerning the horizontal velocity of the water wave as well as the height of the water surface, we have symbolically computed[4] the following:

- Scaling Transformation (8) and Hetero-Bäcklund Transformations (17), from System (1) to an already-investigated linear partial differential equation, under Constraint (16);
- Similarity Reductions (30), each of which from System (1) to a known ODE, under Constraint (16).

We have found the reason for the appearance of two sets of Similarity Reductions (30), i.e., the presence of the "$\pm$" signs.

Both Hetero-Bäcklund Transformations (17) and Similarity Reductions (30) have been seen to be dependent on α, β and γ, the shallow-water coefficients for System (1).

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Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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